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Wavelets, Orlicz spaces, and greedy bases

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Abstract

We study the efficiency of greedy algorithms for N-term wavelet approximation in Orlicz spaces $L^{\Phi}(\mathbb{R}^d)$. We compute the left and right democracy functions in terms of the fundamental function of L^{Φ}, recovering a recent result of Wojtaszczyk [P. Wojtaszczyk, Greediness of the Haar system in rearrangement invariant spaces, in: T. Figiel, A. Kamont (Eds.), Approximation and Probability, vol. 72, Banach Center Publications, Warszawa, 2006, pp. 385–395], which establishes that wavelet bases can only be greedy when $L^{\Phi} = L^p$ for some $1 < p < \infty$. In addition, optimal Jackson and Bernstein inequalities are obtained, as well as inclusions for the approximation spaces based on L^{Φ}. These inclusions are expressed in terms of sequence spaces of weighted Lorentz and Marcinkiewicz type, with the weights depending on the fundamental function of L^{Φ}, which in some cases can be described as Besov spaces of generalized smoothness.

Keywords: Greedy algorithm; Non-linear approximation; Orlicz spaces; Wavelets

1. Introduction

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a Banach (or quasi-Banach) space with a countable unconditional basis $\mathcal{B} = \{e_j\colon\ j \in \mathbb{N}\}$; that is, every $x \in \mathbb{B}$ can be uniquely represented as an unconditionally convergent series $x = \sum_{j\in\mathbb{N}} s_j e_j$, for some sequence of scalars $\{s_j\}$. Let Σ_N denote the set of all elements $y \in \mathbb{B}$ with at most N non-null coefficients in the basis representation $y = \sum_{j\in\mathbb{N}} s_j e_j$. For $x \in \mathbb{B}$, the *N-term error of approximation* (with respect to $\mathcal{B}$) is defined by

$$\sigma_N(x)_{\mathbb{B}} = \inf\bigl\{\|x - y\|_{\mathbb{B}}\colon\ y \in \Sigma_N\bigr\}. \tag{1.1}$$

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Two main questions in approximation theory concern the construction of efficient algorithms for N-term approximation, and the characterization of the *approximation spaces*

$$A_q^\alpha(\mathbb{B}) = \left\{ x \in \mathbb{B}\colon \left[\sum_{N \geqslant 1} \left(N^\alpha \sigma_N(x)_{\mathbb{B}}\right)^q \frac{1}{N} \right]^{\frac{1}{q}} < \infty \right\}, \tag{1.2}$$

when $\alpha > 0$ and $0 < q \leqslant \infty$ (with the obvious modification when $q = \infty$).

A computationally efficient method to produce N-term approximations, which has been widely investigated in recent years, is the so-called *greedy algorithm*. If $x = \sum_{j\in\mathbb{N}} s_j e_j$ and we order the basis elements in such a way that

$$\|s_{j_1} e_{j_1}\|_{\mathbb{B}} \geqslant \|s_{j_2} e_{j_2}\|_{\mathbb{B}} \geqslant \|s_{j_3} e_{j_3}\|_{\mathbb{B}} \geqslant \cdots$$

(handling ties arbitrarily), the *greedy algorithm of step N* is defined by the correspondence

$$x = \sum_{j\in\mathbb{N}} s_j e_j \in \mathbb{B} \to G_N(x) = \sum_{k=1}^{N} s_{j_k} e_{j_k} \in \Sigma_N. \tag{1.3}$$

It is clear that $\sigma_N(x)_{\mathbb{B}} \leqslant \|x - G_N(x)\|_{\mathbb{B}}$. A basis $\mathcal{B}$ is said to be *greedy* in $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ if the converse inequality holds up to a constant, that is, for some $c \geqslant 1$

$$\frac{1}{c}\left\|x - G_N(x)\right\|_{\mathbb{B}} \leqslant \sigma_N(x)_{\mathbb{B}}, \quad \forall x \in \mathbb{B},\ N = 1, 2, \ldots.$$

Thus, for such bases the greedy algorithm produces an almost optimal N-term approximation, which leads often to a precise identification of the approximation spaces $A_q^\alpha(\mathbb{B})$. A result of Konyagin and Temlyakov [19] characterizes greedy bases in a Banach space $\mathbb{B}$ as those which are unconditional and *democratic*, the latter meaning that for some constant $C > 0$

$$\left\| \sum_{\gamma\in\Gamma} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \right\|_{\mathbb{B}} \leqslant C \left\| \sum_{\gamma\in\Gamma'} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \right\|_{\mathbb{B}}$$

holds for all finite sets of indices $\Gamma, \Gamma' \subset \mathbb{N}$ with the same cardinality.

Wavelet systems are well known examples of greedy bases for many function and distribution spaces. Indeed, Temlyakov showed in [29] that the Haar basis (and any wavelet system L^p-equivalent to it) is greedy in the Lebesgue spaces $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. When wavelets have sufficient smoothness and decay, they are also greedy bases for the more general Sobolev and Triebel–Lizorkin classes (see, e.g., [11,14]).

The purpose of this paper is to study the efficiency of wavelet greedy algorithms in the class of *Orlicz spaces* $L^\Phi(\mathbb{R}^d)$. We recall that, as M. Soardi proved in [28], wavelet bases are unconditional in every L^Φ with non-trivial Boyd indices (see Section 2 below for definitions and precise statements). It may seem surprising that wavelet bases are not democratic (hence not greedy) in a typical L^Φ space.

Theorem 1.1. *(See Wojtaszczyk [31].) Let $L^\Phi(\mathbb{R}^d)$ be an Orlicz space with non trivial Boyd indices. An admissible wavelet basis is democratic in $L^\Phi(\mathbb{R}^d)$ if and only if $L^\Phi(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ for some $1 < p < \infty$.*

This result makes interesting to understand how far wavelet bases are from being democratic in general L^Φ spaces. To quantify democracy of a basis $\mathcal{B} = \{e_j\}_{j\in\mathbb{N}}$ we shall study the following functions:

$$h_r(N) = \sup_{\mathrm{Card}(\Gamma)=N} \left\| \sum_{\gamma\in\Gamma} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \right\|_{\mathbb{B}} \quad \text{and} \quad h_\ell(N) = \inf_{\mathrm{Card}(\Gamma)=N} \left\| \sum_{\gamma\in\Gamma} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \right\|_{\mathbb{B}},$$

which we call right and left democracy functions of $\mathcal{B}$ (see also [9,16]). Observe that a basis is democratic if and only if these two quantities are comparable for all $N \geqslant 1$. Our main result gives a precise estimate for these functions in terms of intrinsic properties of the space L^Φ. Namely, let $H_\varphi^+(t) = \sup_{s>0} \varphi(ts)/\varphi(s)$ denote the dilation function associated with the *fundamental function φ of L^Φ*, and let H_φ^- be the same quantity with "sup" replaced by "inf" (see Section 2.1 for the precise definitions).

Theorem 1.2. *Let $L^\Phi(\mathbb{R}^d)$ be an Orlicz space with non-trivial Boyd indices. Then,*

$$h_r(N) \simeq H_\varphi^+(N) \quad \text{and} \quad h_\ell(N) \simeq H_\varphi^-(N)$$

where the involved constants are independent of $N \geqslant 1$.

This result will have interesting applications in the study of greedy approximation in Orlicz spaces. We take up this task in the last part of the paper, where we investigate Jackson and Bernstein type estimates and corresponding inclusions for the N-term approximation spaces. In the well-known L^p case, these estimates are naturally given in terms of the class of *discrete Lorentz spaces* $\ell^{\tau,q}$ (see, e.g., [7,11,12,14,17]). In the general Orlicz situation we shall need *weighted Lorentz sequence spaces*, defined by

$$\Lambda_\eta^q = \left\{ \mathbf{s}\colon\ \|\mathbf{s}\|_{\Lambda_\eta^q} = \left[\sum_{k\geqslant 1} \big(\eta_k |s_k^*|\big)^q \frac{1}{k} \right]^{\frac{1}{q}} < \infty \right\},$$

where $\{s_k^*\}$ is the non-increasing rearrangement of $\mathbf{s}$ and the weight $\eta = \{\eta_k\}$ is a fixed increasing and doubling sequence (see Section 6 below). In particular, $\Lambda_\eta^q = \ell^{\tau,q}$ when $\eta_k = k^{1/\tau}$. Weighted Lorentz spaces have already been used in the study of approximation spaces associated with multivariate Haar systems (see, e.g., [16]). To state our result we use the notation

$$\mathfrak{s}\big(L^\Phi\big) = \big\{ f \in L^\Phi\big(\mathbb{R}^d\big)\colon\ \big\{ \langle f, \psi_Q\rangle \|\psi_Q\|_{L^\Phi} \big\}_Q \in \mathfrak{s} \big\}, \tag{1.4}$$

for any fixed sequence space $\mathfrak{s}$, indexed on the set of dyadic cubes in $\mathbb{R}^d$.

Theorem 1.3. *Let $L^\Phi(\mathbb{R}^d)$ be an Orlicz space with Boyd indices $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$, and let $\alpha > 0$ and $0 < q \leqslant \infty$. Then*

$$\Lambda^q_{k^\alpha h_r(k)}\big(L^\Phi\big) \hookrightarrow A_q^\alpha\big(L^\Phi\big) \hookrightarrow \Lambda^q_{k^\alpha h_\ell(k)}\big(L^\Phi\big). \tag{1.5}$$

These embeddings are optimal, in the sense that the largest and smallest weighted Lorentz spaces that one can place on the left- and right-hand side of (1.5) are respectively $\Lambda^q_{k^\alpha h_r(k)}$ and $\Lambda^q_{k^\alpha h_\ell(k)}$. We point out that a necessary and sufficient condition for these two spaces to be equal is that $h_r(N) \simeq h_\ell(N)$, in which case the basis is necessarily greedy and $L^\Phi = L^p$. Theorem 1.3 leads also to the following inclusions in terms of classical Lorentz spaces.

Corollary 1.4. *Under the same hypotheses of Theorem* 1.3 *we have*:

(a) $A_q^\alpha(L^\Phi) \hookrightarrow \ell^{\tau,q_1}(L^\Phi)$, *for all* $\frac{1}{\tau} < \alpha + \underline{\pi}_{L^\Phi}$ *and* $q_1 \in (0,\infty]$.
(b) $\ell^{\tau,q_1}(L^\Phi) \hookrightarrow A_q^\alpha(L^\Phi)$, *for all* $\frac{1}{\tau} > \alpha + \overline{\pi}_{L^\Phi}$ *and* $q_1 \in (0,\infty]$.

Finally, we point out that some of these inclusions can be described in terms of *Besov spaces of generalized smoothness* [15,22], namely,

$$B_q^\Psi\big(L^\tau\big) = \big\{ f\colon\ \big\{ \Psi\big(2^j\big) \|f * \psi_j\|_\tau \big\} \in \ell^q(\mathbb{Z}) \big\},$$

for suitable increasing functions $\Psi(t)$. We refer to Section 6.4 below for precise statements and explicit results in the particular case of the Zygmund classes $L^p(\log L)^\gamma(\mathbb{R}^d)$.

The organization of the paper is a follows. Section 2 contains definitions and results concerning Orlicz spaces, wavelet bases and the greedy algorithm. Some examples of Orlicz spaces with non-democratic wavelet bases are given in Section 3. Sections 4 and 5 are devoted to the proofs of Theorems 1.2 and 1.1, respectively. Jackson and Bernstein type estimates, as well as the inclusions described in Theorem 1.3 and Corollary 1.4 are given in Section 6.

Remark 1.5. In 2006, after the manuscript of this paper was completed, we discovered an earlier preprint of P. Wojtaszczyk [31] where a more general result than Theorem 1.1 is proved; namely, wavelet bases are actually not greedy in any rearrangement invariant space distinct from L^p. Since our approach to this problem has been independent and different from [31], we have included our original proof of Theorem 1.1, based on the stronger result stated in Theorem 1.2.

2. Preliminaries

2.1. Basics on Orlicz spaces

In this section we recall some basic facts about Orlicz spaces, referring to [27] and [3] for a complete account on this topic.

A *Young function* is a convex non-decreasing function $\Phi:[0,\infty)\to[0,\infty]$ so that $\lim_{t\to 0^+}\Phi(t)=0$ and $\lim_{t\to+\infty}\Phi(t)=\infty$. Throughout this paper we shall assume that Φ is strictly increasing and everywhere finite,[1] so that it is a continuous bijection of $[0,\infty)$. Given such Φ, the Orlicz space $L^\Phi(\mathbb{R}^d)$ is the set of all measurable functions $f:\mathbb{R}^d\to\mathbb{C}$ so that $\Phi(|f(x)|/\lambda)\in L^1(\mathbb{R}^d)$ for some $\lambda>0$. It is well known that $L^\Phi(\mathbb{R}^d)$ becomes a rearrangement invariant Banach function space when endowed with the *Luxemburg norm*

$$\|f\|_{L^\Phi(\mathbb{R}^d)}=\inf\left\{\lambda>0:\int_{\mathbb{R}^d}\Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x\leqslant 1\right\} \tag{2.1}$$

(see, e.g., [3, p. 269]). The *fundamental function* of a rearrangement invariant space $\mathbb{X}$ in $\mathbb{R}^d$ is defined by $\varphi(t)=\|\chi_A\|_{\mathbb{X}}$, where $A\subset\mathbb{R}^d$ is any measurable set with Lebesgue measure $|A|=t$. In the particular case of Orlicz spaces $\mathbb{X}=L^\Phi(\mathbb{R}^d)$, the fundamental function can be computed explicitly in terms of Φ, by means of the formula

$$\varphi(t)=\frac{1}{\Phi^{-1}(1/t)},\quad t>0 \tag{2.2}$$

(see [3, p. 276]). Observe that φ is a continuous strictly increasing bijection of $[0,\infty)$. Moreover, it can be shown that φ is a *quasi-concave* function, that is, $\varphi(t)/t$ is non-increasing [3, p. 67].

The *Boyd indices*, $\underline{\pi}_{\mathbb{X}},\overline{\pi}_{\mathbb{X}}$ of a rearrangement invariant function space $\mathbb{X}$ are usually defined in terms of the norms of the so-called "dilation operators" [3, p. 149]. However, in the special case of Orlicz spaces $\mathbb{X}=L^\Phi$, the Boyd indices can be computed directly from the fundamental function φ. More precisely, if we denote the *dilation function associated with* φ by

$$H_\varphi^+(t)=\sup_{s>0}\frac{\varphi(st)}{\varphi(s)},\quad t>0, \tag{2.3}$$

then the *lower and upper Boyd indices of* $L^\Phi(\mathbb{R}^d)$ are given by

$$\begin{aligned}\underline{\pi}_{L^\Phi}=i_\varphi&=\lim_{t\to 0^+}\frac{\log H_\varphi^+(t)}{\log t}=\sup_{0<t<1}\frac{\log H_\varphi^+(t)}{\log t},\\ \overline{\pi}_{L^\Phi}=I_\varphi&=\lim_{t\to\infty}\frac{\log H_\varphi^+(t)}{\log t}=\inf_{1<t<\infty}\frac{\log H_\varphi^+(t)}{\log t}\end{aligned} \tag{2.4}$$

(see [3, p. 277], [20, p. 54]). In particular, $0\leqslant i_\varphi\leqslant I_\varphi\leqslant 1$. Assuming further that $i_\varphi>0$ it follows that

$$\varphi(st)\leqslant C_\varepsilon\max\left\{s^{i_\varphi-\varepsilon},s^{I_\varphi+\varepsilon}\right\}\varphi(t),\quad s,t>0 \tag{2.5}$$

and

$$\varphi(st)\geqslant C_\varepsilon\min\left\{s^{i_\varphi-\varepsilon},s^{I_\varphi+\varepsilon}\right\}\varphi(t),\quad s,t>0 \tag{2.6}$$

for every $\varepsilon>0$ and some constant $C_\varepsilon>0$ (see, e.g., [18, p. 3]).

In our applications we shall only consider Orlicz spaces with non-trivial Boyd indices, that is, $0<\underline{\pi}_{L^\Phi}\leqslant\overline{\pi}_{L^\Phi}<1$. In this case, from (2.5) and (2.6) we see that

$$\lim_{s\to 0^+}\frac{\varphi(s)}{s}=\lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty\quad\text{and}\quad\lim_{s\to\infty}\frac{\varphi(s)}{s}=\lim_{t\to 0^+}\frac{\Phi(t)}{t}=0.$$

Thus, with the terminology of [27], Φ will be an *N-function* (or "nice" Young function).

[1] This restriction avoids a few pathological cases which fall outside the scope of this paper.

Finally we shall denote by Δ_2 the set of all non-negative functions $h(t)$ in $[0,\infty)$ which are *doubling*, i.e., $0 \leqslant h(2t) \leqslant Ch(t)$ for some constant $C > 0$ and all $t > 0$. It is not difficult to see from (2.2)–(2.6) that $\underline{\pi}_{L^\Phi} > 0$ is actually equivalent to $\Phi \in \Delta_2$. In fact, if (Φ, Ψ) is a pair of *complementary Young functions* (see, e.g., [27, p. 6] for the precise definition), then $\Phi, \Psi \in \Delta_2$ is equivalent to say that (L^Φ, L^Ψ) is a pair of reflexive Orlicz spaces with $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$. Some of these properties will be used below without further mention.

Example 2.1. When $\Phi(t) = t^p$, $1 \leqslant p < \infty$, then $L^\Phi(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ and $\varphi(t) = t^{1/p}$. Hence, $H^+_\varphi(t) = t^{1/p}$, which implies $\underline{\pi}_{L^\Phi} = \overline{\pi}_{L^\Phi} = 1/p$.

Example 2.2. When $\Phi(t) = t^p[\log(e+t)]^\alpha$, with $\alpha > 0$ and $1 \leqslant p < \infty$, then L^Φ is the classical Zygmund space $L^p(\log L)^\alpha$. In this case, $\varphi(t) \simeq t^{1/p}(1+\log^+ 1/t)^{\alpha/p}$ and $H^+_\varphi(t) \simeq t^{1/p}(1+\log^+ 1/t)^{\alpha/p}$, which implies $\underline{\pi}_{L^\Phi} = \overline{\pi}_{L^\Phi} = 1/p$.

Example 2.3. Let $\Phi(t) \simeq t^p[\log(e+t)]^\alpha$, with $\alpha < 0$ and $1 < p < \infty$. Then $\varphi(t) \simeq t^{1/p}(1+\log^+ 1/t)^{\alpha/p}$ and $H^+_\varphi(t) \simeq t^{1/p}/(1+\log^+ t)^{\alpha/p}$, which implies $\underline{\pi}_{L^\Phi} = \overline{\pi}_{L^\Phi} = 1/p$.

Example 2.4. Consider the Young function

$$\Phi(t) = \begin{cases} t^2 & \text{if } 0 \leqslant t \leqslant 1, \\ t^4 & \text{if } t \geqslant 1. \end{cases}$$

In this case one has $L^\Phi = L^2 \cap L^4$ with equivalence of norms: $\|f\|_{L^\Phi} \simeq \|f\|_{L^2\cap L^4} = \max\{\|f\|_{L^2}, \|f\|_{L^4}\}$. Moreover, it is not difficult to see from this identity and the definition of fundamental function that $\varphi(t) = H^+_\varphi(t) = t^{1/4}\chi_{[0,1)}(t) + t^{1/2}\chi_{[1,\infty)}(t)$. Therefore $\underline{\pi}_{L^\Phi} = 1/4$, $\overline{\pi}_{L^\Phi} = 1/2$.

Example 2.5. Consider now the Young function

$$\Phi(t) = \begin{cases} t^4 & \text{if } 0 \leqslant t \leqslant 1 \\ (2t-1)^2 & \text{if } t \geqslant 1 \end{cases} \simeq \begin{cases} t^4 & \text{if } 0 \leqslant t \leqslant 1, \\ t^2 & \text{if } t \geqslant 1. \end{cases}$$

Then $L^\Phi = L^2 + L^4$ with equivalence of norms: $\|f\|_{L^\Phi} \simeq \|f\|_{L^2+L^4} = \inf\{\|g\|_{L^2} + \|h\|_{L^4}\}$, where the infimum is taken over all decompositions $f = g + h$ with $g \in L^2$ and $h \in L^4$. The fundamental function is given by $\varphi(t) \simeq t^{1/2}\chi_{[0,1)}(t) + t^{1/4}\chi_{[1,\infty)}(t)$, while $H^+_\varphi(t)$ is comparable to the one given in the previous example. Thus we obtain again $\underline{\pi}_{L^\Phi} = 1/4$, $\overline{\pi}_{L^\Phi} = 1/2$.

Remark 2.6. In the last two examples the exponents 2 and 4 can be replaced by any $p, q \in [1,\infty)$, leading to the Orlicz spaces $L^p \cap L^q$ and $L^p + L^q$, which satisfy analogous properties after obvious modifications.

2.2. Wavelet bases and Orlicz spaces

Let $\mathcal{D} = \{Q_{j,k} = 2^{-j}([0,1)^d + k)\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ denote the set of all dyadic cubes in $\mathbb{R}^d$. We say that a finite collection of functions $\{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^d)$ is an *orthonormal wavelet family* if the system

$$\big\{\psi^\ell_{Q_{j,k}}(x) = 2^{jd/2}\psi^\ell\big(2^j x - k\big)\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d,\ \ell = 1, \dots, L\big\} \tag{2.7}$$

forms an orthonormal basis of $L^2(\mathbb{R}^d)$. We will say that the wavelet family is *admissible* if in addition the system in (2.7) is an unconditional basis of $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$. The reader can consult [13,23] for constructions, examples and properties of orthonormal wavelets. Admissible wavelets include the d-dimensional Haar system, wavelets arising from r-regular multiresolution analyses (see [23, p. 22]), wavelets belonging to the regularity class $\mathcal{R}^0$ (as defined in [13, p. 64] for $d = 1$), and actually any orthonormal wavelet in $L^2(\mathbb{R}^d)$ with very mild decay conditions (see [26,30]).

M. Soardi proved in [28] that an admissible wavelet basis $\{\psi_Q^\ell\}_{Q\in\mathcal{D},\ \ell=1,\dots,L}$ is also an unconditional basis for any Orlicz space $L^\Phi(\mathbb{R}^d)$ with non-trivial Boyd indices $0<\underline{\pi}_{L^\Phi}\leqslant\overline{\pi}_{L^\Phi}<1$. That is, every function $f\in L^\Phi(\mathbb{R}^d)$ can be written in the form

$$f=\sum_{\ell=1}^{L}\sum_{Q\in\mathcal{D}}\langle f,\psi_Q^\ell\rangle\psi_Q^\ell, \tag{2.8}$$

with unconditional convergence in $L^\Phi(\mathbb{R}^d)$, and moreover

$$\|f\|_{L^\Phi(\mathbb{R}^d)}\simeq\left\|\left(\sum_{\ell=1}^{L}\sum_{Q\in\mathcal{D}}\left|\langle f,\psi_Q^\ell\rangle\right|^2|Q|^{-1}\chi_Q(\cdot)\right)^{1/2}\right\|_{L^\Phi(\mathbb{R}^d)}. \tag{2.9}$$

This result was derived from the corresponding wavelet characterization of Lebesgue spaces $L^p(\mathbb{R}^d)$, $1<p<\infty$, by applying Boyd's interpolation theorem for sublinear operators.

In view of (2.9), we will denote by $\mathfrak{f}^\Phi$ the space of all sequences of complex numbers $\mathbf{s}=\{s_Q^\ell\}_{Q\in\mathcal{D},\ \ell=1,\dots,L}$ such that

$$\|\mathbf{s}\|_{\mathfrak{f}^\Phi}=\left\|\left(\sum_{\ell=1}^{L}\sum_{Q\in\mathcal{D}}\left|s_Q^\ell\right|^2|Q|^{-1}\chi_Q(\cdot)\right)^{1/2}\right\|_{L^\Phi(\mathbb{R}^d)}<\infty. \tag{2.10}$$

Thus, the correspondence $f\mapsto\{s_Q^\ell\}=\{\langle f,\psi_Q^\ell\rangle\}_{Q\in\mathcal{D},\ \ell=1,\dots,L}$ defines an isomorphism from L^Φ onto $\mathfrak{f}^\Phi$. As usual, this will reduce our research about N-term approximation in Orlicz spaces to prove the corresponding results on the sequence spaces $\mathfrak{f}^\Phi$ (see Section 6 below).

Remark 2.7. For the sake of simplicity, we shall assume throughout the paper that the number $L=1$. Our theorems will remain valid for any $L\geqslant 1$, since the finite sum appearing in the definition of $\mathfrak{f}^\Phi$ is completely harmless in our computations.

2.3. Greedy bases and democracy

We defined in the introduction the notion of *greedy basis* in a quasi-normed Banach space $(\mathbb{B},\|\ \|_{\mathbb{B}})$. We also mentioned the result of Konyagin and Temlyakov [19] characterizing greedy bases as those which are unconditional and *democratic*. For simplicity, given a basis $\mathcal{B}=\{e_j\}_{j\geqslant 1}$ in $\mathbb{B}$ we shall denote the *normalized characteristic function* of a set of indices $\Gamma\subset\mathbb{N}$ by

$$\tilde{\mathbf{1}}_\Gamma=\tilde{\mathbf{1}}_\Gamma^{(\mathbb{B},\mathcal{B})}=\sum_{j\in\Gamma}\frac{e_j}{\|e_j\|_{\mathbb{B}}}.$$

Thus, $\mathcal{B}$ is democratic in $\mathbb{B}$ if there exists $C\geqslant 1$ such that

$$\|\tilde{\mathbf{1}}_\Gamma\|_{\mathbb{B}}\leqslant C\|\tilde{\mathbf{1}}_{\Gamma'}\|_{\mathbb{B}} \tag{2.11}$$

for all finite sets of indices $\Gamma,\Gamma'\subset\mathbb{N}$ with $\operatorname{Card}\Gamma=\operatorname{Card}\Gamma'$. Quite often one can show democracy by finding a function $h:\mathbb{N}\to\mathbb{R}^+$ for which

$$\frac{1}{C}h(\operatorname{Card}\Gamma)\leqslant\|\tilde{\mathbf{1}}_\Gamma\|_{\mathbb{B}}\leqslant Ch(\operatorname{Card}\Gamma),\quad\forall\Gamma\subset\mathcal{D}. \tag{2.12}$$

In the case of wavelet bases, many classical function and distribution spaces satisfy (2.12) with $h(N)=N^{1/p}$. Indeed, this is the situation for Lebesgue spaces $L^p(\mathbb{R}^d)$ when $1<p<\infty$ (see [29]); for Hardy spaces $H^p(\mathbb{R}^d)$, $0<p\leqslant 1$ and Sobolev spaces $\dot{W}^{s,p}(\mathbb{R}^d)$, $1<p<\infty$ (see [14]); and more generally for the family of Triebel–Lizorkin spaces $\dot{F}^s_{p,r}(\mathbb{R}^d)$ with $0<p<\infty$, $s\in\mathbb{R}$, $0<r\leqslant\infty$ (under the usual decay and smoothness assumptions, and with the standard modification of the basis in the case of inhomogeneous spaces; see [11]). Thus, wavelet bases are democratic and hence greedy in all these spaces.

Wavelet bases, however, are not democratic in other classical spaces, such as *BMO*, the Besov classes $\dot{B}^\alpha_{p,q}$ with $p\neq q$, and as we shall see below, Orlicz spaces L^Φ distinct from L^p. To deal with these cases the following notion will be useful.

Definition 2.8. Let $\mathcal{B}$ be an unconditional basis in a quasi-Banach space $\mathbb{B}$. The *right-democracy function associated with* $\mathcal{B}$ is defined by

$$h_r(N) = \sup_{\mathrm{Card}(\Gamma)=N} \|\tilde{\mathbf{1}}_\Gamma\|_{\mathbb{B}}, \quad N = 1, 2, \ldots.$$

Analogously, the *left-democracy function associated with* $\mathcal{B}$ is defined by

$$h_\ell(N) = \inf_{\mathrm{Card}(\Gamma)=N} \|\tilde{\mathbf{1}}_\Gamma\|_{\mathbb{B}}, \quad N = 1, 2, \ldots.$$

Observe that $\mathcal{B}$ is democratic in $\mathbb{B}$ if and only if $h_r(N) \leqslant C h_\ell(N)$ for all $N \geqslant 1$ and some $C > 0$. Also, if the ρ-triangle inequality holds in $\mathbb{B}$ and $\mathcal{B}$ is an unconditional basis we have

$$c \leqslant h_l(N) \leqslant h_r(N) \leqslant N^{1/\rho}, \quad \forall N \geqslant 1,$$

for some $c > 0$ (we thank an anonymous referee for pointing out this fact).

3. Examples

We show with a few examples that, in general, admissible wavelet bases are not democratic in Orlicz spaces. In order to do so one needs to estimate $\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi}$ in terms of $\mathrm{Card}\,\Gamma$. This can be easily done when Γ is a collection of *pairwise disjoint dyadic cubes of equal size*.

Lemma 3.1. *Let $L^\Phi(\mathbb{R}^d)$ be an Orlicz space with $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$, and let $\mathcal{B} = \{\psi_Q\colon\ Q \in \mathcal{D}\}$ be an admissible wavelet basis. If $\Gamma = \{Q_1, Q_2, \ldots, Q_N\} \subset \mathcal{D}$ is a pairwise disjoint family then*

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \simeq \left\| \sum_{Q\in\Gamma} \frac{\chi_Q(\cdot)}{\varphi(|Q|)} \right\|_{L^\Phi(\mathbb{R}^d)}. \tag{3.1}$$

If we further assume that all the cubes in Γ are of the same size, say $|Q| = 2^{kd}$ for all $Q \in \Gamma$ and some $k \in \mathbb{Z}$, then

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \simeq \frac{\varphi(N2^{kd})}{\varphi(2^{kd})}. \tag{3.2}$$

Proof. For a single element ψ_Q of the basis $\mathcal{B}$ we have, by (2.9),

$$\|\psi_Q\|_{L^\Phi(\mathbb{R}^d)} \simeq \left\| \left(\frac{\chi_Q(\cdot)}{|Q|} \right)^{1/2} \right\|_{L^\Phi(\mathbb{R}^d)} = \frac{\varphi(|Q|)}{|Q|^{1/2}}. \tag{3.3}$$

Thus, using again the expression of the norm in (2.9) it follows that

$$\begin{aligned} \|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} &= \left\| \sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{L^\Phi(\mathbb{R}^d)}} \right\|_{L^\Phi(\mathbb{R}^d)} \simeq \left\| \left(\sum_{Q\in\Gamma} \frac{1}{\|\psi_Q\|^2_{L^\Phi(\mathbb{R}^d)}} \frac{\chi_Q(\cdot)}{|Q|} \right)^{1/2} \right\|_{L^\Phi(\mathbb{R}^d)} \\ &\simeq \left\| \left(\sum_{Q\in\Gamma} \frac{\chi_Q(\cdot)}{\varphi(|Q|)^2} \right)^{1/2} \right\|_{L^\Phi(\mathbb{R}^d)} = \left\| \sum_{Q\in\Gamma} \frac{\chi_Q(\cdot)}{\varphi(|Q|)} \right\|_{L^\Phi(\mathbb{R}^d)}, \end{aligned}$$

where in the last equality we have used that the cubes in Γ are pairwise disjoint. Assuming further that $|Q| = 2^{kd}$ for every $Q \in \Gamma$, we obtain

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \simeq \frac{1}{\varphi(2^{kd})} \left\| \sum_{Q\in\Gamma} \chi_Q(\cdot) \right\|_{L^\Phi(\mathbb{R}^d)} = \frac{1}{\varphi(2^{kd})} \varphi\left(\left| \bigcup_{Q\in\Gamma} Q \right| \right) = \frac{\varphi(N2^{kd})}{\varphi(2^{kd})}. \qquad \square$$

Remark 3.2. Defining

$$h^+_\varphi(t) = \sup_{k\in\mathbb{Z}} \frac{\varphi(t2^{kd})}{\varphi(2^{kd})}, \qquad h^-_\varphi(t) = \inf_{k\in\mathbb{Z}} \frac{\varphi(t2^{kd})}{\varphi(2^{kd})}, \tag{3.4}$$

it follows from Lemma 3.1 that if Γ is a family of *disjoint cubes of the same size* then

$$h_\varphi^-(\operatorname{Card}\Gamma) \lesssim \|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \lesssim h_\varphi^+(\operatorname{Card}\Gamma). \tag{3.5}$$

Moreover, this estimate is sharp in the sense that we can find families Γ for which $\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)}$ is comparable to either $h_\varphi^-(\operatorname{Card}\Gamma)$ or $h_\varphi^+(\operatorname{Card}\Gamma)$. Thus, if $h_\varphi^+(N)$ and $h_\varphi^-(N)$ are not comparable for $N \geqslant 1$ it follows that admissible wavelet bases are not democratic in Orlicz spaces.

Proposition 3.3. *For the Orlicz spaces $L^2 \cap L^4$ and $L^2 + L^4$ given in Examples* 2.4 *and* 2.5 *we have that $h_\varphi^-(N) \simeq N^{1/4}$ and $h_\varphi^+(N) \simeq N^{1/2}$ when $N \in \mathbb{N}$. Thus, admissible wavelet bases are not democratic for these spaces.*

Recall that in the previous examples we have $\underline{\pi}_{L^\Phi} \neq \overline{\pi}_{L^\Phi}$. We also show that there are Orlicz spaces with $\underline{\pi}_{L^\Phi} = \overline{\pi}_{L^\Phi}$ for which admissible wavelet bases are not democratic.

Proposition 3.4. *Let $\alpha \in \mathbb{R}$ and $1 < p < \infty$. Then, the Orlicz space $L^p(\log L)^\alpha$ satisfies $h_\varphi^-(N) \simeq N^{1/p}(1 + \log N)^{-\alpha/p}$ and $h_\varphi^+(N) \simeq N^{1/p}$ when $\alpha \geqslant 0$ and $h_\varphi^-(N) \simeq N^{1/p}$ and $h_\varphi^+(N) \simeq N^{1/p}(1 + \log N)^{-\alpha/p}$ when $\alpha < 0$. Thus, admissible wavelet bases are neither democratic nor greedy for $L^p(\log L)^\alpha$ with $\alpha \neq 0$.*

We conclude this section with the simple proof of the previous two propositions.

Proof of Proposition 3.3. We first obtain the desired estimates for h_φ^- and h_φ^+. Let us observe that these expressions are not comparable. Thus, by Remark 3.2, in both cases, we can conclude that admissible wavelet bases are not democratic

We do the case $L^\Phi = L^2 \cap L^4$ where Φ is given in Example 2.4 as the other case can be proved similarly. For $N \in \mathbb{N}$, we have

$$\frac{\varphi(Ns)}{\varphi(s)} = \begin{cases} N^{1/4} & \text{if } s \leqslant 1/N, \\ N^{1/2}s^{1/4} & \text{if } 1/N < s \leqslant 1, \\ N^{1/2} & \text{if } s > 1. \end{cases}$$

Hence,

$$h_\varphi^+(N) = \sup_{k\in\mathbb{Z}} \frac{\varphi(N2^{kd})}{\varphi(2^{kd})} = N^{1/2} \quad \text{and} \quad h_\varphi^-(N) = \inf_{k\in\mathbb{Z}} \frac{\varphi(N2^{kd})}{\varphi(2^{kd})} = N^{1/4}. \qquad \square$$

Proof of Proposition 3.4. We do the case $\alpha \geqslant 0$ as the other case can be proved similarly. As before it suffices to get the desired estimates for h_φ^- and h_φ^+. Recall from Example 2.2 that the fundamental function associated with $L^p(\log L)^\alpha$ is given by $\varphi(t) \simeq t^{1/p}(1 + \log^+ 1/t)^{\alpha/p}$. Then,

$$\frac{\varphi(Ns)}{\varphi(s)} \simeq \begin{cases} N^{1/p}\big(\frac{1+\log\frac{1}{Ns}}{1+\log 1/s}\big)^{\alpha/p} & \text{if } s \leqslant 1/N, \\ N^{1/p}(1 + \log 1/s)^{-\alpha/p} & \text{if } 1/N < s \leqslant 1, \\ N^{1/p} & \text{if } s > 1. \end{cases} \tag{3.6}$$

Thus, $h_\varphi^-(N) \simeq N^{1/p}(1 + \log N)^{-\alpha/p}$ and $h_\varphi^+(N) \simeq N^{1/p}$. $\square$

4. Left and right democracy functions for Orlicz spaces

We saw in (3.5) that for any $\Gamma \subset \mathcal{D}$ consisting of *disjoint cubes of the same size* we have

$$h_\varphi^-(\operatorname{Card}\Gamma) \lesssim \|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \lesssim h_\varphi^+(\operatorname{Card}\Gamma).$$

Our main theorem in this section shows that these inequalities remain true for arbitrary $\Gamma \subset Q$. We state this result in a slightly different way than Theorem 1.2 in the Introduction.

Theorem 4.1. *Let $L^\Phi(\mathbb{R}^d)$ be an Orlicz space with indices $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$ and let $\mathcal{B} = \{\psi_Q\colon\ Q \in \mathcal{D}\}$ be an admissible wavelet basis. Then*

$$h_\varphi^-(\operatorname{Card}\Gamma) \lesssim \|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \lesssim h_\varphi^+(\operatorname{Card}\Gamma), \quad \forall \Gamma \subset \mathcal{D}. \tag{4.1}$$

In particular, the left and right democracy functions associated with $\mathcal{B}$ in $L^\Phi(\mathbb{R}^d)$ satisfy $h_\ell \simeq h_\varphi^-$ and $h_r \simeq h_\varphi^+$.

Remark 4.2. As mentioned in Remark 3.2, the estimates in (4.1) are best possible, as one can obtain comparable quantities on the left- or right-hand sides by considering sets Γ consisting only of disjoint cubes of the same size.

The rest of this section is devoted to prove Theorem 4.1. We first present a very simple argument for the case of pairwise disjoint cubes. The general case is more technical and will require a linearization argument and some combinatorics about dyadic intervals.

4.1. Proof of Theorem 4.1: The case of disjoint cubes

Assume first that $\Gamma = \{Q_1, \dots, Q_N\}$ consists of pairwise disjoint cubes. Let $\lambda = h_\varphi^+(N)$, so that $\varphi(N|Q|) \leqslant \lambda\varphi(|Q|)$, for all $Q \in \Gamma$. Therefore, since the elements of Γ are disjoint and Φ is increasing

$$\begin{aligned}\int_{\mathbb{R}^d} \Phi\left(\frac{\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{\varphi(|Q|)}}{\lambda}\right) dx &= \sum_{Q\in\Gamma} \Phi\left(\frac{1}{\lambda\varphi(|Q|)}\right)|Q| \leqslant \sum_{Q\in\Gamma} \Phi\left(\frac{1}{\varphi(N|Q|)}\right)|Q| \\ &= \sum_{Q\in\Gamma} \Phi\left(\Phi^{-1}\left(\frac{1}{N|Q|}\right)\right)|Q| = 1.\end{aligned}$$

Thus, by (3.1) and (2.1) we have

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \simeq \left\| \sum_{Q\in\Gamma} \frac{\chi_Q(\cdot)}{\varphi(|Q|)} \right\|_{L^\Phi(\mathbb{R}^d)} \leqslant h_\varphi^+(N).$$

The lower estimate is obtained in a similar way: take now $\lambda < h_\varphi^-(N)$ so that $\varphi(N|Q|) > \lambda\varphi(|Q|)$ for all $Q \in \Gamma$. Then, reasoning as above

$$\begin{aligned}\int_{\mathbb{R}^d} \Phi\left(\frac{\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{\varphi(|Q|)}}{\lambda}\right) dx &= \sum_{Q\in\Gamma} \Phi\left(\frac{1}{\lambda\varphi(|Q|)}\right)|Q| > \sum_{Q\in\Gamma} \Phi\left(\frac{1}{\varphi(N|Q|)}\right)|Q| \\ &= \sum_{Q\in\Gamma} \Phi\left(\Phi^{-1}\left(\frac{1}{N|Q|}\right)\right)|Q| = 1.\end{aligned}$$

Thus, (3.1) and (2.1) yield

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \simeq \left\| \sum_{Q\in\Gamma} \frac{\chi_Q(\cdot)}{\varphi(|Q|)} \right\|_{L^\Phi(\mathbb{R}^d)} \geqslant h_\varphi^-(N).$$

4.2. Proof of Theorem 4.1: The general case

In the case of disjoint cubes just considered we have two important features. First, Lemma 3.1 allows us to "linearize" the square function in (2.9). Second, for the estimates obtained in the previous argument it is crucial that the sets involved are disjoint. For general families of cubes we are going to follow the same scheme. First we "linearize" the square function and then we dominate this by an expression involving only disjoint subsets from Γ. This last argument is the most subtle, since it requires a careful selection procedure on dyadic cubes.

4.2.1. Linearization of the square function

Given a finite set $\Gamma \subset \mathcal{D}$, we shall denote

$$S_\Gamma(x) = \left(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{\varphi(|Q|)^2} \right)^{1/2}, \tag{4.2}$$

so that, by (2.9) and (3.3), we have $\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \simeq \|S_\Gamma\|_{L^\Phi(\mathbb{R}^d)}$.

For every $x \in \bigcup_{Q\in\Gamma} Q$, we define Q_x as the smallest (hence unique) cube in Γ containing x. It is clear that

$$S_\Gamma(x) \geqslant \frac{\chi_{Q_x}(x)}{\varphi(|Q_x|)}, \quad \forall x \in \bigcup_{Q\in\Gamma} Q, \tag{4.3}$$

since the left-hand side contains at least the cube Q_x (and possibly more). We now show that the reverse inequality holds with some universal constant. Indeed, if we enlarge the sum to include all dyadic cubes containing Q_x we have

$$S_\Gamma(x)^2 = \sum_{Q\in\Gamma} \frac{\chi_Q(x)}{\varphi(|Q|)^2} \leqslant \sum_{\substack{Q\supset Q_x \\ Q\in\mathcal{D}}} \frac{1}{\varphi(|Q|)^2} = \sum_{j=0}^{\infty} \frac{1}{\varphi(2^{jd}|Q_x|)^2}.$$

Since we are working in an Orlicz space with $i_\varphi > 0$, by (2.6) we can choose $0 < \varepsilon < i_\varphi$ and find $C_\varepsilon > 0$ such that $\varphi(2^{jd}|Q_x|) \geqslant C_\varepsilon 2^{jd(i_\varphi-\varepsilon)}\varphi(|Q_x|)$. Therefore,

$$S_\Gamma(x)^2 \leqslant C \sum_{j=0}^{\infty} \frac{1}{2^{2jd(i_\varphi-\varepsilon)}\varphi(|Q_x|)^2} = C \frac{\chi_{Q_x}(x)}{\varphi(|Q_x|)^2}.$$

This and (4.3) show that

$$S_\Gamma(x) \simeq \frac{\chi_{Q_x}(x)}{\varphi(|Q_x|)}. \tag{4.4}$$

This linearization procedure has been used by other authors in the context of N-term approximation (see, e.g., [6,11, 14]).

Observe from (4.4) that $S_\Gamma(x) \simeq S_{\Gamma_{\min}}(x)$, where $\Gamma_{\min}$ denotes the family of *minimal* cubes in Γ, that is,

$$\Gamma_{\min} = \left\{ Q_x\colon\ x \in \bigcup_{Q\in\Gamma} Q \right\}.$$

Moreover, as we shall see below, the cardinalities of Γ and $\Gamma_{\min}$ are comparable, so that for our purposes only the cubes in $\Gamma_{\min}$ will be relevant. However, we still need a finer selection, since the cubes in $\Gamma_{\min}$ are not necessarily pairwise disjoint.

4.2.2. Shaded and lighted cubes

We start with an example. Suppose we have a family Γ of 10 cubes which have been arranged by generations as in Fig. 1.

Projecting a beam of light as shown in Fig. 2, some parts of a cube Q_i receive light: we call these parts $\mathit{Light}(Q_i)$. Some other portion of the cube Q_i is shaded: we call this portion $\mathit{Shade}(Q_i)$. The shaded parts of the cubes given in Fig. 1 are represented with thicker lines in Fig. 2. Observe that the minimal cubes are those with some portion of light, as $x \in \mathit{Light}(Q_i)$ if and only if $Q_x = Q_i$. In this example, $\Gamma_{\min} = \Gamma \setminus \{Q_6\}$. Notice also that $\{\mathit{Light}(Q)\colon\ Q \in \Gamma_{\min}\}$ is a disjoint collection.

Now we give precise definitions: given a fixed $\Gamma \subset \mathcal{D}$, for any $Q \in \Gamma$ we define the *Shade* of Q as the union of all cubes from Γ strictly contained in Q

$$\mathit{Shade}(Q) = \bigcup \{R\colon\ R \in \Gamma,\ R \subsetneq Q\}.$$

We define the *Light* of Q as

$$\mathit{Light}(Q) = Q \setminus \mathit{Shade}(Q).$$

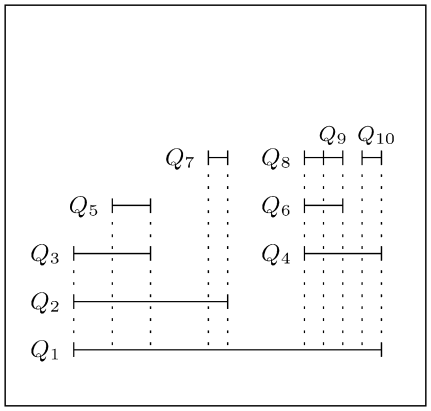

Fig. 1. $\Gamma = \{Q_1, \dots, Q_{10}\}$.

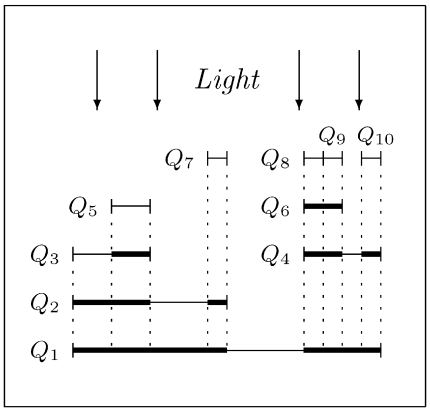

Fig. 2. Shade & Light.

As mentioned above it is clear that $Q \in \Gamma_{\min}$ if and only if $\mathit{Light}(Q) \neq \emptyset$, and moreover

$$\bigcup_{Q\in\Gamma} Q = \bigcup_{Q\in\Gamma_{\min}} \mathit{Light}(Q),$$

where the sets in the last union are pairwise disjoint. Therefore, by (4.4) we can write

$$S_\Gamma(x) \simeq \sum_{Q\in\Gamma_{\min}} \frac{\chi_{\mathit{Light}(Q)}(x)}{\varphi(|Q|)}, \tag{4.5}$$

where in the last sum there is at most one non-zero term.

Next we classify the cubes as shaded if the shade is a big portion of the cube or lighted if this does not happen. Precisely, a cube $Q \in \Gamma$ is called *shaded* if $|\mathit{Shade}(Q)| > \frac{2^d-1}{2^d}|Q|$, and we write Γ_S for the collection of cubes from Γ which are shaded. A cube Q from Γ is called *lighted* if it is not shaded, that is, if $|\mathit{Light}(Q)| \geqslant \frac{1}{2^d}|Q|$. We write Γ_L for the collection of all cubes from Γ that are lighted. Observe that $\Gamma_L \subset \Gamma_{\min}$.

Lemma 4.3. *With the above definitions we have*

$$\frac{2^d-1}{2^d}\operatorname{Card}(\Gamma) \leqslant \operatorname{Card}(\Gamma_L) \leqslant \operatorname{Card}(\Gamma_{\min}) \leqslant \operatorname{Card}(\Gamma), \quad \forall \Gamma \subset \mathcal{D}.$$

Proof. Clearly, as we have observed before $\operatorname{Card}(\Gamma_L) \leqslant \operatorname{Card}(\Gamma_{\min}) \leqslant \operatorname{Card}(\Gamma)$. Thus, we need to prove the left-hand side inequality. Given $Q \in \mathcal{D}$, we write Q^k, $k = 1, 2, \dots, 2^d$ for the 2^d dyadic cubes contained in Q of size $2^{-d}|Q|$. For $Q \in \Gamma_S$ and $k = 1, 2, \dots, 2^d$, let $Q^k_\sharp$ be a biggest cube from Γ with $Q^k_\sharp \subset Q^k$. Notice that the cubes $Q^k_\sharp$ exist for every $Q \in \Gamma_S$: otherwise, if for some $k_0 \in \{1, 2, \dots, 2^d\}$ there is no cube from Γ contained in Q^{k_0} we have that $Q^{k_0} \subset \mathit{Light}(Q)$ and then

$$\big|\mathit{Shade}(Q)\big| \leqslant \big|Q \setminus Q^{k_0}\big| = \big(1 - 2^{-d}\big)|Q| = \frac{2^d-1}{2^d}|Q|,$$

contradicting the definition of Γ_S.

The procedure just described assigns 2^d different cubes from Γ to each $Q \in \Gamma_S$, namely $Q^1_\sharp, Q^2_\sharp, \dots, Q^{2^d}_\sharp$, and neither of them coincides with Q.

We claim that if $Q, R \in \Gamma_S$ and $Q \neq R$, then we necessarily have $Q^k_\sharp \neq R^\ell_\sharp$ for all $1 \leqslant k, \ell \leqslant 2^d$. This is trivially true if $Q \cap R = \emptyset$. Without loss of generality, we may assume $Q \subsetneq R$ and also $Q \subset R^1$. It follows from here that $Q^k_\sharp \neq R^\ell_\sharp$ for all $k = 1, 2, \dots, 2^d$ and all $\ell = 2, 3, \dots, 2^d$ since $Q^k_\sharp \subset R^1$ while $R^\ell_\sharp \subset R^\ell$ for $\ell \neq 1$. Moreover, as $R^1_\sharp$ is the biggest cube in Γ contained in R^1 and $Q \subset R^1$ we have that $Q \subset R^1_\sharp \subset R^1$. Hence, for all $k = 1, \dots, 2^d$ we have $Q^k_\sharp \subsetneq Q \subset R^1_\sharp$ and thus $Q^k_\sharp \neq R^1_\sharp$.

In short, to each $Q \in \Gamma_S$ we have assigned 2^d different cubes in Γ and these are not associated to any other cube in Γ_S. We conclude that $2^d \operatorname{Card}(\Gamma_S) \leqslant \operatorname{Card}(\Gamma)$ and, as desired,

$$\operatorname{Card}(\Gamma_L) = \operatorname{Card}(\Gamma) - \operatorname{Card}(\Gamma_S) \geqslant \operatorname{Card}(\Gamma) - \frac{1}{2^d}\operatorname{Card}(\Gamma) = \frac{2^d-1}{2^d}\operatorname{Card}(\Gamma). \qquad \square$$

4.2.3. Proof of (4.1)

We can now conclude easily the proof of Theorem 4.1. By (4.2) and (4.5), we know that

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \simeq \left\| \sum_{Q\in\Gamma_{\min}} \frac{\chi_{Light(Q)}(x)}{\varphi(|Q|)} \right\|_{L^\Phi(\mathbb{R}^d)}, \tag{4.6}$$

so we only have to estimate this last expression. Let $\lambda = h_\varphi^+(\operatorname{Card}(\Gamma_{\min}))$, so that $\varphi(|Q|\operatorname{Card}(\Gamma_{\min})) \leqslant \lambda\varphi(|Q|)$ for all $Q \in \Gamma_{\min}$. Since $\{Light(Q)\colon\ Q \in \Gamma_{\min}\}$ is a disjoint collection, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \Phi\left(\frac{\sum_{Q\in\Gamma_{\min}} \frac{\chi_{Light(Q)}(x)}{\varphi(|Q|)}}{\lambda}\right)\mathrm{d}x &= \sum_{Q\in\Gamma_{\min}} \Phi\left(\frac{1}{\lambda\varphi(|Q|)}\right)\big|Light(Q)\big| \\ &\leqslant \sum_{Q\in\Gamma_{\min}} \Phi\left(\frac{1}{\varphi(|Q|\operatorname{Card}(\Gamma_{\min}))}\right)|Q| = \sum_{Q\in\Gamma_{\min}} \Phi\left(\Phi^{-1}\left(\frac{1}{|Q|\operatorname{Card}(\Gamma_{\min})}\right)\right)|Q| = 1.\end{aligned}$$

Hence, by (4.6) and Lemma 4.3, and since h_φ^+ is non-decreasing, we have

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \lesssim h_\varphi^+\big(\operatorname{Card}(\Gamma_{\min})\big) \leqslant h_\varphi^+(\operatorname{Card}\Gamma).$$

We next show how to obtain the left-hand side of (4.1). By (4.6), and using that $\Gamma_L \subset \Gamma_{\min}$, we can write

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \gtrsim \left\| \sum_{Q\in\Gamma_L} \frac{\chi_{Light(Q)}(x)}{\varphi(|Q|)} \right\|_{L^\Phi(\mathbb{R}^d)}.$$

Now let $\lambda < h_\varphi^-(2^{-d}\operatorname{Card}(\Gamma_L))$ so that $\lambda\varphi(|Q|) < \varphi(|Q|2^{-d}\operatorname{Card}(\Gamma_L))$ for any $Q \in \Gamma_L$. Proceeding as before, using that $|Light(Q)| \geqslant 2^{-d}|Q|$ for $Q \in \Gamma_L$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d} \Phi\left(\frac{\sum_{Q\in\Gamma_L} \frac{\chi_{Light(Q)}(x)}{\varphi(|Q|)}}{\lambda}\right)\mathrm{d}x &= \sum_{Q\in\Gamma_L} \Phi\left(\frac{1}{\lambda\varphi(|Q|)}\right)\big|Light(Q)\big| \\ &> \sum_{Q\in\Gamma_L} \Phi\left(\frac{1}{\varphi(2^{-d}|Q|\operatorname{Card}(\Gamma_L))}\right)2^{-d}|Q| = 1.\end{aligned}$$

Thus, by (2.1), Lemma 4.3 and by (2.6) with $s = (2^d-1)2^{-2d}$ and $t = \operatorname{Card}(\Gamma)$ we obtain

$$\|\tilde{\mathbf{1}}_\Gamma\|_{L^\Phi(\mathbb{R}^d)} \geqslant h_\varphi^-\big(2^{-d}\operatorname{Card}(\Gamma_L)\big) \geqslant h_\varphi^-\big(\big(2^d-1\big)2^{-2d}\operatorname{Card}(\Gamma)\big) \geqslant Ch_\varphi^-(\operatorname{Card}\Gamma).$$

This completes the proof of Theorem 4.1. $\square$

5. Greediness of wavelet bases in L^Φ

In this section we prove Theorem 1.1. Some of the arguments have been adapted from [27] (see, however, an alternative proof in [31, Section 2]). Throughout the section we shall assume that $\varphi\colon(0,\infty)\to(0,\infty)$ is a non-decreasing function so that $\lim_{t\to0^+}\varphi(t) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$, and, in addition, $\varphi \in \Delta_2$, that is, $\varphi(2t) \leqslant C_0\varphi(t)$, for all $t > 0$.

Recall the definitions of $H_\varphi^+(t)$ and $h_\varphi^\pm(t)$ in (2.3) and (3.4), and let us also introduce

$$H_\varphi^-(t) = \inf_{s>0}\frac{\varphi(st)}{\varphi(s)}, \quad t > 0.$$

The following lemma is a trivial consequence of the doubling property.

Lemma 5.1. *Given φ as above we have*

$$C_0^{-1} h_\varphi^-(t) \leqslant H_\varphi^-(t) \leqslant h_\varphi^-(t) \quad \text{and} \quad h_\varphi^+(t) \leqslant H_\varphi^+(t) \leqslant C_0 h_\varphi^+(t), \quad \forall t > 0. \tag{5.1}$$

Our second lemma follows an argument presented in [27, pp. 31–32] in the context of Young functions, which we have adapted to our situation.

Lemma 5.2. *Let φ be as above and suppose that there exists $C_1 > 0$ such that*

$$H_\varphi^+(N) \leqslant C_1 H_\varphi^-(N), \quad \text{for all } N = 1, 2, 3, \ldots. \tag{5.2}$$

Then, there exist $c_0 \geqslant 1$ and $0 < \alpha < \infty$ such that

$$c_0^{-1} t^\alpha \leqslant \varphi(t) \leqslant c_0 t^\alpha, \quad \text{for all } t > 0. \tag{5.3}$$

Proof. The proof is divided into several steps.

Step 1. *$H_\varphi^+(t) \leqslant C_0 C_1 H_\varphi^-(t)$ for all $t > 0$.*

Let $t \geqslant 1$ and choose N such that $N \leqslant t < N + 1$. Using that φ is non-decreasing, $\varphi \in \Delta_2$ and (5.2), we have

$$H_\varphi^+(t) \leqslant H_\varphi^+(N+1) \leqslant H_\varphi^+(2N) \leqslant C_0 H_\varphi^+(N) \leqslant C_0 C_1 H_\varphi^-(N) \leqslant C_0 C_1 H_\varphi^-(t).$$

The inequality for $t \in (0, 1)$ follows from the previous case and $H_\varphi^+(t) = 1/H_\varphi^-(1/t)$.

Step 2. *There exists $c_0 \geqslant 1$ such that $c_0^{-1}\varphi(t)\varphi(s) \leqslant \varphi(ts) \leqslant c_0\varphi(t)\varphi(s)$ for all $t > 0$ and $s \in (0, 1]$.*

From Step 1 we deduce

$$\frac{\varphi(ts)}{\varphi(s)} \leqslant H_\varphi^+(t) \leqslant C_0 C_1 H_\varphi^-(t) \leqslant C_0 C_1 \frac{\varphi(t \cdot 1)}{\varphi(1)}.$$

On the other hand, Step 1 also implies

$$\varphi(t) = \varphi(1)\frac{\varphi(t \cdot 1)}{\varphi(1)} \leqslant \varphi(1) H_\varphi^+(t) \leqslant \varphi(1) C_0 C_1 H_\varphi^-(t) \leqslant \varphi(1) C_0 C_1 \frac{\varphi(ts)}{\varphi(s)}.$$

Step 3. *There exists $0 \leqslant \alpha < \infty$ such that $\varphi(t) \leqslant c_0 t^\alpha$ for all $t \in (0, 1]$.*

Let $f_1(u) = \log[c_0/\varphi(e^{-u})]$. For all $u, v \geqslant 0$, Step 2 yields

$$f_1(u+v) = \log \frac{c_0}{\varphi(e^{-u}e^{-v})} \leqslant \log \frac{c_0^2}{\varphi(e^{-u})\varphi(e^{-v})} = f_1(u) + f_1(v). \tag{5.4}$$

Let $u \geqslant v > 0$ and choose $n \in \mathbb{N}$ such that $nv \leqslant u < (n+1)v$. Then, by (5.4) and the fact that f_1 is non-decreasing we obtain

$$f_1(u) \leqslant f_1\big((n+1)v\big) \leqslant (n+1) f_1(v).$$

Since $nv + v \leqslant u + v$ we have $(n+1) \leqslant \frac{u+v}{v}$, and hence

$$f_1(u) \leqslant \frac{u+v}{v} f_1(v), \quad u \geqslant v > 0.$$

Thus, for all $v > 0$,

$$\limsup_{u\to\infty} \frac{f_1(u)}{u} \leqslant \limsup_{u\to\infty} \frac{u+v}{u}\frac{f_1(v)}{v} = \frac{f_1(v)}{v}, \tag{5.5}$$

which shows that

$$0 \leqslant \limsup_{u\to\infty} \frac{f_1(u)}{u} \leqslant \liminf_{v\to\infty} \frac{f_1(v)}{v}.$$

Consequently, there exists $\alpha \geqslant 0$ such that $\lim_{u\to\infty} \frac{f_1(u)}{u} = \alpha$. Using (5.5) it follows that $\alpha < \infty$ and also that for $v > 0$, we obtain $\alpha \leqslant \frac{f_1(v)}{v} = \frac{1}{v}\log[c_0/\varphi(e^{-v})]$. This estimate with $t = e^{-v}$, implies that $\varphi(t) \leqslant c_0 t^\alpha$ for all $t \in (0, 1]$, as we wanted to prove.

Step 4. *For all* $t \in (0,1]$, *we have* $t^\alpha \leqslant c_0\varphi(t)$ *and also* $\alpha > 0$.

Let $f_2(u) = \log(1/[c_0\varphi(e^{-u})])$. For all $u, v \geqslant 0$, by Step 2 we have

$$f_2(u) + f_2(v) = \log \frac{1}{c_0^2\varphi(e^{-u})\varphi(e^{-v})} \leqslant \log \frac{1}{c_0\varphi(e^{-(u+v)})} = f_2(u+v). \tag{5.6}$$

For $u \geqslant v > 0$, choose $n \in \mathbb{N}$ such that $nv \leqslant u < (n+1)v$. Then, by (5.6) and the fact that f_2 is non-decreasing

$$nf_2(v) \leqslant f_2(nv) \leqslant f_2(u).$$

Since $u < (n+1)v$ we have $n > \frac{u-v}{v}$, and hence

$$\frac{u-v}{v} f_2(v) \leqslant f_2(u), \quad u \geqslant v > 0.$$

Note that $f_2(u) = 2\log 1/c_0 + f_1(u)$. Hence for all $v > 0$

$$\alpha = \lim_{u\to\infty} \frac{f_2(u)}{u} \geqslant \lim_{u\to\infty} \frac{u-v}{u}\frac{f_2(v)}{v} = \frac{f_2(v)}{v}. \tag{5.7}$$

This implies that $\alpha > 0$. On the other hand, this estimate with $t = e^{-v}$ yields that $\varphi(t) \geqslant \frac{1}{c_0}t^\alpha$ for all $t \in (0,1]$, as we wanted to prove.

Step 5. *The proof of* (5.3).

The previous steps imply that

$$c_0^{-1}t^\alpha \leqslant \varphi(t) \leqslant c_0 t^\alpha, \quad \text{for all } t \in (0,1]. \tag{5.8}$$

Let $t > 1$. By Step 2 and (5.8)

$$c_0^{-1} \leqslant \varphi(1) = \varphi\big(t\cdot t^{-1}\big) \leqslant c_0\varphi(t)\varphi\big(t^{-1}\big) \leqslant c_0^2\varphi(t)t^{-\alpha}.$$

Consequently, $c_0^{-3}t^\alpha \leqslant \varphi(t)$. A similar argument gives

$$c_0 \geqslant \varphi(1) = \varphi\big(t\cdot t^{-1}\big) \geqslant c_0^{-1}\varphi(t)\varphi\big(t^{-1}\big) \geqslant c_0^{-2}\varphi(t)t^{-\alpha}$$

and therefore $\varphi(t) \leqslant c_0^3 t^\alpha$, completing the proof of (5.3). □

Proof of Theorem 1.1. We already mentioned in Section 2.3 that (admissible) wavelet bases are greedy in $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$. Thus, the interesting implication is the converse.

Suppose that a given wavelet basis is democratic in an Orlicz space $L^\Phi(\mathbb{R}^d)$. Then, Theorem 4.1 and Remark 3.2 give

$$h_\varphi^+(N) \leqslant Ch_\varphi^-(N), \quad N = 1,2,3,\ldots,$$

for some constant $C > 0$. Note that the fundamental function φ of L^Φ clearly satisfies the conditions we assumed at the beginning of this section. Hence, Lemma 5.1 implies

$$H_\varphi^+(N) \leqslant C_1 H_\varphi^-(N), \quad N = 1,2,3,\ldots,$$

and therefore Lemma 5.2 leads to $\varphi(t) \simeq t^\alpha$, for some $0 < \alpha < \infty$. Taking $p = 1/\alpha$, we have that $L^\Phi(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ with equivalent norms. Moreover, since $\underline{\pi}_{L^\Phi} = \overline{\pi}_{L^\Phi} = 1/p$, we necessarily have $1 < p < \infty$. □

6. Greedy algorithm and errors of approximation

In this section we prove Theorem 1.3 and Corollary 1.4, concerning the inclusions of the N-term approximation spaces of $L^\Phi(\mathbb{R}^d)$. To do so, it suffices to consider the same problems in the sequence space $\mathfrak{f}^\Phi$ defined in Section 2.2. We recall that $\mathfrak{f}^\Phi$ is the space of all sequences of complex numbers $\mathbf{s} = \{s_Q\}_{Q\in\mathcal{D}}$ such that

$$\|\mathbf{s}\|_{\mathfrak{f}^\Phi} = \left\| \left(\sum_{Q\in\mathcal{D}} |s_Q|^2 |Q|^{-1} \chi_Q(\cdot) \right)^{1/2} \right\|_{L^\Phi(\mathbb{R}^d)} < \infty.$$

In this setting the approximation is performed from the *canonical basis* $\{\mathbf{e}_Q\}_{Q\in\mathcal{D}}$, where each vector $\mathbf{e}_Q$ has entry 1 at the index Q, and 0 otherwise. Observe that the canonical basis is unconditional in $\mathfrak{f}^\Phi$, and in particular that $\mathfrak{f}^\Phi$ satisfies the lattice property

$$|s_Q| \leqslant |t_Q|, \quad \forall Q \in \mathcal{D} \quad \Longrightarrow \quad \big\|\{s_Q\}_{Q\in\mathcal{D}}\big\|_{\mathfrak{f}^\Phi} \leqslant \big\|\{t_Q\}_{Q\in\mathcal{D}}\big\|_{\mathfrak{f}^\Phi}. \tag{6.1}$$

The greedy algorithm in $\mathfrak{f}^\Phi$ takes the following form: given $\mathbf{s} = \{s_Q\}_{Q\in\mathcal{D}} \in \mathfrak{f}^\Phi$, we order the index set in such a way that

$$\|s_{Q_1}\mathbf{e}_{Q_1}\|_{\mathfrak{f}^\Phi} \geqslant \|s_{Q_2}\mathbf{e}_{Q_2}\|_{\mathfrak{f}^\Phi} \geqslant \|s_{Q_3}\mathbf{e}_{Q_3}\|_{\mathfrak{f}^\Phi} \geqslant \cdots \tag{6.2}$$

handling ties arbitrarily. Notice that, as in (3.3)

$$\|\mathbf{e}_Q\|_{\mathfrak{f}^\Phi} = \varphi\big(|Q|\big)/|Q|^{\frac{1}{2}}, \quad Q \in \mathcal{D}. \tag{6.3}$$

The *greedy algorithm of step* $N \geqslant 1$ is given by the correspondence

$$\mathbf{s} = \sum_{Q\in\mathcal{D}} s_Q\mathbf{e}_Q \in \mathfrak{f}^\Phi \to \mathbf{G}_N(\mathbf{s}) = \sum_{k=1}^{N} s_{Q_k}\mathbf{e}_{Q_k}.$$

As usual, when $N = 0$ we set $G_0(\mathbf{s}) = \mathbf{0}$.

We recall the definition of the *approximation spaces*: given $\alpha > 0$ and $0 < q < \infty$

$$A_q^\alpha\big(\mathfrak{f}^\Phi\big) = \left\{\mathbf{s} \in \mathfrak{f}^\Phi\colon \left[\sum_{N\geqslant 1}\big(N^\alpha \sigma_N(\mathbf{s})_{\mathfrak{f}^\Phi}\big)^q \frac{1}{N}\right]^{\frac{1}{q}} < \infty\right\},$$

and

$$\|\mathbf{s}\|_{A_q^\alpha(\mathfrak{f}^\Phi)} = \|\mathbf{s}\|_{\mathfrak{f}^\Phi} + \left[\sum_{N\geqslant 1}\big(N^\alpha \sigma_N(\mathbf{s})_{\mathfrak{f}^\Phi}\big)^q \frac{1}{N}\right]^{\frac{1}{q}}.$$

When $q = \infty$ one modifies these definitions in the standard way:

$$A_\infty^\alpha\big(\mathfrak{f}^\Phi\big) = \Big\{\mathbf{s} \in \mathfrak{f}^\Phi\colon \sup_{N\geqslant 1} N^\alpha \sigma_N(\mathbf{s})_{\mathfrak{f}^\Phi} < \infty\Big\}, \quad \|\mathbf{s}\|_{A_\infty^\alpha(\mathfrak{f}^\Phi)} = \|\mathbf{s}\|_{\mathfrak{f}^\Phi} + \sup_{N\geqslant 1} N^\alpha \sigma_N(\mathbf{s})_{\mathfrak{f}^\Phi}.$$

6.1. Sequence spaces in $\mathcal{D}$

We recall the definition of some classical sequence spaces over the index set $\mathcal{D}$. All of them are subspaces of $\mathbf{c}_0$ and therefore for each sequence $\{s_Q\}_{Q\in\mathcal{D}}$ we can find an enumeration of the index set $\mathcal{D} = \{Q_k\}_{k=1}^\infty$ so that $|s_{Q_1}| \geqslant |s_{Q_2}| \geqslant \cdots$ and in addition $\lim_{k\to\infty} s_{Q_k} = 0$. We shall always assume that $\{s_{Q_k}\}_{k\geqslant 1}$ corresponds to such ordering, which coincides with the *non-increasing rearrangement* $\mathbf{s}^*$ of the sequence $\mathbf{s}$.

Let $\eta = \{\eta_k\}_{k\geqslant 1}$ be a fixed positive increasing sequence so that $\lim_{k\to\infty}\eta_k = \infty$ and η is *doubling* (i.e. $\eta_{2k} \leqslant C\eta_k$, $k \geqslant 1$). Then, for each $0 < r \leqslant \infty$ we define a *discrete Lorentz space* by

$$\Lambda_\eta^r = \left\{\mathbf{s} \in \mathbf{c}_0\colon \|\mathbf{s}\|_{\Lambda_\eta^r} = \left[\sum_{k\geqslant 1}\big(\eta_k|s_{Q_k}|\big)^r \frac{1}{k}\right]^{\frac{1}{r}} < \infty\right\}.$$

Note that for $r = \infty$ one writes $\|\mathbf{s}\|_{\Lambda_\eta^\infty} = \sup_k \eta_k|s_{Q_k}|$. These are quasi-Banach rearrangement invariant spaces, which are Banach when $r \geqslant 1$ and in addition $\{\eta_k^r/k\}_k$ is non-increasing (see [4, p. 28]). When $r = 1$ or $r = \infty$ we shall write, respectively, Λ_η and $\mathbb{M}_\eta$ (the latter called Marcinkiewicz space). The particular case $\{\eta_k = k^{1/\tau}\}$ leads to the classical (discrete) Lorentz spaces $\Lambda_\eta^r = \ell^{\tau,r}(\mathcal{D})$. The spaces Λ_η^r for general η, and in particular their interpolation properties, have been studied, e.g., in [4,22,25]. In our applications we shall use the sequences $\{\eta_k = k^\alpha h_\varphi^\pm(k)\}_{k\geqslant 1}$, for suitable $\alpha \geqslant 0$, which always satisfy the required assumptions.

Remark 6.1. Given a fixed sequence space $\mathfrak{s}$ as above, we define a new sequence space $\mathfrak{s}(\mathfrak{f}^\Phi)$, isomorphic to $\mathfrak{s}$, by

$$\mathfrak{s}\big(\mathfrak{f}^\Phi\big) = \big\{\mathbf{s} = \{s_Q\}_{Q\in\mathcal{D}} \in \mathfrak{f}^\Phi\colon \big\{s_Q\|\mathbf{e}_Q\|_{\mathfrak{f}^\Phi}\big\}_Q \in \mathfrak{s}\big\},$$

with $\|\mathbf{s}\|_{\mathfrak{s}(\mathfrak{f}^\Phi)} = \|\{s_Q\|\mathbf{e}_Q\|_{\mathfrak{f}^\Phi}\}_Q\|_{\mathfrak{s}}$. Such definitions appear naturally in relation with greedy approximation when the basis is not normalized (see, e.g., [11]).

6.2. Jackson's inequalities

In this section we apply our results in Section 4 to obtain Jackson type estimates associated with the greedy algorithm.

Proposition 6.2. *Let Φ be a Young function so that $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$. Then, $\Lambda_{h_\varphi^+}(\mathfrak{f}^\Phi) \hookrightarrow \mathfrak{f}^\Phi$, and moreover, there is a constant $C > 0$ so that*

$$\big\|\mathbf{s} - \mathbf{G}_{N-1}(\mathbf{s})\big\|_{\mathfrak{f}^\Phi} \leqslant C \sum_{k>N/2}^{\infty} \|s_{Q_k}\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi} h_\varphi^+(k)\frac{1}{k}, \quad \forall N \geqslant 1. \tag{6.4}$$

Proof. We show (6.4) for every $N \geqslant 1$ (when $N = 1$, as $G_0(\mathbf{s}) = \mathbf{0}$, this is the embedding $\Lambda_{h_\varphi^+}(\mathfrak{f}^\Phi) \hookrightarrow \mathfrak{f}^\Phi$). By the triangular inequality and (6.1) we have

$$\begin{aligned}
\big\|\mathbf{s} - \mathbf{G}_{N-1}(\mathbf{s})\big\|_{\mathfrak{f}^\Phi} &= \Big\|\sum_{k\geqslant N} s_{Q_k}\mathbf{e}_{Q_k}\Big\|_{\mathfrak{f}^\Phi} \leqslant \sum_{j=0}^{\infty}\Big\|\sum_{2^jN\leqslant k<2^{j+1}N} s_{Q_k}\mathbf{e}_{Q_k}\Big\|_{\mathfrak{f}^\Phi}\\
&\leqslant \sum_{j=0}^{\infty}\|s_{Q_{2^jN}}\mathbf{e}_{Q_{2^jN}}\|_{\mathfrak{f}^\Phi}\Big\|\sum_{2^jN\leqslant k<2^{j+1}N}\frac{\mathbf{e}_{Q_k}}{\|\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi}}\Big\|_{\mathfrak{f}^\Phi}\\
&\leqslant C\sum_{j=0}^{\infty}\|s_{Q_{2^jN}}\mathbf{e}_{Q_{2^jN}}\|_{\mathfrak{f}^\Phi} h_\varphi^+\big(2^jN\big)
\end{aligned}$$

where in the last inequality we have used Theorem 4.1. This estimate can be transformed into (6.4) using that $h_\varphi^+(k)/k$ is non-increasing. Indeed, one just writes the right-hand side as

$$\sum_{j=0}^{\infty}\sum_{2^{j-1}N<k\leqslant 2^jN}\|s_{Q_{2^jN}}\mathbf{e}_{Q_{2^jN}}\|_{\mathfrak{f}^\Phi}\frac{h_\varphi^+(2^jN)}{2^{j-1}N} \leqslant 2\sum_{k>N/2}\|s_{Q_k}\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi}\frac{h_\varphi^+(k)}{k}. \qquad \square$$

Remark 6.3. The inequality in (6.4) is best possible, in the sense that left- and right-hand sides are comparable for certain choices of $\mathbf{s}$. Given $N \geqslant 2$ we take $k \in \mathbb{Z}$ so that

$$\frac{1}{2}h_\varphi^+(N) < \frac{\varphi(N2^{kd})}{\varphi(2^{kd})} \leqslant h_\varphi^+(N). \tag{6.5}$$

Let $\Gamma \subset \mathcal{D}$ be a collection of $2N-1$ pairwise disjoint dyadic cubes of equal size 2^{kd} and set $\mathbf{s} = \tilde{\mathbf{1}}_\Gamma = \sum_{Q\in\Gamma}\mathbf{e}_Q/\|\mathbf{e}_Q\|_{\mathfrak{f}^\Phi}$. Notice that for $Q \in \Gamma$ we have $\|s_Q\mathbf{e}_Q\|_{\mathfrak{f}^\Phi} = 1$. Thus $\mathbf{s} - \mathbf{G}_{N-1}(\mathbf{s}) = \tilde{\mathbf{1}}_{\Gamma'}$ for some $\Gamma' \subset \Gamma$ with $\operatorname{Card}\Gamma' = N$. It is easy to see that

$$\sigma_{N-1}(\mathbf{s})_{\mathfrak{f}^\Phi} = \big\|\mathbf{s} - G_{N-1}(\mathbf{s})\big\|_{\mathfrak{f}^\Phi} = \|\tilde{\mathbf{1}}_{\Gamma'}\|_{\mathfrak{f}^\Phi} = \frac{\varphi(N2^{kd})}{\varphi(2^{kd})} \simeq h_\varphi^+(N), \tag{6.6}$$

where the third equality follows as in Lemma 3.1.

On the other hand, when $\mathbf{s} = \tilde{\mathbf{1}}_\Gamma$, the right-hand side of (6.4) takes the form $\sum_{N/2<k\leqslant 2N-1} h_\varphi^+(k)/k \simeq h_\varphi^+(N)$, by the doubling property of h_φ^+.

Remark 6.4. We should also point out that for certain other sequences $\mathbf{s}$ the estimate in (6.4) may be too "crude." To see this consider the same example as before, but choosing the cubes sizes 2^{kd} so that in place of (6.5) we have $h_\varphi^-(N) \leqslant \frac{\varphi(N2^{kd})}{\varphi(2^{kd})} < 2h_\varphi^-(N)$. Then, $\sigma_{N-1}(\mathbf{s})_{\mathfrak{f}^\Phi} = \|\mathbf{s} - G_{N-1}(\mathbf{s})\|_{\mathfrak{f}^\Phi} \simeq h_\varphi^-(N)$, while the right-hand side of (6.4) is still comparable to $h_\varphi^+(N)$. For non-democratic spaces the gap between these two quantities can be big, as we have seen in the examples in Section 3.

The estimate in (6.4) implies a decay of $\sigma_N(\mathbf{s})_{\mathfrak{f}^\Phi}$ as N growths. For general $\mathbf{s} \in \Lambda_{h_\varphi^+}(\mathfrak{f}^\Phi)$ we do not have further information about the rate of decay. However, restricting $\mathbf{s}$ to appropriate subspaces we can obtain precise rates of convergence.

Corollary 6.5. *Let Φ be a Young function so that $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$, and let $\alpha > 0$. Then, for every $\mathbf{s} \in \mathbb{M}_{k^\alpha h_\varphi^+(k)}(\mathfrak{f}^\Phi)$ we have*

$$\big\|\mathbf{s} - \mathbf{G}_{N-1}(\mathbf{s})\big\|_{\mathfrak{f}^\Phi} \leqslant CN^{-\alpha}\|\mathbf{s}\|_{\mathbb{M}_{k^\alpha h_\varphi^+(k)}(\mathfrak{f}^\Phi)}, \quad \forall N \geqslant 1. \tag{6.7}$$

Proof. By (6.4) and the definition of the Marcinkiewicz space

$$\begin{aligned}\big\|\mathbf{s} - \mathbf{G}_{N-1}(\mathbf{s})\big\|_{\mathfrak{f}^\Phi} &\leqslant C \sum_{k>N/2} \|s_{Q_k}\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi} h_\varphi^+(k)\frac{1}{k} \leqslant C\|\mathbf{s}\|_{\mathbb{M}_{k^\alpha h_\varphi^+(k)}(\mathfrak{f}^\Phi)} \sum_{k>N/2}^{\infty} k^{-\alpha}\frac{1}{k} \\ &\leqslant CN^{-\alpha}\|\mathbf{s}\|_{\mathbb{M}_{k^\alpha h_\varphi^+(k)}(\mathfrak{f}^\Phi)}. \qquad \square\end{aligned}$$

The previous result can be translated as an inclusion of approximation spaces.

Corollary 6.6. *Let $\alpha > 0$. Then*

$$\mathbb{M}_{k^\alpha h_\varphi^+(k)}\big(\mathfrak{f}^\Phi\big) \hookrightarrow A_\infty^\alpha\big(\mathfrak{f}^\Phi\big). \tag{6.8}$$

Moreover, $\mathbb{M}_{k^\alpha h_\varphi^+(k)}$ is the largest $\mathbb{M}_\eta$-space so that $\mathbb{M}_\eta(\mathfrak{f}^\Phi) \hookrightarrow A_\infty^\alpha(\mathfrak{f}^\Phi)$.

Proof. The inclusion (6.8) is obvious from (6.7) and the definition of $A_\infty^\alpha(\mathfrak{f}^\Phi)$. To see the optimality, assume that $\mathbb{M}_\eta(\mathfrak{f}^\Phi) \hookrightarrow A_\infty^\alpha(\mathfrak{f}^\Phi)$, and let $\mathbf{s} = \tilde{\mathbf{1}}_\Gamma$ be as in Remark 6.3. Then, by (6.6) we have $\|\mathbf{s}\|_{A_\infty^\alpha(\mathfrak{f}^\Phi)} \gtrsim N^\alpha h_\varphi^+(N)$. On the other hand, $\|\mathbf{s}\|_{\mathbb{M}_\eta(\mathfrak{f}^\Phi)} = \sup_{1\leqslant k\leqslant 2N-1}\eta_k = \eta_{2N-1}$. Thus, the assumed inclusion and the doubling property give $N^\alpha h_\varphi^+(N) \lesssim \eta_N$, which shows $\mathbb{M}_\eta(\mathfrak{f}^\Phi) \hookrightarrow \mathbb{M}_{k^\alpha h_\varphi^+(k)}(\mathfrak{f}^\Phi)$. $\square$

As a particular case we obtain the following inclusions in terms of classical Lorentz spaces.

Corollary 6.7. *Let $\alpha > 0$. Then, we have the inclusion*

$$\ell^{\tau,\infty}\big(\mathfrak{f}^\Phi\big) \hookrightarrow A_\infty^\alpha\big(\mathfrak{f}^\Phi\big), \quad \textit{whenever } \frac{1}{\tau} > \alpha + \overline{\pi}_{L^\Phi}. \tag{6.9}$$

Proof. By (2.5), we know that $h_\varphi^+(t) \leqslant C_\varepsilon t^{\overline{\pi}_{L^\Phi}+\varepsilon}$, $\forall t \geqslant 1$. Choosing $\varepsilon = \frac{1}{\tau} - \alpha - \overline{\pi}_{L^\Phi}$ this gives $k^\alpha h_\varphi^+(k) \lesssim k^{\frac{1}{\tau}}$, $k \geqslant 1$, which in turn implies $\ell^{\tau,\infty} \hookrightarrow \mathbb{M}_{k^\alpha h_\varphi^+(k)}$. The result then follows from (6.8). $\square$

Remark 6.8. Let us observe that from the proof of Corollary 6.6, if (6.9) is valid for $\frac{1}{\tau} = \alpha + \overline{\pi}_{L^\Phi}$, then it follows that $h_\varphi^+(N) \lesssim N^{\overline{\pi}_{L^\Phi}}$. Also, Lemma 5.1 and (2.4) imply that $h_\varphi^+(N) \gtrsim N^{\overline{\pi}_{L^\Phi}}$ and therefore $h_\varphi^+(N) \simeq N^{\overline{\pi}_{L^\Phi}}$ for $N \geqslant 1$. Conversely, if one assumes that $h_\varphi^+(N) \simeq N^{\overline{\pi}_{L^\Phi}}$ for $N \geqslant 1$, Corollary 6.6 gives $\ell^{\tau,\infty}(\mathfrak{f}^\Phi) \hookrightarrow A_\infty^\alpha(\mathfrak{f}^\Phi)$ with $\frac{1}{\tau} = \alpha + \overline{\pi}_{L^\Phi}$. This shows that for (6.9) to be valid at the endpoint $\frac{1}{\tau} = \alpha + \overline{\pi}_{L^\Phi}$, it is necessary and sufficient that $h_\varphi^+(N) \simeq N^{\overline{\pi}_{L^\Phi}}$, $N \geqslant 1$. In our examples in Section 2.1, this is the case for the Young functions associated with $L^2 + L^4$, $L^2 \cap L^4$ or $L^p(\log L)^\alpha$ with $\alpha > 0$, but may fail in other cases, such as for the spaces $L^p(\log L)^\alpha$ with $\alpha < 0$ (see Example 2.3).

6.3. Bernstein's inequalities

Bernstein type estimates are useful to obtain converse inclusions for approximation spaces.

Proposition 6.9. *Let Φ be a Young function so that $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$. Then, $\mathfrak{f}^\Phi \hookrightarrow \mathbb{M}_{h_\varphi^-}(\mathfrak{f}^\Phi)$ and there is a constant $C > 0$ so that*

$$\big\|\mathbf{G}_N(\mathbf{s})\big\|_{\mathbb{M}_{h_\varphi^-}(\mathfrak{f}^\Phi)} = \sup_{1\leqslant k\leqslant N} \|s_{Q_k}\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi} h_\varphi^-(k) \leqslant C\big\|\mathbf{G}_N(\mathbf{s})\big\|_{\mathfrak{f}^\Phi}, \quad \forall N \geqslant 1. \tag{6.10}$$

Proof. As before, it suffices to show (6.10), since the embedding $\mathfrak{f}^\Phi \hookrightarrow \mathbb{M}_{h_\varphi^-}(\mathfrak{f}^\Phi)$ follows by letting $N \to \infty$. For fixed $1 \leqslant k \leqslant N$, using Theorem 4.1 and the lattice property (6.1) we have

$$\|s_{Q_k}\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi} h_\varphi^-(k) \leqslant C\|s_{Q_k}\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi} \left\| \sum_{j=1}^k \frac{\mathbf{e}_{Q_j}}{\|\mathbf{e}_{Q_j}\|_{\mathfrak{f}^\Phi}} \right\|_{\mathfrak{f}^\Phi} \leqslant C \left\| \sum_{j=1}^k s_{Q_j}\mathbf{e}_{Q_j} \right\|_{\mathfrak{f}^\Phi} \leqslant C\big\|\mathbf{G}_N(\mathbf{s})\big\|_{\mathfrak{f}^\Phi}. \qquad \square$$

Remark 6.10. As before, one can show the optimality of (6.10) by finding an appropriate $\mathbf{s}$ for which both sides of the inequality are comparable. Indeed, one just needs to choose $\mathbf{s} = \tilde{\mathbf{1}}_\Gamma$, for Γ consisting of N disjoint cubes of equal size 2^{kd} and k such that $h_\varphi^-(N) \leqslant \frac{\varphi(N2^{kd})}{\varphi(2^{kd})} < 2h_\varphi^-(N)$. In this case, as in Lemma 3.1 we have

$$\big\|\mathbf{G}_N(\mathbf{s})\big\|_{\mathfrak{f}^\Phi} = \|\mathbf{s}\|_{\mathfrak{f}^\Phi} = \frac{\varphi(N2^{kd})}{\varphi(2^{kd})} \simeq h_\varphi^-(N). \tag{6.11}$$

On the other hand, as h_φ^- is non-decreasing,

$$\big\|\mathbf{G}_N(\mathbf{s})\big\|_{\mathbb{M}_{h_\varphi^-}(\mathfrak{f}^\Phi)} = \sup_{1\leqslant k\leqslant N} h_\varphi^-(k) = h_\varphi^-(N),$$

and therefore both sides of (6.10) are comparable.

Corollary 6.11. *Let Φ be a Young function so that $0 < \underline{\pi}_{L^\Phi} \leqslant \overline{\pi}_{L^\Phi} < 1$ and let $\alpha > 0$. Then, there exists $C > 0$ so that, for all $N \geqslant 1$,*

$$\|\mathbf{s}\|_{\Lambda_{k^\alpha h_\varphi^-(k)}(\mathfrak{f}^\Phi)} \leqslant CN^\alpha \|\mathbf{s}\|_{\mathfrak{f}^\Phi}, \quad \forall \mathbf{s} \in \Sigma_N. \tag{6.12}$$

Proof. Write $\mathbf{s} = G_N(\mathbf{s}) = \sum_{k=1}^N s_{Q_k}\mathbf{e}_{Q_k}$ with $\|s_{Q_1}\mathbf{e}_{Q_1}\|_{\mathfrak{f}^\Phi} \geqslant \|s_{Q_2}\mathbf{e}_{Q_2}\|_{\mathfrak{f}^\Phi} \geqslant \cdots$. By (6.10) we have

$$\|\mathbf{s}\|_{\Lambda_{k^\alpha h_\varphi^-(k)}(\mathfrak{f}^\Phi)} = \sum_{k=1}^N k^\alpha h_\varphi^-(k)\|s_{Q_k}\mathbf{e}_{Q_k}\|_{\mathfrak{f}^\Phi} \frac{1}{k} \leqslant C\big\|G_N(\mathbf{s})\big\|_{\mathfrak{f}^\Phi} \sum_{k=1}^N \frac{k^\alpha}{k} \leqslant C'N^\alpha \|\mathbf{s}\|_{\mathfrak{f}^\Phi}. \qquad \square$$

As before, the above result can be stated as an inclusion of approximation spaces. Below, the number $\rho = \rho_\alpha \in (0, 1]$ is chosen so that the quasi-normed space $\Lambda_{k^\alpha h_\varphi^-(k)}$ satisfies the ρ-triangular inequality, that is, for every $N \geqslant 1$,

$$\|\mathbf{s}_1 + \mathbf{s}_2\|^\rho_{\Lambda_{k^\alpha h_\varphi^-(k)}} \leqslant \|\mathbf{s}_1\|^\rho_{\Lambda_{k^\alpha h_\varphi^-(k)}} + \|\mathbf{s}_2\|^\rho_{\Lambda_{k^\alpha h_\varphi^-(k)}}. \tag{6.13}$$

Corollary 6.12. *Let $\alpha > 0$. Then*

$$A^\alpha_\rho\big(\mathfrak{f}^\Phi\big) \hookrightarrow \Lambda_{k^\alpha h_\varphi^-(k)}\big(\mathfrak{f}^\Phi\big). \tag{6.14}$$

Moreover, $\Lambda_{k^\alpha h_\varphi^-(k)}$ is the smallest Λ_η-space so that $A^\alpha_q(\mathfrak{f}^\Phi) \hookrightarrow \Lambda_\eta(\mathfrak{f}^\Phi)$ for some $0 < q \leqslant 1$.

Proof. The argument for (6.14) is standard (see, e.g., [8]). It suffices to prove that

$$\|\mathbf{s}\|_{\Lambda_{k^\alpha h_\varphi^-(k)}(\mathfrak{f}^\Phi)} \leqslant C\|\mathbf{s}\|_{A^\alpha_\rho(\mathfrak{f}^\Phi)}, \quad \forall \mathbf{s} \in \Sigma_N,\ N \geqslant 1$$

with a constant $C > 0$ independent of N and one obtains the desired inclusion by letting $N \to \infty$. We may also assume $N = 2^J$. Now, write $\mathbf{s} = \sum_{j=0}^{J}[\mathbf{s}^{(j)} - \mathbf{s}^{(j-1)}]$, where by convention $\mathbf{s}^{(J)} = \mathbf{s}$, $\mathbf{s}^{(-1)} = \mathbf{0}$ and $\mathbf{s}^{(j)} \in \Sigma_{2^j}$ is so that $\|\mathbf{s} - \mathbf{s}^{(j)}\|_{\mathfrak{f}^\Phi} \leqslant 2\sigma_{2^j}(\mathbf{s})_{\mathfrak{f}^\Phi}$, $0 \leqslant j < J$. Then applying (6.13), and (6.12) to $\mathbf{s}^{(j)} - \mathbf{s}^{(j-1)} \in \Sigma_{2^{j+1}}$ we obtain

$$\|\mathbf{s}\|_{\Lambda_{k^\alpha h_\varphi^-(k)}(\mathfrak{f}^\Phi)} \leqslant \left[\sum_{j=0}^{J} \big\|\mathbf{s}^{(j)} - \mathbf{s}^{(j-1)}\big\|^\rho_{\Lambda_{k^\alpha h_\varphi^-(k)}(\mathfrak{f}^\Phi)}\right]^{\frac{1}{\rho}} \leqslant C\left[\sum_{j=0}^{J} 2^{j\alpha\rho}\big\|\mathbf{s}^{(j)} - \mathbf{s}^{(j-1)}\big\|^\rho_{\mathfrak{f}^\Phi}\right]^{\frac{1}{\rho}}.$$

Now, by assumption for $1 \leqslant j \leqslant J$

$$\big\|\mathbf{s}^{(j)} - \mathbf{s}^{(j-1)}\big\|_{\mathfrak{f}^\Phi} \leqslant \big\|\mathbf{s}^{(j)} - \mathbf{s}\big\|_{\mathfrak{f}^\Phi} + \big\|\mathbf{s} - \mathbf{s}^{(j-1)}\big\|_{\mathfrak{f}^\Phi} \leqslant 4\sigma_{2^{j-1}}(\mathbf{s})_{\mathfrak{f}^\Phi}.$$

On the other hand, for $j = 0$ we have

$$\big\|\mathbf{s}^{(0)} - \mathbf{s}^{(-1)}\big\|_{\mathfrak{f}^\Phi} = \big\|\mathbf{s}^{(0)}\big\|_{\mathfrak{f}^\Phi} \lesssim \big\|\mathbf{s}^{(0)} - \mathbf{s}\big\|_{\mathfrak{f}^\Phi} + \|\mathbf{s}\|_{\mathfrak{f}^\Phi} \leqslant 2\sigma_1(\mathbf{s})_{\mathfrak{f}^\Phi} + \|\mathbf{s}\|_{\mathfrak{f}^\Phi}.$$

Hence,

$$\|\mathbf{s}\|_{\Lambda_{k^\alpha h_\varphi^-(k)}(\mathfrak{f}^\Phi)} \leqslant C\left[\|\mathbf{s}\|_{\mathfrak{f}^\Phi} + \sum_{j=0}^{J-1}\big(2^{j\alpha}\sigma_{2^j}(\mathbf{s})_{\mathfrak{f}^\Phi}\big)^\rho\right]^{\frac{1}{\rho}} \simeq \|\mathbf{s}\|_{A^\alpha_\rho(\mathfrak{f}^\Phi)}.$$

To see the optimality, assume that for some sequence η and $q \in (0, 1]$ we have $A^\alpha_q(\mathfrak{f}^\Phi) \hookrightarrow \Lambda_\eta(\mathfrak{f}^\Phi)$, and let $\mathbf{s} = \tilde{\mathbf{1}}_\Gamma$ be as in Remark 6.10. Then by (6.11) we have

$$\|\mathbf{s}\|_{A^\alpha_q(\mathfrak{f}^\Phi)} = \|\mathbf{s}\|_{\mathfrak{f}^\Phi} + \left[\sum_{k=1}^{N}\big(k^\alpha\sigma_k(\mathbf{s})_{\mathfrak{f}^\Phi}\big)^q\frac{1}{k}\right]^{\frac{1}{q}} \lesssim \|\mathbf{s}\|_{\mathfrak{f}^\Phi}\left[\sum_{k=1}^{N}k^{\alpha q-1}\right]^{\frac{1}{q}} \simeq CN^\alpha h_\varphi^-(N).$$

On the other hand, by the doubling property

$$\|\mathbf{s}\|_{\Lambda_\eta(\mathfrak{f}^\Phi)} = \sum_{k=1}^{N}\eta_k\frac{1}{k} \geqslant \sum_{N/2<k\leqslant N}\eta_k\frac{1}{k} \gtrsim \eta_{N/2} \gtrsim \eta_N.$$

Thus, if the assumed inclusion holds, the previous two estimates lead us to $\eta_N \lesssim N^\alpha h_\varphi^-(N)$, which in turn implies $\Lambda_{k^\alpha h_\varphi^-(k)} \hookrightarrow \Lambda_\eta$. □

Corollary 6.13. *Let $\alpha > 0$. Then, we have the inclusions*

$$A^\alpha_\rho\big(\mathfrak{f}^\Phi\big) \hookrightarrow \ell^{\tau,1}\big(\mathfrak{f}^\Phi\big), \quad \textit{whenever } \frac{1}{\tau} < \alpha + \underline{\pi}_{L^\Phi}. \tag{6.15}$$

Proof. From (2.5) we have $h_\varphi^-(t) = 1/h_\varphi^+(1/t) \geqslant C_\varepsilon t^{\underline{\pi}_{L^\Phi}-\varepsilon}$, $t \geqslant 1$. Letting $\varepsilon = \alpha + \underline{\pi}_{L^\Phi} - \frac{1}{\tau}$ we obtain that $k^\alpha h_\varphi^-(k) \gtrsim k^{\frac{1}{\tau}}$, which leads to $\Lambda_{k^\alpha h_\varphi^-(k)} \hookrightarrow \ell^{\tau,1}$. The result then follows from (6.14). □

Remark 6.14. As in Remark 6.8 if (6.15) holds at $\frac{1}{\tau} = \alpha + \underline{\pi}_{L^\Phi}$ it follows that $h_\varphi^-(N) \gtrsim N^{\underline{\pi}_{L^\Phi}}$ and therefore $h_\varphi^+(t) \lesssim t^{\underline{\pi}_{L^\Phi}}$ for $0 < t \leqslant 1$. From Lemma 5.1 and (2.4) we also have that $h_\varphi^+(t) \gtrsim t^{\underline{\pi}_{L^\Phi}}$ for $0 < t \leqslant 1$. This yields that $h_\varphi^+(t) \simeq t^{\underline{\pi}_{L^\Phi}}$ for $0 < t \leqslant 1$. On the other hand, assuming that $h_\varphi^+(t) \simeq t^{\underline{\pi}_{L^\Phi}}$ for $0 < t \leqslant 1$, (6.14) implies (6.15) at $\frac{1}{\tau} = \alpha + \underline{\pi}_{L^\Phi}$. All this shows that a necessary and sufficient condition for the endpoint case $\frac{1}{\tau} = \alpha + \underline{\pi}_{L^\Phi}$ in (6.15) to hold is $h_\varphi^+(t) \simeq t^{\underline{\pi}_{L^\Phi}}$ for $t \in (0, 1]$. In our examples in Section 2.1, this is the case for the Young functions associated with $L^2 + L^4$, $L^2 \cap L^4$ or $L^p(\log L)^\alpha$ with $\alpha < 0$, but such property fails in this last case when $\alpha > 0$ (see Example 2.2).

6.4. Inclusions for the approximation spaces $A_q^\alpha(\mathfrak{f}^\Phi)$

Finally, using real interpolation we can obtain inclusions for the whole family of approximation spaces $A_q^\alpha(\mathfrak{f}^\Phi)$, $0 < q \leqslant \infty$. For this we take into account the interpolation properties of the sequence spaces Λ_η^q, namely,

$$\big(\Lambda^r_{k^{\alpha_0}\eta(k)}, \Lambda^r_{k^{\alpha_1}\eta(k)}\big)_{\theta,q} = \Lambda^q_{k^\alpha \eta(k)}, \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \tag{6.16}$$

for all $0 < q, r \leqslant \infty$, $0 < \theta < 1$ (see, e.g., [25, Proposition 6.2], [22, Theorem 3]).

Corollary 6.15. *Let $\alpha > 0$ and $0 < q \leqslant \infty$. Then*

$$\Lambda^q_{k^\alpha h_\varphi^+(k)}\big(\mathfrak{f}^\Phi\big) \hookrightarrow A_q^\alpha\big(\mathfrak{f}^\Phi\big) \hookrightarrow \Lambda^q_{k^\alpha h_\varphi^-(k)}\big(\mathfrak{f}^\Phi\big). \tag{6.17}$$

Proof. Let $\alpha_0 < \alpha < \alpha_1$, so that $\alpha = (\alpha_0 + \alpha_1)/2$. Then, for every $0 < q, r \leqslant \infty$ we have (see, e.g., [8])

$$A_q^\alpha = \big(A_r^{\alpha_0}, A_r^{\alpha_1}\big)_{1/2,q}.$$

Setting $r = \min\{\rho_{\alpha_0}, \rho_{\alpha_1}\}$ and using (6.14)

$$A_q^\alpha\big(\mathfrak{f}^\Phi\big) = \big(A_r^{\alpha_0}\big(\mathfrak{f}^\Phi\big), A_r^{\alpha_1}\big(\mathfrak{f}^\Phi\big)\big)_{1/2,q} \hookrightarrow \big(\Lambda_{k^{\alpha_0}h_\varphi^-(k)}\big(\mathfrak{f}^\Phi\big), \Lambda_{k^{\alpha_1}h_\varphi^-(k)}\big(\mathfrak{f}^\Phi\big)\big)_{1/2,q} = \Lambda^q_{k^\alpha h_\varphi^-(k)}\big(\mathfrak{f}^\Phi\big),$$

where the last equality follows from (6.16). Similarly, by (6.8)

$$A_q^\alpha\big(\mathfrak{f}^\Phi\big) = \big(A_\infty^{\alpha_0}\big(\mathfrak{f}^\Phi\big), A_\infty^{\alpha_1}\big(\mathfrak{f}^\Phi\big)\big)_{1/2,q} \hookleftarrow \big(\mathbb{M}_{k^{\alpha_0}h_\varphi^+(k)}\big(\mathfrak{f}^\Phi\big), \mathbb{M}_{k^{\alpha_1}h_\varphi^+(k)}\big(\mathfrak{f}^\Phi\big)\big)_{1/2,q} = \Lambda^q_{k^\alpha h_\varphi^+(k)}\big(\mathfrak{f}^\Phi\big). \qquad \square$$

As a consequence of (6.17), and proceeding as in Corollaries 6.7 and 6.13 we obtain the following result.

Corollary 6.16. *For all $\alpha > 0$, $q, q_0, q_1 \in (0, \infty]$ we have*

$$\ell^{\tau_0,q_0}\big(\mathfrak{f}^\Phi\big) \hookrightarrow A_q^\alpha\big(\mathfrak{f}^\Phi\big) \hookrightarrow \ell^{\tau_1,q_1}\big(\mathfrak{f}^\Phi\big),$$

whenever $\frac{1}{\tau_1} < \alpha + \underline{\pi}_{L^\Phi} \leqslant \alpha + \overline{\pi}_{L^\Phi} < \frac{1}{\tau_0}$.

Proof. Pick τ so that $\frac{1}{\tau_0} > \frac{1}{\tau} > \alpha + \overline{\pi}_{L^\Phi}$. Then as in the proof of Corollary 6.7 we observed that for all $t \geqslant 1$ we have $t^\alpha h_\varphi^+(t) \lesssim t^{\frac{1}{\tau}}$. Then (6.17) and the embedding $\ell^{\tau_0,q_0} \hookrightarrow \ell^{\tau,q}$ yield

$$A_q^\alpha\big(\mathfrak{f}^\Phi\big) \hookleftarrow \Lambda^q_{k^\alpha h_\varphi^+(k)}\big(\mathfrak{f}^\Phi\big) \hookleftarrow \Lambda^q_{k^{\frac{1}{\tau}}}\big(\mathfrak{f}^\Phi\big) = \ell^{\tau,q}\big(\mathfrak{f}^\Phi\big) \hookleftarrow \ell^{\tau_0,q_0}\big(\mathfrak{f}^\Phi\big).$$

For the other embedding we choose τ verifying $\frac{1}{\tau_1} < \frac{1}{\tau} < \alpha + \underline{\pi}_{L^\Phi}$. The proof of Corollary 6.13 yields that $t^\alpha h_\varphi^-(t) \gtrsim t^{\frac{1}{\tau}}$. Then (6.17) and the embedding $\ell^{\tau,q} \hookrightarrow \ell^{\tau_1,q_1}$ give

$$A_q^\alpha\big(\mathfrak{f}^\Phi\big) \hookrightarrow \Lambda^q_{k^\alpha h_\varphi^-(k)}\big(\mathfrak{f}^\Phi\big) \hookrightarrow \Lambda^q_{k^{\frac{1}{\tau}}}\big(\mathfrak{f}^\Phi\big) = \ell^{\tau,q}\big(\mathfrak{f}^\Phi\big) \hookrightarrow \ell^{\tau_1,q_1}\big(\mathfrak{f}^\Phi\big). \qquad \square$$

Remark 6.17. Observe that the two results stated in the Introduction, Theorem 1.3 and Corollary 1.4, are straightforward consequences of Corollaries 6.15 and 6.16 and the definition of the spaces $\mathfrak{s}(L^\Phi)$ in (1.4).

Remark 6.18. Notice finally that the inclusions in (6.17) remain as well valid when we replace $A_q^\alpha(\mathfrak{f}^\Phi)$ by the smaller approximation space

$$\mathcal{G}_q^\alpha\big(\mathfrak{f}^\Phi\big) = \left\{\mathbf{s} \in \mathfrak{f}^\Phi\colon \left[\sum_{N\geqslant 1}\big(N^\alpha \big\|\mathbf{s} - G_N(\mathbf{s})\big\|_{\mathfrak{f}^\Phi}\big)^q \frac{1}{N}\right]^{\frac{1}{q}} < \infty\right\}.$$

This is because of our formulation of the Jackson estimate in (6.7). We do not know however whether in general one can have $A_q^\alpha(\mathfrak{f}^\Phi) = \mathcal{G}_q^\alpha(\mathfrak{f}^\Phi)$. See more properties of $\mathcal{G}_q^\alpha(\mathfrak{f}^\Phi)$ in [12].

6.5. Besov spaces of generalized smoothness

Let $\Psi:(0,\infty)\to(0,\infty)$ be a fixed continuous function with $\sup_{s>0}\Psi(ts)/\Psi(s)<\infty$, for all $t>0$. Given $0<\tau,q\leqslant\infty$, we define a *Besov space of Ψ-smoothness*, $\dot{B}^{\Psi}_{\tau,q}(\mathbb{R}^d)$, as the set of all tempered distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{\dot{B}^{\Psi}_{\tau,q}}=\left[\sum_{j\in\mathbb{Z}}\big(\Psi\big(2^j\big)\|f*\chi_j\|_{L^\tau(\mathbb{R}^d)}\big)^q\right]^{\frac{1}{q}}<\infty, \tag{6.18}$$

where $\chi\in\mathcal{S}(\mathbb{R}^d)$ is so that $\chi_{\{|\xi|\leqslant1\}}\leqslant\hat{\chi}(\xi)\leqslant\chi_{\{|\xi|\leqslant2\}}$, and $\chi_j(x)=2^{jd}\chi(2^jx)-2^{(j-1)d}\chi(2^{j-1}x)$. As usual, one takes the quotient of $\dot{B}^{\alpha}_{\tau,q}$ with the set of polynomials to obtain a (quasi)-Banach space.

Besov spaces of generalized smoothness were introduced in [5,22] in the context of real interpolation with function parameters (see also references in [1,10]). The particular case $\Psi(t)=t^\alpha$ corresponds to the usual (homogeneous) Besov space $\dot{B}^{\alpha}_{\tau,q}(\mathbb{R}^d)$. When $\Psi(t)=t^\alpha(1+\log^+t)^\gamma$ one obtains logarithmic Besov spaces $\dot{B}^{(\alpha,\gamma)}_{\tau,q}$, analogous to those studied by Leopold in [21] (see also [24]). Alternative characterizations of these spaces also appear in [2,15]. We point out that most of the above mentioned references only consider the theory of "inhomogeneous spaces" (in which the series in (6.18) is truncated to $j\geqslant0$; see (6.23) below). Minor modifications, however, are necessary to carry out a similar theory in the "homogeneous" setting of $\dot{B}^{\Psi}_{\tau,q}$.

In this paper we shall only use the wavelet characterization of $\dot{B}^{\Psi}_{\tau,q}(\mathbb{R}^d)$ (which we may as well take as definition), similar to the one obtained by Almeida in the inhomogeneous setting (see [1]). As in Section 2.2 we fix a wavelet basis $\{\psi^\ell_Q\}$, which we shall assume to consist of Schwartz functions. For notational simplicity, we shall also drop the super-index ℓ.

Proposition 6.19. *A tempered distribution f belongs to $\dot{B}^{\Psi}_{\tau,q}(\mathbb{R}^d)$ if and only if*

$$\sum_{j\in\mathbb{Z}}\left[\left(\sum_{|Q|=2^{-jd}}\big|\Psi\big(|Q|^{-\frac{1}{d}}\big)|Q|^{\frac{1}{\tau}-\frac{1}{2}}\langle f,\psi_Q\rangle\big|^\tau\right)^{\frac{1}{\tau}}\right]^q<\infty. \tag{6.19}$$

Moreover, this expression is comparable to $\|f\|^q_{\dot{B}^{\Psi}_{\tau,q}}$.

A particular case of this result is given next.

Corollary 6.20. *Let Φ be a Young function with $0<\underline{\pi}_{L^\Phi}\leqslant\overline{\pi}_{L^\Phi}<1$ and $\tau>0$. Define $\Psi(t)=t^{\frac{d}{\tau}}/\Phi^{-1}(t^d)=t^{\frac{d}{\tau}}\varphi(t^{-d})$. Then,*

$$\dot{B}^{\Psi}_{\tau,\tau}=\left\{f\in\mathcal{S}'\big(\mathbb{R}^d\big)\colon\ \sum_Q\big\|\langle f,\psi_Q\rangle_Q\mathbf{e}_Q\big\|^\tau_{\mathfrak{f}^\Phi}<\infty\right\}, \tag{6.20}$$

with the equivalence of norms $\|f\|_{\dot{B}^{\Psi}_{\tau,\tau}}\simeq\big(\sum_Q\big\|\langle f,\psi_Q\rangle_Q\mathbf{e}_Q\big\|^\tau_{\mathfrak{f}^\Phi}\big)^{\frac{1}{\tau}}$.

Proof. From (2.5) and (2.6) it follows that the function $\Psi(t)$ satisfies the conditions required at the beginning of this section. By (6.3) and the definition of Ψ we have

$$\|\mathbf{e}_Q\|_{\mathfrak{f}^\Phi}=|Q|^{-\frac{1}{2}}\varphi\big(|Q|\big)=|Q|^{\frac{1}{\tau}-\frac{1}{2}}\Psi\big(|Q|^{-\frac{1}{d}}\big).$$

Therefore we can write

$$\sum_Q\big\|\langle f,\psi_Q\rangle_Q\mathbf{e}_Q\big\|^\tau_{\mathfrak{f}^\Phi}=\sum_Q\big|\Psi\big(|Q|^{-\frac{1}{d}}\big)|Q|^{\frac{1}{\tau}-\frac{1}{2}}\langle f,\psi_Q\rangle\big|^\tau,$$

which together with Proposition 6.19 complete the proof. □

We now proceed to connect these Besov spaces with the approximation spaces $A^\alpha_r(L^\Phi)$.

Corollary 6.21. *Let $\alpha > 0$ and $0 < q \leqslant \infty$. Then*

$$\dot{B}^{t^{\frac{d}{\tau_0}}\varphi(t^{-d})}_{\tau_0,\tau_0} \hookrightarrow A^\alpha_q\big(L^\Phi\big) \hookrightarrow \dot{B}^{t^{\frac{d}{\tau_1}}\varphi(t^{-d})}_{\tau_1,\tau_1}, \tag{6.21}$$

whenever $\frac{1}{\tau_1} < \alpha + \underline{\pi}_{L^\Phi} \leqslant \alpha + \overline{\pi}_{L^\Phi} < \frac{1}{\tau_0}$.

Remark 6.22. As usual, the first inclusion in (6.21) is understood with the assignment $f \mapsto \sum_Q \langle f, \psi_Q\rangle_Q \psi_Q$, so that polynomials in the Besov space are mapped into the null function of L^Φ (see the proof below).

Proof of Corollary 6.21. We prove the first inclusion. Given $f \in \dot{B}^{t^{\frac{d}{\tau_0}}\varphi(t^{-d})}_{\tau_0,\tau_0}$, by (6.20) the sequence $\{\langle f, \psi_Q\rangle_Q \|\mathbf{e}_Q\|_{\mathfrak{f}^\Phi}\}_{Q\in\mathcal{D}}$ belongs to ℓ^{τ_0}, and since $\frac{1}{\tau_0} > \overline{\pi}_{L^\Phi}$, also to $\Lambda_{h^+_\varphi}$. By Proposition 6.2, this implies that $\mathbf{s} = \{\langle f, \psi_Q\rangle\}_{Q\in\mathcal{D}} \in \mathfrak{f}^\Phi$, and therefore $f^\sharp = \sum_Q \langle f, \psi_Q\rangle_Q \psi_Q \in L^\Phi(\mathbb{R}^d)$ (with convergence of the series in L^Φ). Moreover, by Corollary 6.16, we also have $\mathbf{s} \in \ell^{\tau_0}(\mathfrak{f}^\Phi) \hookrightarrow A^\alpha_q(\mathfrak{f}^\Phi)$. Finally, since $\sigma_N(\mathbf{s})_{\mathfrak{f}^\Phi} = \sigma_N(f^\sharp)_{L^\Phi}$ we easily conclude that $f^\sharp \in A^\alpha_q(L^\Phi)$ and $\|f^\sharp\|_{A^\alpha_q(L^\Phi)} \leqslant C\|f\|_{\dot{B}^\Psi_{\tau_0,\tau_0}}$ as asserted. The second inclusion is proved similarly using the right-hand inclusion of Corollary 6.16. □

Remark 6.23. A special case of the previous proof gives the Sobolev type embedding

$$\dot{B}^{t^{\frac{d}{\tau}}\varphi(t^{-d})}_{\tau,\tau} \hookrightarrow L^\Phi, \quad 0 < \tau < 1/\overline{\pi}_{L^\Phi}.$$

This is a refinement of the classical estimate $\dot{B}^{d(\frac{1}{\tau}-\frac{1}{p})}_{\tau,\tau} \hookrightarrow L^p$, for $0 < \tau < p$.

The special case of Zygmund spaces $L^p(\log L)^{\gamma p}$. Let us now consider the special case of the *Zygmund spaces* $L^\Phi = L^p(\log L)^{\gamma p}$ in Examples 2.2 and 2.3 above. We wish to describe the approximation spaces $A^\alpha_q(L^\Phi)$, for fixed $\alpha > 0$ and $0 < q \leqslant \infty$.

The description is given in terms of the logarithmic Besov spaces $\dot{B}^{(\alpha,\gamma)}_{\tau,\tau}(\mathbb{R}^d)$, i.e. $\dot{B}^\Psi_{\tau,\tau}$ with $\Psi(t) = t^\alpha(1+\log^+ t)^\gamma$. By Corollary 6.20 and the explicit expression $\varphi(t) \simeq t^{\frac{1}{p}}(1+\log^+ 1/t)^\gamma$, we can identify $\dot{B}^{(\alpha,\gamma)}_{\tau,\tau}(\mathbb{R}^d)$ with $\ell^\tau(\mathfrak{f}^\Phi)$ when $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$.

Then, Corollary 6.21 gives

$$\dot{B}^{(\alpha_0,\gamma)}_{\tau_0,\tau_0} \hookrightarrow A^{\frac{\alpha}{d}}_q\big(L^p(\log L)^{\gamma p}\big) \hookrightarrow \dot{B}^{(\alpha_1,\gamma)}_{\tau_1,\tau_1}, \tag{6.22}$$

for all $\alpha_1 < \alpha < \alpha_0$, $\frac{1}{\tau_0} = \frac{\alpha_0}{d} + \frac{1}{p}$, and $\frac{1}{\tau_1} = \frac{\alpha_1}{d} + \frac{1}{p}$.

These inclusions can be slightly improved at the endpoints. More precisely, when $\gamma \geqslant 0$, using Corollary 6.15 and $h^+_\varphi(k) \simeq k^{1/p}$, we can take $\alpha_0 = \alpha$ in (6.22), provided $q \geqslant \tau_0$. On the other hand, if $\gamma \leqslant 0$, one has $h^-_\varphi(k) \simeq k^{1/p}$ and then Corollary 6.15 gives the right-hand inclusion of (6.22) with $\alpha_1 = \alpha$, provided $q \leqslant \tau_1$. Finally, observe that in the special case $\gamma = 0$ we recover the well-known identity $\dot{B}^\alpha_{\tau,\tau} = A^{\frac{\alpha}{d}}_\tau(L^p)$ with $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$ (see, e.g., [11, (6.22)]).

6.6. Truncated wavelet bases

In some applications it may be of interest to replace the wavelet basis $\{\psi_Q\}$ in Section 2.2 by a "truncated basis" of the form

$$\mathcal{B} = \big\{\psi_Q\colon |Q| \leqslant 1\big\} \cup \big\{\psi^{(0)}_Q\colon |Q| = 1\big\},$$

where $\psi^{(0)}$ denotes a suitable scaling function. All the results stated in this paper remain valid for such bases, after standard modifications. More precisely, one considers the characterization

$$\|f\|_{L^\Phi} \simeq \left\| \left(\sum_{|Q|\leqslant 1} \big|\langle f, \psi_Q\rangle\big|^2 |Q|^{-1} \chi_Q(\cdot) \right)^{\frac{1}{2}} \right\|_{L^\Phi} + \left\| \left(\sum_{|Q|=1} \big|\langle f, \psi^{(0)}_Q\rangle\big|^2 \chi_Q(\cdot) \right)^{\frac{1}{2}} \right\|_{L^\Phi}$$

(implicit in the arguments of [28]) and the corresponding sequence space (which is isomorphic to the subspace of all sequences of $\mathfrak{f}^{\Phi}$ supported in $|Q| \leqslant 1$). The arguments presented in Sections 3–5 can be carried out in exactly the same way, except for the fact that $h_\varphi^+(t)$, $h_\varphi^-(t)$ in (3.4) are defined as

$$h_\varphi^+(t) = \sup_{k\leqslant 0} \frac{\varphi(t2^{kd})}{\varphi(2^{kd})} \quad \text{and} \quad h_\varphi^-(t) = \inf_{k\leqslant 0} \frac{\varphi(t2^{kd})}{\varphi(2^{kd})},$$

because of the restriction $|Q| \leqslant 1$. Finally, in Section 6 one uses the "inhomogeneous" version of Besov spaces, $B^{\Psi}_{\tau,q}(\mathbb{R}^d)$, given by the norm

$$\|f\|_{B^{\Psi}_{\tau,q}} = \left[\sum_{j\geqslant 0} \big(\Psi\big(2^j\big)\|f * \chi_j\|_\tau\big)^q\right]^{\frac{1}{q}} \tag{6.23}$$

where χ_j are as in Section 6.4 when $j > 0$, and $\chi_0 = \chi$.

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