CONDITIONALITY CONSTANTS OF QUASI-GREEDY BASES IN SUPER-REFLEXIVE BANACH SPACES

G. GARRIGÓS, E. HERNÁNDEZ, AND M. RAJA

Abstract. We show that in a super-reflexive Banach space, the conditionality constants of a quasi-greedy basis $\mathscr{B}$ grow at most like $k_N(\mathscr{B}) = O(\log N)^{1-\varepsilon}$, for some $\varepsilon > 0$. This extends results by the first author and Wojtaszczyk [11], where this property was shown for quasi-greedy bases in L^p when $1 < p < \infty$.

1. Introduction

Let $\mathbb{X}$ be a Banach space with a countable Schauder basis $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^\infty$, which we shall assume semi-normalized, that is $c_1 \le \|\mathbf{e}_j\| \le c_2$ for some constants $c_2 \ge c_1 > 0$. For $x \in \mathbb{X}$ we write the corresponding basis expansion as $x = \sum_{j=1}^\infty a_j(x)\mathbf{e}_j$.

Associated with $\mathscr{B}$, we consider, for each finite $A \subset \mathbb{N}$, the projection operators

$$x \in \mathbb{X} \longmapsto S_A(x) := \sum_{j\in A} a_j(x)\mathbf{e}_j,$$

and define the sequence

$$k_N = k_N(\mathscr{B}, \mathbb{X}) := \sup_{|A|\le N} \big\|S_A\big\|, \quad N = 1, 2, \ldots$$

Notice that $\mathscr{B}$ is unconditional if and only if $k_N = O(1)$. In general, k_N may grow as fast as $O(N)$, and this sequence may be used to quantify the conditionality of the basis $\mathscr{B}$ in $\mathbb{X}$. It is a consequence of a classical result of Gurarii-Gurarii [12] and James [17] that if $\mathbb{X}$ is a super-reflexive Banach space (ie, isomorphic to a uniformly convex or a uniformly smooth space), then

$$k_N = O(N^{1-\varepsilon}), \quad \text{for some } \varepsilon > 0.$$

In this paper we shall be interested in bases $\mathscr{B}$ which are *quasi-greedy* [19, 25], that is their expansions converge when the summands are rearranged in decreasing order. More precisely, if we define N^{th}-order greedy operators by

$$x \in \mathbb{X} \longmapsto G_N(x) = \sum_{j\in \Lambda_N(x)} a_j(x)\mathbf{e}_j, \tag{1.1}$$

where $\Lambda_N(x)$ is a set of cardinality N such that $\min_{j\in\Lambda_N(x)} |a_j| \ge \max_{j\notin\Lambda_N(x)} |a_j|$, then $\{\mathbf{e}_j\}$ is a *quasi-greedy basis* when $G_N(x) \to x$, for all $x \in \mathbb{X}$. We refer to [22] for background and applications of quasi-greedy bases in the study of non-linear N-term approximation in Banach spaces.

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It follows from a result of Dilworth, Kalton and Kutzarova [6, Lemma 8.2] that quasi-greedy bases cannot be not "too conditional", namely they satisfy

$$k_N(\mathscr{B},\mathbb{X}) = O(\log N), \tag{1.2}$$

see also [8, 10]. Moreover, there are examples of quasi-greedy bases in certain Banach spaces for which the logarithmic growth is actually attained; see [10, §6].

More recently, it was noticed in [11] that (1.2) can be improved to $k_N = O(\log N)^{1-\varepsilon}$, for some $\varepsilon > 0$, at least when $\mathbb{X} = L^p$ and $1 < p < \infty$. The purpose of this note is to show that this improvement continues to hold for any super-reflexive Banach space.

Theorem 1.1. *Let $\mathbb{X}$ be a super-reflexive Banach space, and $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^\infty$ a quasi-greedy basis. Then, there exists some $\varepsilon = \varepsilon(\mathscr{B},\mathbb{X}) > 0$ such that*

$$k_N(\mathscr{B},\mathbb{X}) \;=\; O(\log N)^{1-\varepsilon}.$$

We remark that bounds on the sequence k_N are useful in N-term approximation. In particular, if $\mathscr{B}$ is an *almost-greedy basis* in $\mathbb{X}$, in the sense of [7] (ie, quasi-greedy and democratic), then k_N quantifies the performance of the greedy algorithm versus the best N-term approximation. More precisely, if $\Sigma_N = \{\sum_{\lambda\in\Lambda} c_\lambda \mathbf{e}_\lambda : \ \mathrm{Card}\,\Lambda \le N\}$, we have the following

Corollary 1.2. *Let $\mathbb{X}$ be super-reflexive and $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^\infty$ an almost-greedy basis. Then, there exists $\varepsilon = \varepsilon(\mathscr{B},\mathbb{X}) > 0$ and $c > 0$ such that*

$$\|x - G_N x\| \;\le\; c\,(\log N)^{1-\varepsilon}\,\mathrm{dist}\,(x,\Sigma_N)$$

for all $x \in \mathbb{X}$ and $N = 2, 3, \ldots$

This is a direct consequence of Theorem 1.1 and [23, Thm 2.1] (or [10, Thm 1.1]).

We conclude by recalling some examples where *super-reflexivity* occurs. This is a well known property in Functional Analysis, satisfied by all Banach spaces with an equivalent norm which is either uniformly convex or uniformly smooth [17, 9]. In particular, this is the case for $L^p(\mu)$ with $1 < p < \infty$ over any measure space, but also for most examples of reflexive Banach spaces arising in harmonic and functional analysis. Here we list some of them:

(i) Bochner-Lebesgue spaces $L_p(\mu, X)$ over any measure space are uniformly convex if X is uniformly convex and $1 < p < \infty$. As a consequence, a space $L_p(\mu, X)$ and its subspaces inherit the super-reflexivity from X. That covers the classical families of Sobolev, Besov and Triebel-Lizorkin spaces in $\mathbb{R}^n$ for a wide range of parameters, exactly the ones making them reflexive. The isomorphic embedding into a space of the form $L_p(\mu, L_q(\nu))$ comes from their very definition, see [24], but it is also possible to show isomorphisms with the help of special bases, see for instance [4] for certain Sobolev and Besov spaces.

(ii) Orlicz spaces satisfying Luxemburg's characterizations of reflexivity [20] are super-reflexive; see [1]. We note that Luxemburg assumptions on the measure cover the most usual cases, as Orlicz sequence spaces or function spaces on $\mathbb{R}^n$ with the Lebesgue measure.

(iii) Super-reflexivity has also been studied in Lorentz-type spaces, where its characterization is very close to reflexivity, see for instance [13, 18, 15].

(iv) Uniformly non-square Banach spaces are also super-reflexive. These spaces, introduced by James in [16], are those that satisfy

$$\sup\{\min\{\|x+y\|,\|x-y\|\}:\|x\|=\|y\|=1\}<2.$$

(v) Super-reflexivity is preserved as well by certain operations to produce new spaces such as finite products, quotients, ultrapowers and interpolation. In fact, if one of the spaces of the interpolation pair is super-reflexive then all the intermediate spaces are super-reflexive, either with the real [3] or the complex method [5].

2. Proof of Theorem 1.1

The proof will follow the arguments sketched in [11, §5]. All we shall need from the space $\mathbb{X}$ is a weak variant of the paralellogram law.

Now, assume that $\mathbb{X}$ is a super-reflexive Banach space. As we said above, this notion was introduced by James [17] and has several equivalent formulations, one of which being the existence of an equivalent norm $|||\cdot|||$ in $\mathbb{X}$ which is uniformly convex; see [9]. Moreover, a well-known result of Pisier [21] shows that $|||\cdot|||$ can be chosen so that its associated modulus of convexity

$$\delta(\varepsilon)=\inf\Big\{1-|||\tfrac{x+y}{2}|||\;:\;|||x|||=|||y|||=1,\quad |||x-y|||=\varepsilon\Big\} \tag{2.1}$$

is actually of power type for some $p\geq 2$, that is there exists $c>0$ such that

$$\delta(\varepsilon)\geq c\,\varepsilon^p,\quad \text{for all } \varepsilon>0. \tag{2.2}$$

We need the following result, attributed in the literature to Hoffmann-Jørgensen [14].

Lemma 2.1. *Let $\mathbb{X}$ be a Banach space whose modulus of convexity* (2.1) *satisfies* (2.2) *for some $p\geq 2$ and $c>0$. Then there exists a constant $\eta=\eta(p,c)>0$ such that*

$$|||x+y|||^p+\eta\,|||x-y|||^p\leq 2^{p-1}\big(|||x|||^p+|||y|||^p\big) \tag{2.3}$$

for all $x,y\in\mathbb{X}$.

A proof of this lemma can be found in [2, Proposition 7], but we sketch a direct argument in the appendix. We also remark that a version of (2.3) is already satisfied by the uniformly convex renorming of a super-reflexive space done by Pisier [21, Theorem 3.1(a)].

The property of $\mathscr{B}$ being quasi-greedy (and semi-normalized) is preserved under equivalent norms in $\mathbb{X}$. Thus, from now on we shall use $\|\cdot\|$ instead of $|||\cdot|||$, and assume that the former norm satisfies the weak paralellogram inequality in (2.3). We shall also denote by $\kappa=\kappa(\mathscr{B},\mathbb{X})>0$ the smallest constant such that

$$\|G_Nx\|\leq\kappa\|x\|\quad\text{and}\quad\|x-G_Nx\|\leq\kappa\|x\|,\quad\forall\,x\in\mathbb{X},\ N=1,2,\dots \tag{2.4}$$

for all operators G_N as in (1.1). The existence of such constant is actually equivalent to the quasi-greediness of $\mathscr{B}$; see [25, Thm 1].

Theorem 1.1 will then be a consequence of the following.

Theorem 2.2. *Let $\mathbb{X}$ be a Banach space satisfying* (2.3) *for some $p \geq 2$ and $\eta > 0$. If $\mathscr{B} = \{\mathbf{e}_j\}_{j=1}^\infty$ is a quasi-greedy basis, then there exists $\varepsilon = \varepsilon(\kappa, p, \eta) > 0$ such that*

$$k_N(\mathscr{B}, \mathbb{X}) \;=\; O(\log N)^{1-\varepsilon}.$$

2.1. **Proof of Theorem 2.2.** The proof is a small variation of [11, Theorem 5.1].

We shall use the notation $x \succsim y$ when $x = \sum_{j\in A} x_j \mathbf{e}_j$ and $y = \sum_{k\in B} y_k \mathbf{e}_k$ have disjoint supports (ie, $A \cap B = \emptyset$) and $\min_{j\in A} |x_j| \geq \max_{k\in B} |y_k|$. We first establish the following key lemma.

Lemma 2.3. *Assume that $\mathbb{X}$ satisfies* (2.3) *and $\mathscr{B}$ is quasi-greedy. Then*

$$\big\|x+y\big\|^p \;\leq\; \gamma\,\Big(\|x\|^p + \|y\|^p\Big), \quad \forall\, x \succsim y. \tag{2.5}$$

where $\gamma = 2^{p-1} - \frac{\eta}{2\kappa^p}$.

Proof. Call N the cardinality of $\operatorname{supp} x$. Since $x \succsim y$, a use of (2.4) gives

$$\|x\| = \big\|G_N(x-y)\big\| \;\leq\; \kappa\,\big\|x-y\big\| \quad \text{and} \quad \|y\| = \big\|(I-G_N)(x-y)\big\| \;\leq\; \kappa\,\big\|x-y\big\|. \tag{2.6}$$

Thus,

$$\|x-y\|^p \;\geq\; \tfrac{1}{2\kappa^p}\,\big(\|x\|^p + \|y\|^p\big).$$

Inserting this estimate into the weak parallelogram inequality in (2.3) we obtain

$$\|x+y\|^p \leq 2^{p-1}\big(\|x\|^p + \|y\|^p\big) \;-\; \eta\,\|x-y\|^p \;\leq\; \big(2^{p-1} - \tfrac{\eta}{2\kappa^p}\big)\,\big(\|x\|^p + \|y\|^p\big),$$

as we wished to show. □

Iterating this result one easily proves the following (see [11, Lemma 2.4]).

Lemma 2.4. *With the assumptions of Lemma 2.3, if $x_1 \succsim x_2 \succsim \ldots \succsim x_m$ have pairwise disjoint supports, then*

$$\big\|x_1 + \ldots + x_m\big\|^p \;\leq\; \gamma^{\lceil \log_2 m\rceil} \sum_{j=1}^m \|x_j\|^p. \tag{2.7}$$

We now prove Theorem 2.2. We must show that, for $A \subset \mathbb{N}$ with $|A| = N \geq 2$, and every $x = \sum_i a_i \mathbf{e}_i \in \mathbb{X}$ it holds

$$\|S_A(x)\| \leq\; C\,(\log N)^{1-\varepsilon}\,\|x\|, \tag{2.8}$$

for a suitable $\varepsilon > 0$ (independent of x and N) to be determined. By scaling we may assume that $\max_i |a_i| = 1$ (which using (2.4) implies $\|x\| \geq \frac{1}{\kappa}\|G_1 x\| \geq c_1/\kappa$).

Let $m = \lceil \log_2 N\rceil$, so that $2^{m-1} < N \leq 2^m$. For $\ell = 1, \ldots, m$, we define

$$F_\ell = \{j \;:\; 2^{-\ell} < |a_j| \leq 2^{-(\ell-1)}\} \quad \text{and} \quad F_{m+1} = \{j \;:\; |a_j| \leq 2^{-m}\}.$$

Next write A as a disjoint union of the sets $A_\ell = A \cap F_\ell$, $\ell = 1, \ldots, m+1$. Clearly

$$\|S_{A_{m+1}} x\| \leq \sum_{i\in A_{m+1}} |a_i|\|\mathbf{e}_i\| \leq c_2\, 2^{-m} N \leq c_2 \leq \frac{\kappa c_2}{c_1}\,\|x\|. \tag{2.9}$$

For the other terms we quote Lemmas 5.2 and 5.3 in [10], which use the quasi-greedy property and the fact that $A_\ell \subset \{j \;:\; 2^{-\ell} < |a_j| \leq 2^{-(\ell-1)}\}$ to obtain

$$\|S_{A_\ell} x\| \leq\; C\,\|x\|,$$

for a positive constant C (independent of x and ℓ). Now, Lemma 2.4 gives

$$\|\sum_{\ell=1}^{m} S_{A_\ell}x\|^p \le \gamma^{\lceil \log_2 m\rceil} \sum_{\ell=1}^{m} \|S_{A_\ell}x\|^p \le C^p\,\gamma^{\lceil \log_2 m\rceil}\, m\,\|x\|^p. \tag{2.10}$$

Now we can write

$$\gamma^{\log_2 m}\, m = 2^{\log_2 m\,\log_2\gamma}\, m = m^{1+\log_2\gamma} = m^{p\,\alpha},$$

if we set $\alpha = (1+\log_2\gamma)/p$. Notice that $\alpha < 1$ since $\gamma < 2^{p-1}$, by Lemma 2.3. Thus, combining (2.9) with (2.10) we obtain

$$\|S_A x\| \le C'\, m^\alpha\, \|x\| \le C''\,(\log N)^\alpha\,\|x\|,$$

which implies (2.8) if we set

$$\varepsilon = 1-\alpha = 1-(1+\log_2\gamma)/p = \frac{p-1-\log_2\left(2^{p-1}-\frac{\eta}{2\kappa^p}\right)}{p},$$

which is a positive constant. □

3. Appendix: Proof of Lemma 2.1

Although Lemma 2.1 is well-known in the functional analysis community, we sketch a direct proof which we could not find explicitly in the literature.

We assume that the modulus of convexity satisfies (2.2). Then for all $x, y \in \mathbb{X}$ with $\|x\| = \|y\| = 1$ we have

$$1-\left\|\tfrac{x+y}{2}\right\|^p \ge 1-\left\|\tfrac{x+y}{2}\right\| \ge c\,\|x-y\|^p\,.$$

This implies

$$\left\|\frac{x+y}{2}\right\|^p + b^p\left\|\frac{x-y}{2}\right\|^p \le \frac{\|x\|^p+\|y\|^p}{2}\,,\quad \|x\| = \|y\| = 1\,, \tag{3.1}$$

with a constant $b^p = 2^p c > 0$ (setting $y = -x$ we also see that $b \le 1$). Our goal is to show that (3.1) continues to hold for all $x, y \in \mathbb{X}$, this time with the constant $b^p/(1+b^{p'})^{p-1}$.

By symmetry and homogeneity, we may assume that $1 = \|x\| \le \|y\|$. Consider the unit vector $v = y/\|y\|$. Then, from (3.1) and the triangle inequality we can deduce

$$\begin{aligned}\frac{\|x\|^p+\|y\|^p}{2} &= \frac{\|x\|^p+\|v\|^p}{2} + \frac{\|y\|^p-1}{2}\\ &\ge \left\|\frac{x+v}{2}\right\|^p + b^p\left\|\frac{x-v}{2}\right\|^p + \frac{\|y\|^p-1}{2}\\ &\ge \Big(\frac{\|x+y\|-\|y-v\|}{2}\Big)^p + b^p\left\|\frac{x-v}{2}\right\|^p + \frac{\|y\|^p-1}{2}\,.\end{aligned} \tag{3.2}$$

Let A and B be given by

$$A := \|x+y\| \ge \|y\|-1 = \|y-v\| := B\,.$$

With this notation we can write

$$\Big(\frac{\|x+y\|-\|y-v\|}{2}\Big)^p = \Big(\frac{A-B}{2}\Big)^p = \Big(\frac{A}{2}\Big)^p\Big(1-\frac{B}{A}\Big)^p.$$

A simple argument shows that $(1-x)^p \geq 1 - px + x^2$ if $p \geq 2$ and $x \in [0,1]$. Thus, since $A \geq B$ we have

$$\begin{aligned} \Big(\frac{\|x+y\| - \|y-v\|}{2}\Big)^p &\geq \Big(\frac{A}{2}\Big)^p \Big[1 - p\,\frac{B}{A} + \Big(\frac{B}{A}\Big)^2\Big] \\ &\geq \Big\|\frac{x+y}{2}\Big\|^p - p\,\frac{BA^{p-1}}{2^p} + \Big\|\frac{y-v}{2}\Big\|^p, \end{aligned} \tag{3.3}$$

where in the last step we have used that $A^{p-2}B^2 \geq B^p = \|y-v\|^p$ (since $p \geq 2$). Inserting this into (3.2) we obtain

$$\frac{\|x\|^p + \|y\|^p}{2} \geq \Big\|\frac{x+y}{2}\Big\|^p + D + E\,, \tag{3.4}$$

where

$$D = b^p \Big\|\frac{x-v}{2}\Big\|^p + \Big\|\frac{y-v}{2}\Big\|^p \quad \text{and} \quad E = \frac{\|y\|^p - 1}{2} - \frac{pBA^{p-1}}{2^p}.$$

To estimate D notice that the triangle and Hölder's inequalities give

$$\|x-y\| \leq \|x-v\| + \|v-y\| \leq \big(b^p\|x-v\|^p + \|v-y\|^p\big)^{\frac{1}{p}} (1 + b^{-p'})^{1/p'},$$

and therefore

$$D \;\geq\; \frac{b^p}{(1+b^{p'})^{p-1}} \Big\|\frac{x-y}{2}\Big\|^p.$$

So, it remains to show that $E \geq 0$. Now using $A = \|x+y\| \leq \|y\| + 1$ we can write

$$E \geq \frac{\|y\|^p - 1}{2} - \frac{p(\|y\| - 1)(\|y\| + 1)^{p-1}}{2^p}.$$

It is then enough to prove that

$$\frac{\lambda^p - 1}{2} \geq \frac{p(\lambda - 1)(\lambda + 1)^{p-1}}{2^p}\,, \quad \text{for all } \lambda > 1\,.$$

With the change $\lambda = 1/u$, this is equivalent to show

$$1 - u^p \geq p(1-u)\Big(\frac{1+u}{2}\Big)^{p-1}, \quad 0 < u < 1\,,$$

which can be written as

$$\frac{1}{1-u}\int_u^1 t^{p-1}dt \geq \Big(\frac{1+u}{2}\Big)^{p-1}, \quad 0 < u < 1\,.$$

But this is a consequence of Jensen's inequality, since $\varphi(t) = t^{p-1}$ is convex when $p \geq 2$, and

$$\frac{1}{1-u}\int_u^1 \varphi(t)dt \geq \varphi\Big(\frac{1}{1-u}\int_u^1 t\,dt\Big) = \varphi\Big(\frac{1+u}{2}\Big) = \Big(\frac{1+u}{2}\Big)^{p-1}.$$

□

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Gustavo Garrigós, Departamento de Matemáticas, Universidad de Murcia, 30100 Murcia, Spain
E-mail address: gustavo.garrigos@um.es

Eugenio Hernández, Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, Spain
E-mail address: eugenio.hernandez@uam.es

Matías Raja, Departamento de Matemáticas, Universidad de Murcia, 30100 Murcia, Spain
E-mail address: matias@um.es