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Democracy functions of wavelet bases in general Lorentz spaces

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Abstract

We compute the democracy functions associated with wavelet bases in general Lorentz spaces Λ_w^q and $\Lambda_w^{q,\infty}$, for general weights w and $0 < q < \infty$.

Keywords: Lorentz spaces; Greedy algorithm; Democratic bases; Wavelets

1. Introduction

The *Lorentz space* $\Lambda_w^q(\mathbb{R}^d)$ is defined as the set of all measurable $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{\Lambda_w^q} := \left[\int_0^\infty |f^*(t)|^q w(t)\mathrm{d}t\right]^{1/q} < \infty, \tag{1.1}$$

where f^* is the decreasing rearrangement of f (with respect to the Lebesgue measure) and w is a positive locally integrable function with the property $\int_0^\infty w(s)\mathrm{d}s = \infty$. We shall assume that $q \in (0, \infty)$.

Special examples include the classical $L^{p,q}(\mathbb{R}^d)$ spaces (corresponding to $w(t) = t^{\frac{q}{p}-1}$), and the so called Lorentz–Zygmund spaces $L^{p,q}(\log L)^r, r \in \mathbb{R}$, for which $w(t) =$

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$t^{\frac{q}{p}-1}(1+|\log t|)^{rq}$ (see [1]). More general weights w give rise to larger families such as the Lorentz–Karamata spaces, and various other examples considered in the literature (see e.g. [7]).

In this note we shall be interested in the efficiency of the greedy algorithm [9] for the N-term wavelet approximation of functions in Λ^q_w. It is known that greedy algorithms with wavelet bases are never optimal in rearrangement invariant spaces, except for the L^p classes; see [10]. However, it is possible to quantify the efficiency of the algorithm in a space $\mathbb{X}$ by computing the so called *lower and upper democracy functions*; that is,

$$h_\ell(N) = \inf_{\#\Gamma=N} \left\| \sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|} \right\|_{\mathbb{X}} \quad \text{and} \quad h_r(N) = \sup_{\#\Gamma=N} \left\| \sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|} \right\|_{\mathbb{X}}, \tag{1.2}$$

where $\{\psi_Q\}$ is a wavelet system indexed by the set $\mathcal{D}$ of all dyadic cubes of $\mathbb{R}^d$. Indeed, a precise expression for h_ℓ and h_r gives rise to optimal inclusions for the approximation classes $\mathcal{A}^\alpha_s(\mathbb{X})$ in terms of discrete Lorentz spaces (see [4]).

It is not always an easy matter to compute explicitly the democracy functions h_ℓ and h_r in non-democratic settings. We refer the reader to [3] for the case of Orlicz spaces L^Φ, and to [5] for the Lorentz spaces $L^{p,q}$. The objective of this note is to present the computation of h_ℓ and h_r for the larger family of general Lorentz spaces Λ^q_w.

As usual, using wavelet theory one can transfer the problem to the discrete setting. We define the space λ^q_w consisting of all sequences $\mathbf{s} = \{s_Q\}_{Q\in\mathcal{D}}$ such that

$$\|\mathbf{s}\|_{\lambda^q_w} := \left\| \left(\sum_{Q\in\mathcal{D}} |s_Q|^2 \frac{1}{|Q|} \chi_Q(\cdot) \right)^{1/2} \right\|_{\Lambda^q_w} < \infty. \tag{1.3}$$

It is known that sufficiently regular wavelet bases in $\mathbb{R}^d$ give an isomorphism between Λ^q_w and λ^q_w (when the Boyd indices of Λ^q_w are strictly between 0 and 1; see [8]). Thus studying the democracy of wavelet bases in Λ^q_w is equivalent to determining

$$h_\ell(N) = \inf_{\#\Gamma=N} \left\| \sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{\lambda^q_w}} \right\|_{\lambda^q_w} \quad \text{and} \quad h_r(N) = \sup_{\#\Gamma=N} \left\| \sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{\lambda^q_w}} \right\|_{\lambda^q_w},$$

where $\{\mathbf{e}_Q\}$ denotes the canonical basis in λ^q_w. We shall assume in the rest of the paper that h_ℓ and h_r always refer to these quantities (which are comparable to the ones in (1.2) for $\mathbb{X} = \Lambda^q_w$, at least when the wavelet characterization holds).

To state our results we need some notation. We denote the primitive of w by

$$W(t) := \int_0^t w(s)\mathrm{d}s, \quad t \geq 0.$$

Recall that Λ^q_w is quasi-normed if and only if W is doubling (see e.g. [2, 2.2.13]), so we shall always assume ourselves to be in this situation. Observe also that for all measurable sets $E \subset \mathbb{R}^d$ we have

$$\|\chi_E\|_{\Lambda^q_w} = W(|E|)^{1/q}.$$

That is, $W(t)^{1/q}$ is the fundamental function of the rearrangement invariant function space Λ^q_w. We shall denote by $H^{\pm}_W(t)$ the *dilation functions* associated with W, that is

$$H^+_W(t) := \sup_{s>0} \frac{W(ts)}{W(s)} \quad \text{and} \quad H^-_W(t) := \inf_{s>0} \frac{W(ts)}{W(s)}.$$

Since W is doubling these are finite functions. Observe also that $H^-(t) = 1/H^+(1/t)$. Finally we denote by i_W the lower dilation index of W (see [6] or (2.14) for a precise definition), which we typically assume to be positive. Our results can be stated as follows.

Theorem 1.4. *Assume $i_W > 0$. Then for all $N \in \mathbb{N}$ we have*

$$h_\ell(N) \approx \inf \left\{ \left(\sum_{j\in\mathbb{Z}} \frac{W(n_j 2^{jd})}{W(2^{jd})} \right)^{1/q} : n_j \in \mathbb{N} \cup \{0\} \text{ with } \sum_j n_j = N \right\} \tag{1.5}$$

and

$$h_r(N) \approx \sup \left\{ \left(\sum_{j\in\mathbb{Z}} \frac{W(n_j 2^{jd})}{W(2^{jd})} \right)^{1/q} : n_j \in \mathbb{N} \cup \{0\} \text{ with } \sum_j n_j = N \right\}, \tag{1.6}$$

where the constants involved in "≈" are independent of N.

Our second result gives a more explicit expression for weights which are *monotonic near* 0 *and* ∞, that is, in intervals $(0, a)$ and (b, ∞), for some $a \le b$. Observe that most examples arising in practice do actually satisfy this property.

Theorem 1.7. *Assume that w is monotonic near 0 and ∞, and that $i_W > 0$. Then for all $N \in \mathbb{N}$*

$$h_\ell(N) \approx \min \left\{N, H^-_W(N)\right\}^{1/q} \quad \text{and} \quad h_r(N) \approx \max \left\{N, H^+_W(N)\right\}^{1/q}. \tag{1.8}$$

In particular:

(a) *w increasing implies $h_\ell(N) \approx N^{1/q}$ and $h_r(N) \approx H^+_W(N)^{1/q}$;*
(b) *w decreasing implies $h_\ell(N) \approx H^-_W(N)^{1/q}$ and $h_r(N) \approx N^{1/q}$.*

Finally, we consider the weak versions of the Lorentz spaces Λ^q_w. We write $\Lambda^{q,\infty}_w(\mathbb{R}^d)$ for the set of all f such that

$$\|f\|_{\Lambda^{q,\infty}_w} := \sup_{t>0} t w\{f^* > t\}^{1/q} = \sup_{s>0} f^*(s) W(s)^{1/q} < \infty, \tag{1.9}$$

where $0 < q < \infty$. The corresponding sequence space $\lambda^{q,\infty}_w$ is defined as in (1.3) with $\Lambda^{q,\infty}_w$ in place of Λ^q_w. Then we have the following:

Theorem 1.10. *Assume $i_W > 0$. Then for all $N \in \mathbb{N}$ we have*

$$h_\ell(N; \lambda^{q,\infty}_w) \approx 1 \quad \text{and} \quad h_r(N; \lambda^{q,\infty}_w) \approx H^+_W(N)^{1/q}.$$

Section 2 contains some preliminaries about Λ^q_w spaces. The proofs of the theorems are presented, respectively, in Sections 3–5. Finally, Section 6 contains some examples.

2. Preliminaries

We need a few elementary properties for the spaces Λ^q_w. First of all, it is well known that the (quasi-)norm in Λ^q_w can also be written as

$$\|f\|_{\Lambda^q_w} = \left[\int_0^\infty qt^{q-1} W\left(\lambda_f(t)\right) \mathrm{d}t\right]^{1/q} \tag{2.1}$$

where $\lambda_f(t) = \text{ meas } \{x \in \mathbb{R}^d : |f(x)| \geq t\}$ (see e.g. [2, Prop 2.2.5]). From here it is clear that

$$f \leq g \Longrightarrow \|f\|_{\Lambda^q_w} \leq \|g\|_{\Lambda^q_w}. \tag{2.2}$$

We also need discretized versions of (2.1). Let $\mathbb{A}$ denote the collection of all sequences $\{a_j\}_{j=-\infty}^{\infty}$ of positive real numbers such that

$$\inf \frac{a_{j+1}}{a_j} > 1 \quad \text{and} \quad \sup \frac{a_{j+1}}{a_j} < \infty. \tag{2.3}$$

Clearly $\{a^j\}_{j\in\mathbb{Z}}$ with $a > 1$ satisfies these requirements, but we shall make use of more general examples later on. Observe that, in particular, the left condition in (2.3) implies

$$\lim_{j\to-\infty} a_j = 0 \quad \text{and} \quad \lim_{j\to+\infty} a_j = +\infty. \tag{2.4}$$

Lemma 2.5. *Let* $\{a_j\} \in \mathbb{A}$. *Then*

$$\|f\|_{\Lambda^q_w} \approx \left[\sum_{j\in\mathbb{Z}} a_j^q W\left(\lambda_f(a_j)\right)\right]^{1/q}. \tag{2.6}$$

Proof. Define $m = \inf \frac{a_{j+1}}{a_j}$ and $M = \sup \frac{a_{j+1}}{a_j}$. Then, from (2.1) we obtain

$$\begin{aligned}\|f\|^q_{\Lambda^q_w} &= \sum_{j\in\mathbb{Z}} \int_{a_j}^{a_{j+1}} qt^{q-1} W\left(\lambda_f(t)\right) \mathrm{d}t \leq \sum_{j\in\mathbb{Z}} \int_{a_j}^{a_{j+1}} qt^{q-1} \mathrm{d}t\, W\left(\lambda_f(a_j)\right) \\ &= \sum_{j\in\mathbb{Z}} \left(a^q_{j+1} - a^q_j\right) W\left(\lambda_f(a_j)\right) \leq (M^q - 1) \sum_{j\in\mathbb{Z}} a^q_j W\left(\lambda_f(a_j)\right).\end{aligned}$$

For the converse inequality one argues similarly:

$$\begin{aligned}\|f\|^q_{\Lambda^q_w} &\geq \sum_{j\in\mathbb{Z}} \left(a^q_{j+1} - a^q_j\right) W\left(\lambda_f(a_{j+1})\right) \\ &\geq (1 - m^{-q}) \sum_{j\in\mathbb{Z}} a^q_{j+1} W\left(\lambda_f(a_{j+1})\right). \quad \square\end{aligned}$$

In the next lemma we need to use the doubling property $W(2t) \leq cW(t)$. Since W is increasing, this property is equivalent to the subadditivity of W (with the same constant c):

$$W(s + t) \leq c(W(s) + W(t)), \quad \forall s, t > 0.$$

Denote by D_W the smallest such constant, that is

$$D_W = \sup_{s,t>0} \frac{W(s + t)}{W(s) + W(t)}. \tag{2.7}$$

Also, for a fixed $m > 1$, we shall denote by $\mathbb{A}_m$ the subset of all sequences in $\mathbb{A}$ with

$$\inf_{j\in\mathbb{Z}} \frac{a_{j+1}}{a_j} \geq m. \tag{2.8}$$

Lemma 2.9. *Let $\{a_j\} \in \mathbb{A}_m$ with $m > D_W^{1/q}$. If $f \in \Lambda_w^q$ then*

$$\|f\|_{\Lambda_w^q} \approx \left[\sum_{j\in\mathbb{Z}} a_j^q W\left(\lambda_f(a_j : a_{j+1})\right)\right]^{1/q}, \tag{2.10}$$

where $\lambda_f(a_j : a_{j+1}) =$ meas $\{x \in \mathbb{R}^d : a_j \leq |f(x)| < a_{j+1}\}$.

Proof. Using $\lambda_f(a_j) = \lambda_f(a_j : a_{j+1}) + \lambda_f(a_{j+1})$ and the subadditivity of W we obtain

$$W\left(\lambda_f(a_j)\right) \leq D_W\left[W\left(\lambda_f(a_j : a_{j+1})\right) + W\left(\lambda_f(a_{j+1})\right)\right]. \tag{2.11}$$

Define

$$I = \left(\sum_{j\in\mathbb{Z}} a_j^q W\left(\lambda_f(a_j)\right)\right)^{\frac{1}{q}} \quad \text{and} \quad II = \left(\sum_{j\in\mathbb{Z}} a_j^q W\left(\lambda_f(a_j : a_{j+1})\right)\right)^{\frac{1}{q}}.$$

Clearly $II \leq I$. For the converse, using (2.11) and $\inf a_{j+1}/a_j \geq m$, we see that

$$\begin{aligned} I^q &\leq D_W II^q + D_W \sum_{j\in\mathbb{Z}} a_j^q W\left(\lambda_f(a_{j+1})\right) \\ &\leq D_W II^q + D_W \sum_{j\in\mathbb{Z}} \frac{a_{j+1}^q}{m^q} W\left(\lambda_f(a_{j+1})\right) = D_W II^q + \frac{D_W}{m^q} I^q. \end{aligned}$$

Since we are assuming that $m^q > D_W$ it follows that

$$\left(1 - \frac{D_W}{m^q}\right) I^q \leq D_W II^q.$$

Thus $I \approx II$ and the result follows from Lemma 2.5. □

A similar argument gives:

Lemma 2.12. *Let $\{a_j\} \in \mathbb{A}_m$ with $m > D_W^{1/q}$. If $f \in \Lambda_w^{q,\infty}$ then*

$$\|f\|_{\Lambda_w^{q,\infty}} \approx \sup_{j\in\mathbb{Z}} a_j W\left(\lambda_f(a_j : a_{j+1})\right)^{1/q}. \tag{2.13}$$

Recall from [6, p. 53] that the *lower dilation index of W* is defined by

$$i_W := \sup_{0<t<1} \frac{\log H_W^+(t)}{\log t} = \lim_{t\to 0} \frac{\log H_W^+(t)}{\log t} = \lim_{u\to\infty} \frac{\log H_W^-(u)}{\log u}. \tag{2.14}$$

In this paper we will assume that $i_W > 0$, which implies that for all $\epsilon > 0$

$$W(su) \geq C_\epsilon u^{i_W - \epsilon} W(s), \quad \forall s > 0,\ \forall u \geq 1, \tag{2.15}$$

for some $C_\epsilon > 0$. In Section 3 we shall be interested in applying Lemma 2.9 to the sequence $a_j = W(2^{-jd})^{-1/q}$. This sequence clearly satisfies (2.4) (since we assume that $\int_0^\infty w(s)\mathrm{d}s = \infty$), but the validity of (2.3) depends on the growth of W. We show below how to handle this under the assumption $i_W > 0$.

Proposition 2.16. *Assume that $i_W > 0$. Then the norm equivalences in (2.6), (2.10) and (2.13) hold for the sequence*

$$a_j = \frac{1}{W(2^{-jd})^{1/q}}, \quad j \in \mathbb{Z}.$$

The proposition will be an easy consequence of the following lemma.

Lemma 2.17. *Assume that $i_W > 0$ and fix $m > D_W^{1/q}$. Then there exists $L_0 \in \mathbb{N}$ such that for every subsequence $\{k_j\}_{j\in\mathbb{Z}}$ with the property*

$$k_{j+1} = k_j + L_0, \quad \forall\, j \in \mathbb{Z},$$

the sequence $\{W(2^{-k_j d})^{-1/q}\}_{j\in\mathbb{Z}}$ belongs to $\mathbb{A}_m$.

Proof. Define $b_j = W(2^{-k_j d})^{-1/q}$. By the monotonicity of W and (2.15) we see that

$$\left(\frac{b_{j+1}}{b_j}\right)^q = \frac{W(2^{-k_j d})}{W(2^{-d(k_j+L_0)})} \geq C_\epsilon \left(2^{dL_0}\right)^{i_W-\epsilon}.$$

It suffices to choose $\epsilon = i_W/2$ and L_0 large enough so that the right hand side is $\geq m^q$. The bound from above follows from the doubling property of W. □

Proof of Proposition 2.16. We shall only prove (2.10), since the other cases are similar. Let L_0 be as in the previous lemma. Then, for each $r \in \{0, \ldots, L_0 - 1\}$, the sequence $\mathbf{a}^{(r)} = \left\{a_{jL_0+r} = W(2^{-(jL_0+r)d})^{-1/q}\right\}_{j\in\mathbb{Z}}$ belongs to $\mathbb{A}_m$. Thus, for each such r, Lemma 2.9 implies that

$$\|f\|_{\Lambda_w^q} \approx \left[\sum_{j\in\mathbb{Z}} a_{jL_0+r}^q W\left(\lambda_f(a_{jL_0+r} : a_{(j+1)L_0+r})\right)\right]^{1/q}, \tag{2.18}$$

for every $f \in \Lambda_w^q$. We first show the inequality "$\lesssim$" for which we choose $r = 0$ in (2.18). By the subadditivity of W, there is a constant $C = C(W, L_0)$ such that

$$W\left(\lambda_f(a_{jL_0} : a_{(j+1)L_0})\right) \leq C \sum_{s=0}^{L_0-1} W\left(\lambda_f(a_{jL_0+s} : a_{jL_0+s+1})\right).$$

Inserting this into (2.18) (with $r = 0$) and using $a_{jL_0} \approx a_{jL_0+s}$ (by the doubling property of W) we easily obtain

$$\|f\|_{\Lambda_w^q}^q \lesssim \sum_{s=0}^{L_0-1} \sum_{j\in\mathbb{Z}} a_{jL_0+s}^q W\left(\lambda_f(a_{jL_0+s} : a_{jL_0+s+1})\right) = \sum_{k\in\mathbb{Z}} a_k^q W\left(\lambda_f(a_k : a_{k+1})\right).$$

Conversely, since L_0 is a finite constant, (2.18) implies that

$$\|f\|_{\Lambda_w^q}^q \approx \sum_{r=0}^{L_0-1} \sum_{j\in\mathbb{Z}} a_{jL_0+r}^q W\left(\lambda_f(a_{jL_0+r} : a_{(j+1)L_0+r})\right)$$

$$\gtrsim \sum_{r=0}^{L_0-1} \sum_{j\in\mathbb{Z}} a^q_{jL_0+r} W\left(\lambda_f(a_{jL_0+r} : a_{jL_0+r+1})\right) = \sum_{k\in\mathbb{Z}} a^q_k W\left(\lambda_f(a_k : a_{k+1})\right). \quad \square$$

Finally we state a key "linearization" lemma which holds when $i_W > 0$.

Lemma 2.19. *Suppose $i_W > 0$. For every finite collection $\Gamma \subset \mathcal{D}$, and every $x \in \cup_{Q\in\Gamma} Q$, it holds that*

$$\left(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{W(|Q|)^{\frac{2}{q}}}\right)^{1/2} \approx \frac{\chi_{Q_x}(x)}{W(|Q_x|)^{\frac{1}{q}}} \tag{2.20}$$

where Q_x denotes the smallest cube in Γ containing x.

Such linearization arguments have been used by various authors in the context of N-term wavelet approximation. For an elementary proof and references see e.g. [3, Section 4.2.1].

3. Proof of Theorem 1.4

Let $\Gamma \subset \mathcal{D}$ with $\#\Gamma = N$. We use the notation

$$1_\Gamma = \sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{\lambda^q_w}} \quad \text{and} \quad S_\Gamma(x) = \left(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{W(|Q|)^{\frac{2}{q}}}\right)^{1/2}.$$

Observe from (1.3) that

$$\|\mathbf{e}_Q\|_{\lambda^q_w} = |Q|^{-1/2}\|\chi_Q\|_{\Lambda^q_w} = |Q|^{-1/2}W(|Q|)^{1/q},$$

so we are led to estimate the expression

$$\|1_\Gamma\|_{\lambda^q_w} = \left\| \left(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{W(|Q|)^{\frac{2}{q}}}\right)^{1/2} \right\|_{\Lambda^q_w} = \|S_\Gamma\|_{\Lambda^q_w}.$$

Using (2.6) we see that

$$\|1_\Gamma\|_{\lambda^q_w} \approx \left[\sum_{j\in\mathbb{Z}} a^q_j W\left(|\{|S_\Gamma| \ge a_j\}|\right)\right]^{1/q}.$$

We choose $a_j = W(2^{-jd})^{-1/q}$ and define $\Gamma_j = \{Q \in \Gamma : |Q| = 2^{-jd}\}$, $j \in \mathbb{Z}$. Clearly $S_\Gamma(x) \ge a_j$ for all $x \in \cup_{Q\in\Gamma_j} Q$, which implies

$$\|1_\Gamma\|_{\lambda^q_w} \gtrsim \left[\sum_{j\in\mathbb{Z}} \frac{W\left(|\cup_{Q\in\Gamma_j} Q|\right)}{W(2^{-jd})}\right]^{1/q} = \left[\sum_{j\in\mathbb{Z}} \frac{W(2^{-jd}\#\Gamma_j)}{W(2^{-jd})}\right]^{1/q}.$$

For the estimate from above we use Lemma 2.19 and denote by $F_\Gamma(x)$ the function on the right hand side of (2.20). Then (2.10) gives

$$\|1_\Gamma\|_{\lambda^q_w} \approx \|F_\Gamma\|_{\Lambda^q_w} \approx \left[\sum_{j\in\mathbb{Z}} a^q_j W\left(|\{a_j \le |F_\Gamma| < a_{j+1}\}|\right)\right]^{1/q},$$

where as before we set $a_j = W(2^{-jd})^{-1/q}$. Then the condition $a_j \le F_\Gamma(x) < a_{j+1}$ implies that $x \in \cup_{Q\in\Gamma_j} Q$, and therefore

$$\|1_\Gamma\|_{\lambda^q_w} \lesssim \left[\sum_{j\in\mathbb{Z}} \frac{W(2^{-jd}\#\Gamma_j)}{W(2^{-jd})}\right]^{1/q}.$$

We conclude that

$$\|1_\Gamma\|_{\lambda^q_w} \approx \left[\sum_{j\in\mathbb{Z}} \frac{W(2^{-jd}\#\Gamma_j)}{W(2^{-jd})}\right]^{1/q}, \tag{3.1}$$

and since $\sum \#\Gamma_j = \#\Gamma = N$, this clearly implies (1.6).

4. Proof of Theorem 1.10

The proof for the spaces $\Lambda^{q,\infty}_w$ is similar. First observe from the norm definitions that

$$\|\mathbf{e}_Q\|_{\lambda^{q,\infty}_w} = |Q|^{-1/2}\|\chi_Q\|_{\Lambda^{q,\infty}_w} = |Q|^{-1/2} W(|Q|)^{1/q},$$

so we are led to estimate the expression

$$\|1_\Gamma\|_{\lambda^{q,\infty}_w} = \left\| \left(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{W(|Q|)^{\frac{2}{q}}}\right)^{1/2} \right\|_{\Lambda^{q,\infty}_w} = \|S_\Gamma\|_{\Lambda^{q,\infty}_w}.$$

The lower bound $h_\ell(N) \ge 1$ is trivial. To see the optimality, choose Γ formed by pairwise disjoint cubes all of different sizes. Using (2.13) with $a_j = W(2^{-jd})^{-1/q}$ we easily see that

$$\|1_\Gamma\|_{\lambda^{q,\infty}_w} \approx \sup_{j\in\mathbb{Z}} a_j W\left(|\{a_j \le S_\Gamma(x) < a_{j+1}\}|\right)^{1/q} = 1,$$

which proves the assertion.

To obtain bounds for $h_r(N)$, we use again (2.13) with $a_j = W(2^{-jd})^{-1/q}$, together with Lemma 2.19, so

$$\begin{aligned}\|1_\Gamma\|_{\lambda^{q,\infty}_w} &\approx \sup_{j\in\mathbb{Z}} a_j W\left(|\{a_j \le F_\Gamma(x) < a_{j+1}\}|\right)^{1/q} \le \sup_{j\in\mathbb{Z}} \left[\frac{W(2^{-jd}\#\Gamma_j)}{W(2^{-jd})}\right]^{\frac{1}{q}} \\ &\le \sup_{j\in\mathbb{Z}} H^+_W(\#\Gamma_j)^{1/q} \le H^+_W(N)^{1/q}.\end{aligned}$$

This proves that $h_r(N) \lesssim H^+_W(N)^{1/q}$. For the converse, choose Γ consisting of N pairwise disjoint cubes all of the same size, say s_0. Then,

$$\|1_\Gamma\|_{\lambda^{q,\infty}_w} = \left\| \frac{1}{W(s_0)^{1/q}} \chi_{\cup_{Q\in\Gamma} Q} \right\|_{\Lambda^{q,\infty}_w} = \frac{W(Ns_0)^{1/q}}{W(s_0)^{1/q}}.$$

We can select s_0 such that the last quantity is comparable to $H^+_W(N)^{1/q}$, concluding the proof.

5. Proof of Theorem 1.7

We say that W is of type (A) if for some $c \geq 0$ and $C > 0$ it holds that

$$\begin{cases} \dfrac{W(t_0)}{t_0} \leq C\dfrac{W(t_1)}{t_1}, & \text{for } 0 < t_0 < t_1 \leq 2c \qquad (\mathrm{A}_1) \\ \dfrac{W(t_1)}{t_1} \leq C\dfrac{W(t_0)}{t_0}, & \text{for } c/2 < t_0 < t_1 < \infty. \quad (\mathrm{A}_2) \end{cases}$$

We say that W is of type (B) if for some $c \geq 0$ and $C > 0$,

$$\begin{cases} \dfrac{W(t_1)}{t_1} \leq C\dfrac{W(t_0)}{t_0}, & \text{for } 0 < t_0 < t_1 \leq 2c \qquad (\mathrm{B}_1) \\ \dfrac{W(t_0)}{t_0} \leq C\dfrac{W(t_1)}{t_1}, & \text{for } c/2 < t_0 < t_1 < \infty. \quad (\mathrm{B}_2) \end{cases}$$

These conditions can easily be phrased in terms of convexity of W. Namely, when $c > 0$, type (A) is the same as W being quasi-convex for small t and quasi-concave for large t, and similarly for type (B), with opposite convexities in W. Observe that the exact value of the constant $c > 0$ is irrelevant, since we are assuming that W is doubling. By allowing the case $c = 0$ we consider also the situations when W is everywhere quasi-concave (type A), or everywhere quasi-convex (type B) in the half-line $(0, \infty)$.

Lemma 5.1. *If w is monotonic near 0 and ∞, then W is either of type* (A) *or of type* (B) *for some $c \geq 0$.*

Proof. The proof is standard, using the inequalities

$$\min\left\{\frac{x}{u}, \frac{y}{v}\right\} \leq \frac{x+y}{u+v} \leq \max\left\{\frac{x}{u}, \frac{y}{v}\right\}, \qquad x, y, u, v > 0.$$

Indeed, assume that w is increasing in $(0, a)$. Then for $0 < t_0 < t_1 < a$,

$$\frac{W(t_1)}{t_1} = \frac{\int_0^{t_0} w(s)\mathrm{d}s + \int_{t_0}^{t_1} w(s)\mathrm{d}s}{t_0 + (t_1 - t_0)} \geq \min\left\{\frac{1}{t_0}\int_0^{t_0} w(s)\mathrm{d}s, \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} w(s)\mathrm{d}s\right\} = \frac{W(t_0)}{t_0},$$

where in the last step we use that, by the monotonicity of w,

$$\frac{1}{t_1 - t_0}\int_{t_0}^{t_1} w(s)\mathrm{d}s \geq w(t_0) \geq \frac{1}{t_0}\int_0^{t_0} w(s)\mathrm{d}s.$$

Similarly, if we assume that w is decreasing in (b, ∞) then for $t_1 > t_0$,

$$\frac{W(t_1)}{t_1} \leq \max\left\{\frac{1}{t_0}\int_0^{t_0} w(s)\mathrm{d}s, \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} w(s)\mathrm{d}s\right\},$$

so if we take $t_0 > 2b$ the monotonicity of w gives

$$\frac{1}{t_1 - t_0}\int_{t_0}^{t_1} w(s)\mathrm{d}s \leq w(t_0) \leq \frac{1}{t_0 - b}\int_b^{t_0} w(s)\mathrm{d}s \leq \frac{2}{t_0}\int_0^{t_0} w(s)\mathrm{d}s = 2\frac{W(t_0)}{t_0}.$$

Using the doubling property of W, these inequalities can be extended respectively to the larger intervals $(0, 4b)$ and $(a/4, \infty)$, perhaps with multiplicative constants, from which it follows that W is of type (A). The other cases are proved similarly. □

The main result in this section is the following.

Proposition 5.2. *Assume that W is of type* (A) *or* (B) *for some $c \geq 0$. Then for all N and $n_j \in \mathbb{N} \cup \{0\}$ such that $\sum_{j\in\mathbb{Z}} n_j = N$ we have*

$$\min\left\{N, H_W^-(N)\right\} \lesssim \sum_{j\in\mathbb{Z}} \frac{W(n_j 2^{jd})}{W(2^{jd})} \lesssim \max\left\{N, H_W^+(N)\right\}, \tag{5.3}$$

with the constants involved independent on N and n_j.

Observe that the upper and lower bounds in (5.3) are best possible. Indeed, taking all $n_j \in \{0, 1\}$ the middle expression is exactly equal to N. On the other hand, taking $n_{j_0} = N$ and $n_j = 0$ for $j \neq j_0$, an appropriate choice of j_0 makes the middle expression comparable to $H_W^{\pm}(N)$. Thus, Theorem 1.7 is a consequence of Theorem 1.4 and Proposition 5.2 (see also Remarks 5.6 and 5.7).

5.1. *Proof of Proposition 5.2*

Assume first that W is of type (A) for some $c > 0$. For simplicity, throughout the proof we shall write $\lambda_j = 2^{jd}$. Define the sets of indices

$$J_+ = \left\{j \in \mathbb{Z}: n_j\lambda_j \geq c/2\right\} \quad \text{and} \quad J_- = \left\{j \in \mathbb{Z}: n_j\lambda_j < c/2\right\}. \tag{5.4}$$

Then using (A_2) in the first inequality,

$$C \sum_{j\in J_+} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \geq \sum_{j\in J_+} \frac{n_j W(N\lambda_j)}{NW(\lambda_j)} \geq H^-(N) \sum_{j\in J_+} n_j/N.$$

Similarly, using (A_1) one obtains

$$C \sum_{j\in J_-} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \geq \sum_{j\in J_-} n_j.$$

Since either $\sum_{j\in J_+} n_j \geq N/2$ or $\sum_{j\in J_-} n_j \geq N/2$, it follows that

$$\sum_{j\in\mathbb{Z}} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \geq \frac{1}{2C} \min\left\{N, H^-(N)\right\}.$$

To prove the upper bounds we need three sets of indices:

$$J_a = \left\{j: \lambda_j \geq c\right\}, \qquad J_b = \left\{j: \lambda_j < c/N\right\}, \qquad J_c = \left\{j: c/N \leq \lambda_j < c\right\}. \tag{5.5}$$

As before, using respectively (A_2) and (A_1) we see that

$$\sum_{j\in J_a} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \leq C \sum_{j\in J_a} n_j \quad \text{and}$$

$$\sum_{j\in J_b} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \leq C \sum_{j\in J_b} \frac{n_j W(N\lambda_j)}{NW(\lambda_j)} \leq CH^+(N) \sum_{j\in J_b} n_j/N.$$

For indices $j \in J_c$ we use the cruder estimate

$$\sup_{t>0} W(t)/t \le C W(c)/c,$$

which together with (A_1) in the second step leads to

$$\sum_{j\in J_c} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \le C \sum_{j\in J_c} \frac{n_j\lambda_j W(c)}{cW(\lambda_j)} \le C^2 \sum_{j\in J_c} \frac{n_j W(c)}{NW(c/N)} \le C^2 H^+(N) \sum_{j\in J_c} n_j/N.$$

Combining the three cases we see that

$$\sum_{j\in\mathbb{Z}} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \le C^2 \left(N + H^+(N)\right) \lesssim \max\left\{N, H^+(N)\right\}.$$

Remark 5.6. The proof just given is also valid for W of type (A) with $c = 0$. In fact, in this case the sets J_-, J_b and J_c are empty, so one actually obtains

$$H^-(N) \lesssim \sum_{j\in\mathbb{Z}} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \lesssim N.$$

This corresponds to the case of w decreasing, as stated in (b) of Theorem 1.7.

We now turn to the case where W is of type (B), assuming for simplicity $c > 0$. Using the same sets $J_\pm$ as in (5.4) together with (B_2) and (B_1), respectively, we obtain

$$\sum_{j\in J_+} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \le C \sum_{j\in J_+} \frac{n_j W(N\lambda_j)}{NW(\lambda_j)} \le C H^+(N) \sum_{j\in J_+} n_j/N \quad \text{and}$$

$$\sum_{j\in J_-} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \le C \sum_{j\in J_-} n_j.$$

Summing, we get

$$\sum_{j\in\mathbb{Z}} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \le 2C \max\left\{N, H^+(N)\right\}.$$

We turn to the lower bound, for which we use the sets J_a, J_b and J_c in (5.5). As before, the first two sets are easily handled with (B_2) and (B_1):

$$C \sum_{j\in J_a} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \ge \sum_{j\in J_a} n_j \quad \text{and}$$

$$C \sum_{j\in J_b} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \ge \sum_{j\in J_b} \frac{n_j W(N\lambda_j)}{NW(\lambda_j)} \ge H^-(N) \sum_{j\in J_b} n_j/N.$$

For indices $j \in J_c$ we use

$$C \inf_{t>0} W(t)/t \ge W(c)/c,$$

which together with (B_1) in the second step leads to

$$C \sum_{j\in J_c} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \ge \sum_{j\in J_c} \frac{n_j\lambda_j W(c)}{cW(\lambda_j)} \ge \frac{1}{C} \sum_{j\in J_c} \frac{n_j W(c)}{NW(c/N)} \ge \frac{1}{C} H^-(N) \sum_{j\in J_c} n_j/N.$$

Now, since either $\sum_{j\in J_a} n_j \geq N/2$ or $\sum_{j\in J_b\cup J_c} n_j \geq N/2$, it follows that

$$\sum_{j\in\mathbb{Z}} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \geq \frac{1}{2C^2} \min\left\{N, H^-(N)\right\}.$$

Remark 5.7. As before, the proof is also valid for $c = 0$; we obtain in this case

$$N \lesssim \sum_{j\in\mathbb{Z}} \frac{W(n_j\lambda_j)}{W(\lambda_j)} \lesssim H^+(N).$$

This corresponds to the situation of w increasing, as stated in (a) of Theorem 1.7.

6. Examples

We illustrate some examples of Lorentz weights to which the results of Theorem 1.7 can be applied. Consider the following general class of weights:

$$w(t) = \begin{cases} t^{\alpha_0-1}\left[\log\left(e+1/t\right)\right]^\beta, & 0 < t \leq 1 \\ t^{\alpha_1-1}\left[\log(e+t)\right]^\gamma, & t \geq 1 \end{cases}$$

where $\alpha_0, \alpha_1 > 0$ and $\beta, \gamma \in \mathbb{R}$. These are typical examples of piecewise monotonic weights with different behaviors near 0 and ∞. Observe that

$$W(t) \approx \begin{cases} t^{\alpha_0}\left[\log\left(e+1/t\right)\right]^\beta, & 0 < t \leq 1 \\ t^{\alpha_1}\left[\log(e+t)\right]^\gamma, & t \geq 1. \end{cases}$$

From this expression it is not difficult to compute $H^\pm_W(N)$. Indeed, a straightforward (but slightly tedious) calculation gives:

(a) if $a_0 < \alpha_1$ then $H^-(N) \approx N^{\alpha_0}/[\log(e+N)]^{\beta_+}$ and $H^+(N) \approx N^{\alpha_1}[\log(e+N)]^{\gamma_+}$;
(b) if $a_0 = \alpha_1$ then $H^-(N) \approx N^{\alpha_0}/[\log(e+N)]^{\beta_+ + \gamma_-}$ and $H^+(N) \approx N^{\alpha_0}[\log(e+N)]^{\beta_- + \gamma_+}$;
(c) if $a_0 > \alpha_1$ then $H^-(N) \approx N^{\alpha_1}/[\log(e+N)]^{\gamma_-}$ and $H^+(N) \approx N^{\alpha_0}[\log(e+N)]^{\beta_-}$;

where for a real number x we denote

$$x_+ = \begin{cases} |x|, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad \text{and} \quad x_- = \begin{cases} 0, & \text{if } x \geq 0 \\ |x|, & \text{if } x < 0. \end{cases}$$

See e.g. [3, Section 3] for similar examples. In particular, setting $\alpha_0 = \alpha_1 = q/p$ and $\beta = \gamma = rq$ we obtain for the Lorentz–Zygmund spaces $L^{p,q}(\log L)^r$

$$h_\ell(N) \approx \min\left\{N^{\frac{1}{q}}, N^{\frac{1}{p}}\left[\log(e+N)\right]^{-|r|}\right\} \quad \text{and}$$
$$h_r(N) \approx \max\left\{N^{\frac{1}{q}}, N^{\frac{1}{p}}\left[\log(e+N)\right]^{|r|}\right\}.$$

When $r = 0$ we recover the results for the classical $L^{p,q}$ spaces from [5].

A second class of weights to which Theorem 1.7 is applicable is

$$w(t) = t^{\alpha-1}\exp(|\ln t|^\delta), \quad \alpha > 0 \quad \text{and} \quad \delta \in (0, 1).$$

Observe that the functions $\exp(|\ln t|^\delta)$ grow faster than $|\ln t|^N$ for all N but are smaller than any power t^ε (for t near ∞) or $1/t^\varepsilon$ (for t near 0). It is not difficult to see that[1]

$$W(t) \approx t^\alpha \exp(|\ln t|^\delta). \tag{6.1}$$

From this, one easily computes

$$H_W^+(t) \approx t^\alpha \mathrm{e}^{|\ln t|^\delta} \quad \text{and} \quad H_W^-(t) \approx t^\alpha \mathrm{e}^{-|\ln t|^\delta}, \quad t > 0.$$

In particular, if $\alpha = q/p$ we obtain for the corresponding space Λ_w^q

$$h_\ell(N) \approx \min\left\{N^{\frac{1}{q}}, N^{\frac{1}{p}} \mathrm{e}^{-\frac{|\ln N|^\delta}{q}}\right\} \quad \text{and} \quad h_r(N) \approx \max\left\{N^{\frac{1}{q}}, N^{\frac{1}{p}} \mathrm{e}^{\frac{|\ln N|^\delta}{q}}\right\}.$$

Observe that these spaces Λ_w^q are contained in all the Lorentz–Zygmund spaces $L^{p,q}(\log L)^r$ for all $r > 0$ (and hence also in $L^{p,q}$).

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[1] In fact, if $i_W > 0$ it is always true that $W(t) \approx \int_0^t W(s)s^{-1}\mathrm{d}s$; see e.g. [6, p. 57].