Adv Comput Math (2012) 37:255–283
DOI 10.1007/s10444-011-9197-0

Democracy functions and optimal embeddings for approximation spaces

Gustavo Garrigós · Eugenio Hernández · Maria de Natividade

Received: 25 November 2009 / Accepted: 25 February 2011 /
Published online: 23 September 2011

Abstract We prove optimal embeddings for nonlinear approximation spaces $\mathcal{A}_q^\alpha$, in terms of weighted Lorentz sequence spaces, with the weights depending on the democracy functions of the basis. As applications we recover known embeddings for N-term wavelet approximation in L^p, Orlicz, and Lorentz norms. We also study the "greedy classes" $\mathcal{G}_q^\alpha$ introduced by Gribonval and Nielsen, obtaining new counterexamples which show that $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$ for most non-democratic unconditional bases.

Keywords Non-linear approximation · Greedy algorithm · Democratic bases · Jackson and Bernstein inequalities · Discrete Lorentz spaces · Wavelets

Mathematics Subject Classifications (2010) 41A17 · 42C40

Communicated by Volodya Temlyakov.

Research supported by Grants MTM2007-60952 and MTM2010-16518 (Spain). Research of M. de Natividade supported by Instituto Nacional de Bolsas de Estudos de Angola, INABE.

G. Garrigós · E. Hernández (✉) · M. de Natividade
Departamento de Matemáticas,
Universidad Autónoma de Madrid, 28049, Madrid, Spain
e-mail: eugenio.hernandez@uam.es

G. Garrigós
e-mail: gustavo.garrigos@uam.es

M. de Natividade
e-mail: maria.denatividade@uam.es

1 Introduction

Let $(\mathbb{B}, \|.\|_{\mathbb{B}})$ be a quasi-Banach space with a countable **unconditional** basis $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$. A main question in **Approximation Theory** consists in finding a characterization (if possible) or at least suitable embeddings for the non-linear approximation spaces $\mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$, $\alpha > 0$, $0 < q \le \infty$, defined using the **N-term error of approximation** $\sigma_N(x, \mathbb{B})$ (see Sections 2.2 and 2.3 for definitions). Such characterizations or inclusions are often given in terms of "smoothness classes" of the sort

$$\mathfrak{b}(\mathcal{B}; \mathbb{B}) := \left\{ x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} \ : \ \{\|c_j e_j\|_{\mathbb{B}}\}_{j=1}^{\infty} \in \mathfrak{b} \right\},$$

where $\mathfrak{b}$ is a suitable sequence space whose elements decay at infinity, such as ℓ^τ or more generally the discrete Lorentz classes $\ell^{\tau,q}$.

The simplest result in this direction appears when $\mathcal{B}$ is an orthonormal basis in a Hilbert space $\mathbb{H}$, and was first proved by Stechkin when $\alpha = 1/2$ and $q = 1$ (see [31] or [8] for general α, q).

Theorem 1.1 [8, 31] *Let $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in a Hilbert space $\mathbb{H}$, and $\alpha > 0$, $0 < q \le \infty$. Then*

$$\mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{H}) = \ell^{\tau,q}(\mathcal{B}; \mathbb{H})$$

where τ is defined by $\frac{1}{\tau} = \alpha + \frac{1}{2}$.

Many results have been published in the literature similar to Theorem 1.1 when $\mathbb{H}$ is replaced by a particular space (say, L^p) and the basis $\mathcal{B}$ is a particular one (for example, a wavelet basis). We refer to the survey articles [5, 35, 36] for detailed statements and references.

There are also a number of results for general pairs $(\mathbb{B}, \mathcal{B})$ (even with the weaker notion of quasi-greedy basis [9, 13, 20]). We recall two of them in the setting of unconditional bases which we consider here. For simplicity, in all the statements we assume that the basis is *normalized*, meaning $\|e_j\|_{\mathbb{B}} = 1, \forall\, j \in \mathbb{N}$. The first result can be found in [21] (see also [11]).

Theorem 1.2 [21, Theorem 1], [11, Theorem 6.1] *Let $\mathbb{B}$ be a quasi-Banach space and $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ a (normalized) unconditional basis satisfying the following property: there exists $p \in (0, \infty)$ and a constant $C > 0$ such that*

$$\frac{1}{C}|\Gamma|^{1/p} \le \Big\| \sum_{k\in\Gamma} e_k \Big\|_{\mathbb{B}} \le C|\Gamma|^{1/p} \tag{1.1}$$

for all finite $\Gamma \subset \mathbb{N}$. Then, for $\alpha > 0$ and $0 < q \le \infty$ we have

$$\mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B}) = \ell^{\tau,q}(\mathcal{B}; \mathbb{B})$$

when τ is defined by $\frac{1}{\tau} = \alpha + \frac{1}{p}$.

Condition (1.1) is sometimes referred as $\mathcal{B}$ having the p-Temlyakov property [20], or as $\mathbb{B}$ being a p-space [11, 16]. For instance, wavelet bases in L^p satisfy this property [33]. The second result we quote is proved in [13] (see also [21]).

Theorem 1.3 [13, Theorem 3.1]. *Let $\mathbb{B}$ be a Banach space and $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ a (normalized) unconditional basis with the following property: there exist $1 \le p \le q \le \infty$ and constants $A, B > 0$ such that when $x = \sum_{j\in\mathbb{N}} c_j e_j \in \mathbb{B}$ we have*

$$A\,\|\{c_j\}\|_{\ell^{q,\infty}} \;\le\; \|x\|_{\mathbb{B}} \;\le\; B\,\|\{c_j\}\|_{\ell^{p,1}}\,. \tag{1.2}$$

Then, for $\alpha > 0$ and $0 < s \le \infty$ we have

$$\ell^{\tau_p,s}(\mathcal{B};\mathbb{B}) \hookrightarrow \mathcal{A}_s^{\alpha}(\mathcal{B},\mathbb{B}) \hookrightarrow \ell^{\tau_q,s}(\mathcal{B};\mathbb{B}) \tag{1.3}$$

where $\frac{1}{\tau_p} = \alpha + \frac{1}{p}$ and $\frac{1}{\tau_q} = \alpha + \frac{1}{q}$. Moreover, the inclusions given in (1.3) are best possible in the sense described in Section 4 of [13].

Condition (1.2) is referred in [13] as $(\mathbb{B},\mathcal{B})$ having the (p,q) **sandwich property**, and it is shown to be equivalent to

$$A'|\Gamma|^{1/q} \le \Big\|\sum_{k\in\Gamma} e_k\Big\|_{\mathbb{B}} \le B'|\Gamma|^{1/p} \tag{1.4}$$

for all $\Gamma \subset \mathbb{N}$ finite. Observe that (1.4) coincides with (1.1) when $p = q$.

The purpose of this article is to obtain optimal embeddings for $\mathcal{A}_q^{\alpha}(\mathcal{B},\mathbb{B})$ as in (1.3) when no condition such as (1.4) is imposed. As it may be expected, the notion of "democracy function" will play a crucial role. More precisely, we define the **right** and **left democracy functions** associated with a basis $\mathcal{B}$ in $\mathbb{B}$ by

$$h_r(N;\mathcal{B},\mathbb{B}) \equiv \sup_{|\Gamma|=N}\Big\|\sum_{k\in\Gamma}\frac{e_k}{\|e_k\|_{\mathbb{B}}}\Big\|_{\mathbb{B}} \quad\text{and}\quad h_\ell(N;\mathcal{B},\mathbb{B}) \equiv \inf_{|\Gamma|=N}\Big\|\sum_{k\in\Gamma}\frac{e_k}{\|e_k\|_{\mathbb{B}}}\Big\|_{\mathbb{B}}$$

for $N = 1, 2, 3, \dots$. These functions are implicit in earlier works on greedy approximation (see eg [9, 34, 38]) and explicitly defined in [19], page 203. We refer to Section 5 for various examples where $h_\ell(N)$ and $h_r(N)$ are computed explicitly (modulo multiplicative constants). As usual, when $h_\ell(N) \approx h_r(N)$ for all $N \in \mathbb{N}$ we say that $\mathcal{B}$ is a *democratic basis* in $\mathbb{B}$ (see [23]).

The embeddings will be given in terms of **weighted discrete Lorentz spaces** $\boldsymbol{\ell_\eta^q}$, with quasi-norms defined by

$$\big\|\{c_k\}\big\|_{\ell_\eta^q} \equiv \left(\sum_{k=1}^{\infty}\big|\eta(k)\,c_k^*\big|^q\,\frac{1}{k}\right)^{\frac{1}{q}},$$

where $\{c_k^*\}$ denotes the decreasing rearrangement of $\{|c_k|\}$ and the *weight* $\eta = \{\eta(k)\}_{k=1}^{\infty}$ is a suitable sequence increasing to infinity and satisfying the doubling property (see Section 2.4 for precise definitions and references). In the special case $\eta(k) = k^{1/\tau}$ we recover the classical definition $\ell_\eta^q = \ell^{\tau,q}$.

Theorem 1.4 *Let $\mathbb{B}$ be a quasi-Banach space and $\mathcal{B}$ an unconditional basis. Assume that $h_\ell(N)$ is doubling. Then if $\alpha > 0$ and $0 < q \leq \infty$ we have the continuous embeddings*

$$\ell^q_{k^\alpha h_r(k)}(\mathcal{B};\mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B}) \hookrightarrow \ell^q_{k^\alpha h_\ell(k)}(\mathcal{B};\mathbb{B})\,. \tag{1.5}$$

Moreover, for fixed α and q these inclusions are best possible in the scale of weighted discrete Lorentz spaces ℓ^q_η, in the sense explained in Sections 3, 4 and 6.

Observe that this theorem generalizes Theorems 1.2 and 1.3. In Theorem 1.2 we have $h_r(N) \approx h_\ell(N) \approx N^{1/p}$ and in Theorem 1.3, $h_r(N) \lesssim N^{1/p}$ and $h_\ell(N) \gtrsim N^{1/q}$. When $\mathcal{B}$ is democratic in $\mathbb{B}$, Theorem 1.4 shows that

$$\mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B}) = \ell^q_{k^\alpha h(k)}(\mathcal{B};\mathbb{B})$$

with $h(k) = h_r(k) \approx h_\ell(k)$, which recovers Corollary 1 in [13, Section 6] for greedy bases in a Banach space.

Theorem 1.4 is a consequence of the results proved in Sections 3 and 4. Section 3 deals with the lower embedding in (1.5) and shows the relation to Jackson type inequalities. Section 4 deals with the upper embedding of (1.5) and its relation to Bernstein type inequalities. Section 5 contains various examples of democracy functions and embeddings with precise references; these are all special cases of Theorem 1.4. In Section 6 we apply Theorem 1.4 to estimate the democracy functions h_ℓ and h_r of the approximation space $\mathcal{A}^\alpha_q$.

Finally, the last section of the paper is dedicated to study the "greedy classes" $\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B})$ introduced by Gribonval and Nielsen in [13], and their relations with the approximation spaces $\mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B})$. The classes $\mathcal{G}^\alpha_q$ are defined similarly to the approximation spaces, but with the error of approximation $\sigma_N(x)$ replaced by the quantity $\|x - G_N(x)\|_\mathbb{B}$ (see Section 2.3 for details). It is easy to see that $\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B}) \subset \mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B})$; moreover, since any democratic unconditional basis is greedy (see [23]) if follows that in this case we have $\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B}) = \mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B})$. One may conjecture that for unconditional bases $\mathcal{B}$ the converse is true, that is $\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B}) = \mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B})$ implies that $\mathcal{B}$ is democratic in $\mathbb{B}$. We do not know how to show this, but we can exhibit a fairly general class of non-democratic pairs $(\mathcal{B},\mathbb{B})$ for which $\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B}) \neq \mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B})$ for all $\alpha > 0$ and $q \in (0,\infty]$. This is the case for instance of wavelet bases when $\mathbb{B}$ is equal to $L^p(\log L)^\gamma$ ($\gamma \neq 0$) or $L^{p,r}$ ($p \neq r$). We also illustrate how irregular the classes $\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B})$ can be when $\mathcal{B}$ is not democratic, showing in simple situations that they are not even linear spaces.

2 General setting

2.1 Bases

Since we work in the setting of quasi-Banach spaces $(\mathbb{B}, \|\cdot\|_\mathbb{B})$, we shall often use the ρ-power triangle inequality

$$\|x+y\|^\rho_\mathbb{B} \leq \|x\|^\rho_\mathbb{B} + \|y\|^\rho_\mathbb{B}\,, \tag{2.1}$$

which holds for a sufficiently small $\rho = \rho_{\mathbb{B}} \in (0, 1]$ (and hence for all $\mu \le \rho_{\mathbb{B}}$); see [3, Lemma 3.10.1]. The case $\rho_{\mathbb{B}} = 1$ gives a Banach space.

A sequence of vectors $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ is a basis of $\mathbb{B}$ if every $x \in \mathbb{B}$ can be uniquely represented as $x = \sum_{j=1}^{\infty} c_j e_j$ for some scalars c_j, with convergence in $\|\cdot\|_{\mathbb{B}}$. The basis $\mathcal{B}$ is **unconditional** if the series converges unconditionally, or equivalently if there is some $K > 0$ such that

$$\left\| \sum_{j=1}^{\infty} \lambda_j c_j e_j \right\|_{\mathbb{B}} \le K \left\| \sum_{j=1}^{\infty} c_j e_j \right\|_{\mathbb{B}} \tag{2.2}$$

for every sequence of scalars $\{\lambda_j\}_{j=1}^{\infty}$ with $|\lambda_j| \le 1$ (see e.g. [15, Chapter 5]).

For simplicity in the statements, throughout the paper we shall assume that $\mathcal{B}$ is a **normalized** basis, meaning $\|e_j\|_{\mathbb{B}} = 1$ for all $j \in \mathbb{N}$. We shall also assume that the unconditionality constant in (2.2) is $K = 1$. This can be achieved if necessary introducing an equivalent quasi-norm in $\mathbb{B}$

$$|\!|\!| x |\!|\!|_{\mathbb{B}} = \sup_{\Gamma \text{finite}, |\lambda_j| \le 1} \left\| \sum_{j \in \Gamma} \lambda_j x_j e_j \right\|_{\mathbb{B}}, \qquad \text{if } x = \sum_{j=1}^{\infty} x_j e_j.$$

Observe that with this renorming we still have $|\!|\!| e_j |\!|\!|_{\mathbb{B}} = 1$.

With the above assumptions, the following **lattice property** will be used often below: if $|y_k| \le |x_k|$ for all $k \in \mathbb{N}$ and $x = \sum_{k=1}^{\infty} x_k e_k \in \mathbb{B}$, then the series $y = \sum_{k=1}^{\infty} y_k e_k$ converges in $\mathbb{B}$ and $\|y\|_{\mathbb{B}} \le \|x\|_{\mathbb{B}}$. Also, using (2.2) with $K = 1$ we see that, for every $\Gamma \subset \mathbb{N}$ finite

$$\left(\inf_{j \in \Gamma} |c_j| \right) \left\| \sum_{j \in \Gamma} e_j \right\|_{\mathbb{B}} \le \left\| \sum_{j \in \Gamma} c_j e_j \right\|_{\mathbb{B}} \le \left(\sup_{j \in \Gamma} |c_j| \right) \left\| \sum_{j \in \Gamma} e_j \right\|_{\mathbb{B}}. \tag{2.3}$$

2.2 Non-linear approximation and greedy algorithm

Let $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ be a basis in $\mathbb{B}$. Let Σ_N, $N = 1, 2, 3, \ldots$, be the set of all $y \in \mathbb{B}$ with at most N non-null coefficients in the unique basis representation. For $x \in \mathbb{B}$, the ***N*-term error of approximation** with respect to $\mathcal{B}$ is defined as

$$\sigma_N(x) = \sigma_N(x; \mathcal{B}, \mathbb{B}) \equiv \inf_{y \in \Sigma_N} \|x - y\|_{\mathbb{B}}, \quad N = 1, 2, 3 \ldots$$

We also set $\Sigma_0 = \{0\}$ so that $\sigma_0(x) = \|x\|_{\mathbb{B}}$. Using the lattice property mentioned in Section 2.1 it is easy to see that for $x = \sum_{j=1}^{\infty} c_j e_j$ we actually have

$$\sigma_N(x) = \inf_{|\Gamma| = N} \left\{ \left\| x - \sum_{\gamma \in \Gamma} c_\gamma e_\gamma \right\|_{\mathbb{B}} \right\}, \tag{2.4}$$

that is, only coefficients from x are relevant when computing $\sigma_N(x)$; see e.g. [11, (2.6)].

Given $x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B}$, let π denote any bijection of $\mathbb{N}$ such that

$$\|c_{\pi(j)} e_{\pi(j)}\| \ge \|c_{\pi(j+1)} e_{\pi(j+1)}\|, \quad \text{for all} \quad j \in \mathbb{N}. \tag{2.5}$$

Without loss of generality we may assume that the basis is normalized and then (2.5) becomes $|c_{\pi(j)}| \geq |c_{\pi(j+1)}|, \quad$ for all $\quad j \in \mathbb{N}$. A **greedy algorithm of step** N is a correspondence assigning

$$x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} \longmapsto G_N^\pi(x) \equiv \sum_{j=1}^{N} c_{\pi(j)} e_{\pi(j)}$$

for any π as in (2.5). The **error of greedy approximation** at step N is defined by

$$\gamma_N(x) = \gamma_N(x; \mathcal{B}, \mathbb{B}) \equiv \sup_\pi \|x - G_N^\pi(x)\|_{\mathbb{B}}. \tag{2.6}$$

Notice that $\sigma_N(x) \leq \gamma_N(x)$, but the reverse inequality may not be true in general. It is said that $\mathcal{B}$ is a **greedy basis** in $\mathbb{B}$ when there is a constant $c \geq 1$ such that

$$\gamma_N(x; \mathcal{B}, \mathbb{B}) \leq c\, \sigma_N(x; \mathcal{B}, \mathbb{B}), \quad \forall\, x \in \mathbb{B}, \ N = 1, 2, 3, \ldots$$

A celebrated theorem of Konyagin and Temlyakov characterizes greedy bases as those which are unconditional and democratic [23].

2.3 Approximation spaces and greedy classes

The classical non-linear approximation spaces $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ are defined as follows: for $\alpha > 0$ and $0 < q < \infty$

$$\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) = \left\{ x \in \mathbb{B} : \|x\|_{\mathcal{A}_q^\alpha} \equiv \|x\|_{\mathbb{B}} + \left[\sum_{n=1}^{\infty} \left(N^\alpha \sigma_N(x; \mathcal{B}, \mathbb{B}) \right)^q \frac{1}{N} \right]^{\frac{1}{q}} < \infty \right\}.$$

When $q = \infty$ the definition takes the form:

$$\mathcal{A}_\infty^\alpha(\mathcal{B}, \mathbb{B}) = \left\{ x \in \mathbb{B} : \|x\|_{\mathcal{A}_\infty^\alpha} \equiv \|x\|_{\mathbb{B}} + \sup_{N \geq 1} N^\alpha \sigma_N(x) < \infty \right\}.$$

It is well known that $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ are quasi-Banach spaces (see e.g. [29]). Also, equivalent quasi-norms can be obtained restricting to dyadic N's:

$$\|x\|_{\mathcal{A}_q^\alpha} \approx \|x\|_{\mathbb{B}} + \left[\sum_{k=0}^{\infty} \left(2^{k\alpha} \sigma_{2^k}(x) \right)^q \right]^{\frac{1}{q}}$$

and likewise for $q = \infty$. This is a simple consequence of the monotonicity of $\sigma_N(x)$ (see eg [29, Proposition 2] or [7, (2.3)]).

The **greedy classes** $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ are defined as before replacing the role of $\sigma_N(x)$ by the error of greedy approximation $\gamma_N(x)$ given in (2.6), that is

$$\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) = \left\{ x \in \mathbb{B} : \|x\|_{\mathcal{G}_q^\alpha} \equiv \|x\|_{\mathbb{B}} + \left[\sum_{N=1}^{\infty} \left(N^\alpha \gamma_N(x; \mathcal{B}, \mathbb{B}) \right)^q \frac{1}{N} \right]^{\frac{1}{q}} < \infty \right\} \tag{2.7}$$

(and similarly for $q = \infty$). We also have the equivalence

$$\|x\|_{\mathscr{G}_q^\alpha} \approx \|x\|_{\mathbb{B}} + \left[\sum_{k=0}^{\infty}\left(2^{k\alpha}\gamma_{2^k}(x)\right)^q\right]^{\frac{1}{q}}, \tag{2.8}$$

since $\gamma_N(x)$ is non-increasing by the lattice property in Section 2.1.

Since $\sigma_N(x) \le \gamma_N(x)$ for all $x \in \mathbb{B}$ it is clear that[1]

$$\mathscr{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) . \tag{2.9}$$

When $\mathcal{B}$ is a greedy basis in $\mathbb{B}$ it holds that $\mathscr{G}_q^\alpha(\mathcal{B}, \mathbb{B}) = \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ with equivalent quasi-norms. For non greedy bases, however, the inclusion may be strict, and the classes $\mathscr{G}_q^\alpha$ may not even be linear spaces (see Section 7.1 below).

2.4 Discrete Lorentz spaces

Let $\eta = \{\eta(k)\}_{k=1}^\infty$ be a sequence so that

(a) $0 < \eta(k) \le \eta(k+1)$ for all $k = 1, 2, \ldots$ and $\lim_{k\longrightarrow\infty} \eta(k) = \infty$.

(b) η is *doubling*, that is, $\eta(2k) \le C\eta(k)$ for all $k = 1, 2, \ldots,$ and some $C > 0$.

We shall denote the set of all such sequences by $\mathbb{W}$. If $\eta \in \mathbb{W}$ and $0 < r \le \infty$, the **weighted discrete Lorentz space $\boldsymbol{\ell_\eta^r}$** is defined as

$$\ell_\eta^r = \left\{ \mathbf{s} = \{s_k\}_{k=1}^\infty \in \mathfrak{c}_0 \;:\; \|\mathbf{s}\|_{\ell_\eta^r} \equiv \left[\sum_{k=1}^\infty \left(\eta(k)s_k^*\right)^r \frac{1}{k}\right]^{\frac{1}{r}} < \infty \right\}$$

(with $\|\mathbf{s}\|_{\ell_\eta^\infty} = \sup_{k\in\mathbb{N}} \eta(k)s_k^*$ when $r = \infty$). Here $\{s_k^*\}$ denotes the decreasing rearrangement of $\{|s_k|\}$, that is $s_k^* = |s_{\pi(k)}|$ where π is any bijection of $\mathbb{N}$ such that $|s_{\pi(k)}| \ge |s_{\pi(k+1)}|$ for all $k = 1, 2, \ldots$ (since we are assuming $\lim_{k\to\infty} s_k = 0$ such π's always exist). When $\eta \in \mathbb{W}$ the set ℓ_η^r is a quasi-Banach space (see e.g. [4, Section 2.2]). Equivalent quasi-norms are given by

$$\|\mathbf{s}\|_{\ell_\eta^r} \approx \left[\sum_{j=0}^\infty \left(\eta(\kappa^j)s_{\kappa^j}^*\right)^r\right]^{1/r}, \tag{2.10}$$

for any fixed integer $\kappa > 1$. Particular examples are the classical Lorentz sequence spaces $\ell^{p,r}$ (with $\eta(k) = k^{1/p}$), and the Lorentz–Zygmund spaces $\ell^{p,r}(\log \ell)^\gamma$ (for which $\eta(k) = k^{1/p}\log^\gamma(k+1)$; see e.g. [2, p. 285]).

Occasionally we will need to assume a stronger condition on the weights η. For an increasing sequence η we define

$$M_\eta(m) = \sup_{k\in\mathbb{N}} \frac{\eta(k)}{\eta(mk)}, \quad m = 1, 2, 3, \ldots.$$

[1] Here, as in the rest of the paper, $X \hookrightarrow Y$ means $X \subset Y$ and there exists $C > 0$ such that $\|x\|_Y \le C\|x\|_X$ for all $x \in X$. The equality of spaces $X = Y$ is interpreted as $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

Observe that we always have $M_\eta(m) \le 1$. We shall say that $\eta \in \mathbb{W}_+$ when $\eta \in \mathbb{W}$ and there exists some integer $\kappa > 1$ for which $M_\eta(\kappa) < 1$. This is equivalent to say that the "lower dilation index" $i_\eta > 0$, where we let

$$i_\eta \equiv \sup_{m \ge 1} \frac{\log M_\eta(m)}{-\log m}\,.$$

For example, $\eta = \{k^\alpha \log^\beta(k+1)\}$ has $i_\eta = \alpha$, and hence $\eta \in \mathbb{W}_+$ iff $\alpha > 0$. In general, if η is obtained from a increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\eta(k) = \phi(ak)$, for some fixed $a > 0$, then $i_\eta > 0$ iff $i_\phi > 0$, the latter denoting the standard lower dilation index of ϕ (see e.g. [24, p. 54] for the definition).

Below we will need the following result:

Lemma 2.1 *If* $\eta \in \mathbb{W}_+$ *then there exists a constant* $C > 0$ *such that*

$$\sum_{j=0}^{n} \eta(\kappa^j) \le C\eta(\kappa^n), \quad \forall\, n \in \mathbb{N}, \tag{2.11}$$

where $\kappa > 1$ *is an integer as in the definition of* $\mathbb{W}_+$.

Proof Write $\delta = M_\eta(\kappa) < 1$. By definition $M_\eta(\kappa) \ge \eta(\kappa^j)/\eta(\kappa^{j+1})$, and therefore

$$\eta(\kappa^j) \le \delta\eta(\kappa^{j+1}), \quad \forall\, j = 0, 1, 2, \ldots. \tag{2.12}$$

Iterating (2.12) we deduce that $\eta(\kappa^j) \le \delta^{n-j}\eta(\kappa^n)$, for $j = 0, 1, 2, \ldots, n$ and hence

$$\sum_{j=0}^{n} \eta(\kappa^j) \le \eta(\kappa^n) \sum_{j=0}^{n} \delta^{n-j} \le \eta(\kappa^n)\frac{1}{1-\delta}\,.$$

□

Remark 2.2 If η is increasing and doubling, then $\{k^\alpha\,\eta(k)\} \in \mathbb{W}_+$ for all $\alpha > 0$. Also, if $\eta \in \mathbb{W}_+$ then $\eta^r \in \mathbb{W}_+$, for all $r > 0$.

We now estimate the *fundamental function* of ℓ^r_η. We shall denote the indicator sequence of $\Gamma \subset \mathbb{N}$ by 1_Γ, that is the sequence with entries 1 for $j \in \Gamma$ and 0 otherwise.

Lemma 2.3

(a) *If* $\eta \in \mathbb{W}$ *then*

$$\left\|1_\Gamma\right\|_{\ell^\infty_\eta} = \eta(|\Gamma|), \quad \forall \text{ finite } \Gamma \subset \mathbb{N}.$$

(b) *If* $\eta \in \mathbb{W}_+$ *and* $r \in (0, \infty)$ *then*

$$\left\|1_\Gamma\right\|_{\ell^r_\eta} \approx \eta(|\Gamma|), \quad \forall \text{ finite } \Gamma \subset \mathbb{N}$$

with the constants involved independent of Γ.

Proof Part (a) is trivial since η is increasing. To prove (b) use (2.10) and the previous lemma. □

Finally, as mentioned in Section 1, given a (normalized) basis $\mathcal{B}$ in $\mathbb{B}$ we shall consider the following subspaces

$$\ell^q_\eta(\mathcal{B},\mathbb{B}) := \left\{ x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} \;:\; \{c_j\}_{j=1}^{\infty} \in \ell^q_\eta \right\},$$

endowed with the quasi-norm $\|x\|_{\ell^q_\eta(\mathcal{B},\mathbb{B})} := \|\{c_j\}\|_{\ell^q_\eta}$. These spaces are not necessarily complete, but they are when

$$\Big\| \sum_j c_j e_j \Big\|_{\mathbb{B}} \le C \|\{c_j\}\|_{\ell^q_\eta}, \quad \forall \text{ finite } \{c_j\},$$

a property which holds in certain situations (see e.g. Remark 3.2). When this is the case, the space $\ell^q_\eta(\mathcal{B},\mathbb{B})$ is just an isomorphic copy of ℓ^q_η inside $\mathbb{B}$.

2.5 Democracy functions

Following [23], a (normalized) basis $\mathcal{B}$ in a quasi-Banach space $\mathbb{B}$ is said to be **democratic** if there exists $C > 0$ such that

$$\Big\| \sum_{k\in\Gamma} e_k \Big\|_{\mathbb{B}} \le C \Big\| \sum_{k\in\Gamma'} e_k \Big\|_{\mathbb{B}},$$

for all finite sets $\Gamma, \Gamma' \subset \mathbb{N}$ with the same cardinality. This is a key notion in the theory of greedy approximation, as it allows to characterize greedy bases as those which are both unconditional and democratic (see [23]).

As we recall in Section 5, wavelet bases are well known examples of greedy bases for many function spaces, such as L^p, Sobolev, or more generally, the Triebel–Lizorkin spaces. However, they are not democratic in some other instances such as BMO, or the Orlicz L^Φ and Lorentz $L^{p,q}$ spaces (when these are different from L^p). In fact, it is proved in [39] that the Haar basis is democratic in a rearrangement invariant space $\mathbb{X}$ in $[0,1]$ if and only if $\mathbb{X} = L^p$ for some $p \in (1,\infty)$. An earlier example of non-democratic basis is the multivariate (hyperbolic) Haar system in $L^p(\mathbb{R}^d)$ for $p \ne 2$ and $d > 1$ (see [34] and Example 5.5 below).

Thus, non-democratic bases are also common. To quantify the democracy of a (normalized) system $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ in $\mathbb{B}$ one introduces the following concepts:

$$h_r(N;\mathcal{B},\mathbb{B}) \equiv \sup_{|\Gamma|=N} \Big\| \sum_{k\in\Gamma} e_k \Big\|_{\mathbb{B}} \quad \text{and} \quad h_\ell(N;\mathcal{B},\mathbb{B}) \equiv \inf_{|\Gamma|=N} \Big\| \sum_{k\in\Gamma} e_k \Big\|_{\mathbb{B}},$$

which we shall call the **right and left democracy functions of** $\mathcal{B}$ (see also [9, 12, 19]). We shall omit $\mathcal{B}$ or $\mathbb{B}$ when these are understood from the context.

Some general properties of h_ℓ and h_r are proved in the next proposition.

Proposition 2.4 *Let $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ be a (normalized) unconditional basis in $\mathbb{B}$ with the lattice property from Section 2.1. Then*

(a) $1 \le h_\ell(N) \le h_r(N) \le N^{1/\rho}$, $\forall\, N = 1, 2, \ldots$, *where* $\rho = \rho_{\mathbb{B}}$ *is as in* (2.1).
(b) $h_\ell(N)$ *and* $h_r(N)$ *are non-decreasing in* $N = 1, 2, 3 \ldots$
(c) $h_r(N)$ *is doubling, that is,* $\exists\, c > 0$ *such that* $h_r(2N) \le c\, h_r(N)$, $\forall\, N \in \mathbb{N}$.
(d) *There exists* $c \ge 1$ *such that* $h_\ell(N+1) \le c\, h_\ell(N)$ *for all* $N = 1, 2, 3 \ldots$

Proof

(a) and (b) follow immediately from the lattice property of $\mathcal{B}$ and the ρ-triangular inequality.

(c) Given $N \in \mathbb{N}$, choose $\Gamma \subset \mathbb{N}$ with $|\Gamma| = 2N$ such that $\left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} \ge h_r(2N)/2$. Partitioning arbitrarily $\Gamma = \Gamma' \cup \Gamma''$ with $|\Gamma'| = |\Gamma''| = N$, and using the ρ-power triangle inequality, one easily obtains

$$\tfrac{1}{2} h_r(2N) \le \left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} = \left\| \sum_{k\in\Gamma'} e_k + \sum_{k\in\Gamma''} e_k \right\|_{\mathbb{B}} \le 2^{1/\rho} h_r(N)\,.$$

(d) Given $N \in \mathbb{N}$, choose $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$ such that $\left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} \le 2h_\ell(N)$. Let $\Gamma' = \Gamma \cup \{k_o\}$ for any $k_o \notin \Gamma$. Then

$$h_\ell(N+1) \le \left\| \sum_{k\in\Gamma'} e_k \right\|_{\mathbb{B}} \le \left(\left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}}^{\rho} + 1 \right)^{1/\rho}$$
$$\le \left(2^\rho [h_\ell(N)]^\rho + 1\right)^{1/\rho}.$$

Thus, using (a) we obtain $h_\ell(N+1) \le (2^\rho + 1)^{\frac{1}{\rho}} h_\ell(N) \le 2 \cdot 2^{1/\rho} h_\ell(N)$. □

Remark 2.5 We do not know whether property (d) can be improved to show that $h_\ell(N)$ is actually doubling. This is however the case in all the examples we have considered below (see Section 5).

3 Right democracy and Jackson type inequalities

Our first result deals with inclusions for the greedy classes $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$.

Theorem 3.1 *Let $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0, \infty)$. Then, for any sequence η such that $\{k^\alpha \eta(k)\}_{k=1}^{\infty} \in \mathbb{W}_+$ the following statements are equivalent:*

1. *There exists $C > 0$ such that for all $N = 1, 2, 3, \ldots$*

$$\left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} \le C\eta(N)\,, \quad \forall\, \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N. \tag{3.1}$$

2. *Jackson type inequality for* $\ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})$*:* $\exists\, C_\alpha > 0$ *such that* $\forall\, N = 0, 1, 2 \ldots$

$$\gamma_N(x) \le C_\alpha (N+1)^{-\alpha} \|x\|_{\ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})}, \quad \forall\, x \in \ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B}). \tag{3.2}$$

3. $\ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B}) \hookrightarrow \mathcal{G}^{\alpha}_{\infty}(\mathcal{B},\mathbb{B})$.
4. $\ell^{q}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B}) \hookrightarrow \mathcal{G}^{\alpha}_{q}(\mathcal{B},\mathbb{B})$.
5. *Jackson type inequality for* $\ell^{q}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})$*:* $\exists\, C_{\alpha,q} > 0$ *such that* $\forall\, N = 0, 1, 2, \ldots$

$$\gamma_N(x) \le C_{\alpha,q} (N+1)^{-\alpha} \|x\|_{\ell^{q}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})}, \quad \forall\, x \in \ell^{q}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B}) . \tag{3.3}$$

Proof

$1 \Rightarrow 2$ Let $x = \sum_{k\in\mathbb{N}} c_k e_k \in \ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})$ and let π be a bijection of $\mathbb{N}$ such that

$$|c_{\pi(k)}| \ge |c_{\pi(k+1)}|, \quad k = 1, 2, 3, \ldots \tag{3.4}$$

For fixed $N = 0, 1, 2, \ldots$, denote $\lambda_j = 2^j(N+1)$. Then, the ρ-power triangle inequality and (2.3) give

$$\left\|x - G^{\pi}_N(x)\right\|^{\rho}_{\mathbb{B}} = \left\| \sum_{k=N+1}^{\infty} c_{\pi(k)} e_{\pi(k)} \right\|^{\rho}_{\mathbb{B}} \le \sum_{j=0}^{\infty} \left\| \textstyle\sum_{\lambda_j \le k < \lambda_{j+1}} c_{\pi(k)} e_{\pi(k)} \right\|^{\rho}_{\mathbb{B}}$$

$$\le \sum_{j=0}^{\infty} |c_{\pi(\lambda_j)}|^{\rho} \left\| \textstyle\sum_{\lambda_j \le k < \lambda_{j+1}} e_{\pi(k)} \right\|^{\rho}_{\mathbb{B}}.$$

There are exactly $\lambda_j = 2^j(N+1)$ elements in the interior sum, so using (3.1) we obtain

$$\left\|x - G^{\pi}_N(x)\right\|^{\rho}_{\mathbb{B}} \le C^{\rho} \sum_{j=0}^{\infty} \left(c^*_{\lambda_j} \eta(\lambda_j)\right)^{\rho} = C^{\rho} \sum_{j=0}^{\infty} \left(\lambda_j^{\alpha} c^*_{\lambda_j} \eta(\lambda_j)\right)^{\rho} \lambda_j^{-\alpha\rho}$$

$$\le C^{\rho} \|x\|^{\rho}_{\ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})} (N+1)^{-\alpha\rho} \textstyle\sum_{j=0}^{\infty} 2^{-j\alpha\rho}$$

$$= C_{\alpha,\rho}\, (N+1)^{-\alpha\rho}\, \|x\|^{\rho}_{\ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})} .$$

The result follows taking the supremum over all bijections π satisfying (3.4).

Remark 3.2 The special case $N = 0$ in (3.2) says that

$$\|x\|_{\mathbb{B}} \le C \|x\|_{\ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})}, \tag{3.5}$$

which in particular implies $\ell^{q}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B}) \hookrightarrow \mathbb{B}$, for all $q \in (0,\infty]$.

$2 \Rightarrow 3$ This is immediate from the definition of $\mathcal{G}^{\alpha}_{\infty}$ (and Remark 3.2), since

$$\|x\|_{\mathcal{G}^{\alpha}_{\infty}(\mathcal{B},\mathbb{B})} := \|x\|_{\mathbb{B}} + \sup_{N\ge 1} N^{\alpha} \gamma_N(x) \le C_\alpha \|x\|_{\ell^{\infty}_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})}.$$

3 ⇒ 1 Let $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$. Choose Γ' with $|\Gamma'| = N$ and so that $\Gamma \cap \Gamma' = \emptyset$, and consider $x = \sum_{k\in\Gamma} e_k + \sum_{k\in\Gamma'} 2e_k$. Then

$$\gamma_N(x) = \left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} , \tag{3.6}$$

and therefore

$$N^\alpha \left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} = N^\alpha \gamma_N(x) \le \|x\|_{\mathscr{G}^\alpha_\infty(\mathcal{B},\mathbb{B})}. \tag{3.7}$$

On the other hand, call $\omega(k) = k^\alpha \eta(k)$. By monotonicity, Lemma 2.3 and the doubling property of ω we have

$$\|x\|_{\ell^\infty_\omega(\mathcal{B},\mathbb{B})} \le 2\left\| 1_{\Gamma\cup\Gamma'} \right\|_{\ell^\infty_\omega} = 2\omega(2N) \le c\,\omega(N)\,. \tag{3.8}$$

Combining (3.7) and (3.8) with the inclusion $\ell^\infty_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B}) \hookrightarrow \mathscr{G}^\alpha_\infty(\mathcal{B},\mathbb{B})$ gives (3.1).

5 ⇒ 1 Let $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$, and choose Γ' and x as in the proof of 3 ⇒ 1. As before call $\omega(k) = k^\alpha \eta(k)$. Then Lemma 2.3 and the assumption $\omega \in \mathbb{W}_+$ give

$$\|x\|_{\ell^q_\omega(\mathcal{B},\mathbb{B})} \le 2\left\| 1_{\Gamma\cup\Gamma'} \right\|_{\ell^q_\omega} \approx \omega(2N) \le c\,\omega(N)\,.$$

Since we are assuming 5 we can write (recall (3.6))

$$\left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} = \gamma_N(x) \le C_{\alpha,\rho}(N+1)^{-\alpha}\|x\|_{\ell^q_\omega(\mathcal{B},\mathbb{B})} \lesssim N^{-\alpha}\omega(N) = \eta(N),$$

which proves (3.1).

1 ⇒ 4 The proof is similar to 1 ⇒ 2 with a few modifications we indicate next. Given $x \in \ell^q_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})$ and π as in (3.4) we write $x = \sum_{j=-1}^\infty \sum_{2^j<k\le 2^{j+1}} c_{\pi(k)} e_{\pi(k)}$. Then arguing as before (with $N = 2^m$) we obtain

$$\left\| x - G^\pi_{2^m}(x) \right\|^\mu_{\mathbb{B}} \le \sum_{j=m}^\infty |c_{\pi(2^j)}|^\mu \left\| \textstyle\sum_{2^j<k\le 2^{j+1}} e_{\pi(k)} \right\|^\mu_{\mathbb{B}},$$

where we choose now any $\mu < \min\{q, \rho_{\mathbb{B}}\}$. Taking the supremum over all π's and using (3.1) we obtain

$$\gamma_{2^m}(x;\mathcal{B},\mathbb{B})^\mu \le C^\mu \sum_{j=m}^\infty \left(c^*_{2^j}\,\eta(2^j)\right)^\mu.$$

Therefore

$$\left[\sum_{m=0}^\infty \left(2^{m\alpha}\gamma_{2^m}(x)\right)^q\right]^{\frac{1}{q}} \le C\left[\sum_{m=0}^\infty 2^{m\alpha q}\left(\sum_{j=0}^\infty [c^*_{2^{j+m}}\,\eta(2^{j+m})]^\mu\right)^{q/\mu}\right]^{1/q}.$$

Since $q/\mu > 1$, we can use Minkowski's inequality on the right hand side to obtain

$$\left[\sum_{m=0}^{\infty}\left(2^{m\alpha}\gamma_{2^m}(x)\right)^q\right]^{\frac{1}{q}} \leq C\left[\sum_{j=0}^{\infty}\left(\sum_{m=0}^{\infty}2^{m\alpha q}\left[c^*_{2^{j+m}}\,\eta(2^{j+m})\right]^q\right)^{\mu/q}\right]^{1/\mu}$$

$$= C\left[\sum_{j=0}^{\infty}2^{-j\alpha\mu}\left(\sum_{\ell=j}^{\infty}2^{\ell\alpha q}\left[c^*_{2^\ell}\,\eta(2^\ell)\right]^q\right)^{\mu/q}\right]^{1/\mu}$$

$$\leq C'\,\|\{c_k\}\|_{\ell^q_{k^\alpha\eta(k)}}.$$

This implies the desired estimate

$$\|x\|_{\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B})} \lesssim \|\{c_k\}\|_{\ell^q_{k^\alpha\eta(k)}},$$

using the dyadic expressions for the norms in (2.8) and (2.10) (and Remark 3.2).

4 ⇒ 5 This is trivial since 4 implies $\ell^q_{k^\alpha\eta k}(\mathcal{B},\mathbb{B}) \hookrightarrow \mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B}) \hookrightarrow \mathcal{G}^\alpha_\infty(\mathcal{B},\mathbb{B})$, and this clearly gives (3.3).

□

Remark 3.3 The equivalences 1 to 3 remain true under the weaker assumption $\{k^\alpha\eta(k)\} \in \mathbb{W}$.

Remark 3.4 Observe that if any of the statements in 2 to 5 of Theorem 3.1 holds for one fixed $\alpha > 0$ and $q \in (0,\infty]$, then the assertions remain true for all α and q (as long as $\{k^\alpha\eta(k)\} \in \mathbb{W}_+$), since the statement in 1 is independent of these parameters.

Corollary 3.5 (Optimal inclusions into $\mathcal{G}^\alpha_q$) *Let $\mathcal{B}$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0,\infty]$. Then*

$$\ell^q_{k^\alpha h_r(k)}(\mathcal{B},\mathbb{B}) \hookrightarrow \mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B}). \tag{3.9}$$

Moreover, if $\omega \in \mathbb{W}_+$ then, $\ell^q_\omega(\mathcal{B},\mathbb{B}) \hookrightarrow \mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B})$ if and only if $\omega(k) \gtrsim k^\alpha h_r(k)$.

Proof For $q < \infty$, the inclusion (3.9) is an application of 4 in the theorem with $\eta = h_r$ (after noticing that $\{k^\alpha h_r(k)\} \in \mathbb{W}_+$ by Proposition 2.4 and Remark 2.2). The second assertion is just a restatement of 1 ⇔ 4 with $\eta(k) = \omega(k)/k^\alpha$. For $q = \infty$ use 3 instead of 4. □

We now prove similar results for the approximation spaces $\mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B})$.

Theorem 3.6 *Let $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0, \infty]$. Then, for any sequence $\eta \in \mathbb{W}_+$ the following are equivalent:*

1. *There exists $C > 0$ such that for all $N = 1, 2, 3, \ldots$*

$$\Big\| \sum_{k\in\Gamma} e_k \Big\|_{\mathbb{B}} \le C\eta(N)\,, \quad \forall\, \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N. \tag{3.10}$$

2. $\ell^q_{k^\alpha \eta(k)}(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$.
3. *Jackson type inequality for $\ell^q_{k^\alpha \eta(k)}(\mathcal{B}, \mathbb{B})$: $\exists\, C_{\alpha,q} > 0$ such that $\forall\, N = 0, 1, 2, \ldots$*

$$\sigma_N(x) \le C_{\alpha,q}(N+1)^{-\alpha} \|x\|_{\ell^q_{k^\alpha \eta(k)}(\mathcal{B},\mathbb{B})}, \quad \forall\, x \in \ell^q_{k^\alpha \eta(k)}(\mathcal{B}, \mathbb{B})\,. \tag{3.11}$$

Proof $1 \Rightarrow 2$ follows directly from Theorem 3.1 and $\mathcal{G}^\alpha_q \hookrightarrow \mathcal{A}^\alpha_q$. Also, $2 \Rightarrow 3$ is trivial since $\mathcal{A}^\alpha_q \hookrightarrow \mathcal{A}^\alpha_\infty$, and 3 is equivalent to $\ell^q_{k^\alpha \eta(k)}(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_\infty$.

We must show $3 \Rightarrow 1$. Let $\kappa > 1$ be a fixed integer as in the definition of the class $\mathbb{W}_+$ (and in particular satisfying (2.11)), and denote $1_\Delta = \sum_{k\in\Delta} e_k$ for a set $\Delta \subset \mathbb{N}$. For any $\Gamma_n \subset \mathbb{N}$ with $|\Gamma_n| = \kappa^n$, we can find a subset Γ_{n-1} with $|\Gamma_{n-1}| = \kappa^{n-1}$ such that

$$\|1_{\Gamma_n} - 1_{\Gamma_{n-1}}\|_{\mathbb{B}} \le 2\sigma_{\kappa^{n-1}}(1_{\Gamma_n}).$$

Repeating this argument we choose $\Gamma_{j-1} \subset \Gamma_j$ with $|\Gamma_j| = \kappa^j$ and so that

$$\|1_{\Gamma_j} - 1_{\Gamma_{j-1}}\|_{\mathbb{B}} \le 2\sigma_{\kappa^{j-1}}(1_{\Gamma_j}), \quad \text{for } j = 1, 2 \ldots, n\,.$$

Setting $\Gamma_{-1} = \emptyset$, and using the ρ-power triangle inequality we see that

$$\|1_{\Gamma_n}\|^\rho_{\mathbb{B}} = \Big\| \sum_{j=0}^n 1_{\Gamma_j} - 1_{\Gamma_{j-1}} \Big\|^\rho_{\mathbb{B}} \le \sum_{j=0}^n \|1_{\Gamma_j} - 1_{\Gamma_{j-1}}\|^\rho_{\mathbb{B}} \le 2^\rho \sum_{j=0}^n \sigma_{\kappa^{j-1}}(1_{\Gamma_j})^\rho\,.$$

Now, the hypothesis (3.11) and Lemma 2.3 give

$$\sigma_{\kappa^{j-1}}(1_{\Gamma_j}) \lesssim \kappa^{-j\alpha} \|1_{\Gamma_j}\|_{\ell^q_{k^\alpha \eta(k)}(\mathcal{B},\mathbb{B})} \approx \eta(\kappa^j).$$

Thus, combining these two expressions we obtain

$$\|1_{\Gamma_n}\|_{\mathbb{B}} \lesssim \left[\sum_{j=0}^n \eta(\kappa^j)^\rho \right]^{1\rho} \le C\,\eta(\kappa^n)\,, \tag{3.12}$$

where the last inequality follows from the assumption $\eta \in \mathbb{W}_+$ and Lemma 2.1. This shows (3.10) when $N = \kappa^n$, $n = 1, 2, \ldots$ The general case follows easily using the doubling property of η. □

Remark 3.7 As before, if any of the statements in 2 or 3 holds for one fixed $\alpha > 0$ and $q \in (0, \infty]$, then the assertions remain true for all α and q, since 1 is independent of these parameters.

Remark 3.8 Observe also that $1 \Rightarrow 2 \Rightarrow 3$ hold with the weaker assumption $\{k^\alpha \eta(k)\} \in \mathbb{W}_+$ from Theorem 3.1 (and in particular hold for $\eta = h_r$ as stated in (1.5)). However, the stronger assumption $\eta \in \mathbb{W}_+$ is crucial to obtain $3 \Rightarrow 1$, and cannot be removed as shown in Example 5.6 below.

Corollary 3.9 (Optimality of the inclusions into $\mathcal{A}^\alpha_q$) *Let $\mathcal{B}$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0, \infty]$. Then*

$$\ell^q_{k^\alpha h_r(k)}(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B}). \tag{3.13}$$

If for some $\omega \in \mathbb{W}_+$ we have $\ell^q_\omega(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$, then necessarily $\omega(k) \gtrsim k^\alpha$. Moreover if $\omega(k) = k^\alpha \eta(k)$, with η increasing and doubling, then

(a) *if $i_\eta > 0$, then necessarily $\eta(k) \gtrsim h_r(k)$, and hence $\ell^q_\omega \hookrightarrow \ell^q_{k^\alpha h_r(k)}$.*
(b) *if $i_\eta = 0$, then $\eta(k) \gtrsim h_r(k)/(\log k)^{1/\rho}$ and $\ell^q_{k^\alpha \eta(k)} \hookrightarrow \ell^q_{\{k^\alpha h_r(k)/(\log k)^{1/\rho}\}}$.*

Proof The inclusion (3.13) is actually a consequence of (3.9). Assertion (a) is just $2 \Rightarrow 3 \Rightarrow 1$ in the theorem. For assertion (b) notice that in the last step of the proof of $3 \Rightarrow 1$, the right hand inequality of (3.12) can always be replaced by

$$\|1_{\Gamma_n}\|_{\mathbb{B}} \lesssim \left[\sum_{j=0}^{n} \eta(\kappa^j)^\rho\right]^{1\rho} \lesssim \eta(\kappa^n)\, n^{1/\rho}$$

when η is increasing. Thus $h_r(N) \lesssim \eta(N)(\log N)^{1/\rho}$ holds for $N = \kappa^n$, and by the doubling property also for all $N \in \mathbb{N}$. Finally, if $\ell^q_\omega(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$ for some general $\omega \in \mathbb{W}_+$, then given $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$ we trivially have

$$\omega(N) \approx \|1_\Gamma\|_{\ell^q_\omega} \gtrsim \|1_\Gamma\|_{\mathcal{A}^\alpha_\infty} \geq (N/2)^\alpha\, \sigma_{N/2}(1_\Gamma) \geq (N/2)^\alpha. \qquad \square$$

Remark 3.10 Assertion (b) shows that the inclusion in (3.13) is optimal, except perhaps for a logarithmic loss. The logarithmic loss may actually happen, as there are Banach spaces $\mathbb{B}$ with $h_r(N) \approx \log N$ and so that

$$\mathcal{A}^\alpha_q(\mathbb{B}) = \ell^q_{k^\alpha} = \ell^q_{\{k^\alpha h_r(k)/\log k\}}.$$

See Example 5.6 below.

4 Left democracy and Bernstein type inequalities

It is well known that upper inclusions for the approximation spaces $\mathcal{A}^\alpha_q$, as in (1.5), depend upon Bernstein type inequalities. In this section we show how the left democracy function of $\mathcal{B}$ is linked with these two properties.

We first remark that, for each $\alpha > 0$ and $0 < q \leq \infty$, the approximation classes $\mathcal{A}_q^\alpha$ and $\mathcal{G}_q^\alpha$ satisfy trivial Bernstein inequalities, namely, there exists $C_{\alpha,q} > 0$ such that

$$\|x\|_{\mathcal{A}_q^\alpha(\mathcal{B},\mathbb{B})} \leq \|x\|_{\mathcal{G}_q^\alpha(\mathcal{B},\mathbb{B})} \leq C_{\alpha,q}\, N^\alpha \|x\|_{\mathbb{B}}, \quad \forall\, x \in \Sigma_N, \quad N = 1, 2, \ldots \tag{4.1}$$

This follows easily from the definition of the norms and the trivial estimates $\sigma_N(x) \leq \gamma_N(x) \leq \|x\|_{\mathbb{B}}$.

We start with a preliminary result which is essentially known in the literature (see eg [29]). As usual $\mathcal{B} = \{e_j\}_{j=1}^\infty$ is a fixed (normalized) unconditional basis in $\mathbb{B}$.

Proposition 4.1 *Let $\mathbb{E}$ be a subspace of $\mathbb{B}$, endowed with a quasi-norm $\|.\|_{\mathbb{E}}$ satisfying the ρ-triangle inequality for some $\rho = \rho_{\mathbb{E}}$. For each $\alpha > 0$ the following are equivalent:*

1. $\exists\, C_\alpha > 0$ *such that* $\|x\|_{\mathbb{E}} \leq C_\alpha\, N^\alpha\, \|x\|_{\mathbb{B}}$, $\forall\, x \in \Sigma_N$, $N = 1, 2, \ldots$
2. $\mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathbb{E}$.
3. $\mathcal{G}_\rho^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathbb{E}$.

Proof

$1 \Rightarrow 2$ Given $x \in \mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B})$, by the representation theorem for approximation spaces [29] one can write $x = \sum_{k=0}^\infty x_k$ with $x_k \in \Sigma_{2^k}$, $k = 0, 1, 2, \ldots$, such that

$$\left(\sum_{k=0}^\infty 2^{k\alpha\rho} \|x_k\|_{\mathbb{B}}^\rho \right)^{1/\rho} \leq C \|x\|_{\mathcal{A}_\rho^\alpha(\mathcal{B},\mathbb{B})} .$$

The hypothesis 1 and the $\rho_{\mathbb{E}}$-triangular inequality then give

$$\|x\|_{\mathbb{E}}^\rho \leq \sum_{k=0}^\infty \|x_k\|_{\mathbb{E}}^\rho \leq C_\alpha^\rho \sum_{k=0}^\infty 2^{k\alpha\rho} \|x_k\|_{\mathbb{B}}^\rho \leq C' \|x\|_{\mathcal{A}_\rho^\alpha(\mathcal{B},\mathbb{B})}^\rho .$$

$2 \Rightarrow 3$ This follows from the trivial inclusion $\mathcal{G}_\rho^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B})$.

$3 \Rightarrow 1$ This is immediate using (4.1).

□

Theorem 4.2 *Let $\mathcal{B} = \{e_j\}_{j=1}^\infty$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0, \infty]$. Then, for any increasing and doubling sequence $\{\eta(k)\}$ the following statements are equivalent:*

1. *There exists $C > 0$ such that for all $N = 1, 2, 3, \ldots$*

$$\left\| \sum_{k\in\Gamma} e_k \right\|_{\mathbb{B}} \geq \tfrac{1}{C}\, \eta(N), \quad \forall\, \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N. \tag{4.2}$$

2. *Bernstein type inequality for* $\ell^q_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})$*:* $\exists\, C_{\alpha,q}>0$ *such that*

$$\|x\|_{\ell^q_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})} \le C_{\alpha,q}\, N^\alpha\, \|x\|_{\mathbb{B}}, \quad \forall\, x\in\Sigma_N,\ N=1,2,3,\ldots \tag{4.3}$$

3. $\mathcal{A}^\alpha_q(\mathcal{B},\mathbb{B}) \hookrightarrow \ell^q_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})$.
4. $\mathcal{G}^\alpha_q(\mathcal{B},\mathbb{B}) \hookrightarrow \ell^q_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})$.

Proof

$1\Rightarrow 2$ Let $x=\sum_{k\in\Gamma} c_k e_k\in\Sigma_N$. For any bijection π with $|c_{\pi(k)}|$ decreasing, and any integer $m\in\{1,\ldots,N\}$ we have

$$|c_{\pi(m)}|\,\eta(m)\le C\,|c_{\pi(m)}|\left\|\sum_{j=1}^m e_{\pi(j)}\right\|_{\mathbb{B}} \le C\left\|\sum_{j=1}^m c_{\pi(j)}e_{\pi(j)}\right\|_{\mathbb{B}} \le C\|x\|_{\mathbb{B}}\,,$$

using (2.3) in the second inequality. This gives

$$\|x\|_{\ell^q_{k^\alpha\eta(k)}} = \left[\sum_{m=1}^N (m^\alpha\eta(m)c^*_m)^q\frac{1}{m}\right]^{1/q} \le C\|x\|_{\mathbb{B}}\left[\sum_{m=1}^N m^{\alpha q}\frac{1}{m}\right]^{1/q} \approx \|x\|_{\mathbb{B}}\,N^\alpha.$$

$2\Rightarrow 1$ For any $\Gamma\subset\mathbb{N}$ with $|\Gamma|=N$, applying (4.3) to $1_\Gamma=\sum_{k\in\Gamma}e_k$ we obtain

$$\|1_\Gamma\|_{\mathbb{B}} \ge \tfrac{1}{C_{\alpha,q}}\,N^{-\alpha}\,\|1_\Gamma\|_{\ell^q_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B})} \gtrsim \eta(N),$$

where in the last inequality we have used $\|1_\Gamma\|_{\ell^q_\omega}\gtrsim\omega(N)$, when $\omega\in\mathbb{W}$.

$2\Rightarrow 3$ We have already proved that $1\Leftrightarrow 2$; since 1 does not depend on α, q, then 2 actually holds for all $\tilde\alpha>0$. In particular, from Proposition 4.1, we have

$$\mathcal{A}^{\tilde\alpha}_\rho \hookrightarrow \mathbb{E} := \ell^q_{k^{\tilde\alpha}\eta(k)}(\mathcal{B},\mathbb{B}) \tag{4.4}$$

for $\tilde\alpha\in(\frac{\alpha}{2},\frac{3\alpha}{2})$ and some sufficiently small $\rho>0$. Now, from the general theory developed in [7], the spaces $\mathcal{A}^\alpha_q$ satisfy a reiteration theorem for the real interpolation method, and in particular

$$\mathcal{A}^\alpha_q = \big(\mathcal{A}^{\alpha_0}_{q_0},\mathcal{A}^{\alpha_1}_{q_1}\big)_{1/2,\,q}\,, \tag{4.5}$$

when $\alpha=(\alpha_0+\alpha_1)/2$ with $\alpha_1>\alpha_0>0$, and $q_0,q_1,q\in(0,\infty]$. On the other hand, for the family of weighted Lorentz spaces it is known that

$$\big(\ell^q_{\omega_0},\ell^q_{\omega_1}\big)_{\theta,\,q} = \ell^q_\omega\,, \quad 0<\theta<1, \quad 0<q\le\infty, \tag{4.6}$$

when $\omega_0,\omega_1\in\mathbb{W}_+$ and $\omega=\omega_0^{1-\theta}\omega_1^\theta$ (see e.g. [25, Theorem 3]). Thus, for fixed α and q, we can choose the parameters accordingly, and use the inclusion (4.4), to obtain

$$\mathcal{A}^\alpha_q = \big(\mathcal{A}^{\alpha_0}_\rho,\mathcal{A}^{\alpha_1}_\rho\big)_{1/2,\,q} \hookrightarrow \big(\ell^q_{k^{\alpha_0}\eta(k)},\ell^q_{k^{\alpha_1}\eta(k)}\big)_{1/2,\,q} = \ell^q_{k^\alpha\eta(k)}(\mathcal{B},\mathbb{B}).$$

$3\Rightarrow 4$ This is trivial since $\mathcal{G}^\alpha_q\hookrightarrow\mathcal{A}^\alpha_q$.
$4\Rightarrow 2$ This is trivial from (4.1). □

Remark 4.3 Observe that $3 \Rightarrow 4 \Rightarrow 2 \Leftrightarrow 1$ hold with the weaker assumption $\{k^\alpha \eta(k)\} \in \mathbb{W}$.

Corollary 4.4 (Optimal inclusions of $\mathcal{A}_q^\alpha$ into ℓ_ω^q) *Let $\mathcal{B}$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0, \infty]$.*

(a) *If $h_\ell(N)$ is doubling then $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell^q_{k^\alpha h_\ell(k)}(\mathcal{B}, \mathbb{B})$.*
(b) *If for some $\omega \in \mathbb{W}$ we have $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_\omega^q(\mathcal{B}, \mathbb{B})$ then necessarily $\omega(k) \lesssim k^\alpha h_\ell(k)$, and hence $\ell^q_{k^\alpha h_\ell(k)} \hookrightarrow \ell^q_\omega$.*

Proof Part (a) is an application of $1 \Rightarrow 3$ in the theorem with $\eta = h_\ell$ (which under the doubling assumption satisfies $\{k^\alpha h_\ell(k)\} \in \mathbb{W}_+$ for all $\alpha > 0$). Part (b) is just a restatement of $3 \Rightarrow 1$ in the theorem, setting $\eta(k) = \omega(k)/k^\alpha$ and taking into account Remark 4.3. □

5 Examples and applications

In this section we describe the democracy functions h_ℓ and h_r in various examples which can be found in the literature. Inclusions for $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ and $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ will be obtained inmediately from the results of Sections 3 and 4. The most interesting case appears when $\mathcal{B}$ is a wavelet basis, and $\mathbb{B}$ a function or distribution space in $\mathbb{R}^d$ which can be characterized by such basis (eg, the general Besov or Triebel–Lizorkin spaces, $B^\alpha_{p,q}$ and $F^s_{p,q}$, and also rearrangement invariant spaces as the Orlicz and Lorentz classes, L^Φ and $L^{p,q}$). Such characterizations provide a description of each $\mathbb{B}$ as a sequence space, so for simplicity we shall work in this simpler setting, reminding in each case the original function space framework.

Let $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ denote the family of all dyadic cubes Q in $\mathbb{R}^d$, ie

$$\mathcal{D} = \left\{ Q_{j,k} = 2^{-j}([0,1)^d + k) \ : \ j \in \mathbb{Z},\ k \in \mathbb{Z}^d \right\}.$$

We shall consider sequences indexed by $\mathcal{D}$, $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{D}}$, endowed with quasi-norms of the following form

$$\left\| \left(\sum_{Q \in \mathcal{D}} \left(|Q|^{\gamma - \frac{1}{2}} \, |s_Q| \, \chi_Q(\cdot) \right)^r \right)^{1/r} \right\|_{\mathbb{X}} , \tag{5.1}$$

where $0 < r \le \infty$, $\gamma \in \mathbb{R}$ and $\mathbb{X}$ is a suitable quasi-Banach function space in $\mathbb{R}^d$, such as the ones we consider below. The canonical basis $\mathcal{B}_c = \{\mathbf{e}_Q\}_{Q \in \mathcal{D}}$ is formed by the sequences $\mathbf{e}_Q$ with entry 1 at Q and 0 otherwise. In each of the examples below, the greedy algorithms and democracy functions are considered with respect to the normalized basis $\mathcal{B} = \left\{\mathbf{e}_Q/\|\mathbf{e}_Q\|_{\mathbb{B}}\right\}$. Similarly, when stating the corresponding results for the functional setting we shall write $\mathcal{W}$ for the wavelet basis.

Example 5.1 ($\mathbb{X} = L^p(\mathbb{R}^d), 0 < p < \infty$) In this case, it is customary to consider the sequence spaces $\mathfrak{f}^s_{p,r}$, $s \in \mathbb{R}$, $0 < r \leq \infty$, with quasi-norms given by

$$\|\mathbf{s}\|_{\mathfrak{f}^s_{p,r}} := \left\| \left(\sum_{Q\in\mathcal{D}} \left(|Q|^{-\frac{s}{d}-\frac{1}{2}} |s_Q|\, \chi_Q(\cdot) \right)^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^d)} .$$

It was proved in [11, 16, 18] that, for all $s \in \mathbb{R}$ and $0 < r \leq \infty$,

$$h_\ell(N; \mathfrak{f}^s_{p,r}) \approx h_r(N; \mathfrak{f}^s_{p,r}) \approx N^{1/p} \tag{5.2}$$

and

$$\mathcal{A}^\alpha_q(\mathfrak{f}^s_{p,r}) = \ell^{\tau,q}(\mathfrak{f}^s_{p,r}) = \Big\{ \mathbf{s} \;:\; \{s_Q \|e_Q\|_{\mathfrak{f}^s_{p,r}}\}_Q \in \ell^{\tau,q} \Big\}, \tag{5.3}$$

if $\frac{1}{\tau} = \alpha + \frac{1}{p}$, as asserted in Theorem 1.2.

It is well-known that $\mathfrak{f}^s_{p,r}$ coincides with the coefficient space under a wavelet basis $\mathcal{W}$ of the (homogeneous) Triebel–Lizorkin space $\dot{F}^s_{p,r}(\mathbb{R}^d)$, defined in terms of Littlewood–Paley theory (see e.g. [10, 22, 26]). In particular, under suitable decay and smoothness on the wavelet family (so that it is an unconditional basis of the involved spaces) the statement in (5.3) can be translated into

$$\mathcal{A}^\alpha_q(\mathcal{W}, \dot{F}^s_{p,r}(\mathbb{R}^d)) = \mathcal{G}^\alpha_q(\mathcal{W}, \dot{F}^s_{p,r}(\mathbb{R}^d)) = \dot{B}^{s+\alpha d}_{q,q}(\mathbb{R}^d)$$

when $\frac{1}{q} = \alpha + \frac{1}{p}$. We refer to [5, 11, 16, 17] for details and further results.

Example 5.2 (Weighted Lebesgue spaces $\mathbb{X} = L^p(w), 0 < p < \infty$) For weights $w(x)$ in the Muckenhoupt class $A_\infty(\mathbb{R}^d)$, one can define sequence spaces $\mathfrak{f}^s_{p,r}(w)$ with the quasi-norm

$$\|\mathbf{s}\|_{\mathfrak{f}^s_{p,r}(w)} := \left\| \left(\sum_{Q\in\mathcal{D}} \left(|Q|^{-\frac{s}{d}-\frac{1}{2}} |s_Q|\, \chi_Q(\cdot) \right)^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^d, w)} .$$

Similar computations as in the previous case in this more general situation will also lead to the identities in (5.2) and (5.3), with $\mathfrak{f}^s_{p,r}$ replaced by $\mathfrak{f}^s_{p,r}(w)$. We refer to [21, 27] for details in some special cases.

When $\mathcal{W}$ is a (sufficiently smooth) orthonormal wavelet basis and w is a weight in the Muckenhoupt class $A_p(\mathbb{R}^d)$, $1 < p < \infty$, then $\mathfrak{f}^0_{p,2}(w)$ becomes the coefficient space of the weighted Lebesgue space $L^p(w)$ (see e.g. [1]). One then obtains as special case

$$h_\ell(N; \mathcal{W}, L^p(w)) \approx h_r(N; \mathcal{W}, L^p(w)) \approx N^{\frac{1}{p}} .$$

Moreover, if $\omega \in A_\tau(\mathbb{R}^d)$,

$$\mathcal{A}^\alpha_\tau(\mathcal{W}, L^p(w)) \approx \mathcal{G}^\alpha_\tau(\mathcal{W}, L^p(w)) \approx \dot{B}^{\alpha d}_{\tau,\tau}(w^{\tau/p}), \quad \text{if } \frac{1}{\tau} = \alpha + \frac{1}{p} ,$$

where $\dot{B}^\alpha_{\tau,q}(w)$ denotes a weighted Besov space (see [27] for details).

Example 5.3 (Orlicz spaces $\mathbb{X} = L^{\Phi}(\mathbb{R}^d)$) Following [12], we denote by $\mathfrak{f}^{\Phi}$ the sequence space with quasi-norm

$$\|\mathbf{s}\|_{\mathfrak{f}^{\Phi}} := \left\| \left(\sum_{Q \in \mathcal{D}} \left(|s_Q| \, \frac{\chi_Q(\cdot)}{|Q|^{1/2}} \right)^2 \right)^{1/2} \right\|_{L^{\Phi}(\mathbb{R}^d)},$$

where L^{Φ} is an Orlicz space with non-trivial Boyd indices. If we denote by $\varphi(t) = 1/\Phi^{-1}(1/t)$, the fundamental function of L^{Φ}, then it is shown in [12] that

$$h_\ell(N; \mathfrak{f}^{\Phi}) \approx \inf_{s>0} \frac{\varphi(Ns)}{\varphi(s)} \qquad \text{and} \qquad h_r(N; \mathfrak{f}^{\Phi}) \approx \sup_{s>0} \frac{\varphi(Ns)}{\varphi(s)},$$

with the two expressions being equivalent iff $\varphi(t) = t^{1/p}$ (ie, iff $L^{\Phi} = L^p$). Thus, these are first examples of non-democratic spaces, with a wide range of possibilities for the democracy functions. The theorems in Sections 3 and 4 recover the embeddings obtained in [12] for the approximation classes $\mathcal{A}^{\alpha}_q(\mathfrak{f}^{\Phi})$ and $\mathcal{G}^{\alpha}_q(\mathfrak{f}^{\Phi})$ in terms of weighted discrete Lorentz spaces. When using suitable wavelet bases, these lead to corresponding inclusions for $\mathcal{A}^{\alpha}_q(\mathcal{W}, L^{\Phi})$ and $\mathcal{G}^{\alpha}_q(\mathcal{W}, L^{\Phi})$, some of which can be expressed in terms of Besov spaces of generalized smoothness (see [12] for details).

Example 5.4 (Lorentz spaces $\mathbb{X} = L^{p,q}(\mathbb{R}^d)$, $0 < p, q < \infty$) Consider sequence spaces $\mathfrak{l}^{p,q}$ defined by the following quasi-norms

$$\|\mathbf{s}\|_{\mathfrak{l}^{p,q}} := \left\| \left(\sum_{Q \in \mathcal{D}} \left(|s_Q| \, \frac{\chi_Q(\cdot)}{|Q|^{1/2}} \right)^2 \right)^{1/2} \right\|_{L^{p,q}(\mathbb{R}^d)}.$$

Their democracy functions have been computed in [14], obtaining

$$h_\ell(N; \mathfrak{l}^{p,q}) \approx N^{\frac{1}{\max(p,q)}} \qquad \text{and} \qquad h_r(N; \mathfrak{l}^{p,q}) \approx N^{\frac{1}{\min(p,q)}}.$$

These imply corresponding inclusions for the classes $\mathcal{A}^{\alpha}_s(\mathfrak{l}^{p,q})$ and $\mathcal{G}^{\alpha}_s(\mathfrak{l}^{p,q})$ in terms of discrete Lorentz spaces $\ell^{\tau,s}$ (as described in the theorems of Sections 3 and 4). The spaces $\mathfrak{l}^{p,q}$ characterize, via wavelets, the usual Lorentz spaces $L^{p,q}(\mathbb{R}^d)$ when $1 < p < \infty$ and $1 \le q < \infty$ [32]. Hence inclusions for $\mathcal{A}^{\alpha}_s(\mathcal{W}, L^{p,q})$ and $\mathcal{G}^{\alpha}_s(\mathcal{W}, L^{p,q})$ can be obtained using standard Besov spaces.

Example 5.5 (Hyperbolic wavelets) For $0 < p < \infty$, consider now the sequence space

$$\|\mathbf{s}\|_{\mathfrak{f}^p_{\mathrm{hyp}}} := \left\| \left(\sum_{R} \left(|s_R| \, \frac{\chi_R(\cdot)}{|R|^{1/2}} \right)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}.$$

where R runs over the family of all dyadic rectangles of $\mathbb{R}^d$, that is $R = I_1 \times \ldots \times I_d$, with $I_i \in \mathcal{D}(\mathbb{R})$, $i = 1, \ldots, d$. This gives another example of non-democratic basis. In fact, the following result is proved in [38, Proposition 11] (see also [34]):

(a) If $0 < p \le 2$,

$$h_\ell(N; \mathfrak{f}^p_{\text{hyp}}) \approx N^{1/p}(\log N)^{(\frac{1}{2}-\frac{1}{p})(d-1)} \qquad \text{and} \qquad h_r(N; \mathfrak{f}^p_{\text{hyp}}) \approx N^{1/p}.$$

(b) If $2 \le p < \infty$,

$$h_\ell(N; \mathfrak{f}^p_{\text{hyp}}) \approx N^{1/p} \qquad \text{and} \qquad h_r(N; \mathfrak{f}^p_{\text{hyp}}) \approx N^{1/p}(\log N)^{(\frac{1}{2}-\frac{1}{p})(d-1)}.$$

If $\mathcal{H}_d$ denotes the multidimensional (hyperbolic) Haar basis, then $\mathfrak{f}^p_{\text{hyp}}$ becomes the coefficient space of the usual $L^p(\mathbb{R}^d)$ if $1 < p < \infty$ (and the dyadic Hardy space $H^p(\mathbb{R}^d)$ if $0 < p \le 1$). In this case, one obtains corresponding inclusions for the classes $\mathcal{A}^\alpha_q(\mathcal{H}_d, L^p)$ and $\mathcal{G}^\alpha_q(\mathcal{H}_d, L^p)$ (see also [19, Theorem 5.2]), some of which could possibly be expressed in terms of Besov spaces of bounded mixed smoothness [6, 19].

Example 5.6 (Bounded mean oscillation) Let bmo denote the space of sequences $\mathbf{s} = \{s_I\}_{I\in\mathcal{D}}$ with

$$\|\mathbf{s}\|_{bmo} = \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|} \sum_{J\subset I, J\in\mathcal{D}} |s_J|^2 |J| \right)^{1/2} < \infty. \tag{5.4}$$

This sequence space gives the correct characterization of $BMO(\mathbb{R})$ for sufficiently smooth wavelet bases appropriately normalized(see [10, 16, 37]). Their democracy functions are determined by

$$h_\ell(N; bmo) \approx 1, \quad h_r(N; bmo) \approx \big(\log N\big)^{1/2}. \tag{5.5}$$

The first part of (5.5) is easy to prove, and the second follows, for instance, by an argument similar to the one presented in the proof of [28, Lemma 3]. Our results of Sections 3 and 4 give in this case the inclusions:

$$\ell^q_{k^\alpha\sqrt{\log k}} \hookrightarrow \mathcal{G}^\alpha_q(bmo) \hookrightarrow \mathcal{A}^\alpha_q(bmo) \hookrightarrow \ell^q_{k^\alpha} = \ell^{1/\alpha,q}. \tag{5.6}$$

However, this is not the best one can say for the approximation classes $\mathcal{A}^\alpha_q$. A result proved in [30] (see also Proposition 11.6 in [16]) shows that one actually has

$$\mathcal{A}^\alpha_q(bmo) = \mathcal{A}^\alpha_q(\ell^\infty) = \ell^{1/\alpha,q},$$

for all $\alpha > 0$ and $q \in (0, \infty]$. For $0 < r < \infty$ one can define the space bmo_r replacing the 2 by r in (5.4); it can then be shown that $h_r(N; bmo_r) \approx \big(\log N\big)^{1/r}$ and $\mathcal{A}^\alpha_q(bmo_r) = \ell^{1/\alpha,q}$.

6 Democracy functions for $\mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$ and $\mathcal{G}^\alpha_q(\mathcal{B}, \mathbb{B})$

As usual, we fix a (normalized) unconditional basis $\mathcal{B} = \{e_j\}_{j=1}^\infty$ in $\mathbb{B}$. In this section we compute the democracy functions for the spaces $\mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$ and

$\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$, in terms of the democracy functions in the ambient space $\mathbb{B}$. To distinguish among these notions we shall use, respectively, the notations

$$h_\ell(N; \mathcal{A}_q^\alpha), \quad h_\ell(N; \mathcal{G}_q^\alpha) \quad \text{and} \quad h_\ell(N; \mathbb{B}),$$

and similarly for h_r (recall the definitions in Section 2.5). Since we shall use the embeddings in Sections 3 and 4, observe first that

$$h_\ell(N; \ell^q_\omega(\mathcal{B}, \mathbb{B})) \approx h_r(N; \ell^q_w(\mathcal{B}, \mathbb{B})) \approx \omega(N), \tag{6.1}$$

for all $\omega \in \mathbb{W}_+$ and $0 < q \le \infty$. This is immediate from the definition of the spaces $\ell^q_\omega(\mathcal{B}, \mathbb{B})$ and Lemma 2.3.

Proposition 6.1 *Fix $\alpha > 0$ and $0 < q \le \infty$. If $h_\ell(\,\cdot\,; \mathbb{B})$ is doubling then*

(a) $h_\ell(N; \mathcal{G}_q^\alpha) \approx N^\alpha h_\ell(N; \mathbb{B})$.
(b) $h_r(N; \mathcal{G}_q^\alpha) \approx N^\alpha h_r(N; \mathbb{B})$.

In particular, $\mathcal{B}$ is democratic in $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ if and only if $\mathcal{B}$ is democratic in $\mathbb{B}$.

Proof The inequalities "$\gtrsim$" in (a), and "$\lesssim$" in (b) follow immediately from the embeddings

$$\ell^q_{k^\alpha h_r(k)}(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell^q_{k^\alpha h_\ell(k)}(\mathcal{B}; \mathbb{B})$$

and the remark in (6.1). Thus we must show the converse inequalities. To establish (a), given $N = 1, 2, 3, \ldots$ choose Γ with $|\Gamma| = N$ and so that $\|1_\Gamma\|_\mathbb{B} \le 2h_\ell(N; \mathbb{B})$. Then, using the trivial bound in (4.1) we obtain

$$h_\ell(N; \mathcal{G}_q^\alpha) \;\le\; \|1_\Gamma\|_{\mathcal{G}_q^\alpha} \;\lesssim\; N^\alpha \|1_\Gamma\|_\mathbb{B} \;\approx\; N^\alpha h_\ell(N; \mathbb{B}).$$

We now prove "$\gtrsim$" in (b). Given $N = 1, 2, \ldots$, choose first Γ with $|\Gamma| = N$ and $\|1_\Gamma\|_\mathbb{B} \ge \frac{1}{2} h_r(N; \mathbb{B})$, and then any Γ' disjoint with Γ with $|\Gamma'| = N$. Then

$$h_r(2N; \mathcal{G}_q^\alpha) \;\ge\; \left\|1_{\Gamma\cup\Gamma'}\right\|_{\mathcal{G}_q^\alpha} \;\gtrsim\; N^\alpha \gamma_N(1_{\Gamma\cup\Gamma'}; \mathbb{B}) \gtrsim N^\alpha \left\|1_\Gamma\right\|_\mathbb{B} \;\approx\; N^\alpha h_r(N; \mathbb{B}).$$

The required bound then follows from the doubling property of h_r. □

Proposition 6.2 *Fix $\alpha > 0$ and $0 < q \le \infty$, and assume that $h_\ell(\,\cdot\,; \mathbb{B})$ is doubling. Then*

(a) $h_\ell(N; \mathcal{A}_q^\alpha) \approx N^\alpha h_\ell(N; \mathbb{B})$.
(b) $h_r(N; \mathcal{A}_q^\alpha) \lesssim N^\alpha h_r(N; \mathbb{B})$.

In particular, if $\mathcal{B}$ is democratic in $\mathbb{B}$ then $\mathcal{B}$ is democratic in $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$.

Proof As before, "$\gtrsim$" in (a), and "$\lesssim$" in (b) follow immediately from the embeddings

$$\ell^q_{k^\alpha h_r(k)}(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell^q_{k^\alpha h_\ell(k)}(\mathcal{B}; \mathbb{B}).$$

The converse inequality in (a) follows from the previous proposition and the trivial inclusion $\mathcal{G}_q^\alpha \hookrightarrow \mathcal{A}_q^\alpha$. □

As shown in Example 5.6, the converse to the last statement in Proposition 6.2 is not necessarily true. The space $\mathbb{B} = bmo$ is not democratic, but their approximation classes $\mathcal{A}^\alpha_q(bmo) = \ell^{1/\alpha,q}$ are democratic. Moreover, this example shows that the converse to the inequality in (b) does not necessarily hold, since

$$h_r(N; \mathcal{A}^q_\alpha(bmo)) = N^\alpha \quad \text{but} \quad N^\alpha h_r(N; bmo) \approx N^\alpha (\log N)^{1/2}.$$

Nevertheless, we can give a sufficient condition for $h_r(N; \mathcal{A}^\alpha_q) \approx N^\alpha h_r(N; \mathbb{B})$, which turns out to be easily verifiable in all the other examples presented in §5.

Property (H) We say that $\mathcal{B}$ satisfies the **Property (H)** if for each $n = 1, 2, 3, \ldots$ there exist $\Gamma_n \subset \mathbb{N}$, with $|\Gamma_n| = 2^n$, satisfying the property

$$\|1_{\Gamma'}\|_{\mathbb{B}} \approx h_r(2^{n-1}; \mathbb{B}), \quad \forall\, \Gamma' \subset \Gamma_n \quad \text{with} \quad |\Gamma'| = 2^{n-1}.$$

Proposition 6.3 *Assume that $\mathcal{B}$ satisfies the Property (H). Then, for all $\alpha > 0$ and $0 < q \le \infty$*

$$h_r(N; \mathcal{A}^\alpha_q) \approx N^\alpha h_r(N; \mathbb{B})$$

Proof We must show "$\gtrsim$", for which we argue as in the proof of Proposition 6.1. Given $N = 2^n$, select Γ_n as in the definition of Property (H). Then,

$$h_r(N; \mathcal{A}^\alpha_q) \;\ge\; \left\|1_{\Gamma_n}\right\|_{\mathcal{A}^\alpha_q} \gtrsim N^\alpha\, \sigma_{N/2}(1_{\Gamma_n}).$$

Now, the property (H) (and the remark in (2.4)) give

$$\sigma_{N/2}(1_{\Gamma_n}) \;=\; \inf\left\{\|1_{\Gamma'}\|_{\mathbb{B}} \;:\; \Gamma' \subset \Gamma,\ |\Gamma'| = N/2\right\} \;\approx\; h_r(N/2; \mathbb{B}) \approx\; h_r(N; \mathbb{B}).$$

Combining these two facts the proposition follows for $N = 2^n$. For general N use the result just proved and the doubling property of h_r. □

As an immediate consequence, the property (H) allows to remove the possible logarithmic loss for the embedding $\ell^q_{k^\alpha h_r(k)}(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$ discussed in Corollary 3.9.

Corollary 6.4 (More about optimality for inclusions into $\mathcal{A}^\alpha_q$) *Assume that $(\mathbb{B}, \mathcal{B})$ satisfies property (H). If for some $\alpha > 0$, $q \in (0, \infty]$ and $\omega \in \mathbb{W}_+$ we have $\ell^q_\omega(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, \mathbb{B})$, then necessarily $\omega(k) \gtrsim k^\alpha h_r(k)$, and therefore $\ell^q_\omega \hookrightarrow \ell^q_{k^\alpha h_r(k)}$.*

The following examples show that Property (H) is often satisfied.

Example 6.1 Wavelet bases in Orlicz spaces $L^\Phi(\mathbb{R}^d)$ satisfy the property (H). Indeed, recall from [12, Theorem 1.2] (see also Example 5.3) that

$$h_r(N; L^\Phi) \approx \sup_{s>0} \varphi(Ns)/\varphi(s)\,. \tag{6.2}$$

Moreover, any collection Γ of N pairwise disjoint dyadic cubes *with the same fixed size* $a > 0$ satisfies

$$\|1_\Gamma\|_{L^\Phi} \approx \varphi(Na)/\varphi(a)\,, \tag{6.3}$$

(see eg [12, Lemma 3.1]). Thus, for each $N = 2^n$, we first select $a_n = 2^{j_n d}$ so that $h_r(2^n; L^\Phi) \approx \varphi(2^n a_n)/\varphi(a_n)$, and then we choose as Γ_n any collection of 2^n pairwise disjoint cubes with constant size a_n. Then, any subfamily $\Gamma' \subset \Gamma_n$ with $|\Gamma'| = N/2$, satisfies

$$\|1_{\Gamma'}\|_{L^\Phi} \approx \varphi((N/2)a_n)/\varphi(a_n) \approx \varphi(Na_n)/\varphi(a_n) \approx h_r(N) \approx h_r(N/2),$$

by (6.3) and the doubling property of φ and h_r.

Example 6.2 Wavelet bases in Lorentz spaces $L^{p,q}(\mathbb{R}^d)$, $1 < p, q < \infty$. These also satisfy the property (H). Indeed, it can be shown that any set Γ consisting of N *disjoint cubes of the same size* has

$$\|1_\Gamma\|_{L^{p,q}} \approx N^{\frac{1}{p}}\,,$$

while sets Δ consisting of N disjoint cubes *all having different sizes* satisfy

$$\|1_\Delta\|_{L^{p,q}} \approx N^{\frac{1}{q}}\,.$$

(see [14, (3.6) and (3.8)]). Since $h_r(N) \approx N^{1/(p\wedge q)}$, we can define the Γ_n's with sets of the first type when $p \le q$, and with sets of the second type when $q < p$, to obtain in both cases a collection satisfying the hypotheses of property (H).

Example 6.3 The hyperbolic Haar system in $L^p(\mathbb{R}^d)$ from Example 5.5 also satisfies property (H). In this case, again, any set Γ consisting of N disjoint rectangles has

$$\|1_\Gamma\|_{L^p(\mathbb{R}^d)} = N^{\frac{1}{p}}\,.$$

On the other hand, if Δ_n denotes the set of all the dyadic rectangles in the unit cube with fixed size 2^{-n}, then

$$\|1_{\Delta_n}\|_{L^p(\mathbb{R}^d)} \approx 2^{n/p}\, n^{(d-1)/2} \approx |\Delta_n|^{1/p} (\log|\Delta_n|)^{(d-1)(\frac{1}{2}-\frac{1}{p})}\,. \tag{6.4}$$

Moreover, it is not difficult to show that any $\Delta' \subset \Delta_n$ with $|\Delta'| = |\Delta_n|/2$ also satisfies (6.4) (with Δ_n replaced by Δ'). Hence, combining these two cases and using the description of $h_r(N)$ in Example 5.5, one easily establishes the property (H).

7 Counterexamples for the classes $\mathscr{G}_q^\alpha(\mathcal{B}, \mathbb{B})$

7.1 Conditions for $\mathscr{G}_q^\alpha \ne \mathcal{A}_q^\alpha$

Recall from Section 2.3 that $\mathscr{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$, with equality of the spaces when $\mathcal{B}$ is a greedy basis. It is known that there are some *conditional* democratic bases for which $\mathcal{G}_q^\alpha = \mathcal{A}_q^\alpha$ (see [13, Remark 6.2]). For unconditional bases, however, one could ask whether non-democracy necessarily implies that $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$. We do not know how to prove such a general result, but we can show that the inclusion $\mathcal{A}_q^\alpha \hookrightarrow \mathcal{G}_q^\alpha$ must fail whenever the gap between $h_\ell(N)$ and $h_r(N)$ is at least logarithmic (and even less than that). More precisely, we have the following.

Proposition 7.1 *Let $\mathcal{B}$ be an unconditional basis in $\mathbb{B}$ and $\alpha > 0$. Suppose that there exist integers $p_N \geq q_N \geq 1$, $N = 1, 2, \ldots$ such that*

$$\lim_{N\to\infty} \frac{p_N}{q_N} = \infty \qquad \text{and} \qquad \frac{h_r(q_N)}{h_\ell(p_N)} \gtrsim \left(\frac{p_N}{q_N}\right)^\alpha . \tag{7.1}$$

Then the inclusion $\mathcal{A}_\tau^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_\tau^\alpha(\mathcal{B}, \mathbb{B})$ does not hold for any $\tau \in (0, \infty]$.

Proof For each N, choose $\Gamma_l, \Gamma_r \subset \mathbb{N}$ with $|\Gamma_l| = p_N$, $|\Gamma_r| = q_N$, and such that

$$\|1_{\Gamma_l}\|_{\mathbb{B}} \leq 2h_\ell(p_N), \qquad \|1_{\Gamma_r}\|_{\mathbb{B}} \geq \tfrac{1}{2}\, h_r(q_N)\,. \tag{7.2}$$

Set $x_N = \mathbf{1}_{\Gamma_r} + 2 \cdot \mathbf{1}_{\Gamma_l - \Gamma_l \cap \Gamma_r}$. Since $\#(\Gamma_l - \Gamma_l \cap \Gamma_r) \geq p_N - q_N$, when $k \in [1, p_N - q_N]$ we have

$$\|x_N - G_k(x_N)\|_{\mathbb{B}} \geq \|1_{\Gamma_r}\|_{\mathbb{B}} \geq \tfrac{1}{2}\, h_r(q_N)\,.$$

Therefore, using $p_N - q_N > p_N/2$ (since $p_N/q_N > 2$ for N large), we obtain that

$$\|x_N\|_{\mathcal{G}_\tau^\alpha(\mathcal{B},\mathbb{B})} \geq \tfrac{1}{2} \left[\sum_{k=1}^{p_N/2} \left(k^\alpha h_r(q_N)\right)^\tau \tfrac{1}{k}\right]^{\frac{1}{\tau}} \gtrsim h_r(q_N)\, p_N^\alpha\,. \tag{7.3}$$

On the other hand, we can estimate the norm of x_N as follows:

$$\|x_N\|_{\mathbb{B}} \lesssim \|1_{\Gamma_r}\|_{\mathbb{B}} + \|1_{\Gamma_l - \Gamma_l\cap\Gamma_r}\|_{\mathbb{B}} \leq h_r(q_N) + 2h_\ell(p_N) \lesssim h_r(q_N) \tag{7.4}$$

where the last inequality is true for N large due to (7.1). Thus

$$\sigma_k(x_N) \leq \|x_N\|_{\mathbb{B}} \lesssim h_r(q_N)\,. \tag{7.5}$$

Next, if $k \geq q_N$, by (7.2)

$$\sigma_k(x_N) \leq 2\|1_{\Gamma_l - \Gamma_l\cap\Gamma_r}\|_{\mathbb{B}} \leq 2\|1_{\Gamma_l}\|_{\mathbb{B}} \lesssim h_\ell(p_N)\,. \tag{7.6}$$

Combining (7.4), (7.5) and (7.6) we see that

$$\begin{aligned}
\|x_N\|_{\mathcal{A}_\tau^\alpha(\mathcal{B},\mathbb{B})} &\lesssim h_r(q_N) + \left[\sum_{k=1}^{q_N-1} \left(k^\alpha h_r(q_N)\right)^\tau \frac{1}{k} + \sum_{k=q_N}^{p_N+q_N} \left(k^\alpha h_\ell(p_N)\right)^\tau \frac{1}{k}\right]^{\frac{1}{\tau}} \\
&\lesssim h_r(q_N) + \left[h_r(q_N)^\tau (q_N)^{\alpha\tau} + h_\ell(p_N)^\tau (p_N)^{\alpha\tau}\right]^{\frac{1}{\tau}} \\
&\lesssim h_r(q_N) + h_r(q_N)(q_N)^\alpha \lesssim h_r(q_N)(q_N)^\alpha
\end{aligned} \tag{7.7}$$

where in the second inequality we have used the elementary fact $\sum_{k=a}^{a+b} k^{\gamma-1} \lesssim b^{\gamma}$ if $b \geq a$, and the third inequality is due to (7.1). Therefore, from (7.3) and (7.7) we deduce

$$\frac{\|x_N\|_{\mathcal{G}_\tau^\alpha}}{\|x_N\|_{\mathcal{A}_\tau^\alpha}} \gtrsim \frac{h_r(q_N)(p_N)^\alpha}{h_r(q_N)(q_N)^\alpha} = \left(\frac{p_N}{q_N}\right)^\alpha \longrightarrow \infty$$

as $N \to \infty$. This shows the desired result. □

Corollary 7.2 *Let $\mathcal{B}$ be an unconditional basis such that $h_\ell(N) \lesssim N^{\beta_0}$ and $h_r(N) \gtrsim N^{\beta_1}$, for some $\beta_1 > \beta_0 \geq 0$. Then, $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$ for all $\alpha > 0$ and all $q \in (0, \infty]$.*

Proof Choose $r, s \in \mathbb{N}$, such that $\frac{\alpha+\beta_0}{\alpha+\beta_1} < \frac{r}{s} < 1$. Take $p_N = N^s$ and $q_N = N^r$. Then, $\lim_{N\to\infty} \frac{p_N}{q_N} = \lim_{N\to\infty} N^{s-r} = \infty$ and

$$\frac{h_r(q_N)}{h_\ell(p_N)} \gtrsim \frac{N^{r\beta_1}}{N^{s\beta_0}} > N^{\alpha(s-r)} = \left(\frac{N^s}{N^r}\right)^\alpha = \left(\frac{p_N}{q_N}\right)^\alpha,$$

which proves (7.1) in this case, so that we can apply Proposition 7.1. □

Corollary 7.3 *Let $\mathcal{B}$ be an unconditional basis such that for some $\beta \geq 0$ and $\gamma > 0$ we have either*

(i) $h_r(N) \gtrsim N^\beta(\log N)^\gamma$ *and* $h_\ell(N) \lesssim N^\beta$, *or*
(ii) $h_r(N) \gtrsim N^\beta$ *and* $h_\ell(N) \lesssim N^\beta(\log N)^{-\gamma}$.
Then, $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$ for all $\alpha > 0$ and all $q \in (0, \infty]$.

Proof i) Choose $a, b \in \mathbb{N}$ such that $0 < \frac{a}{b} < \frac{\gamma}{\alpha+\beta}$. Let $p_N = N^a 2^{N^b}$ and $q_N = 2^{N^b}$. Then, $\lim_{N\to\infty} \frac{p_N}{q_N} = \lim_{N\to\infty} N^a = \infty$ and

$$\frac{h_r(q_N)}{h_\ell(p_N)} \gtrsim \frac{(2^{N^b})^\beta(\log 2^{N^b})^\gamma}{N^{a\beta}(2^{N^b})^\beta} \approx \frac{N^{b\gamma}}{N^{a\beta}} = N^{b\gamma-a\beta} > N^{a\alpha} = \left(\frac{p_N}{q_N}\right)^\alpha$$

which proves (7.1) in this case, so that we can apply Proposition 7.1 to conclude the result. The proof of ii) is similar with the same choice of p_N and q_N. □

7.2 Non linearity of $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$

We conclude by showing with simple examples that $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ may not even be a linear space when the basis $\mathcal{B}$ is not democratic.

Let $\mathbb{B} = \ell^p \oplus_{\ell^1} \ell^q$, $0 < q < p < \infty$; that is, $\mathbb{B}$ consists of pairs $(a, b) \in \ell^p \times \ell^q$, endowed with the quasi-norm $\|a\|_{\ell^p} + \|b\|_{\ell^q}$. We consider the canonical basis in $\mathbb{B}$.

Now, set $\beta = \alpha + \frac{1}{p}$ and $x = \{(k^{-\beta}, 0)\}_{k\in\mathbb{N}} \in \mathbb{B}$. For $N = 1, 2, 3, \ldots$ we have

$$\gamma_N(x) = \left(\sum_{k>N} \frac{1}{k^{\beta p}}\right)^{1/p} \approx \left(\frac{1}{N^{\beta p-1}}\right)^{1/p} = N^{-\alpha}.$$

This shows that $x \in \mathcal{G}^\alpha_\infty(\mathcal{B}, \mathbb{B})$. Similarly, if we let $\gamma = \alpha + \frac{1}{q}$, then $y = \{(0, j^{-\gamma})\}_{j\in\mathbb{N}}$ belongs to $\mathcal{G}^\alpha_\infty$. We will show, however, that $x + y \notin \mathcal{G}^\alpha_\infty$. In fact, we will find a subsequence N_J of natural numbers so that

$$\gamma_{N_J}(x + y) \approx \frac{1}{N_J^{\alpha\beta/\gamma}} \tag{7.8}$$

(notice that $\beta < \gamma$ since we chose $q < p$). To prove (7.8) let $A_1 = \{1\}$ and

$$A_j = \left\{k \in \mathbb{N} : \frac{1}{j^\gamma} \le \frac{1}{k^\beta} < \frac{1}{(j-1)^\gamma}\right\}, \quad j = 2, 3, \ldots$$

The number of elements in A_j is

$$|A_j| \approx j^{\gamma/\beta} - (j-1)^{\gamma/\beta} \approx j^{\frac{\gamma}{\beta}-1}, \quad j = 1, 2, 3, \ldots \tag{7.9}$$

For $J = 2, 3, 4, \ldots$ let $N_J = \sum_{j=1}^J |A_j| + J$. From (7.9) we obtain

$$N_J \approx \sum_{j=1}^{J} j^{\frac{\gamma}{\beta}-1} + J \approx J^{\frac{\gamma}{\beta}} + J \approx J^{\frac{\gamma}{\beta}},$$

since $\gamma > \beta$. Thus,

$$\gamma_{N_J}(x+y) \approx \left(\sum_{k>J^{\frac{\gamma}{\beta}}} k^{-\beta p}\right)^{1/p} + \left(\sum_{j>J} j^{-\gamma q}\right)^{1/q} \approx \left[(J^{\gamma/\beta})^{-\beta p+1}\right]^{1/p} + \left[J^{-\gamma q+1}\right]^{1/q}$$
$$= J^{-\alpha\gamma/\beta} + J^{-\alpha} \approx J^{-\alpha} \approx (N_J)^{-\alpha\beta/\gamma},$$

proving (7.8).

A simple modification of the above construction can be used to show that the set $\mathcal{G}^\alpha_s(\mathcal{B}, \mathbb{B})$ is not linear, for any $\alpha > 0$ and any $s \in (0, \infty)$.

Note added in Proof C. Cabrelli and U. Molter have pointed out to us that the conditions in Proposition 7.1 hold for every $\alpha > 0$ as long as $\lim_{N\to\infty} h_r(N)/h_l(N) = \infty$, or even if one only assumes $\limsup_{N\to\infty} h_r(N)/h_l(N) = \infty$ and h_l doubling. A proof of these facts will appear elsewhere.

References

1. Aimar, H.A., Bernardis, A.L., Martín-Reyes, F.J.: Multiresolution approximation and wavelet bases of weighted Lebesgue spaces. J. Fourier Anal. Appl. **9**(5), 497–510 (2003)
2. Bennett, C., Sharpley, R.: Interpolation of operators. Academic Press Inc (1988)
3. Bergh, J., Löfström, J.: Interpolation spaces. An introduction, no. 223. Springer-Verlag, New York (1976)
4. Carro, M.J., Raposo, J., Soria, J.: Recent developments in the theory of Lorentz spaces and weighted inequalities. Mem. Am. Math. Soc. **877**, 187 (2007)
5. DeVore, R.A.: Nonlinear approximation. Acta Numer. **7**, 51–150 (1998)
6. DeVore, R., Konyagin, S., Temlyakov, V.: Hyperbolic wavelet approximation. Constr. Approx. **14**, 1–26 (1998)

7. DeVore, R., Popov, V.A.: Interpolation spaces and nonlinear approximation. Function Spaces and Applications (Lund, 1986). Lecture Notes in Math., vol. 1302. Springer, Berlin, pp. 191–205 (1988)
8. DeVore, R.A., Temlyakov, V.N.: Some remarks on greedy algorithms. Adv. Comput. Math. **5**(2–3), 113–187 (1996)
9. Dilworth, S.J., Kalton, N.J., Kutzarova, D., Temlyakov, V.N.: The thresholding greedy algorithm, greedy bases, and duality. Constr. Approx. **19**, 575–597 (2003)
10. Frazier, M., Jawerth, B.: A discrete transform and decomposition of distribution spaces. J. Funct. Anal. **93**, 34–170 (1990)
11. Garrigós, G., Hernández, E.: Sharp Jackson and Bernstein inequalities for n-term approximation in sequence spaces with applications. Indiana Univ. Math. J. **53**, 1739–1762 (2004)
12. Garrigós, G., Hernández, E., Martell, J.M.: Wavelets, Orlicz spaces and greedy bases. Appl. Comput. Harmon. Anal. **24**, 70–93 (2008)
13. Gribonval, R., Nielsen, M.: Some remarks on non-linear approximation with Schauder bases. East. J. Approx. **7**(2), 1–19 (2001)
14. Hernández, E., Martell, J.M., de Natividade, M.: Quantifying democracy of wavelet bases in Lorentz spaces. Constr. Approx. **33**, 1–14 (2011). doi:10.1007/s00365-010-9113-8
15. Hernández, E., Weiss, G.: A first course on wavelets. CRC Press, Boca Raton, FL (1996)
16. Hsiao, C., Jawerth, B., Lucier, B.J., Yu, X.M.: Near optimal compression of almost optimal wavelet expansions. Wavelets: Mathemathics and Applications, Stud. Adv. Math., vol. 133, pp. 425–446. CRC, Boca Raton, FL (1994)
17. Jawerth, B., Milman, M.: Wavelets and best approximation in Besov spaces. In: Interpolation Spaces and Related Topics (Haifa, 1990), pp. 107–112, Israel Math. Conf. Proc., vol. 5. Bar-Ilan University, Ramat Gan (1992)
18. Jawerth, B., Milman, M.: Weakly rearrangement invariant spaces and approximation by largest elements. In: Interpolation Theory and Applications, pp. 103–110, Contemp. Math., vol. 445. Amer. Math. Soc., Providence, RI (2007)
19. Kamont, A., Temlyakov, V.N.: Greedy approximation and the multivariate Haar system. Stud. Math. **161**(3), 199–223 (2004)
20. Kerkyacharian, G., Picard, D.: Entropy, universal coding, approximation, and bases properties. Constr. Approx. **20**, 1–37 (2004). doi:10.1007/s00365-003-0556-z
21. Kerkyacharian, G., Picard, D.: Nonlinear approximation and Muckenhoupt weights. Constr. Approx. **24**, 123–156 (2006). doi:10.1007/s00365-005-0618-5
22. Kyriazis, G. Multilevel characterization of anisotropic function spaces. SIAM J. Math. Anal. **36**, 441–462 (2004)
23. Konyagin, S.V., Temlyakov, V.N.: A remark on greedy approximation in Banach spaces. East. J. Approx. **5**, 365–379 (1999)
24. Krein, S., Petunin, J., Semenov, E.: Interpolation of Linear Operators. Translations Math. Monographs, vol. 55. Amer. Math. Soc., Providence (1992)
25. Merucci, C.: Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces. Interpolation Spaces and Allied Topics in Analysis. Lecture Notes in Math., vol. 1070, pp. 183–201. Springer, Berlin (1984)
26. Meyer, Y.: Ondelettes et Operateurs. I: Ondelettes. Hermann, Paris, (1990). [English translation: Wavelets and Operators. Cambridge University Press (1992)]
27. de Natividade, M.: Best approximation with wavelets in weighted Orlicz spaces. Monatsh. Math. **164**, 87–114 (2011). doi:10.1007/s00605-010-0244-6
28. Oswald, P.: Greedy algorithms and best m-term approximation with respect to biorthogonal systems. J. Fourier Anal. Appl. **7**(4), 325–341 (2001)
29. Pietsch, A.: Approximation spaces. J. Approx. Theory **32**, 113–134 (1981)
30. Rochberg, R., Taibleson, M.: An averaging operator on a tree. In: Harmonic analysis and partial differential equations (El Escorial, 1987), pp. 207–213, Lecture Notes in Math., vol. 1384. Springer, Berlin (1989)
31. Stechkin, S.B.: On absolute convergence of orthogonal series. Dokl. Akad. Nauk SSSR **102**, 37–40 (1955)
32. Soardi, P.: Wavelet bases in rearrangement invariant function spaces. Proc. Am. Math. Soc. **125**(12), 3669–3973 (1997)
33. Temlyakov, V.N.: The best m-term approximation and greedy algorithms. Adv. Comput. Math. **8**, 249–265 (1998)

34. Temlyakov, V.N.: Nonlinear m-term approximation with regard to the multivariate Haar system. East. J. Approx. **4**, 87–106 (1998)
35. Temlyakov, V.N.: Nonlinear methods of approximation. Found. Comput. Math. **3**(1), 33–107 (2003)
36. Temlyakov, V.N.: Greedy approximation. Acta Numer., vol. 17, pp. 235–409, Cambridge University Press (2008)
37. Wojstaszczyk, P.: The Franklin system is an unconditional basis in H^1. Ark. Mat. **20**, 293–300 (1982)
38. Wojstaszczyk, P.: Greedy algorithm for general biorthogonal systems. J. Approx. Theory **107**, 293–314 (2000)
39. Wojstaszczyk, P.: Greediness of the Haar system in rearrangement invariant spaces. Banach Cent. Publ., Warszawa **72**, 385–395 (2006)