The Journal of Fourier Analysis and Applications

Volume 3, Special Issue, 1997

A Characterization of Functions that Generate Wavelet and Related Expansion

Michael Frazier, Gustavo Garrigós, Kunchuan Wang, and Guido Weiss

1. Introduction

An *orthonormal wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that $\{ \psi_{j,k} \mid j \in \mathbb{Z}, \;\; k \in \mathbb{Z} \}$ is an orthonormal basis for $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) \equiv 2^{j/2}\psi\left(2^j x - k\right), \quad j, k \in \mathbb{Z} .$$

In this case we have the equality

$$f = \sum_{j,k\in\mathbb{Z}} (f, \psi_{j,k})\psi_{j,k} \tag{1.1}$$

for each $f \in L^2(\mathbb{R})$, where the series converges in the $L^2(\mathbb{R})$-norm and $(g, h) = \int_{\mathbb{R}} g\bar{h}$. There are several extensions of these notions that have commanded considerable interest during the past decade. If $\mathbb{R}$ is replaced by $\mathbb{R}^n$, n-dimensional Euclidean space, the definition of the family $\{ \psi_{j,\mathbf{k}} \mid j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^n \}$ is

$$\psi_{j,\mathbf{k}}(\mathbf{x}) \equiv 2^{jn/2}\psi\left(2^j\mathbf{x} - \mathbf{k}\right) , \tag{1.2}$$

where $j \in \mathbb{Z}$ and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. This is clearly a natural extension of the 1-dimensional case since the dilations in (1.2) preserve the $L^2(\mathbb{R}^n)$-norm of ψ and the translations must involve points in $\mathbb{R}^n$. The function ψ is then said to be an orthonormal wavelet in $L^2(\mathbb{R}^n)$ if the family defined by (1.2) is an orthonormal basis for this space. It is well-known that many different kinds of wavelets exist in $L^2(\mathbb{R})$. In the higher dimensional case, however, the situation is more complicated. If one imposes some "relatively mild" conditions of smoothness and decrease at infinity on the Fourier transform

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(\mathbf{x})e^{-i\mathbf{x}\cdot\xi}\, d\mathbf{x} \qquad \xi \in \mathbb{R}^n , \tag{1.3}$$

it can be shown that a single function $\psi \in L^2(\mathbb{R}^n)$ cannot generate an orthonormal basis by forming the family defined in (1.2) when $n \geq 2$. In this case one needs at least $L = 2^n - 1$ such generating

Math Subject Classifications: 42, 46

Keywords and Phrases: Wavelets, φ-transform, ψ-transform

Gustavo Garrigós: Research partially supported by a grant from the Southwestern Bell Telephone Company.

Guido Weiss: Research supported by AFOSR Grant F 49620-95-1-0.

ISSN 1069-5869

functions $\psi^1, \psi^2, \dots, \psi^L \in L^2(\mathbb{R}^n)$ (see [1, p. 215], [7, p. 339], [9, p. 93]). We encounter, therefore, orthonormal bases of the form $\{\psi^\ell_{j,\mathbf{k}}\}$, where $\ell = 1, 2, \dots, L$, $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$ with $L \geq 1$. Such bases produce expansions of the form

$$f = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \psi^\ell_{j,\mathbf{k}}\right) \psi^\ell_{j,\mathbf{k}} \tag{1.4}$$

for any $f \in L^2(\mathbb{R}^n)$, where the multiple series converges in the norm of $L^2(\mathbb{R}^n)$. Recently, however, several investigators have constructed examples of (single) wavelets in $L^2(\mathbb{R}^n)$ (see [3, 11]). Thus, the case $L = 1$ in (1.4) is realized in some cases. In general, when $\{\psi^\ell_{j,\mathbf{k}}\}$ is an orthonormal basis, we call the family $\Psi = \{\psi^1, \dots, \psi^L\}$ a *family of orthonormal wavelets*.

More general representations of functions in $L^2(\mathbb{R}^n)$, sharing the same dyadic dilation and translation structure with these expansions, have been studied and effectively applied. Frazier and Jawerth (see [4] and [5]) introduced expansions of the form

$$f = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \phi_{j,\mathbf{k}}\right) \psi_{j,\mathbf{k}} \tag{1.5}$$

where $\phi_{j,\mathbf{k}}$, $\psi_{j,\mathbf{k}}$ are defined by equality (1.2) and ϕ and ψ is an appropriate pair of functions in $L^2(\mathbb{R}^n)$. The convergence of the series in (1.5) is, again, in the $L^2(\mathbb{R}^n)$-norm. More general function spaces and different types of convergence are of great interest; however, in this article we limit our attention to the $L^2(\mathbb{R}^n)$ case.

A feature of expansions of the form given by (1.5) is that one of the functions, ϕ, provides a system of "analyzing" functions; that is, the needed information about f is obtained by calculating the inner products $(f, \phi_{j,\mathbf{k}})$. The other function, ψ, provides a "synthesizing" system which enables us to reconstruct f from this information via the series in (1.5). We observe that this situation has much in common with expansions involving frames and those associated with bi-orthogonal wavelets.

The purpose of this work is to characterize those pair of families $\Phi = \{\phi^1, \dots, \phi^L\}$ and $\Psi = \{\psi^1, \dots, \psi^L\}$ in $L^2(\mathbb{R}^n)$ having the property that for each $f \in L^2(\mathbb{R}^n)$

$$f = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \phi^\ell_{j,\mathbf{k}}\right) \psi^\ell_{j,\mathbf{k}}, \tag{1.6}$$

where the convergence of the series on the right is in the norm of $L^2(\mathbb{R}^n)$ or in the weak sense. We will show that equality (1.6) (or a variant of this representation of f) is, in a sense, equivalent to the fact that the Fourier transforms $\{\hat\phi^1, \dots, \hat\phi^L\}$, $\{\hat\psi^1, \dots, \hat\psi^L\}$ satisfy the following two equations:

$$\left.\begin{array}{lll}
(i) & \displaystyle\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \hat\phi^\ell\left(2^j\xi\right)\overline{\hat\psi^\ell\left(2^j\xi\right)} = 1\,, & a.e.\ \xi \in \mathbb{R}^n \\
(ii) & t_{\mathbf{q}}(\xi) \equiv \displaystyle\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \hat\phi^\ell\left(2^j\xi\right)\overline{\hat\psi^\ell\left(2^j(\xi + 2\pi\mathbf{q})\right)} = 0\,, & a.e.\ \xi \in \mathbb{R}^n,\ \forall \mathbf{q} \in \mathcal{O}^n
\end{array}\right\} \tag{1.7}$$

where $\mathcal{O}^n = \mathbb{Z}^n \setminus (2\mathbb{Z})^n = \{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n :$ at least one component k_j is odd$\}$.

We do encounter the problem of determining the meaning of the series in these two equations. It is easy to see that $t_{\mathbf{q}}$ is a well-defined function in $L^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \left|t_{\mathbf{q}}(\xi)\right| d\xi \leq \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \left|\hat\phi^\ell(2^j\xi)\right| \left|\hat\psi^\ell\left(2^j(\xi + 2\pi\mathbf{q})\right)\right| d\xi =$$

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\left|\hat{\phi}^\ell(\eta)\right|\left|\hat{\psi}^\ell\left(\eta+2^{j+1}\pi\mathbf{q}\right)\right| d\eta \le \left(\sum_{\ell=1}^{L}\left\|\hat{\phi}^\ell\right\|_2\left\|\hat{\psi}^\ell\right\|_2\right)\left(\sum_{j=0}^{\infty}2^{-jn}\right)$$
$$=\frac{2^{2n}\pi^n}{2^n-1}\left(\sum_{\ell=1}^{L}\left\|\phi^\ell\right\|_2\left\|\psi^\ell\right\|_2\right)<\infty .$$

On the other hand, the convergence of the series in (1.7)(i) requires a further assumption about the two families Φ and Ψ. In order to explain this matter better, we first consider the case when $\Phi = \Psi$. In this case, the two equations (1.7)(i) and (ii) have the form

$$\left.\begin{array}{lll}(i) & \displaystyle\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^\ell\left(2^j\xi\right)\right|^2=1\,, & a.e.\ \xi\in\mathbb{R}^n \\ (ii) & \displaystyle t_{\mathbf{q}}(\xi)\equiv\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}^\ell\left(2^j\xi\right)\overline{\hat{\psi}^\ell\left(2^j(\xi+2\pi\mathbf{q})\right)}=0\,, & a.e.\ \xi\in\mathbb{R}^n,\ \forall\mathbf{q}\in\mathcal{O}^n\end{array}\right\}\qquad(1.8)$$

Since all the summands are non-negative, the convergence of the series in equation (i) is well-defined (if we include the possibility that the sum is infinity for some of the ξ). Thus, in this case the meaning of the series appearing in the two equations is clear.

These equations are very useful for the construction of wavelets as well as explicit, more general solutions that lead to expansions of the form (1.1), (1.4), or (1.5) (see [8]). We shall also see in the development below that a considerable amount of information is contained in these two equations.

The following simple example of a function that satisfies the two equations illustrates some important features of the general solution. Let us first consider the one-dimensional case for (1.8) with $L = 1$. We choose a non-negative even function b supported in $[-\pi, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \pi]$ such that

$$b^2(\xi)+b^2\left(\frac{\xi}{2}\right)=1,\qquad \xi\in\left[\frac{\pi}{2},\pi\right]. \qquad (1.9)$$

It is easy to see that there exist C^∞ functions b with these properties. We let ψ be chosen so that $|\hat{\psi}| = b$. Equation (1.8)(i) in this case is $\sum_{j\in\mathbb{Z}}|\hat{\psi}^\ell(2^j\xi)|^2 = 1$. Because of the assumption we made about the support of b, this sum contains at most two non-zero terms and equality (1.9) shows that (1.8)(i) is satisfied by $\hat{\psi}$. The second equation, (1.8)(ii), is also clearly satisfied since the two points, $2^j\xi$ and $2^j(\xi + 2\pi q)$, at which the products are evaluated, are at distance from each other that is at least 2π (since $j \ge 0$ and q is odd) which is the diameter of the support of $\hat{\psi}$.

Thus, it follows from the results that we shall prove that equality (1.1) is true for all $f \in L^2(\mathbb{R})$ when ψ is chosen so that $|\hat{\psi}| = b$. The family $\{\psi_{j,k}\}$, however, is *not* an orthonormal basis for $L^2(\mathbb{R})$. First of all,

$$\|\psi\|_2^2=\frac{1}{2\pi}\int_{\mathbb{R}}\left|\hat{\psi}(\xi)\right|^2 d\xi \le \frac{meas\,(\operatorname{supp}\hat{\psi})}{2\pi}\le\frac{3}{4}<1$$

since, as is the case for any function $\hat{\psi}$ satisfying (1.8)(i), $|\hat{\psi}(\xi)| \le 1$ $a.e.$; moreover, a simple re-normalization cannot convert this system to an orthonormal basis. The discussion in the next section will clarify this matter. It is also clear that a radial version of this example provides us with a similar example in $\mathbb{R}^n$, $n > 1$.

Let us now pass to the precise statement and proof of our principal result when the families Φ and Ψ coincide.

2. The First Theorem

We begin by making some observations about expansions of the type we are considering in a general Hilbert space $\mathcal{H}$ endowed with an inner product $(\cdot, \cdot)$. Let $\mathcal{E} = \{e_j\}$ be a family of vectors in $\mathcal{H}$. For the sake of simplicity, we let j range through the natural numbers, $\mathbb{N}$; however, our statements apply when the indexing set is $\mathbb{Z} \times \mathbb{Z}^n$ or $\{1, \ldots, L\} \times \mathbb{Z} \times \mathbb{Z}^n$.

Lemma 1.

Let $\mathcal{E} = \{e_j\} \subset \mathcal{H}$, $j \in \mathbb{N}$, then the following two properties are equivalent:

(i) $\|f\|^2 = (f, f) = \sum_{j=1}^{\infty} |(f, e_j)|^2$ holds for all $f \in \mathcal{H}$

(ii) $f = \sum_{j=1}^{\infty}(f, e_j)e_j$, with convergence in $\mathcal{H}$, for all $f \in \mathcal{H}$.

Moreover, if $\|e_j\| \geq 1$, for all $j \in \mathbb{N}$, (i) or (ii) is equivalent to the fact that $\mathcal{E}$ is an orthonormal basis of $\mathcal{H}$.

The proof of this lemma is elementary and can be found in Chapter 7 of [8] (Theorems (1.7) and (1.8)). A more general version of this result, involving two sets $\mathcal{E} = \{e_j\}$ and $\mathcal{F} = \{f_j\}$ in $\mathcal{H}$, is stated and proved in Section 4 (see Lemma 8). The following result is also proved in Chapter 7 of [8] (see Lemma (1.10)):

Lemma 2.

Suppose $\mathcal{E} = \{e_j\} \subset \mathcal{H}$, $j \in \mathbb{N}$, is a family for which equality (i) of Lemma 1 holds for all f belonging to a dense subset $\mathcal{D} \subset \mathcal{H}$, then this equality holds for all $f \in \mathcal{H}$.

Again, a more general version of Lemma 2, involving two systems $\mathcal{E}$ and $\mathcal{F}$, is stated and proved in Section 4 (see Lemma 7). The following is our main result in case $\Phi = \Psi$:

Theorem 1.

Suppose $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$, then

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \left|\left(f, \psi^{\ell}_{j,\mathbf{k}}\right)\right|^2 \tag{2.1}$$

for all $f \in L^2(\mathbb{R}^n)$ if and only if the functions in Ψ satisfy (1.8)(i) and (ii).

Proof. Let us make a few observations before embarking on the proof of this theorem:

Remark 1.

Because of Lemma 1 we see that equality (2.1) for all $f \in L^2(\mathbb{R}^n)$ is equivalent to equality (1.4) for all $f \in L^2(\mathbb{R}^n)$. Thus, Theorem 1 gives us the desired characterization of those families $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ for which (1.4) holds for all $f \in L^2(\mathbb{R}^n)$.

Remark 2.

The last sentence in Lemma 1 leads us to the characterization of all orthonormal wavelets in $L^2(\mathbb{R}^n)$: $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ is a family of orthonormal wavelets in $L^2(\mathbb{R}^n)$ if and only if $\|\psi^1\|_2 = \ldots = \|\psi^L\|_2 = 1$ and this family satisfies (1.8)(i) and (ii) (since, in this case, $\|\psi^{\ell}_{j,\mathbf{k}}\|_2 = \|\psi^{\ell}\|_2 \geq 1$ for all $\ell = 1, \ldots, L$, $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$). This characterization has been obtained independently by Gripenberg [6] and Wang [12], in the case $n = 1$ (see [8] for an historical account of this matter).

Remark 3.

Because of Lemma 2, it suffices to show that (1.8)(i) and (ii) imply that (2.1) holds for all f belonging to a dense subset, $\mathcal{D}$, of $L^2(\mathbb{R}^n)$. The dense subset we will choose in our proof is

$$\mathcal{D} = \left\{ f \in L^2\left(\mathbb{R}^n\right) \,\middle|\, \hat{f} \in L^{\infty}\left(\mathbb{R}^n\right) \text{ and supp } \hat{f} \text{ is a compact subset of } \mathbb{R}^n \setminus \{\mathbf{0}\} \right\}. \tag{2.2}$$

Unless stated otherwise, therefore, from now on, all functions f that we shall consider will belong to $\mathcal{D}$.

A basic step in the proof of Theorem 1 is to decompose the series I on the right in (2.1) into two sums so that

$$(2\pi)^n I = I_0 + I_1 , \tag{2.3}$$

where

$$I_0 = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}^\ell(2^j \xi)\right|^2 d\xi$$

and

$$I_1 = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{p \in \mathbb{Z}} \sum_{\mathbf{q} \in \mathcal{O}^n} \hat{f}\left(\xi + 2\pi 2^p \mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi$$

This is done by first applying the Plancherel theorem to the inner products $(f, \psi^\ell_{j,\mathbf{k}})$, which allows us to obtain

$$\begin{aligned}
(2\pi)^{2n} \left|(f, \psi^\ell_{j,\mathbf{k}})\right|^2 &= \left|(\hat{f}, \hat{\psi}^\ell_{j,\mathbf{k}})\right|^2 = 2^{-jn} \left|\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{\psi}^\ell(2^{-j}\xi)e^{i2^{-j}\mathbf{k}\cdot\xi}}\, d\xi\right|^2 \\
&= 2^{jn} \left|\int_{\mathbb{R}^n} \hat{f}\left(2^j \xi\right) \overline{\hat{\psi}^\ell(\xi)} e^{i\mathbf{k}\cdot\xi}\, d\xi\right|^2 .
\end{aligned}$$

We used the fact that the Fourier transform of $\psi^\ell_{j,\mathbf{k}}$ is $\hat{\psi}^\ell_{j,\mathbf{k}}(\xi) = 2^{-jn/2}\hat{\psi}^\ell(2^{-j}\xi)e^{-i2^{-j}\mathbf{k}\cdot\xi}$, when $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, $\ell = 1, ..., L$. Thus,

$$(2\pi)^{2n} I = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} 2^{jn} \left|\int_{\mathbb{R}^n} \hat{f}\left(2^j \xi\right) \overline{\hat{\psi}^\ell(\xi)} e^{i\mathbf{k}\cdot\xi}\, d\xi\right|^2 .$$

The decomposition (2.3) will be obtained by first using a periodization argument that provides us with the identity

$$\begin{aligned}
&\sum_{\mathbf{k} \in \mathbb{Z}^n} \left|\int_{\mathbb{R}^n} \hat{f}\left(2^j \xi\right) \overline{\hat{\psi}^\ell(\xi)} e^{i\mathbf{k}\cdot\xi}\, d\xi\right|^2 \\
&= (2\pi)^n \int_{\mathbb{R}^n} \overline{\hat{f}\left(2^j \xi\right) \hat{\psi}^\ell(\xi)} \left\{ \sum_{\mathbf{m} \in \mathbb{Z}^n} \hat{f}\left(2^j(\xi + 2\mathbf{m}\pi)\right) \overline{\hat{\psi}^\ell(\xi + 2\mathbf{m}\pi)} \right\} d\xi .
\end{aligned} \tag{2.4}$$

Multiplying both sides by 2^{jn} and then summing over $j \in \mathbb{Z}$ and $\ell = 1, \ldots, L$ we obtain a new expression for I. In this last expression, we separate the terms with $\mathbf{m} = \mathbf{0}$, obtaining I_0, and the remaining terms, obtaining I_1. Thus, we first establish (2.4); after this we manipulate the expression involving the terms with $\mathbf{m} \neq \mathbf{0}$ and obtain the equality that we need for the definition of I_1.

Let us fix ℓ and j and put $F(\xi) = F^\ell_j(\xi) \equiv \hat{f}(2^j \xi)\overline{\hat{\psi}^\ell(\xi)}$. We remind the reader that $f \in \mathcal{D}$; thus, F is compactly supported in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and belongs to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Moreover,

$$\int_{\mathbb{R}^n} \hat{f}\left(2^j \xi\right) \overline{\hat{\psi}^\ell(\xi)} e^{i\mathbf{k}\cdot\xi}\, d\xi = \int_{\mathbb{R}^n} F(\xi) e^{i\mathbf{k}\cdot\xi}\, d\xi = \hat{F}(-\mathbf{k}) . \tag{2.5}$$

Hence,

$$\hat{F}(-\mathbf{k}) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \int_{\mathbb{T}^n + 2\mathbf{m}\pi} F(\xi) e^{i\mathbf{k}\cdot\xi}\, d\xi = \int_{\mathbb{T}^n} e^{i\mathbf{k}\xi} \left\{ \sum_{\mathbf{m} \in \mathbb{Z}^n} F(\xi + 2\mathbf{m}\pi) \right\} d\xi ,$$

where $\mathbb{T}^n$ is the n-torus, which we may identify with $\{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid 0 \le \xi_\ell < 2\pi,\ \ell = 1, \ldots, n\}$. The fact that F is compactly supported implies that the series $\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\xi + 2\pi\mathbf{m})$ involves only a finite number of terms. This, together with the 2π periodicity of $e^{i\mathbf{k}\cdot\xi}$, justifies the interchange of summation and integration (over $\mathbb{T}^n$) that gives us the last equality. We see, therefore, that the numbers $(2\pi)^{-n}\hat{F}(-\mathbf{k})$ are the Fourier coefficients of the periodic function $\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\xi + 2\pi\mathbf{m})$, which is in $L^2(\mathbb{T}^n)$ because the sum only involves a finite number of $\mathbf{m}$s when ξ is in $\mathbb{T}^n$. Thus, by the Plancherel theorem for Fourier series,

$$\begin{aligned} \frac{1}{(2\pi)^n}\sum_{\mathbf{k}\in\mathbb{Z}^n}\left|\hat{F}(-\mathbf{k})\right|^2 &= \int_{\mathbb{T}^n}\left(\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\xi + 2\mathbf{m}\pi)\right)\left(\sum_{\mathbf{p}\in\mathbb{Z}^n}\overline{F(\xi + 2\mathbf{p}\pi)}\right) d\xi \\ &= \int_{\mathbb{R}^n}\left(\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\xi + 2\mathbf{m}\pi)\right)\overline{F(\xi)}\, d\xi \,. \end{aligned}$$

Again, the fact that F is compactly supported and the 2π-periodicity of the series justifies the interchange of summation and integration (over $\mathbb{T}^n$) that gives us the last equality. This last equality, written in terms of f and ψ by using the identity

$$\overline{F(\xi)}\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\xi + 2\pi\mathbf{m}) = \overline{\hat{f}\left(2^j\xi\right)\hat{\psi}^\ell(\xi)}\sum_{\mathbf{m}\in\mathbb{Z}^n}\hat{f}\left(2^j(\xi + 2\mathbf{m}\pi)\right)\overline{\hat{\psi}^\ell(\xi + 2\mathbf{m}\pi)}\,,$$

together with (2.5), gives us (2.4).

Thus,

$$\begin{aligned} (2\pi)^n I &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} 2^{jn}\int_{\mathbb{R}^n}\left|\hat{f}(2^j\xi)\right|^2\left|\hat{\psi}^\ell(\xi)\right|^2 d\xi \\ &+ \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} 2^{jn}\int_{\mathbb{R}^n}\overline{\hat{f}\left(2^j\xi\right)\hat{\psi}^\ell(\xi)}\left\{\sum_{\mathbf{m}\neq\mathbf{0}}\hat{f}\left(2^j(\xi + 2\mathbf{m}\pi)\right)\overline{\hat{\psi}^\ell(\xi + 2\mathbf{m}\pi)}\right\} d\xi \,. \end{aligned}$$

The first of these summands is I_0 (after a change of variables $\eta = 2^j\xi$). In order to justify the manipulations that show that the second summand equals I_1, as defined immediately after (2.3), we shall prove the following:

Lemma 3.

For every $f \in \mathcal{D}$ and $\psi \in L^2(\mathbb{R}^n)$, then:

$$\sum_{j\in\mathbb{Z}} 2^{jn}\int_{\mathbb{R}^n}\left|\hat{f}\left(2^j\xi\right)\hat{\psi}(\xi)\right|\sum_{\mathbf{m}\neq\mathbf{0}}\left|\hat{f}\left(2^j(\xi + 2\mathbf{m}\pi)\right)\hat{\psi}(\xi + 2\mathbf{m}\pi)\right| d\xi < \infty \,.$$

Remark 4.

In addition to allowing us to make all the changes in the order of summation and integration that will give us the desired expresion for I_1, this lemma shows that the sum of the squares of the "coefficients" $(f, \psi^\ell_{j,\mathbf{k}})$ (which we denoted by I) is finite if and only if $I_0 < \infty$, for each $f \in \mathcal{D}$. By varying f in $\mathcal{D}$ (for example, letting $\hat{f} = \chi_C$, where C is a compact subset of $\mathbb{R}^n \setminus \{\mathbf{0}\}$) we see that $I_0 < \infty$ if and only if $\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}^\ell(2^j\xi)|^2$ is locally integrable in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ (that is, integrable over each compact $C \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$). This is, of course, clearly true when (1.8)(i) is valid. As we shall see, this local integrability property will furnish us with an appropiate condition that guarantees the convergence of the series (1.7)(i).

In order to establish Lemma 3 we observe that since

$$2|\hat{\psi}(\xi)|\,|\hat{\psi}(\xi+2\mathbf{m}\pi)| \le \left|\hat{\psi}(\xi)\right|^2 + \left|\hat{\psi}(\xi+2\mathbf{m}\pi)\right|^2 ,$$

it suffices to show that

$$\int_{\mathbb{R}^n} \left\{ \sum_{j\in\mathbb{Z}} \sum_{\mathbf{m}\neq\mathbf{0}} 2^{jn} \left|\hat{f}(2^j\xi)\right| \left|\hat{f}\left(2^j(\xi+2\mathbf{m}\pi)\right)\right| \right\} \left|\hat{\psi}(\xi)\right|^2 d\xi < \infty \tag{2.6}$$

(observe that the sum involving $|\hat{\psi}(\xi+2\mathbf{m}\pi)|^2$ reduces to (2.6) via the changes of variable $\eta = \xi + 2\mathbf{m}\pi$). But (2.6) is an immediate consequence of the following:

Lemma 4.

Suppose $0 < a < b < \infty$, $\hat{f} \in L^\infty(\mathbb{R}^n)$, *supp*$\hat{f} \subseteq \{\xi : a < |\xi| < b\}$ *and* $\delta =$ *diam(supp* $\hat{f}$*), then*

$$\sigma(\xi) \equiv \sum_{j\in\mathbb{Z}} \sum_{\mathbf{m}\neq\mathbf{0}} 2^{jn} \left|\hat{f}\left(2^j\xi\right)\right| \left|\hat{f}\left(2^j(\xi+2\mathbf{m}\pi)\right)\right| \le C\delta^n \left\|\hat{f}\right\|_\infty^2$$

for a.e. $\xi \in \mathbb{R}^n$, *where* $C = \left(\frac{3}{2\pi}\right)^n \left(1+\log_2 \frac{b}{a}\right)$.

Proof. If $\delta < 2^j 2\pi$, then at most one of the points $2^j\xi$ and $2^j\xi + 2^j 2\mathbf{m}\pi$ lies in supp $\hat{f}$, since $\mathbf{m}$ is a non-zero n-tuple with integer components. Thus, in the sum defining $\sigma(\xi)$ we need only consider $j \le j_0$, where j_0 is the greatest integer satisfying $2^{j_0} \le \frac{\delta}{2\pi}$. We claim that the sum of the terms, in the series defining $\sigma(\xi)$, that involves each such j does not exceed $\left(\frac{3\delta}{2\pi}\right)^n \|\hat{f}\|_\infty^2$. To see this, we first observe that $2^{jn}|\hat{f}(2^j\xi)|\,|\hat{f}(2^j(\xi+2\mathbf{m}\pi))| \le 2^{jn}\|\hat{f}\|_\infty^2$. We then observe that, for j and ξ fixed, the number of lattice points $\mathbf{m} \neq \mathbf{0}$ for which $\hat{f}(2^j(\xi+2\mathbf{m}\pi)) \neq 0$ is not larger than $(1+\frac{2^{-j}\delta}{\pi})^n$. To see this suppose $\mathbf{m}_0$ is a lattice point such that $\hat{f}(2^j(\xi+2\mathbf{m}_0\pi)) \neq 0$. Then, since $\delta = \text{diam}(\text{supp } f)$, if $\hat{f}(2^j(\xi+2\mathbf{m}\pi)) \neq 0$, we must have $\delta \ge |2^j(\xi+2\mathbf{m}\pi) - 2^j(\xi+2\mathbf{m}_0\pi)| = 2^j|\mathbf{m}-\mathbf{m}_0|2\pi$. Thus, $\mathbf{m}$ must lie within the sphere about $\mathbf{m}_0$ of radius $\frac{2^{-j}\delta}{2\pi}$. This sphere is contained in the n-dimensional "cube" of sidelength $\frac{2^{-j}\delta}{\pi}$ centered at $\mathbf{m}_0$. But the number of lattice points within this cube does not exceed $\left(1+\frac{\delta}{\pi}2^{-j}\right)^n$. Putting these estimates together we see that for $j \le j_0$

$$\sum_{\mathbf{m}\neq\mathbf{0}} 2^{jn} \left|\hat{f}\left(2^j\xi\right)\right| \left|\hat{f}\left(2^j(\xi+2\mathbf{m}\pi)\right)\right| \le$$

$$\left(1+\frac{\delta}{\pi}2^{-j}\right)^n 2^{jn} \left\|\hat{f}\right\|_\infty^2 = \left(2^j+\frac{\delta}{\pi}\right)^n \left\|\hat{f}\right\|_\infty^2 \le \left(2^{j_0}+\frac{\delta}{\pi}\right)^n \left\|\hat{f}\right\|_\infty^2 \le \left(\frac{3\delta}{2\pi}\right)^n \left\|\hat{f}\right\|_\infty^2 .$$

Finally, we observe that for $\hat{f}(2^j\xi)$ to be non-zero we must have $a \le 2^j|\xi| \le b$. Thus, j must lie in the interval $\left[\log_2\frac{a}{|\xi|}, \log_2\frac{b}{|\xi|}\right]$ to produce a non-zero summand in the series defining $\sigma(\xi)$. But there are at most $1+\log_2\frac{b}{a}$ integers in this interval. Together with the last estimate, this gives us

$$\sigma(\xi) \le \left(1+\log_2\frac{b}{a}\right)\left(\frac{3}{2\pi}\right)^n \delta^n \left\|\hat{f}\right\|_\infty^2 .$$

This completes the proof of Lemmas 3 and 4. □

We now turn our attention to showing that I_1 has the form announced after equality (2.3). We have shown that

$$(2\pi)^n I =$$

$$I_0 + \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} \overline{\hat{f}(2^j \xi)} \hat{\psi}^\ell(\xi) \left\{ \sum_{\mathbf{m} \neq \mathbf{0}} \hat{f}\left(2^j(\xi + 2\mathbf{m}\pi)\right) \overline{\hat{\psi}^\ell(\xi + 2\mathbf{m}\pi)} \right\} d\xi . \tag{2.7}$$

If $\mathbf{m} = (m_1, m_2, \dots, m_n) \neq (0, 0, \dots, 0) = \mathbf{0}$, then there exists a unique non-negative integer r such that $\mathbf{m} = 2^r \mathbf{q}$ with $\mathbf{q} \in \mathcal{O}^n$. Since by Lemma 3 the integrand in the second summand of (2.7) is absolutely convergent, the following equalities that allow us to isolate the terms involving $t_{\mathbf{q}}(\xi)$ are valid and the second term in (2.7) equals

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{\psi}^\ell\left(2^{-j}\xi\right) \sum_{\mathbf{m} \neq \mathbf{0}} \hat{f}\left(\xi + 2^j 2\mathbf{m}\pi\right) \overline{\hat{\psi}^\ell\left(2^{-j}\xi + 2\mathbf{m}\pi\right)} d\xi =$$

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{\psi}^\ell\left(2^{-j}\xi\right) \sum_{r \geq 0} \sum_{\mathbf{q} \in \mathcal{O}^n} \hat{f}\left(\xi + 2^j 2\pi 2^r \mathbf{q}\right) \overline{\hat{\psi}^\ell\left(2^{-j}\xi + 2\pi 2^r \mathbf{q}\right)} d\xi =$$

$$\sum_{\ell=1}^{L} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q} \in \mathcal{O}^n} \sum_{r \geq 0} \sum_{j \in \mathbb{Z}} \hat{\psi}^\ell\left(2^r\left(2^{-r-j}\xi\right)\right) \hat{f}\left(\xi + 2^{j+r} 2\pi \mathbf{q}\right) \overline{\hat{\psi}^\ell\left(2^r\left(2^{-r-j}\xi + 2\mathbf{q}\pi\right)\right)} d\xi =$$

$$\sum_{\ell=1}^{L} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q} \in \mathcal{O}^n} \sum_{r \geq 0} \sum_{p \in \mathbb{Z}} \hat{\psi}^\ell\left(2^r\left(2^{-p}\xi\right)\right) \overline{\hat{\psi}^\ell\left(2^r\left(2^{-p}\xi + 2\pi\mathbf{q}\right)\right)} \hat{f}\left(\xi + 2^p 2\pi \mathbf{q}\right) d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q} \in \mathcal{O}^n} \sum_{p \in \mathbb{Z}} \hat{f}\left(\xi + 2^p 2\pi \mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi .$$

This proves (2.3). It is now clear that the "if" part of Theorem 1 is true. Indeed, if the system Ψ satisfies (1.8)(i) and (ii), then $I_0 = \|\hat{f}\|^2$ and $I_1 = 0$. By (2.3) we then have

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \left|\left(f, \psi_{j,\mathbf{k}}^\ell\right)\right|^2 = I = (2\pi)^{-n}(I_0 + I_1) = (2\pi)^{-n} \left\|\hat{f}\right\|^2 + 0 = \|f\|^2 ,$$

which is equality (2.1) for all $f \in \mathcal{D}$. By Lemma 2, we then can conclude that (2.1) holds for all $f \in L^2(\mathbb{R})$.

We shall now prove the converse. Let us assume that (2.1) holds for all $f \in \mathcal{D}$. As we explained in Remark 4, this implies that

$$\tau(\xi) \equiv \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left|\hat{\psi}^\ell\left(2^j \xi\right)\right|^2$$

is locally integrable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Thus, almost every point in $\mathbb{R}^n$ is a point of differentiability of the integral of τ. Let us choose such a $\xi_0 \neq \mathbf{0}$; that is, if Ω_n is the volume of the unit ball in $\mathbb{R}^n$,

$$\lim_{\delta \to 0+} \frac{1}{\Omega_n \delta^n} \int_{|\xi - \xi_0| \leq \delta} \tau(\xi)\, d\xi = \tau\left(\xi_0\right) . \tag{2.8}$$

Let us fix $\delta > 0$ such that $B_\delta(\xi_0) = \left\{\xi : |\xi - \xi_0| \leq \delta\right\} \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ and choose $f_\delta \in \mathcal{D}$ by letting

$$\hat{f}_\delta(\xi) = \frac{1}{\sqrt{\Omega_n \delta^n}} \chi_{B_\delta(\xi_0)}(\xi) .$$

Using the notation in (2.3), and adding the superscript δ to denote the dependence on this choice of f_δ, we have

$$(2\pi)^n I = (2\pi)^n I^\delta = I_0^\delta + I_1^\delta .$$

Thus, $I^\delta = \|f_\delta\|^2 = (2\pi)^{-n}\|\hat{f}_\delta\|^2 = (2\pi)^{-n}$ and we have

$$1 = \frac{1}{\Omega_n \delta^n} \int_{B_\delta(\xi_0)} \tau(\xi)\, d\xi + I_1^\delta ,$$

for every δ small enough. From this we see that if we show that I_1^δ tends to 0 as $\delta \to 0+$, we have $\tau(\xi_0) = 1$ (by (2.8)) and equality (1.8)(i) is satisfied by the system Ψ, since almost all points of $\mathbb{R}^n$ are such ξ_0s.

Arguing as we did when we established Lemma 3, we see that $|I_1^\delta|$ is dominated by the sum of two terms:

$$\int_{\mathbb{R}^n} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{m}\neq\mathbf{0}} 2^{jn} \left|\hat{f}_\delta\left(2^j\xi\right)\right| \left|\hat{f}_\delta\left(2^j(\xi + 2\mathbf{m}\pi)\right)\right| \left|\hat{\psi}^\ell(\xi)\right|^2 d\xi$$

and another term in which $\hat{\psi}^\ell(\xi)$ is replaced by $\hat{\psi}^\ell(\xi + 2\mathbf{m}\pi)$ (which, after the change of variables $\eta = \xi + 2\mathbf{m}\pi$, reduces to the first term). Letting ψ denote any one of the L functions in Ψ, it suffices to show, therefore, that

$$I_1^{\delta,\sharp} = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{m}\neq\mathbf{0}} 2^{jn} \left|\hat{f}_\delta\left(2^j\xi\right)\right| \left|\hat{f}_\delta\left(2^j(\xi + 2\mathbf{m}\pi)\right)\right| \left|\hat{\psi}(\xi)\right|^2 d\xi$$

tends to 0 as $\delta \to 0+$. The diameter of the support of $\hat{f}_\delta$ is 2δ; hence, since $\mathbf{m} \neq \mathbf{0}$ we must have

$$\hat{f}_\delta\left(2^j\xi\right)\, \hat{f}_\delta\left(2^j(\xi + 2\mathbf{m}\pi)\right) = 0$$

if $2^j > \frac{\delta}{\pi}$. Let j_0 be the largest integer such that $2^{j_0} \leq \frac{\delta}{\pi}$; then we need only consider $j \leq j_0$ in the sum defining $I_1^{\delta,\sharp}$. Also, if $\hat{f}_\delta(2^j\xi) \neq 0$, we must have $|2^j\xi - \xi_0| \leq \delta$ and this, in turn, implies $|\xi_0| - \delta \leq 2^j|\xi|$. Since $B_\delta(\xi_0) \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$, we must have $|\xi_0| - \delta > 0$ as well. Hence,

$$|\xi| \geq 2^{-j}\left(|\xi_0| - \delta\right) \geq 2^{-j_0}\left(|\xi_0| - \delta\right) \geq \frac{\pi}{\delta}\left(|\xi_0| - \delta\right) > 0 .$$

Thus, applying Lemma 4 to f_δ, with $a = |\xi_0| - \delta$, $b = |\xi_0| + \delta$, we have

$$I_1^{\delta,\sharp} \leq \int_{|\xi| \geq (|\xi_0| - \delta)\frac{\pi}{\delta}} \sigma_\delta(\xi) \left|\hat{\psi}(\xi)\right|^2 d\xi \leq$$

$$\left(\frac{3}{2\pi}\right)^n \left(1 + \log_2 \frac{|\xi_0| + \delta}{|\xi_0| - \delta}\right) (2\delta)^n \left\|\hat{f}_\delta\right\|_\infty^2 \int_{|\xi| \geq (|\xi_0| - \delta)\frac{\pi}{\delta}} \left|\hat{\psi}(\xi)\right|^2 d\xi \leq$$

$$\Omega_n^{-1} \left(\frac{3}{\pi}\right)^n \left(1 + \log_2 \frac{|\xi_0| + \delta}{|\xi_0| - \delta}\right) \int_{|\xi| \geq (|\xi_0| - \delta)\frac{\pi}{\delta}} \left|\hat{\psi}(\xi)\right|^2 d\xi .$$

It is clear that this last expression tends to 0 as $\delta \to 0+$. We can conclude, therefore, that equality (1.8)(i) is satisfied by Ψ. This also shows that $I_1 = 0$ for all $f \in \mathcal{D}$ since I_0 must, then, equal $\|\hat{f}\|^2 = (2\pi)^n\|f\|^2$ and, thus, $\|f\|^2 = I = (2\pi)^{-n}(I_0 + I_1) = \|f\|^2 + (2\pi)^{-n}I_1$. That is,

$$0 = I_1 = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}\left(\xi + 2^p 2\pi\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi$$

for all $f \in \mathcal{D}$. An application of the polarization identity then gives us

$$\int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{g}\left(\xi + 2^p 2\pi\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi = 0 \tag{2.9}$$

for all $f, g \in \mathcal{D}$.

Let us fix $\mathbf{q}_0 \in \mathcal{O}^n$ and choose a point ξ_0 of differentiability of the integral of $t_{\mathbf{q}_0}$ such that $\xi_0 \neq \mathbf{0} \neq \xi_0 + 2\pi\mathbf{q}_0$. Since $t_{\mathbf{q}_0} \in L^1(\mathbb{R}^n)$ (see the argument that follows (1.7) almost all points of $\mathbb{R}^n$ have these properties. We need only consider $\delta > 0$ sufficiently small so that both $B_\delta(\xi_0)$ and $B_\delta(\xi_0 + 2\pi\mathbf{q}_0)$ lie within $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Let f_δ and g_δ in $\mathcal{D}$ be functions such that

$$\hat{f}_\delta(\xi) = \frac{1}{\sqrt{\Omega_n \delta^n}} \chi_{B_\delta(\xi_0)}(\xi) \text{ and } \hat{g}_\delta(\xi) = \frac{1}{\sqrt{\Omega_n \delta^n}} \chi_{B_\delta(\xi_0 + 2\pi\mathbf{q}_0)}(\xi) \,.$$

(observe that $\hat{g}_\delta(\xi) = \hat{f}_\delta(\xi - 2\pi\mathbf{q}_0)$). Then

$$\overline{\hat{f}_\delta(\xi)}\hat{g}_\delta(\xi + 2\pi\mathbf{q}_0) = \frac{1}{\Omega_n \delta^n} \chi_{B_\delta(\xi_0)}(\xi)$$

and this allows us to write (2.9) in the form

$$\begin{aligned} 0 &= \frac{1}{\Omega_n \delta^n} \int_{B_\delta(\xi_0)} t_{\mathbf{q}_0}(\xi)\, d\xi + \sum_{\substack{(p,\mathbf{q}) \in \mathbb{Z} \times \mathcal{O}^n \\ (p,\mathbf{q}) \neq (0,\mathbf{q}_0)}} \int_{\mathbb{R}^n} \overline{\hat{f}_\delta(\xi)}\hat{g}_\delta\left(\xi + 2^p 2\pi\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi \\ &= \frac{1}{\left|B_\delta(\xi_0)\right|} \int_{B_\delta(\xi_0)} t_{\mathbf{q}_0}(\xi)\, d\xi + J_\delta \,. \end{aligned}$$

In order to show that equality (1.8)(ii) is satisfied at ξ_0 (and, thus, *a.e.*) it suffices to prove that $\lim_{\delta \to 0+} J_\delta = 0$. We therefore examine the sum defining J_δ more closely.

If $\overline{\hat{f}_\delta(\xi)}\hat{g}_\delta(\xi + 2^p 2\pi\mathbf{q}) \neq 0$, we must have $|\xi - \xi_0| < \delta$ and $|\xi_0 + 2\pi\mathbf{q}_0 - \xi - 2^p 2\pi\mathbf{q}| < \delta$. Thus,

$$\left|\mathbf{q}_0 - 2^p\mathbf{q}\right| = \frac{1}{2\pi} \left|(\xi_0 + 2\pi\mathbf{q}_0) - (\xi + 2^p 2\pi\mathbf{q}) + (\xi - \xi_0)\right| < \frac{\delta}{\pi} \,. \tag{2.10}$$

Since we are interested in $\lim_{\delta \to 0+} J_\delta$ we can assume $\frac{\delta}{\pi} < 1$. If $p > 0$ we must have $|\mathbf{q}_0 - 2^p\mathbf{q}| \geq 1 > \frac{\delta}{\pi}$ since $\mathbf{q}_0 \in \mathcal{O}^n$. If $p = 0$ we must have $|\mathbf{q}_0 - 2^p\mathbf{q}| = |\mathbf{q}_0 - \mathbf{q}| \geq 1 > \frac{\delta}{\pi}$ when $\mathbf{q}_0 \neq \mathbf{q}$. Finally, if $p < 0$ we must have $|\mathbf{q}_0 - 2^p\mathbf{q}| = 2^p|2^{-p}\mathbf{q}_0 - \mathbf{q}| \geq 2^p$ since $\mathbf{q} \in \mathcal{O}^n$. Hence, if j_0 is the largest integer such that $2^{j_0} \leq \frac{\delta}{\pi}$, we have $j_0 < 0$ and

$$J_\delta = \sum_{p \leq j_0} \sum_{\mathbf{q} \in \mathcal{O}^n} \int_{\mathbb{R}^n} \overline{\hat{f}_\delta(\xi)}\hat{g}_\delta\left(\xi + 2^p 2\pi\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi \,.$$

Making the change of variables $\eta = 2^{-p}\xi$ we obtain

$$|J_\delta| \leq \sum_{p \leq j_0} \sum_{\mathbf{q} \in \mathcal{O}^n} 2^{pn} \int_{\mathbb{R}^n} \left|\hat{f}_\delta\left(2^p\eta\right)\right| \left|\hat{g}_\delta\left(2^p(\eta + 2\pi\mathbf{q})\right)\right| \left|t_{\mathbf{q}}(\eta)\right| d\eta \,.$$

Since

$$2\left|t_{\mathbf{q}}(\xi)\right| \leq \sum_{\ell=1}^{L} \left\{ \sum_{m \geq 0} \left|\hat{\psi}^\ell\left(2^m\xi\right)\right|^2 + \sum_{m \geq 0} \left|\hat{\psi}^\ell\left(2^m(\xi + 2\pi\mathbf{q})\right)\right|^2 \right\} ,$$

we can reduce our problem to estimating

$$J_\delta^1 = \sum_{p \leq j_0} \sum_{\mathbf{q} \in \mathcal{O}^n} 2^{pn} \int_{\mathbb{R}^n} \left|\hat{f}_\delta\left(2^p\xi\right)\right| \left|\hat{g}_\delta\left(2^p(\xi + 2\pi\mathbf{q})\right)\right| \left\{ \sum_{m \geq 0} \left|\hat{\psi}^\ell\left(2^m\xi\right)\right|^2 \right\} d\xi \,.$$

and an analogous term, J_δ^2, in which $\hat{\psi}^\ell(2^m\xi)$ is replaced by $\hat{\psi}^\ell(2^m(\xi+2\pi\mathbf{q}))$ (this is the same argument we used prior to our introduction of $I_1^{\delta,\sharp}$). Once we have shown that $\lim_{\delta\to 0+} J_\delta^1 = 0$, it will be easy to see how we can modify the argument to obtain $\lim_{\delta\to 0+} J_\delta^2 = 0$. J_δ^1 (and J_δ^2) depends on ℓ; we do not indicate this dependence in the sequel because there are only a finite number of these terms.

Let $T(\xi) = \sum_{m\geq 0} \left|\hat{\psi}^\ell(2^m\xi)\right|^2$. The argument in the first section that showed that $t_{\mathbf{q}} \in L^1(\mathbb{R}^n)$ applies here to show $T \in L^1(\mathbb{R}^n)$. Furthermore, since $\left|B_\delta(\xi_0)\right|^{1/2} \hat{f}_\delta = \chi_{B_\delta(\xi_0)}(\xi)$ we have

$$J_\delta^1 = \sum_{p\leq j_0}\sum_{\mathbf{q}\in\mathcal{O}^n} \frac{2^{pn}}{\sqrt{\Omega_n\delta^n}} \int_{|2^p\xi-\xi_0|<\delta} T(\xi)\left|\hat{g}_\delta\left(2^p(\xi+2\pi\mathbf{q})\right)\right| d\xi\ .$$

If $\hat{g}_\delta(2^p(\xi+2\pi\mathbf{q}))\neq 0$ we must have $|2^p(\xi+2\pi\mathbf{q})-(\xi_0+2\pi\mathbf{q}_0)|\leq\delta$ and this inequality, together with $|2^p\xi-\xi_0|<\delta$, implies (as in the case (2.10))

$$\left|\mathbf{q}_0-2^p\mathbf{q}\right| = \frac{1}{2\pi}\left|2^p(\xi+2\pi\mathbf{q})-\left(\xi_0+2\pi\mathbf{q}_0\right)+\left(\xi_0-2^p\xi\right)\right| \leq \frac{\delta}{\pi}\ . \tag{2.11}$$

In our case $p\leq j_0<0$; hence, from (2.11) we have $|2^{-p}\mathbf{q}_0-\mathbf{q}|\leq\frac{2^{-p}\delta}{\pi}$ and $2^{-p}\mathbf{q}_0$ is a lattice point. The number of lattice points $\mathbf{q}$ satisfying this inequality cannot exceed $\left(1+\frac{2^{-p}2\delta}{\pi}\right)^n$ since they must lie within the n-dimensional cube centered at $2^{-p}\mathbf{q}_0$ of side length at most $\frac{2^{-p}2\delta}{\pi}$. Consequently,

$$\sum_{\mathbf{q}\in\mathcal{O}^n}\left|\hat{g}_\delta\left(2^p(\xi+2\pi\mathbf{q})\right)\right| \leq \left(1+\frac{2^{-p}2\delta}{\pi}\right)^n \left\|\hat{g}_\delta\right\|_\infty\ .$$

Using this estimate together with $\|\hat{g}_\delta\|_\infty = \frac{1}{\sqrt{\Omega_n\delta^n}}$ in the above expression for J_δ^1 we obtain

$$\begin{aligned}
J_\delta^1 &\leq \sum_{p\leq j_0}\left(2^p+\frac{2\delta}{\pi}\right)^n \frac{1}{\Omega_n\delta^n}\int_{|2^p\xi-\xi_0|<\delta} T(\xi)\,d\xi \\
&\leq \sum_{p\leq j_0}\left(2^{j_0}+\frac{2\delta}{\pi}\right)^n \frac{1}{\Omega_n\delta^n}\int_{|2^p\xi-\xi_0|<\delta} T(\xi)\,d\xi \\
&\leq \sum_{p\leq j_0}\left(\frac{3}{\pi}\right)^n \frac{1}{\Omega_n}\int_{|2^p\xi-\xi_0|<\delta} T(\xi)\,d\xi\ .
\end{aligned}$$

But, $\left\{\xi : |2^p\xi-\xi_0|<\delta\right\} \subset \left\{\xi : 2^{-p}(|\xi_0|-\delta)<|\xi|<2^{-p}(|\xi_0|+\delta)\right\} \equiv \mathcal{A}_p$. If $3\delta<|\xi_0|$ the sets $\mathcal{A}_p$, $p=j_0, j_0-1, j_0-2,\ldots$, are mutually disjoint. Thus,

$$\begin{aligned}
J_\delta^1 &\leq \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \sum_{p\leq j_0}\int_{\mathcal{A}_p} T(\xi)\,d\xi = \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \int_{\cup_{p\leq j_0}\mathcal{A}_p} T(\xi)\,d\xi \\
&\leq \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \int_{2^{-j_0}(|\xi_0|-\delta)<|\xi|} T(\xi)\,d\xi \leq \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \int_{\frac{\pi}{\delta}(|\xi_0|-\delta)<|\xi|} T(\xi)\,d\xi\ .
\end{aligned}$$

But the last integral tends to 0 as $\delta\to 0+$ since $T\in L^1(\mathbb{R}^n)$. Thus, $J_\delta^1\to 0$ as $\delta\to 0+$. As mentioned before, a similar argument shows $\lim_{\delta\to 0+} J_\delta^2 = 0$. In fact, the change of variables $\eta = \xi+2\pi\mathbf{q}$ in the integrals defining J_δ^2 convert this quantity to, essentially, J_δ^1 except that the roles of f_δ and g_δ are interchanged. Because of this we can let the point $\xi_0+2\pi\mathbf{q}_0$ play the role of ξ_0 in the argument we just gave in order to show that $\lim_{\delta\to 0+} J_\delta^2 = 0$. We thus obtain the desired result $\lim_{\delta\to 0+} J_\delta = 0$ and, therefore, Equation (1.8)(ii) is satisfied by the system Ψ almost everywhere. This establishes Theorem 1. □

3. The Second Theorem

We now consider the case when the "analyzing" family $\Phi = \{\phi^1, \phi^2, \dots, \phi^L\}$ differs from the "synthesizing" system $\Psi = \{\psi^1, \psi^2, \dots, \psi^L\}$. As indicated in the first section, the result we shall establish "essentially" asserts that Equation (1.6) is satisfied for all $f \in L^2(\mathbb{R}^n)$ if and only if Equations (1.7)(i) and (ii) are satisfied *a.e.* by Φ and Ψ. There are some basic differences between the general case, however, and the one we just presented in Section 2. The local integrability in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ of the expressions $\sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(2^j\xi)|^2$, first discussed in Remark 4, played an important role in our arguments and arose in a natural way from our arguments. We did not, however, need to assume this property for the system Ψ when we announced Theorem 1, the principal result in Section 2. If we do assume that this property holds for both systems Φ and Ψ, the proof that the two equations (1.7)(i) and (ii) are equivalent to the equality

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \phi^\ell_{j,\mathbf{k}}\right)\left(\psi^\ell_{j,\mathbf{k}}, f\right) \tag{3.1}$$

for all $f \in \mathcal{D}$ is essentially the same as the proof we presented for Theorem 1. We will be more precise about this below. From this we do obtain a "weak version" of the representation (1.6) for all such f. This is a consequence of the fact that, by polarization, (3.1) implies

$$(f, g) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \phi^\ell_{j,\mathbf{k}}\right)\left(\psi^\ell_{j,\mathbf{k}}, g\right) \tag{3.2}$$

for all f and g in $\mathcal{D}$. We cannot, however, establish the $L^2(\mathbb{R}^n)$-convergence of the series (1.6).

If we do not make the local integrability assumptions for each of the two series

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}^\ell\left(2^j\xi\right)\right|^2 \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \left|\hat{\phi}^\ell\left(2^j\xi\right)\right|^2, \quad \ell = 1, 2, \dots, L \tag{3.3}$$

we can still establish the equivalence of (1.7) and some form of (3.1). We will present the arguments for this result in the fourth section. It is also clear from our presentation that the case $L > 1$ offers no more complications than the case $L = 1$; hence, we will not use the upper index ℓ from now on.

We begin by presenting an example that illustrates some of the assertions we have made. Let $I = [-2\pi, -\pi) \cup (\pi, 2\pi]$ and θ be the function satisfying $\hat{\theta} = \chi_I$. Thus, θ is the *Shannon orthonormal wavelet* (see [8]). We then define ψ by $\hat{\psi}(\xi) = \hat{\theta}(2\xi) = \chi_{[-\pi,-\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]}(\xi)$ or, equivalently, by $\psi(x) = 2^{-1}\theta(2^{-1}x)$. Let ϕ be the *scaling function* associated with the Shannon wavelet; that is, $\hat{\phi} = \chi_{[-\pi,\pi]}$.

Proposition 1.

The pair (ϕ, ψ) satisfy equalities (1.7)(i) and (ii); that is,

$$(i) \qquad \sum_{j\in\mathbb{Z}} \hat{\phi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j\xi\right)} = 1 \qquad \textit{for all} \quad \xi \neq 0,$$

$$(ii) \qquad t_q(\xi) = \sum_{m=0}^{\infty} \hat{\phi}\left(2^m\xi\right)\overline{\hat{\psi}\left(2^m(\xi + 2\pi q)\right)} = 0 \qquad \textit{for a.e. } \xi \in \mathbb{R},\ \forall q \in 2\mathbb{Z}+1\,.$$

Proof. Since $\operatorname{supp}\hat{\psi} \subset \operatorname{supp}\hat{\phi}$ we have $\hat{\phi}\,\overline{\hat{\psi}} = \hat{\psi}$ (because $\hat{\psi} = \chi_{\frac{1}{2}I}$ is real-valued) and (i) is an immediate consequence of the fact that $\{2^j I\}$, $j \in \mathbb{Z}$, is partition of $\mathbb{R} \setminus \{0\}$. Equality ($ii$) follows from the fact that the supports of $\hat{\phi}$ and $\hat{\psi}(\cdot + 2\pi q)$ are disjoint (except for a set of Lebesgue measure zero) since q is an odd integer and, thus, $|q| \geq 1$. □

Proposition 2.
$\left\{\sqrt{2}\,\psi_{j,2\ell}\right\}$ *and* $\left\{\sqrt{2}\,\psi_{j,2\ell+1}\right\}$, $j, \ell \in \mathbb{Z}$, *are, each, an orthonormal basis for* $L^2(\mathbb{R})$.

Proof. The identity $\sqrt{2}\,\psi_{j,2\ell} = \theta_{j-1,\ell}$, $j, \ell \in \mathbb{Z}$, and the fact that θ is an orthonormal wavelet show that $\left\{\sqrt{2}\,\psi_{j,2\ell}\right\}$, $j, \ell \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$.

To see that the family $\left\{\sqrt{2}\,\psi_{j,2\ell+1}\right\}$, $j, \ell \in \mathbb{Z}$, is an orthonormal family, we first observe that, since ψ satisfies (1.8)(i) and (ii)[1], equality (2.1) is true for all $f \in L^2(\mathbb{R})$; that is,

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \left|(f, \psi_{j,k})\right|^2 \tag{3.4}$$

for all $f \in L^2(\mathbb{R})$. Breaking up the sum on the right into even and odd ks, and using the fact that $\left\{\sqrt{2}\,\psi_{j,2\ell}\right\}$ $j, \ell \in \mathbb{Z}$ is an orthonormal basis we obtain, by (3.4),

$$\begin{aligned}\|f\|^2 &= \sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \left\{\left|(f, \psi_{j,2\ell})\right|^2 + \left|(f, \psi_{j,2\ell+1})\right|^2\right\} \\ &= \frac{1}{2}\,\|f\|^2 + \sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \left|(f, \psi_{j,2\ell+1})\right|^2 .\end{aligned}$$

Hence,

$$\|f\|^2 = \sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \left|\left(f, \sqrt{2}\,\psi_{j,2\ell+1}\right)\right|^2 \tag{3.5}$$

for all $f \in L^2(\mathbb{R})$. Since $\|\sqrt{2}\,\psi_{j,2\ell+1}\| = 1$ for all $j, \ell \in \mathbb{Z}$, it follows from Lemma 1 and (3.5) that the system $\left\{\sqrt{2}\,\psi_{j,2\ell+1}\right\}$, $j, \ell \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$. □

This example illustrates why the unconditional $L^2(\mathbb{R}^n)$-convergence of the series (1.6) is not true in general when the two equations in Proposition 1 are satisfied. Let us be more precise. We continue using the functions ϕ and ψ we just introduced. We observed that $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$ for all $\xi \in \mathbb{R} \setminus \{0\}$; however, $\sum_{j\in\mathbb{Z}} |\hat{\phi}(2^j\xi)|^2 = \infty$ for all $\xi \in \mathbb{R}$ (in particular this last sum does not define a locally integrable function in $\mathbb{R} \setminus \{0\}$). Thus, we are in the situation described in (3.3) and we shall show that a "weak version" of (1.6) is, indeed, true. The $L^2(\mathbb{R})$-convergence of the series (1.6), however, is not unconditional. To see some of the difficulties we encounter in this case let us choose a function $f \in V_0$, the space generated by the integral translates of the scaling function ϕ. In fact, let us choose $f = \phi$. Since $V_0 \subset V_j$ for $j \ge 0$, we have

$$\phi = \sum_{k\in\mathbb{Z}} (\phi, \phi_{j,k})\, \phi_{j,k}$$

for each $j \ge 0$. Since $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j

$$\|\phi\|^2 = \sum_{k\in\mathbb{Z}} \left|(\phi, \phi_{j,k})\right|^2 \tag{3.6}$$

[1]The function ψ we are considering now is a particular case of the class of functions ψ satisfying $\left|\hat{\psi}\right| = b$, where b is as in (1.9). Proposition 2 clearly shows that a "simple re-normalization" cannot convert the system $\left\{\psi_{j,k}\right\}$ to an orthonormal basis, as we indicated at the end of Section 1.

for $j \geq 0$.

Consider the series $\sum_{k\in\mathbb{Z}}(\phi, \phi_{j,k})\psi_{j,k}$ for each j. Because of Proposition 2, this series is the sum of two orthogonal expansions

$$u_j + v_j = \sum_{\ell\in\mathbb{Z}} \left(\phi, \phi_{j,2\ell}\right) \psi_{j,2\ell} + \sum_{\ell\in\mathbb{Z}} \left(\phi, \phi_{j,2\ell+1}\right) \psi_{j,2\ell+1}$$

and, from (3.6) we have

$$\begin{aligned}\|u_j\|^2 + \|v_j\|^2 &= \frac{1}{2}\sum_{\ell\in\mathbb{Z}} |(\phi, \phi_{j,2\ell})|^2 + \frac{1}{2}\sum_{\ell\in\mathbb{Z}} |(\phi, \phi_{j,2\ell+1})|^2 \\ &= \frac{1}{2}\|\phi\|^2\end{aligned}$$

for each $j \geq 0$. Hence, for infinitely many j either $\|u_j\|^2$ or $\|v_j\|^2$ exceeds $\frac{1}{4}\|\phi\|^2 = \frac{1}{4}$. Thus, if, say, $\|u_j\|^2 \geq \frac{1}{4}$ infinitely often, then

$$\left\|\sum_{j\geq 0}\sum_{\ell\in\mathbb{Z}} \left(\phi, \phi_{j,2\ell}\right) \psi_{j,2\ell}\right\|_2^2 = \sum_{j\geq 0} \|u_j\|^2 = \infty$$

(since $u_j \perp u_{j'}$ if $0 \leq j < j'$). Clearly, then, the series $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(\phi, \phi_{j,k})\psi_{j,k}$ cannot converge unconditionally to ϕ in the $L^2(\mathbb{R})$-norm.

We now turn our attention to the extension of Theorem 1 to the case of a pair ϕ, ψ of "generating functions". We first establish the following.

Theorem 2.

Suppose that $\phi, \psi \in L^2(\mathbb{R}^n)$ are such that the functions defined by the series in (3.3) are locally integrable in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Then ϕ and ψ satisfy the equations

$$(i) \quad \sum_{j\in\mathbb{Z}} \hat{\phi}\left(2^j\xi\right) \overline{\hat{\psi}\left(2^j\xi\right)} = 1, \qquad \text{for a.e. } \xi \in \mathbb{R}^n,$$

$$(ii) \quad t_{\mathbf{q}}(\xi) = \sum_{m=0}^{\infty} \hat{\phi}\left(2^m\xi\right) \overline{\hat{\psi}\left(2^m(\xi + 2\pi\mathbf{q})\right)} = 0, \quad \text{for a.e. } \xi \in \mathbb{R}^n, \text{ when } \mathbf{q} \in \mathcal{O}^n, \tag{3.7}$$

if and only if

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \phi_{j,\mathbf{k}}\right)\left(\psi_{j,\mathbf{k}}, f\right) \tag{3.8}$$

for all $f \in \mathcal{D}$. The convergence of all these series is absolute and, thus, unconditional.

Since $|(f, \psi_{j,\mathbf{k}})|\,|(\phi_{j,\mathbf{k}}, f)| \leq |(f, \psi_{j,\mathbf{k}})|^2 + |(f, \phi_{j,\mathbf{k}})|^2$, the decomposition (2.3) applied to ψ and ϕ separately (with $f \in \mathcal{D}$), and the observations made in Remark 4, give us the absolute (and unconditional) convergence of the series appearing in Theorem 2 (we use, of course, the local integrability of the series in (3.3)).

Proof. Let us now indicate which modifications in the argument we presented in Section 2 are needed to provide a proof of this theorem. Here I denotes the sum on the right in (3.8). We begin by establishing the analog of the decomposition (2.3):

$$(2\pi)^n I = I_0 + I_1, \tag{3.9}$$

where

$$\begin{aligned} I_0 &= \sum_{j\in\mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} \left|\hat{f}\left(2^j\xi\right)\right|^2 \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\, d\xi \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \hat{\phi}\left(2^{-j}\xi\right)\overline{\hat{\psi}\left(2^{-j}\xi\right)}\, d\xi \end{aligned}$$

and

$$\left.\begin{aligned} I_1 &= \sum_{j\in\mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} \overline{\hat{f}\left(2^j\xi\right)\hat{\phi}(\xi)} \sum_{\mathbf{k}\neq\mathbf{0}} \hat{f}\left(2^j(\xi+2\mathbf{k}\pi)\right)\overline{\hat{\psi}(\xi+2\mathbf{k}\pi)}\, d\xi \\ &= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}\left(\xi+2\pi 2^p\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right)\, d\xi \end{aligned}\right\} \tag{3.10}$$

where $t_{\mathbf{q}}$ is defined by (ii) in the statement of Theorem 2 and involves both ϕ and ψ.

The proof of the decomposition (3.9) follows the same line as the one we gave for (2.3). The changes that are needed are obvious: the Plancherel theorem gives us the product

$$\left(\int_{\mathbb{R}^n} \hat{f}\left(2^j\xi\right)\overline{\hat{\psi}(\xi)}e^{i\mathbf{k}\xi}\, d\xi\right)\overline{\left(\int_{\mathbb{R}^n} \hat{f}\left(2^j\xi\right)\overline{\hat{\phi}(\xi)}e^{i\mathbf{k}\xi}\, d\xi\right)}$$

instead of the absolute value squared of the first factor. This leads us to the introduction of the function $G_j(\xi) = \hat{f}(2^j\xi)\overline{\hat{\phi}(\xi)}$ along with $F_j(\xi) = \hat{f}(2^j\xi)\overline{\hat{\psi}(\xi)}$. We then "periodize" both F_j and G_j to obtain the equality

$$(2\pi)^{-n} \sum_{\mathbf{k}\in\mathbb{Z}^n} \hat{F}_j(\mathbf{k})\overline{\hat{G}_j(\mathbf{k})} = \int_{\mathbb{R}^n} \left\{\sum_{\mathbf{m}\in\mathbb{Z}^n} F_j(\xi+2\mathbf{m}\pi)\right\} \overline{G_j(\xi)}\, d\xi$$

which leads us to the first expression for I_1 in (3.10) (see the argument preceding Lemma 3).

In order to prove that I_1 equals the second expression for I_1 in (3.10) we need to establish the analogs of Lemmas 3 and 4. This last estimate, of course, does not involve the functions ϕ and ψ and no change is needed. For the analog of Lemma 3 the problem can clearly be reduced to establishing the inequality (2.6).

Having established the decomposition (3.9), it is then immediate that equalities (i) and (ii), of Theorem 2 imply (3.8) for all $f \in \mathcal{D}$. We must, therefore, show that the converse is true. Again, the argument we gave in Section 2 for the corresponding part in (2.1) applies here if we make some simple and natural modifications. In (2.8) we now must choose $\tau(\xi) = \sum_{j\in\mathbb{Z}} \hat{\phi}(2^j\xi)\overline{\hat{\psi}(2^j\xi)}$. Then, the same choice of $f_\delta \in \mathcal{D}$ and the inequality

$$2\left|\hat{\psi}(\xi)\right| \left|\hat{\phi}(\xi+2\mathbf{m}\pi)\right| \le \left|\hat{\psi}(\xi)\right|^2 + \left|\hat{\phi}(\xi+2\mathbf{m}\pi)\right|^2 ,$$

leads us to the equality $1 = \frac{1}{\Omega_n\delta^n} \int_{B_\delta(\xi_0)} \tau(\xi)\, d\xi + I_1^\delta$, where $\lim_{\delta\to 0+} I_1^\delta = 0$. When ξ_0 is a point of differentiability of the integral of τ we obtain equality (i) in Theorem 2 at ξ_0. As in Section 2, this also gives us the equality (2.9) with $t_{\mathbf{q}}(\xi)$ as defined in (ii) of Theorem 2. Again, we choose f_δ and g_δ as before. The rest of proof given at the end of Section 2 applies here and this establishes Theorem 2. □

4. Some Other Results

In the last section we obtained the equivalence between equality (3.8) and the two equations (i) and (ii) announced in Theorem 2 provided the two series in (3.3) are locally integrable in $\mathbb{R}^n\setminus\{\mathbf{0}\}$. We also gave an example of two functions $\phi, \psi \in L^2(\mathbb{R})$ that satisfy (i) and (ii) but $\sum_{j\in\mathbb{Z}} |\hat{\phi}(2^j\xi)|^2$ is not locally integrable in $\mathbb{R} \setminus \{0\}$. We left open the question of the validity of (3.8) in this case but did observe that (1.6) cannot be interpreted in terms of unconditional convergence in $L^2(\mathbb{R})$ (which clearly implies (3.8)). In this section we examine other interpretations of (3.8) and its connection with equalities (i) and (ii) without assuming local integrability of the series in (3.3).

Let us first examine the convergence of the series in (3.8) when ϕ and ψ are functions in $L^2(\mathbb{R}^n)$. Toward this end we establish the following result:

Lemma 5.

Let J be a fixed integer, then the series

$$S_J = S_J(f) = \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \psi_{j,\mathbf{k}}\right)\left(\phi_{j,\mathbf{k}}, f\right)$$

converges absolutely for each $f \in \mathcal{D}$ when $\phi, \psi \in L^2(\mathbb{R}^n)$.

Proof. By Schwarz's inequality it suffices to show that

$$\sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n} \left|\left(f, \psi_{j,\mathbf{k}}\right)\right|^2 < \infty$$

for $f \in \mathcal{D}$. We argue as we did in the proof of the decomposition of (2.3) to obtain

$$(2\pi)^n \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n} \left|\left(f, \psi_{j,\mathbf{k}}\right)\right|^2 = \sum_{j\le J}\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}\left(2^{-j}\xi\right)\right|^2 d\xi +$$

$$\sum_{j\le J} 2^{jn} \int_{\mathbb{R}^n} \overline{\hat{f}\left(2^j\xi\right)\hat{\psi}(\xi)} \sum_{\mathbf{m}\neq\mathbf{0}} \hat{f}\left(2^j(\xi + 2\mathbf{m}\pi)\right) \overline{\hat{\psi}(\xi + 2\mathbf{m}\pi)}\, d\xi\,.$$

Lemma 3 assures us that the second summand represents an absolutely convergent series that is integrable. Moreover,

$$\sum_{j\le J} 2^{jn} \int_{\mathbb{R}^n} \left|\hat{f}\left(2^j\xi\right)\right| \left|\hat{\psi}(\xi)\right| \sum_{\mathbf{m}\neq\mathbf{0}} \left|\hat{f}\left(2^j(\xi + 2\mathbf{m}\pi)\right)\right| \left|\hat{\psi}(\xi + 2\mathbf{m}\pi)\right| d\xi \le C < \infty$$

where C is independent of J. In order to estimate the first summand, we use the fact that $f \in \mathcal{D}$ and, thus, supp $\hat{f}$ lies in an annular region of the form $\left\{\xi \in \mathbb{R}^n \,|\, 2^{-L}\pi \le |\xi| \le 2^L\pi\right\}$. Hence, we can estimate this first summand as follows:

$$\int_{2^{-L}\pi\le|\xi|\le 2^L\pi} \left|\hat{f}(\xi)\right|^2 \sum_{j\le J} \left|\hat{\psi}\left(2^{-j}\xi\right)\right|^2 d\xi \le \left\|\hat{f}\right\|_\infty^2 \sum_{\ell=-L}^{L-1} \int_{2^\ell\pi\le|\xi|\le 2^{\ell+1}\pi} \sum_{j\le J} \left|\hat{\psi}\left(2^{-j}\xi\right)\right|^2 d\xi$$

$$= \left\|\hat{f}\right\|_\infty^2 \sum_{\ell=-L}^{L-1} \sum_{j\le J} 2^{jn} \int_{2^{\ell-j}\pi\le|\eta|\le 2^{(\ell-j)+1}\pi} \left|\hat{\psi}(\eta)\right|^2 d\eta \le 2^{nJ} \left\|\hat{f}\right\|_\infty^2 \sum_{\ell=-L}^{L-1} \int_{2^{\ell-J}\pi\le|\eta|} \left|\hat{\psi}(\eta)\right|^2 d\eta$$

$$\le 2^{nJ} 2L \left\|\hat{f}\right\|_\infty^2 \left\|\hat{\psi}\right\|^2 < \infty$$

and Lemma 5 is proved. □

The double sum on the right of (3.8) corresponds to the expression I in Section 2. In analogy with (2.3) we shall consider a similar decomposition for the partial sums S_J of this double series:

$$(2\pi)^n S_J = I_0^J + I_1^J\,,\ J \in \mathbb{Z}, \tag{4.1}$$

where

$$I_0^J = \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \sum_{j\le J} \hat{\phi}\left(2^{-j}\xi\right)\overline{\hat{\psi}\left(2^{-j}\xi\right)}\, d\xi$$

and

$$I_1^J = \sum_{j\le J} 2^{jn} \int_{\mathbb{R}^n} \overline{\hat{f}\left(2^j\xi\right)}\hat{\phi}(\xi) \sum_{\mathbf{k}\ne\mathbf{0}} \hat{f}\left(2^j(\xi+2\mathbf{k}\pi)\right)\overline{\hat{\psi}(\xi+2\mathbf{k}\pi)}\, d\xi\,.$$

The observations we made that showed how to obtain (3.9) and the first equality in (3.10) are valid here and provide us with (4.1). The argument we made in the proof of Lemma 5 shows that $\sum_{j\le J}|\hat{\phi}(2^{-j}\xi)|\,|\hat{\psi}(2^{-j}\xi)|$ is locally integrable in $\mathbb{R}^n\setminus\{\mathbf{0}\}$. Hence, I_0^J is well-defined for each $f\in\mathcal{D}$. Reasoning as we did at the end of Section 3, we shall show that I_1^J has an expression involving the functions $t_{\mathbf{q}}$ (as in the term following the second equality in (3.10)); in fact, for J sufficiently large, we have, as in (3.10),

$$I_1^J = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}\left(\xi+2\pi 2^p\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right)\, d\xi = I_1 \tag{4.2}$$

(the size of J for this to be true depends on the diameter of the support of $\hat{f}$, where $f\in\mathcal{D}$). To see this we repeat the arguments we gave before, but need to take into account that the sum in the index j is limited by J:

$$\begin{aligned}
I_1^J &= \sum_{j\le J}\int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{\phi}\left(2^{-j}\xi\right)\sum_{\mathbf{m}\ne\mathbf{0}}\hat{f}\left(\xi+2^j 2\mathbf{m}\pi\right)\overline{\hat{\psi}\left(2^{-j}\xi+2\mathbf{m}\pi\right)}\, d\xi\\
&= \sum_{j\le J}\int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{\phi}\left(2^{-j}\xi\right)\sum_{r\ge 0}\sum_{\mathbf{q}\in\mathcal{O}^n}\hat{f}\left(\xi+2^{j+r}2\pi\mathbf{q}\right)\overline{\hat{\psi}\left(2^{-j}\xi+2\pi 2^r\mathbf{q}\right)}\, d\xi\\
&= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{r\ge 0}\sum_{j\le J}\hat{\phi}\left(2^r\left(2^{-(r+j)}\xi\right)\right)\hat{f}\left(\xi+2^{j+r}2\pi\mathbf{q}\right)\overline{\hat{\psi}\left(2^r\left(2^{-(r+j)}\xi+2\mathbf{q}\pi\right)\right)}\, d\xi\\
&= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{r\ge 0}\sum_{p\le J+r}\hat{\phi}\left(2^r 2^{-p}\xi\right)\overline{\hat{\psi}\left(2^r\left(2^{-p}\xi+2\pi\mathbf{q}\right)\right)}\hat{f}\left(\xi+2^p 2\pi\mathbf{q}\right)\, d\xi\\
&= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{p\le J}\sum_{r\ge 0}\hat{\phi}\left(2^r 2^{-p}\xi\right)\overline{\hat{\psi}\left(2^r\left(2^{-p}\xi+2\pi\mathbf{q}\right)\right)}\hat{f}\left(\xi+2^p 2\pi\mathbf{q}\right)\, d\xi\\
&+ \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{p> J}\sum_{r\ge p-J}\hat{\phi}\left(2^r 2^{-p}\xi\right)\overline{\hat{\psi}\left(2^r\left(2^{-p}\xi+2\pi\mathbf{q}\right)\right)}\hat{f}\left(\xi+2^p 2\pi\mathbf{q}\right)\, d\xi\,.
\end{aligned}$$

The first summand equals

$$\int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{p\le J}\hat{f}\left(\xi+2^p 2\pi\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right)\, d\xi\,.$$

If the diameter of the support of $\hat{f}$ does not exceed $2^{J+1}2\pi$, then either ξ or $\xi+2^p 2\pi\mathbf{q}$ must lie outside $\operatorname{supp}\hat{f}$ if $p>J$ (since $\mathbf{q}\in\mathcal{O}^n$). Thus, $\overline{\hat{f}(\xi)}\,\hat{f}(\xi+2^p 2\pi\mathbf{q})=0$ and the second term is 0. But, in this case we also have

$$\int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{p> J}\hat{f}\left(\xi+2^p 2\pi\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right)\, d\xi = 0\,.$$

We have shown, therefore, that (4.2) is true if $J+2 \geq \log_2 \left\{ \frac{\text{diam (supp } f)}{\pi} \right\}$. Thus, together with (4.1), this gives us the equivalence $\lim_{J\to\infty} S_J$ exists if and only if $\lim_{J\to\infty} I_0^J$ exists.

In the present context we have not yet considered the two equations, (i) and (ii), in (3.7); however, these observations can be used to obtain the following version of Theorem 2 when we do not assume the local integrability of the series (3.3) in $\mathbb{R}^n \setminus \{\mathbf{0}\}$:

Theorem 3.

Suppose $\phi, \psi \in L^2(\mathbb{R}^n)$. *Then*

$$\lim_{J\to\infty} S_J = \lim_{J\to\infty} \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}}) (\phi_{j,\mathbf{k}}, f) = \|f\|^2 \tag{4.3}$$

for every $f \in \mathcal{D}$ *if and only if*

$$\left.\begin{array}{ll} (i) & \lim_{J\to\infty} \sum_{j\le J} \hat{\phi}\left(2^{-j}\xi\right) \overline{\hat{\psi}\left(2^{-j}\xi\right)} = \mathbf{1} \\ & \textit{weakly in } L^1(K) \textit{ whenever } K \textit{ is a compact subset of } \mathbb{R}^n \setminus \{\mathbf{0}\}, \\ & \textit{where } \mathbf{1} \textit{ is the constant function that equals 1 on } \mathbb{R}^n, \\ (ii) & t_{\mathbf{q}}(\xi) = \sum_{r=0}^{\infty} \hat{\phi}\left(2^r\xi\right) \overline{\hat{\psi}\left(2^r(\xi + 2\pi\mathbf{q})\right)} = 0, \textit{ for a.e. } \xi \in \mathbb{R}^n, \ \forall \mathbf{q} \in \mathcal{O}^n \end{array}\right\} \tag{4.4}$$

Proof. We first show that (4.4) implies (4.3). Since $t_{\mathbf{q}}(\xi) = 0$ for *a.e.* ξ, (4.1) and (4.2) imply

$$S_J(f) = S_J = (2\pi)^{-n} I_0^J = (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \sum_{j\le J} \hat{\phi}\left(2^{-j}\xi\right) \overline{\hat{\psi}\left(2^{-j}\xi\right)} d\xi .$$

But, by (i) and the fact that $K = \text{supp } \hat{f}$ is a compact subset of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ we have

$$\lim_{J\to\infty} S_J = (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 d\xi = \|f\|^2$$

and, thus, (4.3) is true.

To establish the converse we first show that (4.4)(ii) is a consequence of (4.3). The argument is much like the one we gave for Theorem 1. We select a point ξ_0 of differentiability of the integral of $t_{\mathbf{q}_0}$ such that neither ξ_0 nor $\xi_0 + 2\pi\mathbf{q}_0$ is 0, and $\delta > 0$ such that $B_\delta(\xi_0)$, $B_\delta(\xi_0 + 2\pi\mathbf{q}_0)$ are disjoint balls in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Again we choose f_δ and g_δ such that $\hat{f}_\delta = \frac{1}{\sqrt{\Omega_n\delta^n}} \chi_{B_\delta(\xi_0)}$ and $\hat{g}_\delta = \frac{1}{\sqrt{\Omega_n\delta^n}} \chi_{B_\delta(\xi_0+2\pi\mathbf{q}_0)}$. From (4.1) and polarization we have, for large J (see the discussion before Theorem 3),

$$\begin{aligned} &(2\pi)^n \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f_\delta, \psi_{j,\mathbf{k}}) (\phi_{j,\mathbf{k}}, g_\delta) \\ &= \int_{\mathbb{R}^n} \hat{f}_\delta(\xi)\overline{\hat{g}_\delta(\xi)} \sum_{j\le J} \hat{\phi}\left(2^{-j}\xi\right) \overline{\hat{\psi}\left(2^{-j}\xi\right)} d\xi \\ &\quad + \int_{\mathbb{R}^n} \overline{\hat{g}_\delta(\xi)} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}_\delta\left(\xi + 2\pi 2^p\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi \\ &= \int_{\mathbb{R}^n} \overline{\hat{g}_\delta(\xi)} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}_\delta\left(\xi + 2\pi 2^p\mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi . \end{aligned}$$

since $\hat{f}_\delta \overline{\hat{g}_\delta} = 0$ because $B_\delta(\xi_0) \cap B_\delta(\xi_0 + 2\pi \mathbf{q}_0) = \emptyset$ (we observe that it suffices to choose J so that $2^{J+2}\pi >$ diam(supp $(\hat{f}_\delta + \hat{g}_\delta)$)). On the other hand, by polarization and (4.3) we have

$$\lim_{J\to\infty} \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f_\delta, \psi_{j,\mathbf{k}}\right)\left(\phi_{j,\mathbf{k}}, g_\delta\right) = (f_\delta, g_\delta) = 0$$

(using, again, $\hat{f}_\delta \overline{\hat{g}_\delta} = 0$ and the Plancherel theorem). But we have just shown that for J large enough

$$(2\pi)^n \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f_\delta, \psi_{j,\mathbf{k}}\right)\left(\phi_{j,\mathbf{k}}, g_\delta\right) =$$
$$\int_{\mathbb{R}^n} \overline{\hat{g}_\delta(\xi)} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}_\delta\left(\xi + 2\pi 2^p \mathbf{q}\right) t_{\mathbf{q}}\left(2^{-p}\xi\right) d\xi \equiv A_\delta$$

and the last expression is independent of J. It follows that $A_\delta = 0$.

But the argument that was presented at the end of Section 2, which showed $\lim_{\delta\to 0+} A_\delta = t_{\mathbf{q}_0}(\xi_0)$, applies here. Consequently, $t_{\mathbf{q}_0}(\xi_0) = 0$ and (4.4)(ii) is established since almost every point in $\mathbb{R}^n$ is such ξ_0, a point of differentiability of the integral of $t_{\mathbf{q}_0}$.

It is now particularly easy to show that (4.4)(i) is also a consequence of (4.3). In fact, if we use (4.1), (4.2), and (4.4)(ii) we have

$$S_J = (2\pi)^{-n} I_0^J = (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \sum_{j\le J} \hat{\phi}\left(2^{-j}\xi\right) \overline{\hat{\psi}\left(2^{-j}\xi\right)}\, d\xi\,.$$

Thus, by (4.3) we have

$$\lim_{J\to\infty} (2\pi)^{-n} \int_K g(\xi) \sum_{j\le J} \hat{\phi}\left(2^{-j}\xi\right) \overline{\hat{\psi}\left(2^{-j}\xi\right)}\, d\xi = (2\pi)^{-n} \int_K g(\xi)\, d\xi$$

for every $g \ge 0$, $g \in L^\infty(\mathbb{R}^n)$ with supp g a compact subset K of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Writing $g = g_1 + ig_2 = g_1^+ - g_1^- + i(g_2^+ - g_2^-)$ for the general $g \in L^\infty(K)$ we obtain (4.4)(i). □

Remark 5.

The weak convergence of the sequence in (4.4)(i) can be also expressed as

$$\lim_{J\to\infty} \sum_{j\le J} \hat{\phi}\left(2^{-j}\xi\right) \overline{\hat{\psi}\left(2^{-j}\xi\right)} = \mathbf{1} \quad in\ \sigma\left(L^1_{\text{loc}}\left(\mathbb{R}^n \setminus \{\mathbf{0}\}\right), L^\infty_c\left(\mathbb{R}^n \setminus \{\mathbf{0}\}\right)\right),$$

the ω-topology of the Fréchet space $L^1_{\text{loc}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ with respect to its dual $L^\infty_c(\mathbb{R}^n \setminus \{\mathbf{0}\})$. The same reasoning applies in (4.3), in which we may say that

$$f = \lim_{J\to\infty} \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \psi_{j,\mathbf{k}}\right)\phi_{j,\mathbf{k}} = \lim_{J\to\infty} \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \phi_{j,\mathbf{k}}\right)\psi_{j,\mathbf{k}} \quad in\ \sigma(\mathcal{D}^*, \mathcal{D}) \qquad (4.5)$$

where the dense space $\mathcal{D}$ has the topology inherited by the Fréchet space $\widehat{\mathcal{D}} = L^\infty_c(\mathbb{R}^n \setminus \{\mathbf{0}\})$ i.e., $f_n \to f$ in $\mathcal{D}$ if and only if $\hat{f}_n \to \hat{f}$ in $L^\infty_c(\mathbb{R}^n \setminus \{\mathbf{0}\})$ and the convergence of the sequences in (4.5) is in the ω^*-topology of the dual space $\mathcal{D}^*$ of $\mathcal{D}$.

This result, Theorem 3, applies to the example of the pair of functions, ϕ and ψ, we introduced in Section 3. In fact, Proposition 1 tells us that, in particular, (4.4)(i) and (ii) are satisfied by this pair of functions. Moreover, if K is a compact set of $\mathbb{R}^n \setminus \{\mathbf{0}\}$, then (4.4)$(i)$ is a finite sum (in j) when $\xi \in K$. Thus, the pair ϕ and ψ satisfy (4.3) which, by polarization, is equivalent to

$$\lim_{J\to\infty} \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \psi_{j,\mathbf{k}}\right)\left(\phi_{j,\mathbf{k}}, g\right) = (f, g)$$

for all $f, g \in \mathcal{D}$ (in this case the series above is again a finite sum over j). This is a weak form of the representation

$$f = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \psi_{j,\mathbf{k}}\right) \phi_{j,\mathbf{k}} \tag{4.6}$$

for $f, g \in \mathcal{D}$, as we indicated in (4.5). We have already discussed why we cannot expect (4.6) to be true in $L^2(\mathbb{R}^n)$ as an unconditionally convergent series. Another inconvenient feature of our results is that we have established them only for $f \in \mathcal{D}$. Of course, we assumed very little about ϕ and ψ besides the equations (4.4)(i) and (ii). We shall end this article with a result that involves the $L^2(\mathbb{R}^n)$-convergence of the series in (4.6) based on a "natural" hypothesis about the systems $\{\phi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$.

Theorem 4.

Suppose $\phi, \psi \in L^2(\mathbb{R}^n)$. Then

$$f = \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} \left(f, \psi_{j,\mathbf{k}}\right) \phi_{j,\mathbf{k}} = \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} \left(f, \phi_{j,\mathbf{k}}\right) \psi_{j,\mathbf{k}} \tag{4.7}$$

for all $f \in L^2(\mathbb{R}^n)$ with both series converging unconditionally in $L^2(\mathbb{R}^n)$, is equivalent to the following three properties. There exists a constant $C > 0$ such that

$$\left.\begin{aligned} \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} \left|\left(f, \psi_{j,\mathbf{k}}\right)\right|^2 \le C \|f\|^2 \\ \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} \left|\left(f, \phi_{j,\mathbf{k}}\right)\right|^2 \le C \|f\|^2 \end{aligned}\right\} \quad \text{for all } f \in L^2(\mathbb{R}^n)\,, \tag{4.8}$$

$$\sum_{j\in\mathbb{Z}} \hat{\psi}\left(2^j\xi\right) \overline{\hat{\phi}\left(2^j\xi\right)} = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n\,, \tag{4.9}$$

$$t_{\mathbf{q}}(\xi) = \sum_{m=0}^{\infty} \hat{\phi}\left(2^m\xi\right) \overline{\hat{\psi}\left(2^m(\xi + 2\pi\mathbf{q})\right)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n,\ \forall\, \mathbf{q} \in \mathcal{O}^n\,, \tag{4.10}$$

where the series in (4.9) and (4.10) are absolutely convergent for a.e. $\xi \in \mathbb{R}^n$.

Remark 6.

A system $\{e_j\}$ of vectors in a Hilbert space $\mathcal{H}$ such that $\sum_j |(f, e_j)|^2 \le C\|f\|^2$ for all $f \in \mathcal{H}$ is called a Bessel sequence. Thus, condition (4.8) asserts that $\{\phi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$ are each a Bessel sequence. A simple condition that guarantees that ψ generates a Bessel sequence $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, (see [2]) is the following: Let θ be any non-negative function on $[0, \infty)$ that is increasing on $[0, 1)$ and decreasing on $[1, \infty)$; suppose, in addition, that

$$\int_0^\infty \theta(w)\left(1 + \frac{1}{w}\right) dw < \infty\,.$$

Then, if $\psi \in L^2(\mathbb{R})$ satisfies $\left|\hat{\psi}(w)\right| \le \theta(|w|)$ for $w \in \mathbb{R}$, $\{\psi_{j,k}(x)\} = \{a^{j/2}\psi(a^j x - bk)\}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}$, is a Bessel sequence whenever $a > 1$ and $b > 0$. *An n-dimensional version of this result is easy to obtain. We cite the result in [2] to show that (4.8) is not very restrictive; the couples ϕ, ψ introduced by Frazier and Jawerth (see [4, 5]) satisfy these conditions.*

Remark 7.

It will be shown in the course of the proof of Theorem 4 that if (4.8) holds, then there exists a positive constant C such that

$$\sum_{j\in\mathbb{Z}} \left|\hat{\phi}\left(2^j\xi\right)\right|^2 \le C \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(2^j\xi\right)\right|^2 \le C \quad \text{a.e. } \xi \in \mathbb{R}^n \tag{4.11}$$

In particular, the series in (4.9) is absolutely convergent for a.e. ξ.

Remark 8.

Each of the systems $\{\phi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$, is a frame. *In fact, suppose the first inequality in (4.8) is satisfied and*

$$\|f\|^2 = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}}) (\phi_{j,\mathbf{k}}, f)$$

for all $f \in L^2(\mathbb{R}^n)$ (which follows from (4.7)). Then

$$\begin{aligned} \|f\|^2 &= \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}}) (\phi_{j,\mathbf{k}}, f) \\ &\le \left(\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |(f, \psi_{j,\mathbf{k}})|^2 \right)^{1/2} \left(\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |(f, \phi_{j,\mathbf{k}})|^2 \right)^{1/2} \\ &\le \sqrt{C}\, \|f\|_2 \left(\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |(f, \phi_{j,\mathbf{k}})|^2 \right)^{1/2} . \end{aligned}$$

Dividing by $\sqrt{C}\|f\|_2$ we obtain

$$\frac{1}{C} \|f\|^2 \le \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |(f, \phi_{j,\mathbf{k}})|^2 .$$

Thus, together with the second inequality in (4.8) shows that $\{\phi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$, is a frame.

Proof. We first show that (4.7) implies (4.8), (4.9), and (4.10). We will use the following result for a general Hilbert space $\mathcal{H}$ (see [10], Vol I, Lemma (14.9)(b) on page 425):

Lemma 6.

Let $\{x_i\}_{i=1}^{\infty}$ be a system of vectors in $\mathcal{H}$. If $\sum_{i=1}^{\infty} x_i$ converges unconditionally in $\mathcal{H}$, then

$$\sum_{i=1}^{\infty} \|x_i\|^2 \le C = C(\{x_i\}) < \infty .$$

We apply this lemma to the sequence $\{(f, \phi_{j,\mathbf{k}})\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$: $\sum_{j,\mathbf{k}}(f, \phi_{j,\mathbf{k}})\psi_{j,\mathbf{k}}$ converges unconditionally by (4.7); thus, since ϕ and ψ are fixed and non-zero in this discussion,

$$\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |(f, \phi_{j,\mathbf{k}})|^2 = \|\psi\|^{-2} \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} \left\|(f, \phi_{j,\mathbf{k}})\ \psi_{j,\mathbf{k}}\right\|_2^2 \le C\ \|\psi\|^{-2} = C_f . \qquad (4.12)$$

Consider the linear operator defined on $L^2(\mathbb{R}^n)$ by

$$Tf = \{(f, \phi_{j,\mathbf{k}})\} , \qquad (j, \mathbf{k}) \in \mathbb{Z} \times \mathbb{Z}^n$$

Inequality (4.12) shows that it maps $L^2(\mathbb{R}^n)$ into $\ell^2(\mathbb{Z}\times\mathbb{Z}^n)$. Suppose the pair $(f, \boldsymbol{\ell}$, with $\boldsymbol{\ell} = \{\ell_{j,\mathbf{k}}\}$, is a limit point of its graph. Then there exists a sequence $\{f_m\}$, $m \in \mathbb{N}$, such that $(f_m, Tf_m) \to (f, \boldsymbol{\ell})$ in the graph norm as $m \to \infty$. In particular, $\lim_{m\to\infty} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} |(f_m, \phi_{j,\mathbf{k}}) - \ell_{j,\mathbf{k}}|^2 = 0$ showing that $\ell_{j,\mathbf{k}} = \lim_{m\to\infty}(f_m, \phi_{j,\mathbf{k}}) = (f, \phi_{j,\mathbf{k}})$ for each $(j, \mathbf{k}) \in \mathbb{Z} \times \mathbb{Z}^n$. Thus, the graph of T is closed and, as a consequence, T is bounded: there exists a constant $C > 0$, independent of $f \in L^2(\mathbb{R}^n)$, such that

$$\sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} |(f, \phi_{j,\mathbf{k}})|^2 \le C\, \|f\|^2 .$$

The same argument shows that

$$\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n}\left|\left(f,\psi_{j,\mathbf{k}}\right)\right|^2 \le C\,\|f\|^2$$

for all $f\in L^2(\mathbb{R}^n)$, and this establishes (4.8).

Let ρ denote either ϕ or ψ and let us apply the decomposition (2.3) for $f\in\mathcal{D}$ and $t_{\mathbf{q}}(\xi)=\sum_{m=0}^{\infty}\hat\rho(2^m\xi)\overline{\hat\rho(2^m(\xi+2\pi\mathbf{q}))}$:

$$(2\pi)^n\sum_{(j,\mathbf{k})\in\mathbb{Z}\times\mathbb{Z}^n}\left|\left(f,\rho_{j,\mathbf{k}}\right)\right|^2=\int_{\mathbb{R}^n}\left|\hat f(\xi)\right|^2\sum_{j\in\mathbb{Z}}\left|\hat\rho\left(2^j\xi\right)\right|^2 d\xi$$
$$+\int_{\mathbb{R}^n}\overline{\hat f(\xi)}\sum_{p\in\mathbb{Z}}\sum_{\mathbf{q}\in\mathcal{O}^n}\hat f\left(\xi+2\pi 2^p\mathbf{q}\right)t_{\mathbf{q}}\left(2^{-p}\xi\right)d\xi\,.$$

By the observation we made in Remark 4 and (4.8), we see that $\sum_{j\in\mathbb{Z}}|\hat\rho(2^j\xi)|^2=\tau(\xi)$ is locally integrable in $\mathbb{R}^n\setminus\{0\}$. We choose ξ_0 to be a point of differentiability of the integral of $\tau(\xi)$ and f_δ so that $\hat f_\delta=\frac{1}{\sqrt{\Omega_n\delta^n}}\chi_{B_\delta(\xi_0)}$ (see (2.8) and the sentence that follows). Applying the argument we presented after (2.8) we have

$$\frac{1}{\Omega_n\delta^n}\int_{B_\delta(\xi_0)}\tau(\xi)\,d\xi+I_1^\delta=\int_{\mathbb{R}^n}\left|\hat f_\delta(\xi)\right|^2\tau(\xi)\,d\xi+I_1^\delta=(2\pi)^n I^\delta$$
$$=(2\pi)^n\sum_{(j,\mathbf{k})\in\mathbb{Z}\times\mathbb{Z}^n}\left|\left(f_\delta,\rho_{j,\mathbf{k}}\right)\right|^2\le(2\pi)^n C\,\|f_\delta\|_2^2=C\,,$$

where the inequality follows from (4.8). Letting $\delta\to 0+$, since $\lim_{\delta\to 0+}I_1^\delta=0$, we obtain $\tau(\xi_0)\le C$. That is, (4.11) is satisfied and the series in (4.9) is absolutely convergent for *a.e.* ξ. The equality part of (4.9) and equality (4.10) are consequences of Theorem 2.

We now turn to the converse. We begin by proving the two extensions of Lemmas 1 and 2 that involve two systems $\mathcal{E}=\{e_j\}$, $\mathcal{F}=\{f_j\}$, $j\in\mathbb{N}$, of vectors in a Hilbert space $\mathcal{H}$. These are general versions of the systems $\{\phi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n$, which, for simplicity, we index by the natural numbers $\mathbb{N}$. In this context, (4.8) becomes

$$(i)\quad\sum_{i\in\mathbb{N}}|(h,e_i)|^2\le C\,\|h\|^2\qquad (ii)\quad\sum_{i\in\mathbb{N}}|(h,f_i)|^2\le C\,\|h\|^2 \tag{4.13}$$

for all $h\in\mathcal{H}$.

Lemma 7.

Suppose $\mathcal{E}=\{e_j\}$ and $\mathcal{F}=\{f_j\}$ satisfy (4.13) and for all h in a dense subset $\mathcal{D}$ of $\mathcal{H}$

$$\|h\|^2=\sum_{i\in\mathbb{N}}(h,e_i)(f_i,h)\,. \tag{4.14}$$

Then equality (4.14) is valid for all $h\in\mathcal{H}$.

Proof. Let $h\in\mathcal{H}$; (4.13) implies that the series in (4.14) is absolutely convergent. Let $\{h_n\}\subset\mathcal{D}$ be a sequence such that $\|h_n-h\|\to 0$ as $n\to\infty$. Then, by Schwarz's inequality and (4.13),

$$\left|\sum_{j\in\mathbb{N}}\left(h,e_j\right)\left(f_j,h\right)-\|h\|^2\right|\le\left|\sum_{j\in\mathbb{N}}\left(h-h_n,e_j\right)\left(f_j,h\right)+\left(h_n,e_j\right)\left(f_j,h-h_n\right)\right|$$
$$+\left|\,\|h_n\|^2-\|h\|^2\right|\le C\,\|h-h_n\|\;\|h\|+C\,\|h_n\|\;\|h-h_n\|+\left|\,\|h_n\|^2-\|h\|^2\right|$$

which goes to 0 as $n \to \infty$. □

Lemma 8.

Suppose $\mathcal{E} = \{e_j\}$ and $\mathcal{F} = \{f_j\}$ satisfy (4.13). Then the following two properties are equivalent

$$(i) \qquad \|h\|^2 = \sum_{j\in\mathbb{N}} \left(h, e_j\right)\left(f_j, h\right) \quad \textit{for all } f \in \mathcal{H}$$

$$(ii) \qquad h = \sum_{j\in\mathbb{N}} \left(h, e_j\right) f_j = \sum_{j\in\mathbb{N}} \left(h, f_j\right) e_j \quad \textit{for all } f \in \mathcal{H} \textit{ with convergence in } \mathcal{H}\,.$$

In this case, the convergence of all the series is unconditional.

Proof. That (ii) implies (i) is trivial. Let us, then, assume (i). By polarization we have

$$(g, h) = \sum_{j\in\mathbb{N}} \left(g, e_j\right)\left(f_j, h\right) \quad \text{for all } g, h \in \mathcal{H}\,. \tag{4.15}$$

If we can show that the partial sums of $\sum_{j=1}^{\infty}(h, e_j)f_j$ (or the second series in (ii)) form a Cauchy sequence, it follows that (ii) must be true. Indeed, if $u = \sum_{j=1}^{\infty}(h, e_j)f_j$ then, using (4.15), we must have

$$(u, g) = \sum_{j=1}^{\infty} \left(h, e_j\right)\left(f_j, g\right) = (h, g)$$

for all $g \in \mathcal{H}$ and, thus, $u = h$. But

$$\left\| \sum_{j=M}^{N} \left(h, e_j\right) f_j \right\| = \sup_{\|g\|=1} \left| \sum_{j=M}^{N} \left(h, e_j\right)\left(f_j, g\right) \right| \le$$

$$\sup_{\|g\|=1} \left(\sum_{j=M}^{N} \left|\left(h, e_j\right)\right|^2 \right)^{1/2} \left(\sum_{j=M}^{N} \left|\left(g, f_j\right)\right|^2 \right)^{1/2} \le$$

$$\sup_{\|g\|=1} \left(\sum_{j=M}^{N} \left|\left(h, e_j\right)\right|^2 \right)^{1/2} C^{1/2}\|g\| = C^{1/2} \left(\sum_{j=M}^{N} \left|\left(h, e_j\right)\right|^2 \right)^{1/2},$$

where the last inequality is a consequence of (4.13)(ii). Since the series $\sum_{j=1}^{\infty} |(h, e_j)|^2$ is convergent (by (4.13)(i)) we see that the partial sums in question do form a Cauchy sequence. □

We can now easily finish the proof of Theorem 4. Equalities (4.9), (4.10), and inequality (4.8) (which as we indicated implies (4.11)) permit us to apply Theorem 2 to obtain (3.8) for all $f \in \mathcal{D}$. An application of Lemma 7 then gives us the equality

$$\sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(f, \psi_{j,\mathbf{k}}\right)\left(\phi_{j,\mathbf{k}}, f\right) = \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$ and, then, by Lemma 8 the desired equalities (4.7). □

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Dept. of Mathematics
Michigan State University
East Lansing, MI 48824

Dept. of Mathematics
Washington University
St. Louis, MO 63130

Buddhist Tzu-Chi College of Nursing
Hualien, Taiwan

Dept. of Mathematics
Washington University
St. Louis, MO 63130