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A.e. convergence and 2-weight inequalities for Poisson-Laguerre semigroups

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Abstract We find optimal decay estimates for the Poisson kernels associated with various Laguerre-type operators L. From these, we solve two problems about the Poisson semigroup $e^{-t\sqrt{L}}$. First, we find the largest space of initial data f so that $e^{-t\sqrt{L}} f(x) \to f(x)$ at $a.e.\, x$. Secondly, we characterize the largest class of weights w which admit 2-weight inequalities of the form $\| \sup_{0<t\leq t_0} |e^{-t\sqrt{L}} f| \|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$, for some other weight v.

Keywords Laguerre expansions · Poisson integral · Heat semigroup · 2-weight problem · Fractional laplacian

Mathematics Subject Classification 33C45 · 35C15 · 40A10 · 42C10 · 47D06

1 Introduction

In this paper, we continue the research, started in [6,7], about Poisson integrals associated with certain differential operators L, say symmetric and positive in $L^2(\Omega, \mu)$. Namely, we are interested in the behavior of

$$u(t, x) = e^{-t\sqrt{L}} f(x)$$

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as a solution of the elliptic differential equation

$$(\mathrm{P}) \qquad \begin{cases} u_{tt} - Lu = 0 \\ u(0,x) = f(x), \end{cases} \quad \text{in the half-plane } (0,\infty)\times\Omega.$$

We shall study two questions which are closely related

(i) find the **largest** class of functions f for which $\lim_{t\to 0^+} u(t,x) = f(x),\ a.e.\, x\in\Omega$;
(ii) establish 2-weight inequalities of the form

$$\Big\| \sup_{0<t\le t_0} |u(t,x)| \Big\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$$

for the **largest** class of weights w for which a suitable v with this property exists.

In [6], we considered these questions in full detail when L is the Hermite operator. In this paper, we intend to do the same for the various *Laguerre operators*. We remark that to solve (i) or (ii) we shall need **optimal** decay estimates of the Poisson kernels $e^{-t\sqrt{L}}(x,y)$, at least in the y variable. These are new in the literature and in particular produce new results on a.e. convergence compared to those of Muckenhoupt [10] and Stempak [15], and also larger weight classes compared to those of Nowak in [11]. The kernel estimates may also have an independent interest in other problems; see, e.g., recent work by Liu and Sjögren [9].

We now state our results. For $\alpha > -1$ (fixed throughout the paper), we first consider the (Hermite-type) Laguerre operator in $L^2(\mathbb{R}_+) := L^2((0,\infty), dy)$ given by

$$L = -\partial_{yy} + \Big[y^2 + \frac{\alpha^2 - \frac14}{y^2}\Big] + 2\mu, \quad \text{where } \mu \ge -(\alpha+1). \tag{1.1}$$

The parameter μ will be useful later, when transferring the results to other Laguerre operators; see Table 1. We shall also consider a slightly more general family of partial differential equations, namely

$$\begin{cases} u_{tt} + \frac{1-2\nu}{t} u_t = Lu \\ u(t,0) = f, \end{cases} \qquad \text{where } \nu > 0. \tag{1.2}$$

These pde's appear in relation to the *fractional powers* of L, $f \mapsto L^\nu f$ (see, e.g., [16]). When $\nu = 1/2$, we recover the original equation (P).

As discussed in [16], a candidate solution to (1.2) is given by the *Poisson-like integral*

$$P_t f(x) := \tfrac{t^{2\nu}}{4^\nu\Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4u}} \big[e^{-uL} f\big](x)\, \frac{du}{u^{1+\nu}}, \quad t>0, \tag{1.3}$$

which is subordinated to the "heat" semigroup $\{e^{-uL}\}_{u>0}$. Our first goal is to find the most general conditions on a function $f : \mathbb{R}_+ \to \mathbb{C}$ so that $P_t f$ is a meaningful solution of (1.2). These are determined by the following key function

$$\Phi(y) = \begin{cases} \dfrac{y^{\alpha+\frac12}\, e^{-\frac{y^2}{2}}}{(1+y)^{\mu+\alpha+1}[\ln(e+y)]^{1+\nu}} & \text{if } \mu > -(\alpha+1) \\[2ex] \dfrac{y^{\alpha+\frac12}\, e^{-\frac{y^2}{2}}}{[\ln(e+y)]^{\nu}} & \text{if } \mu = -(\alpha+1). \end{cases} \tag{1.4}$$

We write $L^1(\Phi)$ for the space of measurable functions f in $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} |f|\Phi < \infty$.

Theorem 1.1 *Let $\alpha > -1$, $\nu > 0$, L as in (1.1) and Φ as in (1.4). Then, for every $f \in L^1(\Phi)$ the function $u(t,x) = P_t f(x)$ in (1.3) is defined by an absolutely convergent integral such that*

(i) *$u(t,x) \in C^\infty((0,\infty)\times\mathbb{R}_+)$ and satisfies the pde (1.2)*
(ii) *$\lim_{t\to 0^+} u(t,x) = f(x)$ at every Lebesgue point x of f.*

Conversely, if a function $f \geq 0$ is such that the integral in (1.3) is finite for some $(t,x) \in (0,\infty)\times\mathbb{R}_+$, then f must necessarily belong to $L^1(\Phi)$.

Our second question concerns more "quantitative" bounds for the solutions of (1.2), expressed in terms of the following *local maximal operators*

$$P^*_{t_0} f(x) := \sup_{0<t\leq t_0} \left| P_t f(x) \right|, \quad \text{with } t_0 > 0 \text{ fixed.} \tag{1.5}$$

We shall prove estimates of the Poisson kernels, as in (1.9) below, which lead to local bounds of the form

$$P^*_{t_0} : L^1(\Phi) \to L^s_{\rm loc} \quad \text{if } s < 1, \quad \text{and} \quad P^*_{t_0} : L^1(\Phi)\cap L^p_{\rm loc} \to L^p_{\rm loc} \quad \text{if } p > 1.$$

However, our main interest is to obtain global bounds in x, which we shall phrase through the following problem.

Problem 1 **A 2-weight problem for the operator** $P^*_{t_0}$. *Given $1 < p < \infty$, characterize the class of weights $w(x) > 0$ such that $P^*_{t_0}$ maps $L^p(w) \to L^p(v)$ boundedly, for some other weight $v(x) > 0$.*

Our second main result gives a complete answer to Problem 1. For $p \in (1,\infty)$, we define the class of weights

$$D_p(\Phi) = \Big\{ w(y) > 0 \;:\; \big\| w^{-\frac{1}{p}}\,\Phi \big\|_{L^{p'}(\mathbb{R}_+)} < \infty \Big\}. \tag{1.6}$$

Observe that $L^p(w) \subset L^1(\Phi)$ if and only if $w \in D_p(\Phi)$, so in view of Theorem 1.1, this is a necessary condition for Problem 1. Our second theorem shows that it is also sufficient.

Theorem 1.2 *Let $1 < p < \infty$ and $t_0 > 0$ be fixed. Then, for a weight $w(x) > 0$ the condition $w \in D_p(\Phi)$ is equivalent to the existence of some other weight $v(x) > 0$ such that*

$$P^*_{t_0} : L^p(w) \to L^p(v) \quad \textit{boundedly}. \tag{1.7}$$

Moreover, for every $\varepsilon > 0$, we can choose an explicit weight $v \in D_{p+\varepsilon}(\Phi)$ satisfying (1.7).

We remark that Problem 1 is only a "one-side" problem, in contrast to the (more difficult) question of characterizing all pairs of weights (w, v) for which (1.7) holds. One-side problems were considered in the early 80s by Rubio de Francia [13] and Carleson and Jones [2] for various classical operators. Here, we shall follow the approach by the latter, which has the advantage of giving **explicit** expressions for the second weight $v(x)$ (see Remark 6.2 below). This is also a novelty compared to [6], which was based on the non-constructive method of Rubio de Francia.

We can now briefly describe our approach to the proofs in this paper. Most of our work will be employed in deriving *precise decay estimates* for the kernel $P_t(x,y)$ of the operator in (1.3). First, we shall show that, for fixed t and x

$$c_1(t,x)\,\Phi(y) \;\leq\; P_t(x,y) \;\leq\; c_2(t,x)\,\Phi(y), \tag{1.8}$$

for suitable c_1 and c_2; see Proposition 5.1. In a second step, we give more precise bounds, uniform in t (see Proposition 5.2), which lead to the following control of the operator $P^*_{t_0}$

$$P^*_{t_0} f(x) \lesssim C(x) \Big[\mathcal{M}^{\text{loc}}(f\Phi)(x) + \int_{\mathbb{R}_+} |f|\,\Phi \Big], \tag{1.9}$$

for a reasonably well behaved $C(x)$. Here, $\mathcal{M}^{\text{loc}} = \mathcal{M}^{\text{loc}}_M$ denotes a *local* Hardy–Littlewood maximal operator in $\mathbb{R}_+$, given by

$$\mathcal{M}^{\text{loc}}_M f(x) := \sup_{r>0} \frac{1}{|I(x,r)|} \int_{I(x,r)} |f(y)|\, \chi_{\{\frac{x}{2} \le y \le Mx\}}\, dy \tag{1.10}$$

for a suitable $M > 1$. We also use the notation $I(x,r) = (x-r, x+r) \cap \mathbb{R}_+$.

Concerning the kernel estimates, they are necessarily more technical than in the Hermite setting treated in [5,6] (which is essentially the case $\alpha = -1/2$). The term $1/y^2$ in (1.1) produces an additional singularity when $y \to 0$ which must be handled separately from $y \to \infty$. This dual behavior already appears in the Bessel function I_α which is part of the kernel expression of e^{-uL}; see (2.2). New difficulties also arise when $\alpha \in (-1, -1/2)$, related to the fact that the associated Laguerre functions blow up when $y \to 0$.

Finally, we consider the same problem for other classical families of Laguerre operators. These are listed in Table 1 below, together with their eigenfunctions. As we shall see in Sect. 7, using suitable transformations (as, e.g., in [1]) we can "transfer" the statements of Theorems 1.1 and 1.2 from L to each of these differential operators, provided we choose the function Φ as in Table 1. More precisely, we have the following

Theorem 1.3 *Let $\alpha > -1$, $\mu \ge -(\alpha+1)$ and $\nu > 0$. Then, if L is replaced by any of the operators listed in Table 1, both Theorems 1.1 and 1.2 hold with the expression of the function Φ given in the table.*

Table 1 Table of Laguerre-type operators, and their corresponding Φ-functions

Differential operators (for $\mu \ge -\alpha - 1$)	Orthonormal eigenfunctions	Function[a] Φ
$L = -\partial_{yy} + y^2 + \frac{\alpha^2 - \frac14}{y^2} + 2\mu$ in $L^2(dy)$	$\varphi^\alpha_n(y) = \sqrt{2}\, y^{\alpha+\frac12} e^{-\frac{y^2}{2}} L^\alpha_n(y^2)$	$\dfrac{y^{\alpha+\frac12}\, e^{-y^2/2}}{(1+y)^{1+\alpha+\mu}\,[\ln(e+y)]^{1+\nu}}$
$\mathfrak{L} = -y\partial_{yy} - \partial_y + \frac14\Big[y + \frac{\alpha^2}{y}\Big] + \frac{\mu}{2}$ in $L^2(dy)$	$\mathfrak{L}^\alpha_n(y) = y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} L^\alpha_n(y)$	$\dfrac{y^{\frac{\alpha}{2}}\, e^{-y/2}}{(1+y)^{\frac{1+\alpha+\mu}{2}}\,[\ln(e+y)]^{1+\nu}}$
$\mathscr{L} = -y\partial_{yy} - (\alpha+1)\partial_y + \frac{y}{4} + \frac{\mu}{2}$ in $L^2(y^\alpha dy)$	$\ell^\alpha_n(y) = e^{-\frac{y}{2}} L^\alpha_n(y)$	$\dfrac{y^\alpha e^{-y/2}}{(1+y)^{\frac{1+\alpha+\mu}{2}}\,[\ln(e+y)]^{1+\nu}}$
$\Lambda = -\partial_{yy} - \frac{2\alpha+1}{y}\partial_y + y^2 + 2\mu$ in $L^2(y^{2\alpha+1} dy)$	$\psi^\alpha_n(y) = \sqrt{2} e^{-\frac{y^2}{2}} L^\alpha_n(y^2)$	$\dfrac{y^{2\alpha+1}\, e^{-y^2/2}}{(1+y)^{1+\alpha+\mu}\,[\ln(e+y)]^{1+\nu}}$
$\mathbb{L} = -y\,\partial_{yy} - (\alpha+1-y)\,\partial_y + \frac{\mu+\alpha+1}{2}$ in $L^2(y^\alpha e^{-y} dy)$	$L^\alpha_n(y)$	$\dfrac{y^\alpha e^{-y}}{(1+y)^{\frac{1+\alpha+\mu}{2}}\,[\ln(e+y)]^{1+\nu}}$

[a] We shall use the agreement, as in (1.4), that in the extremal case $\mu = -(\alpha+1)$ the logarithmic factor in the denominator becomes $[\ln(e+y)]^\nu$

As an application, in the case of the standard Laguerre operator $\mathbb{L}$, with $m = 0$ and $\nu = 1/2$, we recall that Muckenhoupt, in his classical paper [10], proved the pointwise convergence of $P_t f$ for all data f in the space $L^1(y^\alpha e^{-y}\,dy)$. Our results show that one can enlarge the space to

$$L^1(y^\alpha e^{-y}/\sqrt{\ln(e+y)}\,dy),$$

and include new initial data such as $f(y) = e^y/[(1+y)^{\alpha+1}\ln(e+y)]$.

The outline of the paper will be the following. In Sect. 2, we consider a version of Theorem 1.1 for *heat integrals* $u(t,x) = e^{-tL}f(x)$, which are solutions of the equation

$$u_t + Lu = 0 \quad \text{in } (0,\mathrm{T}) \times \mathbb{R}_+, \quad \text{with} \quad u(0,x) = f(x).$$

Heat integrals are easier to handle, and the explicit expression of the heat kernel, $e^{-tL}(x,y)$, makes more transparent the behavior we shall later encounter in Poisson kernels. In Sect. 3, we study 2-weight inequalities for the local maximal operator $\mathcal{M}^{\text{loc}}$. In Sect. 4, we apply these to prove a version of Theorem 1.2 for heat integrals. In Sect. 5, we take up the study of Poisson integrals, splitting in various subsections the detailed kernel estimates leading to (1.9). In Sect. 6, we shall give the proof of Theorems 1.1 and 1.2 for the operator L. Finally, in Sect. 7 we show how to transfer the results to each of the Laguerre operators in Table 1.

Throughout the paper $\alpha > -1$ is fixed, as are the parameters $\mu \geq -(\alpha+1)$ and $\nu > 0$ in the differential operators. The notation $A \lesssim B$ will mean $A \leq c\,B$, for a constant $c > 0$ which may depend on α, μ, ν and other parameters like p, M, t_0, ε, but not on t, x, y. If needed, we shall stress the latter dependence by $c(x)$, $c(t,x)$, ... Finally, if $1 < p < \infty$ we set $p' = p/(p-1)$.

2 The simpler model of heat integrals

In this section, we set $\mu = 0$ in (1.1) and consider

$$L = -\partial_{yy} + \left[y^2 + \frac{\alpha^2 - \frac{1}{4}}{y^2} \right]. \tag{2.1}$$

The corresponding eigenfunctions $\{\varphi_n^\alpha\}_{n=0}^\infty$ satisfy

$$L\varphi_n^\alpha = (4n + 2\alpha + 2)\varphi_n^\alpha, \quad n = 0, 1, 2, \dots$$

and form an orthonormal basis of $L^2(0,\infty)$; see [17, (5.1.2)]. The kernel of the associated heat semigroup e^{-tL}, in terms of the new variable $s = \operatorname{th} t$, has the explicit expression[1]

$$\begin{aligned} e^{-tL}(x,y) &= \sum_{n=0}^{\infty} e^{-(4n+2\alpha+2)t}\varphi_n^\alpha(x)\varphi_n^\alpha(y) \\ &= \sqrt{\frac{1-s^2}{2s}}\,\bar{I}_\alpha\left(\frac{(1-s^2)xy}{2s}\right) e^{-\frac{(x-y)^2}{4s}}\, e^{-\frac{s(x+y)^2}{4}}. \end{aligned} \tag{2.2}$$

Here, we have used the convenient notation $\bar{I}_\alpha(z) = \sqrt{z}e^{-z}I_\alpha(z)$, so that from known properties of the Bessel functions I_α (see, e.g., [8, (5.7.1), (5.11.10)]), we have $\bar{I}_\alpha(z) \approx \langle z\rangle^{\alpha+\frac{1}{2}}$, with $\langle z\rangle = \min\{z, 1\}$.

[1] This formula is easily derived from the more classical [17, (5.1.15)] or [8, (4.17.6)]; see, e.g., [3, p. 341].

2.1 A.e. convergence of heat integrals

We wish to establish the pointwise convergence of $e^{-tL}f(x)$ with the weakest possible conditions on f. We claim that

$$e^{-tL}(x,y) \lesssim \left\langle \frac{xy}{s} \right\rangle^{\alpha+\frac{1}{2}} \frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{s}}\, e^{-\frac{sy^2}{4}}. \tag{2.3}$$

To produce this bound from (2.2), one disregards x in the last exponential and uses $1-s^2 \le 1$ when $\alpha \ge -\frac{1}{2}$. If $\alpha \in (-1, -\frac{1}{2})$, note that $\langle \lambda z\rangle \ge \lambda \langle z\rangle$ for $\lambda \le 1$, so one can leave outside a power $(1-s^2)^{\alpha+1} \le 1$.

Theorem 2.1 *Let $\alpha > -1$ be fixed, and f be such that*

$$\int_0^\infty |f(y)| e^{-ay^2} \langle y\rangle^{\alpha+\frac{1}{2}}\, dy < \infty, \quad \textit{for some (possibly large) } a > 0. \tag{2.4}$$

Then,

$$\lim_{t\to 0^+} e^{-tL}f(x) = f(x), \quad a.e.\, x \in \mathbb{R}_+.$$

Proof For each fixed $N \ge 2$, it suffices to show that $\lim_{t\to 0^+} e^{-tL}f(x) = f(x)$ for $a.e.\, x \in (1/N, N)$. We split

$$f = f\chi_{\{0<y\le 2N\}} + f\chi_{\{y>2N\}} = f_1 + f_2.$$

The function f_1 has bounded support and belongs to $L^1(y^{\alpha+\frac{1}{2}} e^{-\frac{y^2}{2}}\, dy)$, so we can apply the results of Muckenhoupt [10] (with a suitable change of variables[2], as indicated by Stempak [15]) to obtain

$$\lim_{t\to 0^+} e^{-tL}f_1(x) = f_1(x) = f(x), \quad a.e.\, x \in \left[\tfrac{1}{N}, N\right].$$

Next, we shall show that, under the hypothesis (2.4),

$$\lim_{t\to 0^+} e^{-tL}f_2(x) = 0, \quad \forall x \in \left[\tfrac{1}{N}, N\right].$$

Since $t \to 0$, we may assume that $s = \operatorname{th} t \le s_0$ for some $s_0 < \frac{1}{10}$ (which we shall make precise below). Note that $\frac{1}{N} \le x \le N$ and $y > 2N$ imply that

$$\left\langle \frac{xy}{s} \right\rangle^{\alpha+\frac{1}{2}} = 1, \quad \forall s < 1.$$

So, by (2.3), in this region we have a gaussian bound for the kernel

$$e^{-tL}(x,y) \lesssim s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4s}} \le s^{-\frac{1}{2}} e^{-\frac{y^2}{16s}},$$

using in the last step that $|x-y| \ge y/2$. Choosing $s_0 < \frac{1}{32a}$ (with a as in (2.4)), we see that for all $y > 2N$,

$$e^{-tL}(x,y) \lesssim s^{-\frac{1}{2}} e^{-\frac{y^2}{32s}}\, e^{-ay^2} \le s^{-\frac{1}{2}} e^{-\frac{N^2}{8s}}\, e^{-ay^2}$$

[2] See also Sects. 7.2 and 7.4 below for the explicit change of variables.

and therefore

$$e^{-tL} f_2(x) \lesssim s^{-\frac{1}{2}} e^{-\frac{N^2}{8s}} \int_{y>2N} |f(y)|\, e^{-ay^2}\, dy \longrightarrow 0, \quad \text{as } s \to 0^+.$$

□

2.2 Heat kernel estimates

The elementary bound (2.3) was enough for Theorem 2.1. Below, in order to handle maximal operators, we shall need the following more precise bound.

Proposition 2.2 *Let* $\alpha > -1$. *Then, for every* $\gamma > 1$ *there is some* $M = M_\gamma > 1$ *such that*

$$e^{-tL}(x,y) \le C_\gamma \begin{cases} \dfrac{e^{-\frac{|x-y|^2}{4s}}}{\sqrt{s}} \left\langle \dfrac{xy}{s} \right\rangle^{\alpha+\frac{1}{2}} & \text{if } \frac{x}{2} \le y \le Mx \\ c(x)\, \langle y\rangle^{\alpha+\frac{1}{2}}\, e^{-\frac{y^2}{2\gamma \operatorname{th}(2t)}} & \text{if } y < \frac{x}{2} \text{ or } y > Mx \end{cases}, \tag{2.5}$$

for all $x, y \in \mathbb{R}_+$ *and* $s = \operatorname{th} t \in (0,1)$. *Here we can set* $c(x) = 1/\langle x\rangle^{\alpha+\frac{3}{2}}$.

Proof Clearly, (2.3) implies the estimate in the local part $y \in [\frac{x}{2}, Mx]$, so we shall look at the complementary range. Given x, y and s, for simplicity we write $z = \frac{xy}{s}$, so that (2.3) becomes

$$e^{-tL}(x,y) \lesssim \langle z\rangle^{\alpha+\frac{1}{2}}\, s^{-\frac{1}{2}}\, e^{-\frac{(x-y)^2}{4s}}\, e^{-\frac{sy^2}{4}}. \tag{2.6}$$

Below we shall show that

$$A_s(x,y) := \langle z\rangle^{\alpha+\frac{1}{2}} \frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{s}} \le C_\gamma \frac{\langle y\rangle^{\alpha+\frac{1}{2}}}{\langle x\rangle^{\alpha+\frac{3}{2}}} e^{-\frac{y^2}{4\gamma s}}, \quad \text{if } y < \frac{x}{2} \text{ or } y > Mx, \tag{2.7}$$

for some $M = M_\gamma$. This estimate inserted into (2.6) establishes (2.5), since

$$e^{-\frac{y^2}{4\gamma s}}\, e^{-\frac{sy^2}{4}} \le e^{-\frac{y^2}{4\gamma}\left(s+\frac{1}{s}\right)} = e^{-\frac{y^2}{2\gamma \operatorname{th}(2t)}},$$

as $s + s^{-1} = 2/\operatorname{th}(2t)$ when $s = \operatorname{th} t$.

To prove (2.7), we need to separate the cases $z \le 1$ and $z \ge 1$. We begin with $z \ge 1$. In the region $y > Mx$, we may use $|x-y| \ge (1-\frac{1}{M})y$ to obtain

$$A_s(x,y) \lesssim \frac{e^{-\left(\frac{M-1}{M}\right)^2 \frac{y^2}{4s}}}{\sqrt{s}} \le c_M \frac{e^{-\left(\frac{M-1}{M}\right)^3 \frac{y^2}{4s}}}{y} \le c_M\, e^{-\frac{y^2}{4s\gamma}} \frac{\langle y\rangle^{\alpha+\frac{1}{2}}}{\langle x\rangle^{\alpha+\frac{3}{2}}}, \tag{2.8}$$

where in the last step we select $M = M_\gamma$ sufficiently large so that $(\frac{M}{M-1})^3 \le \gamma$, and have used the trivial estimate

$$\frac{1}{y} \le \frac{\langle y\rangle^{\alpha+\frac{1}{2}}}{\langle y\rangle^{\alpha+\frac{3}{2}}} \le \frac{\langle y\rangle^{\alpha+\frac{1}{2}}}{\langle x\rangle^{\alpha+\frac{3}{2}}}, \quad \text{if } y \ge x. \tag{2.9}$$

On the other hand, if $y < x/2$ we have $|x-y| \ge x/2$, which leads to

$$A_s(x,y) \lesssim \frac{e^{-\frac{(x/2)^2}{4s}}}{\sqrt{s}} \le c_\gamma \frac{e^{-\frac{(x/2)^2}{4s\gamma}}}{x} \le c_\gamma \frac{e^{-\frac{y^2}{4\gamma s}}}{x}. \tag{2.10}$$

In the case $\alpha \in (-1, -\frac{1}{2})$, this can be combined with

$$\frac{1}{x} \le \frac{\langle x \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}} \le \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}}, \quad \text{since } x \ge y. \tag{2.11}$$

If on the contrary $\alpha \ge -\frac{1}{2}$, we can insert $1 \le z^{\alpha+\frac{1}{2}}$ in the first step of (2.10) to obtain

$$A_s(x, y) \lesssim \frac{e^{-\frac{(x/2)^2}{4s}}}{\sqrt{s}} \left(\tfrac{xy}{s}\right)^{\alpha+\frac{1}{2}} = \left(\tfrac{x^2}{s}\right)^{\alpha+1} e^{-\frac{(x/2)^2}{4s}} \frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}} \le c_\gamma \, e^{-\frac{y^2}{4s\gamma}} \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}}, \tag{2.12}$$

where in the last step we have used

$$\frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}} \le \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}}, \quad \text{if } x \ge y$$

(this is clear if $y \le 1$, and follows from $y^{\alpha+\frac{1}{2}}/x^{\alpha+\frac{3}{2}} \le 1/y \le 1$ if $y \ge 1$). This completes the proof of (2.7) when $z \ge 1$.

We turn to the case $z \le 1$, for which we have to estimate

$$A_s(x, y) = s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4s}} z^{\alpha+\frac{1}{2}}. \tag{2.13}$$

When $\alpha \ge -\frac{1}{2}$, the last factor helps, so some of the previous arguments also lead to (2.7); namely one can disregard z in the region $y > Mx$, and must keep it when $y < \frac{x}{2}$ and argue as in (2.12). We are left with the case $\alpha \in (-1, -\frac{1}{2})$, which makes $z^{\alpha+\frac{1}{2}} \ge 1$. In the region $y > Mx$, this can be absorbed by the exponentials as follows

$$\begin{aligned} A_s(x, y) &\lesssim \left(\tfrac{xy}{s}\right)^{\alpha+\frac{1}{2}} s^{-\frac{1}{2}} e^{-\left(\frac{M-1}{M}\right)^2 \frac{y^2}{4s}} = \left(\tfrac{y^2}{s}\right)^{\alpha+1} e^{-\left(\frac{M-1}{M}\right)^2 \frac{y^2}{4s}} \frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}} \\ &\le c_M \, e^{-\left(\frac{M-1}{M}\right)^3 \frac{y^2}{4s}} \frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}} \le c_M \, e^{-\frac{y^2}{4s\gamma}} \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}}, \end{aligned}$$

where in the last step we first argue as in (2.8), and then use the elementary inequalities

$$\frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}} \le \frac{\langle x \rangle^{\alpha+\frac{1}{2}} \langle y \rangle^{\alpha+\frac{1}{2}}}{\langle y \rangle^{2\alpha+2}} \le \frac{\langle x \rangle^{\alpha+\frac{1}{2}} \langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{2\alpha+2}}, \qquad \text{if } y \ge x$$

(the first one should be clear if $x \le 1$, and follows from $x^{\alpha+\frac{1}{2}}/y^{\alpha+\frac{3}{2}} \le 1/x \le 1$ if $x \ge 1$). Finally, in the region $y < \frac{x}{2}$ the inequalities in (2.12) remain also valid, so we have completed the proof of (2.7), and hence of Proposition 2.2. □

Using the local Hardy–Littlewood maximal function $\mathcal{M}_M^{\text{loc}}$ defined in (1.10), we can easily deduce from Proposition 2.2 the following

Corollary 2.3 *Let $\alpha > -1$ and $\gamma > 1$. Then, there is some $M = M_\gamma > 1$ such that*

$$\sup_{0<t\le t_0} \left| e^{-tL} f(x) \right| \lesssim \mathcal{M}_M^{\text{loc}} f(x) + c(x) \int_{\mathbb{R}_+} |f(y)| \, \langle y \rangle^{\alpha+\frac{1}{2}} \, e^{-\frac{y^2}{2\gamma \operatorname{th}(2t_0)}} \, dy, \tag{2.14}$$

for every $x, t_0 > 0$ and $c(x) = 1/\langle x \rangle^{\alpha+\frac{3}{2}}$.

Proof Let $f = f_1 + f_2$ with $f_1 = f\chi_{[\frac{x}{2}, Mx]}$. From Proposition 2.2, (2.14) is clear for f_2. To treat f_1, we first assume $s = \operatorname{th} t \le x^2$. Then, $z = \frac{xy}{s} \gtrsim 1$ (since $x \approx y$ when $y \in \operatorname{supp} f_1$), so the first bound in (2.5) and a standard slicing argument easily lead to

$$\left|e^{-tL} f_1(x)\right| \lesssim \frac{1}{\sqrt{s}} \int_{\mathbb{R}_+} e^{-\frac{(x-y)^2}{4s}} |f(y)| \chi_{\{\frac{x}{2} \le y \le Mx\}} \, dy \lesssim \mathcal{M}_M^{\text{loc}} f(x).$$

If we assume instead $s = \operatorname{th} t \ge x^2$, then using again $x \approx y$ and (2.5) we have

$$\left|e^{-tL} f_1(x)\right| \lesssim \frac{x^{2\alpha+1}}{s^{\alpha+1}} \int_{\frac{x}{2}}^{Mx} |f(y)| \, dy \lesssim \frac{x^{2\alpha+1}}{x^{2\alpha+2}} \int_{\frac{x}{2}}^{Mx} |f(y)| \, dy \lesssim \mathcal{M}_M^{\text{loc}} f(x).$$

□

Remark 2.4 An estimate quite similar to (2.14), with a slightly worse bound for the exponential inside the integral, was obtained by Chicco-Ruiz and Harboure in [3, §5].

3 2-weight inequalities for $\mathcal{M}^{\text{loc}}$

In this section, we prove 2-weight inequalities for the *local maximal operator* in $\mathbb{R}_+$

$$\mathcal{M}^{\text{loc}} f(x) := \sup_{r>0} \frac{1}{|I(x,r)|} \int_{I(x,r)} |f(y)| \, \chi_{\{\frac{x}{2} < y < Mx\}} \, dy,$$

where $M > 1$ is a fixed parameter (which for simplicity we avoid in the notation). The 1-weight theory of (a related version of) this operator was considered in [12, §6].

3.1 The Carleson–Jones theorem for $\mathcal{M}^{\text{loc}}$

For each $p \in (1, \infty)$, we consider the following class of weights in $\mathbb{R}_+$

$$D_p^{\text{loc}} = \left\{ W(x) > 0 \;:\; \int_J W^{-\frac{p'}{p}} < \infty, \quad \forall\, J \Subset (0, \infty) \right\}.$$

Associated with each $W \in D_p^{\text{loc}}$, we define a collection of weights $\{V_\varepsilon\}_{\varepsilon>0}$ by

$$V_\varepsilon(x) = V(x)\, \rho_\varepsilon\big[V(x)\big], \quad \text{where } V(x) := \left[\mathcal{M}^{\text{loc}}\left(W^{-\frac{p'}{p}}\right)(x)\right]^{-\frac{p}{p'}}, \tag{3.1}$$

and we set $\rho_\varepsilon(x) := \min\left\{x^\varepsilon, x^{-\varepsilon}\right\}$. Observe that $V_{\varepsilon_2} \le V_{\varepsilon_1} \le V \le W$ if $\varepsilon_1 \le \varepsilon_2$. This is a slight variant of the weight family used by Carleson and Jones in [2] (for the *classical* Hardy–Littlewood operator).

Theorem 3.1 *Let $1 < p < \infty$ and $W \in D_p^{\text{loc}}$. Then, for every $\varepsilon > 0$*

$$\mathcal{M}^{\text{loc}} : L^p(W) \to L^p(V_\varepsilon) \quad \textit{boundedly,}$$

where V_ε is defined as in (3.1).

Proof We use the ideas in [2], with some modifications required for the local operator $\mathcal{M}^{\text{loc}}$; see also [5, Prop. 4.2]. Call $E_n = \{x \in \mathbb{R}_+ \;:\; \mathcal{M}^{\text{loc}}(W^{-\frac{p'}{p}})(x) < 2^n\}$, $n \in \mathbb{Z}$, and define the operators

$$T_n g(x) := \chi_{E_n} \mathcal{M}^{\text{loc}}\left(W^{-\frac{p'}{p}} g\right)(x). \tag{3.2}$$

Note that $T_n : L^1(W^{-\frac{p'}{p}}) \to L^{1,\infty}$, with a uniform bound in n, since

$$\left|\left\{T_n g(x) > R\right\}\right| \le \left|\left\{\mathcal{M}\left(W^{-\frac{p'}{p}} g\right)(x) > R\right\}\right| \le \frac{c_0}{R} \int_{\mathbb{R}_+} W^{-\frac{p'}{p}} |g|, \tag{3.3}$$

using in the last step the weak-1 boundedness of the classical Hardy–Littlewood maximal operator $\mathcal{M}$. Similarly, $T_n : L^\infty(W^{-\frac{p'}{p}}) \to L^\infty$ with $\|T_n\| \le 2^n$, since

$$\|T_n g\|_\infty = \sup_{x \in E_n} \left|\mathcal{M}^{\rm loc}\left(W^{-\frac{p'}{p}} g\right)(x)\right| \le 2^n \|g\|_\infty. \tag{3.4}$$

Thus, by the Marcinkiewicz interpolation theorem we obtain

$$\int_{E_n} |T_n(g)|^p \le c_p \, 2^{\frac{np}{p'}} \int_{\mathbb{R}_+} |g|^p \, W^{-\frac{p'}{p}}, \quad n \in \mathbb{Z}. \tag{3.5}$$

Setting $g = f\, W^{\frac{p'}{p}}$ in the above inequality, this is the same as

$$\int_{E_n} |\mathcal{M}^{\rm loc}(f)|^p \le c_p \, 2^{\frac{np}{p'}} \int_{\mathbb{R}_+} |f|^p \, W, \quad n \in \mathbb{Z}. \tag{3.6}$$

Now, modulo null sets $\mathbb{R}_+ = \cup_{n\in\mathbb{Z}}\big[E_n \setminus E_{n-1}\big]$ (since $0 < \mathcal{M}^{\rm loc}(W^{-\frac{p'}{p}})(x) < \infty$ at $a.e.\, x$), and we have

$$V_\varepsilon(x) \approx 2^{-\frac{np}{p'}} 2^{-\frac{\varepsilon|n|p}{p'}}, \text{ if } x \in E_n \setminus E_{n-1}.$$

Therefore, we obtain

$$\begin{aligned}\int_{\mathbb{R}_+} |\mathcal{M}^{\rm loc} f|^p \, V_\varepsilon &\lesssim \sum_{n\in\mathbb{Z}} 2^{-\frac{np}{p'}} 2^{-\frac{\varepsilon|n|p}{p'}} \int_{E_n} |\mathcal{M}^{\rm loc} f|^p \\ &\overset{\text{(by (3.6))}}{\lesssim} \left(\sum_{n\in\mathbb{Z}} 2^{-\frac{\varepsilon|n|p}{p'}}\right) \int_{\mathbb{R}_+} |f|^p W,\end{aligned}$$

as we wished to show. □

3.2 Properties of the weights V_ε

The weights V_ε inherit some of the integrability behavior of W if ε is sufficiently small.

Proposition 3.2 *Let $1 < p < \infty$ and $W \in D_p^{\rm loc}$. Then, for each $\varepsilon \in (0,1)$, the weight defined in (3.1) satisfies $V_\varepsilon \in D_q^{\rm loc}$ for all $q > p + \varepsilon p/p'$.*

Proof Observe that, for each $x \in \mathbb{R}_+$,

$$V_\varepsilon(x)^{-\frac{q'}{q}} = \max_{\pm} \left[\mathcal{M}^{\rm loc}\left(W^{-\frac{p'}{p}}\right)(x)\right]^{\frac{p-1}{q-1}(1\pm\varepsilon)}. \tag{3.7}$$

The assumption $q > p + \frac{\varepsilon p}{p'}$ implies that $s = \frac{(p-1)(1+\varepsilon)}{q-1} < 1$. Then, given $J = [a,b] \Subset \mathbb{R}_+$,

$$\begin{aligned}\int_J \left[\mathcal{M}^{\rm loc}\left(W^{-\frac{p'}{p}}\right)\right]^s &\lesssim |J|^{1-s} \left\|\mathcal{M}\left(W^{-\frac{p'}{p}} \chi_{J^*}\right)\right\|^s_{L^{1,\infty}} \\ &\lesssim c_J \left(\int_{J^*} W^{-\frac{p'}{p}}\right)^s < \infty,\end{aligned}$$

with $J^* = [a/2, Mb] \Subset (0, \infty)$. The same applies if we set $s = \frac{(p-1)(1-\varepsilon)}{q-1}$ (which is also < 1), so we deduce from (3.7) that $\int_J V_\varepsilon^{-q'/q} < \infty$. □

Our later applications suggest to define the following classes of weights

$$D_p^0(\beta) = \left\{ W \in D_p^{\text{loc}} \ : \ \int_0^1 W^{-\frac{p'}{p}}(y)\,\langle y\rangle^{\beta p'}\,\mathrm{d}y < \infty \right\}, \quad \text{for } \beta > -1,$$

$$D_p^{\exp_\lambda}(a) = \left\{ W \in D_p^{\text{loc}} \ : \ \int_1^\infty W^{-\frac{p'}{p}}(y)\,e^{-ay^\lambda p'}\,\mathrm{d}y < \infty \right\}, \quad \text{for } a, \lambda > 0.$$

Proposition 3.3 *Let* $1 < p < \infty$, $\varepsilon \in (0,1)$ *and* $q > p + \varepsilon p/p'$. *Then, for* $\beta, \beta_1 > -1$ *and* $a, a_1, \lambda > 0$ *we have the following*

(i) $W \in D_p^0(\beta)$ *implies* $V_\varepsilon \in D_q^0(\beta_1)$, *provided* $q > \frac{1+\beta}{1+\beta_1}p + \varepsilon\,\frac{p}{p'}\,\frac{|1+\beta p'|}{1+\beta_1}$

(ii) $W \in D_p^{\exp_\lambda}(a)$ *implies* $V_\varepsilon \in D_q^{\exp_\lambda}(a_1)$, *provided* $q > p(1+\varepsilon)M^\lambda a/a_1$.

Proof With the same notation in the proof of Proposition 3.2, we set $s = \frac{(p-1)(1+\varepsilon)}{q-1} < 1$. Then, denoting $I_j = [2^{-j-1}, 2^{-j}]$, we have

$$\begin{aligned}
\int_0^1 \left[\mathcal{M}^{\text{loc}}\left(W^{-\frac{p'}{p}}\right)\right]^s \langle y\rangle^{\beta_1 q'}\,\mathrm{d}y &\lesssim \sum_{j=0}^\infty 2^{-j\beta_1 q'}\,|I_j|^{1-s}\,\left\|\mathcal{M}\left(W^{-\frac{p'}{p}}\chi_{I_j^*}\right)\right\|_{L^{1,\infty}}^s \\
&\lesssim \sum_{j=0}^\infty 2^{-j\beta_1 q'}2^{-j(1-s)}\left(\int_{2^{-j-2}}^{M2^{-j}} W^{-\frac{p'}{p}}\right)^s \\
&\lesssim \sum_{j=0}^\infty 2^{-j[\beta_1 q' - \beta p' s + 1 - s]}\left(\int_0^M W^{-\frac{p'}{p}}(y)\,\langle y\rangle^{\beta p'}\,\mathrm{d}y\right)^s.
\end{aligned}$$

This is a finite expression when $W \in D_p^0(\beta)$, provided

$$\beta_1 q' - \beta p' s + 1 - s > 0\,.$$

Using the value of $s = \frac{(p-1)(1+\varepsilon)}{q-1}$ and solving for q, this is equivalent to

$$q > \frac{1+\beta}{1+\beta_1}p + \frac{\varepsilon p(1+\beta p')}{p'(1+\beta_1)}.$$

In order to have $\int_0^1 V_\varepsilon^{-q'/q}\,\langle y\rangle^{\beta_1 q'}\,\mathrm{d}y < \infty$, the previous relation must also hold with ε replaced by $-\varepsilon$, so a sufficient condition is

$$q > \frac{1+\beta}{1+\beta_1}p + \frac{\varepsilon p\,|1+\beta p'|}{p'(1+\beta_1)},$$

as we wished to show.

We now prove (ii). Let $\gamma > 1$ (to be precised later), and as before set $I_j = [\gamma^j, \gamma^{j+1}]$ and $s = \frac{(p-1)(1+\varepsilon)}{q-1} < 1$. Then

$$\begin{aligned}
&\int_1^\infty \left[\mathcal{M}^{\text{loc}}\left(W^{-\frac{p'}{p}}\right)\right]^s e^{-a_1 y^\lambda q'}\,\mathrm{d}y \\
&\lesssim \sum_{j=0}^\infty e^{-a_1\gamma^{\lambda j} q'}\,\gamma^{(1-s)j}\,\left\|\mathcal{M}\left(W^{-\frac{p'}{p}}\chi_{I_j^*}\right)\right\|_{L^{1,\infty}}^s \\
&\lesssim \sum_{j=0}^\infty e^{-a_1\gamma^{\lambda j} q'}\,\gamma^{(1-s)j}\,\left(\int_{\gamma^j/2}^{M\gamma^{j+1}} W^{-\frac{p'}{p}}\,\mathrm{d}y\right)^s \\
&\leq \sum_{j=0}^\infty \gamma^{(1-s)j}\,e^{-\gamma^{\lambda j}[a_1 q' - p' a M^\lambda \gamma^\lambda s]}\left(\int_{1/2}^\infty W^{-\frac{p'}{p}}\,e^{-p' a y^\lambda}\,\mathrm{d}y\right)^s.
\end{aligned}$$

This is now a finite expression provided

$$a_1 q' > p' a M^\lambda \gamma^\lambda s,$$

which using the value of s and solving for q gives

$$q > p(1+\varepsilon)M^\lambda\gamma^\lambda a/a_1.$$

Clearly, we can choose a $\gamma > 1$ with this property under the assumption

$$q > p(1+\varepsilon)M^\lambda a/a_1.$$

Since this also implies the validity of the estimates with ε replaced by $-\varepsilon$, we may conclude that $V_\varepsilon \in D_q^{\exp_\lambda}(a_1)$, as desired. □

4 2-weight inequalities for local maximal heat operators

Let L be as in (2.1), and for each $t_0 > 0$, consider

$$h_{t_0}^* f(x) := \sup_{0<t\leq t_0} \left|e^{-tL} f(x)\right|.$$

By Corollary 2.3, given any $\mathtt{T} > t_0$, this operator is well defined for all functions $f \in L^1(\varphi_\mathtt{T})$, where

$$\varphi_\mathtt{T}(y) = \langle y\rangle^{\alpha+\frac{1}{2}}\,e^{-\frac{y^2}{2\text{th}\,(2\mathtt{T})}}.$$

We wish to study 2-weight inequalities for $h_{t_0}^*$ over subspaces $L^p(w) \subset L^1(\varphi_\mathtt{T})$. By duality, the class of weights for which such inclusion holds is given by

$$D_p(\varphi_\mathtt{T}) := \left\{w > 0 \;:\; \left\|w^{-\frac{1}{p}}\varphi_\mathtt{T}\right\|_{p'} < \infty\right\}.$$

Here, we show that for all such weights the operator $h_{t_0}^*$ satisfies a 2-weight inequality.

Theorem 4.1 *Let $\mathtt{T} > t_0 > 0$ and $1 < p < \infty$. Then, for every $w \in D_p(\varphi_\mathtt{T})$ there exists another weight $v(x) > 0$ such that*

$$h_{t_0}^* : L^p(w) \to L^p(v), \quad \textit{boundedly}.$$

Moreover, if $q > p$ and t_0 is sufficiently small (depending on q/p and $\mathtt{T}$), then we can select $v \in D_q(\varphi_{\mathtt{T}})$.

Remark 4.2 The second weight $v(x)$ will be constructed explicitly; see (4.2), (4.5) and (4.6) below. Observe that v depends on α, p, t_0, $\mathtt{T}$ and of course w.

Proof of Theorem 4.1 The crucial estimate was already given in Corollary 2.3. We shall use it with the parameter $\gamma = \operatorname{th}(2\mathtt{T})/\operatorname{th}(2t_0) > 1$, which produces a suitable $M = M_\gamma > 1$ such that

$$h^*_{t_0} f(x) \lesssim \mathcal{M}^{\mathrm{loc}}_M f(x) + c(x) \int_0^\infty |f(y)|\,\varphi_{\mathtt{T}}(y)\,\mathrm{d}y. \tag{4.1}$$

The last integral is bounded by $\|f\|_{L^p(w)} \|w^{-\frac{1}{p}}\varphi_{\mathtt{T}}\|_{p'}$, so the second term will be fine for any weight $v(x)$ such that $c(x) = 1/\langle x\rangle^{\alpha+\frac{3}{2}} \in L^p(v)$. For instance, we may take

$$v_2(x) = \frac{\langle x\rangle^{\left(\alpha+\frac{3}{2}\right)p-1}}{[\ln(e/\langle x\rangle)]^2\,(1+x)^p} \tag{4.2}$$

which clearly satisfies

$$\int_0^\infty |c(x)|^p\, v_2(x)\,\mathrm{d}x = \int_0^\infty \frac{\mathrm{d}x}{\langle x\rangle[\ln(e/\langle x\rangle)]^2\,(1+x)^p} < \infty.$$

Further, we claim that $v_2 \in D_q(\varphi_{\mathtt{T}})$ iff $q > p$. Indeed, since $\varphi_{\mathtt{T}}(x)$ decays exponentially, it suffices to check the integrability for x near 0. Writing $\beta = \alpha + \frac{1}{2}$ so that

$$\varphi_{\mathtt{T}}(x) \approx \langle x\rangle^\beta \quad \text{and} \quad v_2(x) \approx \frac{\langle x\rangle^{(\beta+1)p-1}}{[\ln(e/\langle x\rangle)]^2},$$

we easily see that

$$\int_0^1 v_2(x)^{-\frac{q'}{q}}\,\langle x\rangle^{\beta q'}\,\mathrm{d}x \approx \int_0^1 \frac{[\ln(e/x)]^{2q'/q}}{x^{1-q'(\beta+1)\left(1-\frac{p}{q}\right)}}\,\mathrm{d}x < \infty. \tag{4.3}$$

For the first term in (4.1), we shall use the results in Sect. 3. We first note that

$$w \in D_p(\varphi_{\mathtt{T}}) \iff w \in D^0_p(\alpha + \tfrac{1}{2}) \cap D^{\exp_2}_p(a), \quad \text{with } a = 1/(2\operatorname{th} 2\mathtt{T}), \tag{4.4}$$

where the weight classes $D^0_p(\beta)$ and $D^{\exp_2}_p(a)$ were defined in §3.2. Then, for every $\varepsilon > 0$ Theorem 3.1 gives

$$\|\mathcal{M}^{\mathrm{loc}}_M f\|_{L^p(v_{1,\varepsilon})} \lesssim \|f\|_{L^p(w)},$$

provided

$$v_{1,\varepsilon}(x) = \mathscr{V}(x)\rho_\varepsilon\Big(\mathscr{V}(x)\Big), \quad \text{where } \mathscr{V}(x) = \left[\mathcal{M}^{\mathrm{loc}}_M\left(w^{-\frac{p'}{p}}\right)(x)\right]^{-\frac{p}{p'}} \tag{4.5}$$

(or $v_{1,\varepsilon} = V_\varepsilon$ in the notation of (3.1)). Hence, setting

$$v(x) = \min\{v_{1,\varepsilon}(x), v_2(x)\} \tag{4.6}$$

with $v_{1,\varepsilon}$ and v_2 defined as in (4.5) and (4.2), we have proved that $h^*_{t_0} : L^p(w) \to L^p(v)$.

It remains to verify the last statement in Theorem 4.1. We already know that, for every $q > p$, we have $v_2 \in D_q(\varphi_{\mathtt{T}})$. Concerning $v_{1,\varepsilon}$, from the equivalence in (4.4) it suffices to prove that $V_\varepsilon \in D_q^0(\alpha+\frac{1}{2}) \cap D_q^{\exp_2}(a)$ for a sufficiently small ε and $a = 1/(2\text{th}\, 2\mathrm{T})$. The first assertion is immediate from (i) in Proposition 3.3. However, (ii) in the same proposition only gives $V_\varepsilon \in D_\rho^{\exp_2}(a)$ if $\rho > p(1+\varepsilon)M^2$, where $M = M_\gamma$ is the parameter obtained in Proposition 2.2 by the rule $\left(\frac{M}{M-1}\right)^3 = \gamma = \text{th}\,(2\mathrm{T})/\text{th}\,(2t_0)$. If we allow both ε and t_0 to be sufficiently small (so that M becomes close enough to 1), then we can set $\rho = q$ and hence conclude that $v_{1,\varepsilon} \in D_q(\varphi_\mathrm{T})$ as desired. □

5 Poisson kernel estimates

In this section, we fix $\alpha > -1$ and $\mu \geq -(\alpha+1)$ and recall that

$$L = -\partial_{yy} + \left[y^2 + \frac{\alpha^2 - \frac{1}{4}}{y^2} \right] + 2\mu. \tag{5.1}$$

The eigenfunctions $\{\varphi_n^\alpha\}_{n=0}^\infty$ form an orthonormal basis of $L^2(0,\infty)$, and satisfy

$$L\varphi_n^\alpha = (4n + 2(\alpha+1+\mu))\,\varphi_n^\alpha, \quad n = 0, 1, 2, \ldots$$

They can be expressed in terms of the (normalized) Laguerre polynomials L_n^α by

$$\varphi_n^\alpha(y) \;=\; \sqrt{2}\, y^{\alpha+\frac{1}{2}}\, e^{-\frac{y^2}{2}}\, L_n^\alpha(y^2), \tag{5.2}$$

although we shall not use this formula here. The kernel of the associated heat semigroup, e^{-tL}, can be written explicitly in various forms

$$e^{-tL}(x,y) = \sum_{n=0}^\infty e^{-[4n+2(\alpha+1+\mu)]t} \varphi_n^\alpha(x)\varphi_n^\alpha(y)$$

$$(r = e^{-2t}) = r^\mu \sqrt{\tfrac{2r}{1-r^2}}\; \bar{I}_\alpha\left(\frac{2rxy}{1-r^2}\right) e^{-\frac{(x-ry)^2}{1-r^2}}\, e^{\frac{x^2-y^2}{2}} \tag{5.3}$$

$$(s = \text{th}\, t) = \left(\tfrac{1-s}{1+s}\right)^\mu \sqrt{\tfrac{1-s^2}{2s}}\; \bar{I}_\alpha\left(\frac{(1-s^2)xy}{2s}\right) e^{-\frac{(x-y)^2}{4s}}\, e^{-\frac{s(x+y)^2}{4}} \tag{5.4}$$

where as before we have set $\bar{I}_\alpha(z) = \sqrt{z}e^{-z} I_\alpha(z)$. Thus, using the notation $\langle z\rangle = \min\{z,1\}$, we shall have $\bar{I}_\alpha(z) \approx \langle z\rangle^{\alpha+\frac{1}{2}}$. Both expressions of the heat kernel will be useful in our later computations. For instance, (5.3) is good when $r \approx 0$, as it isolates correctly the decaying factor $\langle y\rangle^{\alpha+\frac{1}{2}}\, e^{-y^2/2}$. On the other hand, (5.4) will be useful when $s \approx 0$ (hence $r \approx 1$), since it makes transparent the gaussian behavior of the singularity $s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4s}}$.

Using the subordination formula in (1.3), the Poisson-like kernel associated with L becomes

$$P_t(x,y) := \frac{t^{2\nu}}{4^\nu \Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4u}} \left[e^{-uL}(x,y)\right] \frac{\mathrm{d}u}{u^{1+\nu}}, \quad t > 0. \tag{5.5}$$

Changing variables $r = e^{-2u}$ (i.e., $u = \frac{1}{2}\ln\frac{1}{r}$), one sees that

$$P_t(x,y) \;\approx\; t^{2\nu}\, e^{\frac{x^2-y^2}{2}} \int_0^1 e^{-\frac{t^2}{2\ln\frac{1}{r}}}\, r^{\mu+\frac{1}{2}}\, \frac{e^{-\frac{(x-ry)^2}{1-r^2}}}{\sqrt{1-r}}\, \frac{\left\langle \frac{rxy}{1-r^2}\right\rangle^{\alpha+\frac{1}{2}}}{\left(\ln\frac{1}{r}\right)^{1+\nu}}\, \frac{dr}{r}.$$

We shall consider two regions of integration according to the behavior of $z := \frac{rxy}{1-r^2}$. The regions will be separated by the number

$$r_0(xy) = \begin{cases} \frac{1}{2xy} & , \text{ if } xy \geq 1 \\ 1 - \frac{xy}{2} & , \text{ if } xy \leq 1. \end{cases}$$

Indeed, it is elementary to check that

(1) If $0 < r \leq r_0(xy)$ then $z \leq 1$.
(2) If $r_0(xy) \leq r < 1$ then $z \geq 1/2$.

Thus we can write

$$\begin{aligned} P_t(x,y) &\approx t^{2\nu} e^{\frac{x^2-y^2}{2}} \left[\int_0^{r_0(xy)} \cdots \left(\frac{rxy}{1-r^2} \right)^{\alpha+\frac{1}{2}} \frac{dr}{r} + \int_{r_0(xy)}^1 \cdots \frac{dr}{r} \right] \\ &= B_t(x,y) + A_t(x,y), \end{aligned}$$

where in $A_t(x,y)$ we have used that $\langle z \rangle^{\alpha+\frac{1}{2}} \approx 1$. The next two propositions summarize the estimates we shall need to handle these kernels. We shall make extensive use of the function Φ in (1.4), which we can equivalently write as

$$\Phi(y) \approx \frac{\langle y \rangle^{\alpha+\frac{1}{2}} e^{-y^2/2}}{(1+y)^{\mu+\frac{1}{2}} [\ln(y+e)]^{1+\nu}}, \tag{5.6}$$

with the agreement that in the extreme case $\mu = -(\alpha+1)$ the logarithm in the denominator is just $[\ln(y+e)]^{\nu}$. The first result gives, for fixed t and x, the *optimal decay* of $y \mapsto P_t(x,y)$ in terms of the function $\Phi(y)$.

Proposition 5.1 *Given $t, x > 0$, there exist $c_1(t,x) > 0$ and $c_2(t,x) > 0$ such that*

$$c_1(t,x)\,\Phi(y) \leq P_t(x,y) \leq c_2(t,x)\,\Phi(y), \quad y \in \mathbb{R}_+. \tag{5.7}$$

The second result is a refinement of the upper bound in (5.7) with a few advantages: it is uniform in the variable t, isolates in the “local part” the singularities of the kernel $P_t(x,y)$, and finally provides “reasonable” bounds for the constant's dependence on x.

Proposition 5.2 *There exists $M > 1$ such that the following holds for all $t, x, y > 0$*

$$P_t(x,y) \lesssim C_1(x) \frac{t^{2\nu} e^{-y^2/2}}{\big(t+|x-y|\big)^{1+2\nu}} \chi_{\left\{\frac{x}{2}<y<Mx\right\}} + C_2(x)\,(t \vee 1)^{2\nu}\,\Phi(y), \tag{5.8}$$

where $C_1(x) = (1+x)^{2\nu} e^{\frac{x^2}{2}}$ and $C_2(x) = [\ln(e+x)]^{1+\nu} (1+x)^{|\mu+\frac{1}{2}|} e^{\frac{x^2}{2}} / \langle x \rangle^{\alpha+\frac{3}{2}}$.

If we consider, for fixed $M > 1$, the *local maximal operator* $\mathcal{M}_M^{\text{loc}}$ in (1.10), then we may express (5.8) as follows.

Corollary 5.3 *Let $t_0 > 0$ be fixed. Then, there is some $M > 1$ such that*

$$P_{t_0}^* f(x) \lesssim C_1(x)\,\mathcal{M}_M^{\text{loc}}\big(f e^{-\frac{y^2}{2}}\big)(x) + C_2(x)\,\big\|f\big\|_{L^1(\Phi)}, \quad x \in \mathbb{R}_+, \tag{5.9}$$

with $C_1(x)$ and $C_2(x)$ as in Proposition 5.2.

This is the key estimate from which we shall deduce the theorems claimed in Sect. 1 for the operator L. The reader willing to skip the technical proofs of the propositions in the next subsections may pass directly to Sect. 6 for the proof of the theorems.

5.1 Estimates from below for $B_t(x, y)$

Recall that

$$B_t(x,y) \approx t^{2\nu}\,(xy)^{\alpha+\frac12}\,e^{\frac{x^2-y^2}{2}}\int_0^{r_0(xy)}\frac{r^{\alpha+\mu+1}}{(1-r)^{\alpha+1}\,(\ln\frac1r)^{1+\nu}}\,e^{-\frac{t^2}{2\ln\frac1r}}\,e^{-\frac{(x-ry)^2}{1-r^2}}\,\frac{dr}{r}\,. \tag{5.10}$$

The lower bound in Proposition 5.1 will be a consequence of $P_t(x, y) \gtrsim B_t(x, y)$ and the estimate in the next lemma.

Lemma 5.4 *For fixed $t, x > 0$, it holds*

$$B_t(x,y) \ge c_1(t,x)\,\Phi(y), \quad y \in \mathbb{R}_+,$$

for a suitable function $c_1(t, x) > 0$.

Proof We first look at $y < x \wedge \frac1x$. Then, $xy < 1$, and hence $r_0(xy) > 1/2$. So, we can estimate $B_t(x, y)$ by an integral over $0 < r < \frac12$, which disregarding irrelevant terms becomes

$$B_t(x,y) \gtrsim t^{2\nu}\,(xy)^{\alpha+\frac12}\,e^{\frac{x^2-y^2}{2}}\int_0^{\frac12}\frac{r^{\alpha+\mu+1}}{(\ln\frac1r)^{1+\nu}}\,e^{-\frac{t^2}{2\ln\frac1r}}\,e^{-\frac{(x-ry)^2}{1-r^2}}\,\frac{dr}{r}.$$

We can get rid of the first two exponentials using

$$e^{\frac{x^2-y^2}{2}} \ge 1 \quad (\text{since } y \le x) \quad \text{and} \quad e^{-\frac{t^2}{2\ln\frac1r}} \ge e^{-\frac{t^2}{2\ln 2}} \quad \left(\text{since } r \le \frac12\right).$$

For the last exponential, notice that $0 < x - ry \le x$, and hence $e^{-\frac{(x-ry)^2}{1-r^2}} \ge e^{-\frac43 x^2}$. This leaves a convergent integral in r, so we conclude that

$$B_t(x,y) \gtrsim c_1(t,x)\,\langle y\rangle^{\alpha+\frac12}\,,$$

with $c_1(t, x) = t^{2\nu}x^{\alpha+\frac12}e^{-\frac{t^2}{2\ln 2}}e^{-\frac43 x^2}$. Notice that $y \le 1$ in this range, so we find the required expression for $\Phi(y)$.

Suppose now that $y \ge x \vee \frac1x$. Then, $xy \ge 1$, and hence $r_0(xy) = \frac{1}{2xy} \le 1/2$. Arguing as before, we can estimate $B_t(x, y)$ by

$$B_t(x,y) \gtrsim t^{2\nu}\,(xy)^{\alpha+\frac12}\,e^{-\frac{y^2}{2}}\,e^{-\frac{t^2}{2\ln 2}}\int_0^{\frac{1}{2xy}}\frac{r^{\alpha+\mu+1}}{\left(\ln\frac1r\right)^{1+\nu}}\,e^{-\frac{(x-ry)^2}{1-r^2}}\,\frac{dr}{r}.$$

This time, we get rid of the exponential inside the integral using

$$|x-ry| \le x+ry \le x+\frac{1}{2x} \quad \left(\text{since } r \le \frac{1}{2xy}\right),$$

which implies $e^{-\frac{(x-ry)^2}{1-r^2}} \ge e^{-\frac43(x+\frac1x)^2}$. We can easily compute the integral

$$\int_0^{\frac{1}{2xy}}\frac{r^{\alpha+\mu+1}}{\left(\ln\frac1r\right)^{1+\nu}}\,\frac{dr}{r} \approx \frac{1}{(xy)^{\alpha+\mu+1}\,[\ln 2xy]^{1+\nu}}, \quad \text{if } \alpha+\mu+1 > 0,$$

with the right hand side becoming $1/[\ln 2xy]^\nu$ in the extreme case $\alpha + 1 + \mu = 0$. Since $y > \max\{x, 1\}$, note that

$$\ln(2xy) \le \ln(2y^2) \approx \ln(y+e).$$

Thus, combining all the previous estimates we conclude that

$$B_t(x, y) \gtrsim c_1(t, x) \frac{e^{-\frac{y^2}{2}}}{y^{\mu+\frac{1}{2}} [\ln (y + e)]^{1+\nu}},$$

which, since $y \geq 1$, is the required expression for $\Phi(y)$ (with the usual agreement when $\mu + \alpha + 1 = 0$). In this part we have set $c_1(t, x) = t^{2\nu} e^{-\frac{t^2}{2\ln 2}} x^{-(\mu+\frac{1}{2})} e^{-\frac{4}{3}(x+\frac{1}{x})^2}$.

Finally, since the function $y \mapsto B_t(x, y)/\Phi(y)$ is continuous and positive, it is also bounded from below by some $c_1(t, x)$ when y belongs to the compact set $[x \wedge \frac{1}{x}, x \vee \frac{1}{x}]$. □

5.2 Estimates from above for $B_t(x, y)$

The next lemma, combined with the previous one, shows that for fixed t and x, the function $B_t(x, y)$ essentially behaves like $\Phi(y)$.

Lemma 5.5 *For every $t, x, y > 0$, it holds*

$$B_t(x, y) \lesssim c(x) \max\{t^{2\nu}, 1\} \Phi(y), \tag{5.11}$$

with $c(x) = 1/\langle x\rangle^{\alpha+\frac{3}{2}}$.

Proof We first notice that the two exponential factors in (5.3) can be written as

$$e^{-\frac{(x-ry)^2}{1-r^2}} e^{\frac{x^2-y^2}{2}} = e^{-\frac{1+r^2}{1-r^2}\frac{x^2+y^2}{2}} e^{\frac{2rxy}{1-r^2}} \lesssim e^{-\frac{x^2+y^2}{2}}, \tag{5.12}$$

since $\frac{1+r^2}{1-r^2} \geq 1$ and in the region of integration of $B_t(x, y)$ the exponent $z = \frac{2rxy}{1-r^2} \lesssim 1$. We now separate cases.

(i) *Case* $xy \geq 1$: then $r_0(xy) = \frac{1}{2xy} \leq \frac{1}{2}$ and

$$B_t(x, y) \lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} \int_0^{\frac{1}{2xy}} \frac{r^{\alpha+\mu+1}}{\left(\ln \frac{1}{r}\right)^{\nu+1}} \frac{dr}{r}. \tag{5.13}$$

The last integral is approximately given by

$$\int_0^{\frac{1}{2xy}} \frac{r^{\alpha+\mu+1}}{\left(\ln \frac{1}{r}\right)^{\nu+1}} \frac{dr}{r} \approx \left(\frac{1}{xy}\right)^{\alpha+\mu+1} \frac{1}{[\ln (2xy)]^{1+\nu}}$$

(with the usual convention when $\alpha + \mu + 1 = 0$ of reducing one power of the logarithm). This is a good estimate if we assume that $x \geq 1/2$, since we may use

$$\ln (2xy) \gtrsim \ln (y \vee 2) \approx \ln (y + e),$$

and overall obtain

$$B_t(x, y) \lesssim t^{2\nu} x^{-(\mu+\frac{1}{2})} e^{-\frac{x^2}{2}} \frac{y^{\alpha+\frac{1}{2}} e^{-y^2/2}}{y^{\alpha+\mu+1}[\ln (y + e)]^{1+\nu}} \lesssim t^{2\nu} \Phi(y).$$

When $x \leq 1/2$, we need a refinement to obtain the $c(x)$ in the statement of the lemma. We split the integral bounding $B_t(x, y)$ in (5.13) as

$$B_t(x, y) \lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} \left[\int_0^{\frac{2x}{y}} \cdots + \int_{\frac{2x}{y}}^{r_0(xy)} \cdots\right] = I + II, \tag{5.14}$$

noticing that the partition point $\frac{2x}{y} \le r_0(xy) = \frac{1}{2xy}$. Since $x \le \frac{1}{2}$ and $xy \ge 1$, we also have $y \ge 2$. Now, the first summand can be bound as above by

$$I \lesssim t^{2\nu}\,(xy)^{\alpha+\frac{1}{2}}\,e^{-\frac{x^2+y^2}{2}}\,\left(\frac{x}{y}\right)^{\alpha+\mu+1}\,\frac{1}{\left[\ln\left(\frac{y}{2x}\right)\right]^{1+\nu}}$$
$$\lesssim t^{2\nu}\,\langle x\rangle^{\alpha+\frac{1}{2}}\,x^{\alpha+\mu+1}\,e^{-\frac{x^2}{2}}\,\frac{e^{-y^2/2}}{y^{\mu+\frac{1}{2}}[\ln y]^{1+\nu}} \lesssim t^{2\nu}\,\langle x\rangle^{\alpha+\frac{1}{2}}\,\Phi(y).$$

since in this range $y \ge 2$. This implies the stated estimate because $\langle x\rangle^{\alpha+\frac{1}{2}} \le 1/\langle x\rangle^{\alpha+\frac{3}{2}} = c(x)$. To handle II, we need a different bound for the exponentials in (5.12), noticing that

$$r > \frac{2x}{y} \quad\Rightarrow\quad |x-ry| = ry - x \ge \frac{ry}{2} \quad\Rightarrow\quad e^{-\frac{(x-ry)^2}{1-r^2}} \le e^{-\frac{r^2y^2}{4}}. \tag{5.15}$$

Thus

$$II \lesssim t^{2\nu}\,(xy)^{\alpha+\frac{1}{2}}\,e^{\frac{x^2-y^2}{2}}\int_{\frac{2x}{y}}^{\frac{1}{2}}\frac{r^{\alpha+\mu+1}\,e^{-\frac{(ry)^2}{4}}}{\left(\ln\frac{1}{r}\right)^{\nu+1}}\,\frac{dr}{r}. \tag{5.16}$$

Changing variables $ry = u$, the latter integral can be estimated by

$$y^{-(\alpha+\mu+1)}\int_0^{\frac{y}{2}}\frac{u^{\alpha+\mu+1}\,e^{-\frac{u^2}{4}}}{\left(\ln\frac{y}{u}\right)^{\nu+1}}\,\frac{\mathrm{d}u}{u} \approx \frac{1}{y^{\alpha+\mu+1}\,[\ln y]^{1+\nu}}$$

since the major contribution happens when $u \approx 1$ (with the usual convention of reducing one power of the logarithm if $\alpha+\mu+1=0$). Inserting this into (5.16) (and using $y \ge 2$ and $x \le 1/2$), we obtain once again

$$II \lesssim t^{2\nu}\,\langle x\rangle^{\alpha+\frac{1}{2}}\,\Phi(y).$$

This concludes the proof of the case $xy \ge 1$.

(ii) *Case* $xy \le 1$: this time $r_0(xy) = 1 - \frac{xy}{2} \ge \frac{1}{2}$, so we can write

$$B_t(x,y) \lesssim t^{2\nu}\,(xy)^{\alpha+\frac{1}{2}}\,e^{-\frac{x^2+y^2}{2}}\left[\int_0^{\frac{1}{2}}\cdots + \int_{\frac{1}{2}}^{r_0(xy)}\cdots\right] = B_1 + B_2$$

The first term can be handled essentially as in the previous case. Namely, if $y \le 2$ we use a similar bound to (5.13)

$$B_1 \lesssim t^{2\nu}\,(xy)^{\alpha+\frac{1}{2}}\,e^{-\frac{x^2+y^2}{2}}\int_0^{\frac{1}{2}}\frac{r^{\alpha+\mu+1}}{\left(\ln\frac{1}{r}\right)^{\nu+1}}\,\frac{dr}{r} \approx t^{2\nu}\,x^{\alpha+\frac{1}{2}}\,e^{-\frac{x^2}{2}}\,\langle y\rangle^{\alpha+\frac{1}{2}}$$
$$\lesssim t^{2\nu}\,\langle x\rangle^{\alpha+\frac{1}{2}}\,\Phi(y).$$

If $y \ge 2$, then $x \le \frac{1}{2}$ and $\frac{2x}{y} \le \frac{1}{2}$, so we may split

$$B_1 \le \int_0^{\frac{2x}{y}}\cdots + \int_{\frac{2x}{y}}^{\frac{1}{2}}\cdots$$

and exactly the same computations we used in (5.14), give us the bound

$$B_1 \lesssim t^{2\nu}\,\langle x\rangle^{\alpha+\frac{1}{2}}\,\Phi(y).$$

Thus we are left with the integral corresponding to B_2, that is the range $\frac{1}{2} < r < 1 - \frac{xy}{2}$. First of all, observe that

$$\ln \frac{1}{r} \approx 1 - r, \quad r \in [1/2, 1] \quad \Rightarrow \quad e^{-\frac{t^2}{2 \ln \frac{1}{r}}} \leq e^{-\frac{ct^2}{1-r}},$$

for a suitable $c > 0$. Next, we need once again more precise bounds for the exponentials in (5.12). We claim that if $r \in [1/2, 1]$, then

$$e^{-\frac{1+r^2}{1-r^2}\frac{x^2+y^2}{2}} \leq e^{-\gamma \frac{x^2+y^2}{1-r}} e^{-(1+\gamma)\frac{x^2+y^2}{2}}$$

for a small constant $\gamma > 0$. This is easily obtained using the fact that $\frac{1+r^2}{1-r^2} \geq \frac{5}{3}$ in this interval. With these exponential bounds, we can control the integral B_2 as follows

$$\begin{aligned} B_2 &\lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-(1+\gamma)\frac{x^2+y^2}{2}} \int_{1/2}^{1-\frac{xy}{2}} \frac{e^{-\frac{ct^2+\gamma(x^2+y^2)}{1-r}}}{(1-r)^{\alpha+\nu+1}} \frac{dr}{1-r} \\ &\lesssim \frac{t^{2\nu} (xy)^{\alpha+\frac{1}{2}}}{[t^2+x^2+y^2]^{\alpha+\nu+1}} e^{-(1+\gamma)\frac{x^2+y^2}{2}} \int_0^\infty e^{-u} u^{\alpha+\nu+1} \frac{du}{u}, \end{aligned} \tag{5.17}$$

after changing variables $u = [ct^2 + \gamma(x^2+y^2)]/(1-r)$. The last integral is a finite constant (because $\alpha + \nu + 1 > 0$), so we observe two possible cases:

(1) if $\max\{y, t, x\} \geq 1$, we can disregard the denominator and obtain

$$B_2 \lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-(1+\gamma)\frac{x^2+y^2}{2}} \lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} \Phi(y),$$

since the exponential decay in y is actually better than $\Phi(y)$.

(2) if $\max\{y, t, x\} \leq 1$, we bound the denominator in the two obvious ways to obtain

$$B_2 \lesssim \frac{t^{2\nu} (xy)^{\alpha+\frac{1}{2}}}{t^{2\nu} x^{2(\alpha+1)}} = \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}}, \tag{5.18}$$

which is precisely the upper bound stated in (5.11). Observe that when $x \to 0$ this piece gives the largest contribution to $B_t(x, y)$.

□

Remark 5.6 It is also possible to obtain a bound

$$B_t(x, y) \lesssim c'(x)\, t^{2\nu}\, \Phi(y), \tag{5.19}$$

with perhaps a worse function $c'(x)$, but without the loss produced by $\max\{1, t^{2\nu}\}$. This loss appeared when $t, x, y \leq 1$ in (5.18) above. Looking at (5.17), we may replace that bound by

$$B_2 \lesssim \frac{t^{2\nu} (xy)^{\alpha+\frac{1}{2}}}{x^{2(\alpha+\nu+1)}},$$

which implies (5.19) with $c'(x) = 1/\langle x \rangle^{\alpha+\frac{3}{2}+2\nu}$. This estimate will also be useful later.

5.3 Upper estimates for $A_t(x, y)$: integrals over $r \le 1/2$

Recall that

$$A_t(x,y) \approx t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{r_0(xy)}^{1} \frac{r^{\mu+\frac{1}{2}}}{\sqrt{1-r}\,\left(\ln\frac{1}{r}\right)^{1+\nu}} e^{-\frac{t^2}{2\ln\frac{1}{r}}} e^{-\frac{(x-ry)^2}{1-r^2}} \frac{dr}{r}. \tag{5.20}$$

It will be convenient to split

$$A_t(x,y) = \int_{r_0(xy)\wedge\frac{1}{2}}^{\frac{1}{2}} \cdots + \int_{r_0(xy)\vee\frac{1}{2}}^{1} \cdots = A1 + A2.$$

When $xy < 1$, since $r_0(xy) = 1 - \frac{xy}{2} > \frac{1}{2}$, we have $A1 = 0$ (and $A_t(x,y) = A2$). When $xy \ge 1$, since $r_0(xy) = \frac{1}{2xy} \le \frac{1}{2}$, both terms $A1$ and $A2$ play a role. In this section, we shall only estimate $A1$.

Lemma 5.7 *If $xy \ge 1$, then*

$$A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}}}{\left(\ln\frac{1}{r}\right)^{1+\nu}} e^{-\frac{(x-ry)^2}{1-r^2}} \frac{dr}{r} \lesssim c(x)\, t^{2\nu}\, \Phi(y), \tag{5.21}$$

where $c(x) = [\ln(e+x)]^{\nu+1}\,(1+x)^{|\mu+\frac{1}{2}|}\,\exp(x^2/2)$.

Proof We shall distinguish cases

(i) *Case* $y \le 4x$. In this region, we essentially disregard the exponential factor $e^{-\frac{(x-ry)^2}{1-r^2}}$ inside the integral, and directly estimate

$$A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}}}{(\ln\frac{1}{r})^{1+\nu}} \frac{dr}{r}. \tag{5.22}$$

Notice however that when $y \le x$ such exponential produces an additional gain, due to

$$ry \le \tfrac{x}{2} \quad\Rightarrow\quad |x - ry| \ge \tfrac{x}{2} \quad\Rightarrow\quad e^{-\frac{(x-ry)^2}{1-r^2}} \le e^{-\frac{x^2}{4}}. \tag{5.23}$$

In particular, we can use it when $y \le 1$ (since then $x \ge xy \ge 1$) to control some of the powers of x appearing below.

We now evaluate the integral in (5.22), depending on the sign of $\mu + \frac{1}{2}$.

(1) If $\mu + \frac{1}{2} \ge 0$ the integral is bounded by a constant, and hence

$$A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}}.$$

We shall enlarge this value to match (5.21) as follows. Since $xy \ge 1$, in this range we have $x \ge 1/2$. So if $1 \le y \le 4x$ we may use

$$1 \lesssim \frac{(1+x)^{\mu+\frac{1}{2}}[\ln(e+x)]^{\nu+1}}{(1+y)^{\mu+\frac{1}{2}}[\ln(e+y)]^{\nu+1}}.$$

If $\frac{1}{x} \le y \le 1$, we use instead

$$1 \lesssim \max\left\{x^{\alpha+\frac{1}{2}}, 1\right\} \langle y\rangle^{\alpha+\frac{1}{2}},$$

which in this region can be combined with the extra exponential in factor in (5.23). In both cases, we obtain $A1 \lesssim t^{2\nu} c(x)\Phi(y)$, as wished.

(2) If $\mu + \frac{1}{2} < 0$, the integral diverges near 0, but we still obtain

$$\int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}}}{\left(\ln \frac{1}{r}\right)^{1+\nu}} \frac{dr}{r} \lesssim \frac{1}{(xy)^{\mu+\frac{1}{2}} (\ln 2xy)^{\nu+1}}.$$

Thus, using the inequality $\ln(2xy) \gtrsim \max\{\ln y, \ln 2\}$ we arrive at

$$A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \frac{(1+x)^{|\mu+\frac{1}{2}|}}{(1+y)^{\mu+\frac{1}{2}} [\ln(e+y)]^{\nu+1}} \leq t^{2\nu} c(x)\Phi(y), \quad \text{if } 1 \leq y \leq 4x$$

$$A1 \lesssim t^{2\nu} \frac{e^{\frac{x^2}{4}} y^{\alpha+\frac{1}{2}}}{x^{\mu+\frac{1}{2}} y^{\mu+\alpha+1}} \leq t^{2\nu} e^{\frac{x^2}{4}} x^{\alpha+\frac{1}{2}} \langle y\rangle^{\alpha+\frac{1}{2}} \leq t^{2\nu} c(x)\Phi(y), \quad \text{if } \frac{1}{x} \leq y \leq 1,$$

after inserting in the first step of this last case the additional exponential gain in (5.23).

(ii) *Case* $y \geq 4x$. This is the same as $\frac{2x}{y} \leq \frac{1}{2}$, and remember from (5.15) that when $r \in \left[\frac{2x}{y}, \frac{1}{2}\right]$ a better bound for the exponential is available, namely

$$e^{-\frac{(x-ry)^2}{1-r^2}} \leq e^{-\frac{r^2y^2}{4}}. \tag{5.24}$$

Thus we may consider two subcases, depending on whether $\frac{2x}{y}$ is above or below $r_0(xy)$.

- *Subcase* $\frac{2x}{y} \leq r_0(xy) = \frac{1}{2xy} \leq \frac{1}{2}$. Using (5.24) we obtain

$$A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}} e^{-\frac{(ry)^2}{4}}}{(\ln \frac{1}{r})^{1+\nu}} \frac{dr}{r}$$

$$(ry = u) = \frac{t^{2\nu} e^{\frac{x^2-y^2}{2}}}{y^{\mu+\frac{1}{2}}} \int_{\frac{1}{2x}}^{y/2} \frac{u^{\mu+\frac{1}{2}} e^{-\frac{u^2}{4}}}{\left(\ln \frac{y}{u}\right)^{1+\nu}} \frac{du}{u}$$

Observe that $x \leq \frac{1}{2}$ (and $y \geq 2$), so the latter integral reaches its major contribution at $u = \frac{1}{2x}$

$$\int_{\frac{1}{2x}}^{y/2} \frac{u^{\mu+\frac{1}{2}} e^{-\frac{u^2}{4}}}{\left(\ln \frac{y}{u}\right)^{1+\nu}} \frac{du}{u} \lesssim \frac{(1/x)^{\mu-\frac{1}{2}} e^{-c/x^2}}{(\ln 2xy)^{1+\nu}} \lesssim \frac{1}{(\ln y)^{1+\nu}},$$

using in the last step the elementary bound of logarithms

$$\ln(2xy) \gtrsim \frac{\ln(y+e)}{\ln\left(\frac{1}{x}+e\right)}, \quad \text{if } y \geq \max\{1, 1/x\},$$

which is easily verified considering the cases $x > 1/\sqrt{y}$ and $x \leq 1/\sqrt{y}$. Thus we conclude that

$$A1 \lesssim t^{2\nu} e^{\frac{x^2}{2}} \Phi(y).$$

- *Subcase* $r_0(xy) < \frac{2x}{y} \leq \frac{1}{2}$. Here we split

$$A1 = \int_{\frac{2x}{y}}^{\frac{1}{2}} \cdots + \int_{r_0(xy)}^{\frac{2x}{y}} \cdots = I + II.$$

The first term is similar to the previous subcase, except that now $x > \frac{1}{2}$ (and $y \geq 4x \geq 2$)

$$I \lesssim \frac{t^{2\nu}\, e^{\frac{x^2-y^2}{2}}}{y^{\mu+\frac{1}{2}}} \int_{2x}^{y/2} \frac{u^{\mu+\frac{1}{2}}\, e^{-\frac{u^2}{4}}}{\left(\ln \frac{y}{u}\right)^{1+\nu}} \frac{\mathrm{d}u}{u}$$

and the last integral is bounded by constant times

$$\frac{x^{\mu-\frac{1}{2}}\, e^{-cx^2}}{\left(\ln \frac{y}{2x}\right)^{1+\nu}} \lesssim x^{\mu-\frac{1}{2}}\, e^{-cx^2} \frac{[\ln (e+x)]^{1+\nu}}{[\ln (e+y)]^{1+\nu}} \lesssim \frac{1}{[\ln (e+y)]^{1+\nu}}.$$

Finally, we consider II. Here, there is no exponential gain, and similarly to (5.22) we have

$$II \lesssim t^{2\nu}\, e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{2x}{y}} \frac{r^{\mu+\frac{1}{2}}}{\left(\ln \frac{1}{r}\right)^{1+\nu}} \frac{dr}{r} = \frac{t^{2\nu}\, e^{\frac{x^2-y^2}{2}}}{y^{\mu+\frac{1}{2}}} \int_{\frac{1}{2x}}^{2x} \frac{u^{\mu+\frac{1}{2}}}{\left(\ln \frac{y}{u}\right)^{1+\nu}} \frac{\mathrm{d}u}{u}.$$

Now, the last integral can easily be analyzed (depending on the sign of $\mu + \frac{1}{2}$) to obtain

$$\int_{\frac{1}{2x}}^{2x} \frac{u^{\mu+\frac{1}{2}}}{\left(\ln \frac{y}{u}\right)^{1+\nu}} \frac{\mathrm{d}u}{u} \lesssim \frac{x^{|\mu+\frac{1}{2}|}[\ln (x+e)]^{1+\nu}}{[\ln (y+e)]^{1+\nu}}.$$

Thus, overall we conclude that in this subcase

$$A1 \lesssim I + II \lesssim t^{2\nu}\, (1+x)^{|\mu+\frac{1}{2}|}[\ln (x+e)]^{1+\nu}\, e^{\frac{x^2}{2}}\, \Phi(y).$$

□

5.4 Upper estimates for $A2$ when $y \leq x/2$ or $y \geq Mx$

In view of the previous subsection, it only remains to estimate

$$\begin{aligned} A2 &\approx t^{2\nu}\, e^{\frac{x^2-y^2}{2}} \int_{\max\left\{r_0(xy),\, \frac{1}{2}\right\}}^{1} \frac{e^{-\frac{t^2}{2\ln \frac{1}{r}}}}{\sqrt{1-r}\, \left(\ln \frac{1}{r}\right)^{1+\nu}}\, e^{-\frac{(x-ry)^2}{1-r^2}}\, dr \\ &\lesssim t^{2\nu}\, e^{\frac{x^2-y^2}{2}} \int_{\max\left\{1-\frac{xy}{2},\, \frac{1}{2}\right\}}^{1} \frac{e^{-\frac{ct^2}{1-r}}}{(1-r)^{\nu+\frac{3}{2}}}\, e^{-\frac{(x-ry)^2}{1-r^2}}\, dr \end{aligned} \tag{5.25}$$

noticing that $\ln \frac{1}{r} \approx 1-r$ when $r \in [\frac{1}{2}, 1]$. In this region, however, it is more convenient to use the write-up for the heat kernel in (5.4), in terms of the parameter s. This gives a more reasonable expression for the exponentials, namely

$$e^{\frac{x^2-y^2}{2}}\, e^{-\frac{(x-ry)^2}{1-r^2}} = e^{-\frac{1}{4}\left[\frac{(x-y)^2}{s} + s(x+y)^2\right]}.$$

Since the parameters r and s are related by $s = \frac{1-r}{1+r}$ (or $r = \frac{1-s}{1+s}$), either from (5.4) or directly from (5.25), we obtain that

$$A2 \lesssim t^{2\nu} \int_0^{\min\left\{\frac{1}{3},\, \frac{xy}{3}\right\}} \frac{e^{-\frac{ct^2}{s}}\, e^{-\frac{1}{4}\left[\frac{(x-y)^2}{s} + s(x+y)^2\right]}}{s^{\nu+\frac{3}{2}}}\, ds, \tag{5.26}$$

after perhaps slightly enlarging the range of integration. Our first result shows that when y is far from x this can also be controlled by the function $\Phi(y)$.

Lemma 5.8 *There exists $M > 1$ such that, if $y \le \frac{x}{2}$ or $y \ge Mx$, then*

$$A2 \lesssim t^{2\nu} \int_0^{\min\left\{\frac{1}{3}, \frac{xy}{3}\right\}} \frac{e^{-\frac{ct^2}{s}} e^{-\frac{1}{4}\left[\frac{(x-y)^2}{s}+s(x+y)^2\right]}}{s^{\nu+\frac{3}{2}}} ds \lesssim c(x) \max\{t^{2\nu}, 1\} \Phi(y),$$

with $c(x) = 1/\langle x\rangle^{\alpha+\frac{3}{2}}$.

Proof We claim that, in the assumed range of x and y, there is some $\gamma > 0$ such that

$$A2 \lesssim t^{2\nu} e^{-\left(\frac{1}{2}+\gamma\right)y^2} \int_0^{\min\left\{\frac{1}{3}, \frac{xy}{3}\right\}} e^{-\gamma \frac{t^2+(x-y)^2}{s}} s^{-(\nu+\frac{3}{2})} ds. \tag{5.27}$$

This is just a bound of the exponentials. Indeed, if we distinguish the two cases

(1) *case $y \ge Mx$*: this implies $|y-x| \ge (1-\frac{1}{M})y$, so for any $\eta < 1$ we have

$$e^{-\frac{ct^2}{s}} e^{-\frac{1}{4}\left[\frac{(x-y)^2}{s}+s(x+y)^2\right]} \le e^{-\frac{ct^2+\frac{\eta}{4}(x-y)^2}{s}} e^{-\frac{1-\eta}{4}\left(\frac{M-1}{M}\right)^2\left(\frac{1}{s}+s\right)y^2}$$

which implies the required assertion using that $\frac{1}{s}+s \ge \frac{10}{3}$, when $s \in (0, \frac{1}{3})$, and choosing M sufficiently large and η sufficiently small.

(2) *case $y \le x/2$*: this time $|x-y| \ge \frac{x}{2} \ge y$, so we have

$$e^{-\frac{ct^2}{s}} e^{-\frac{1}{4}\left[\frac{(x-y)^2}{s}+s(x+y)^2\right]} \le e^{-\frac{ct^2+\frac{\eta}{4}(x-y)^2}{s}} e^{-\frac{1-\eta}{4}(\frac{1}{s}+s)y^2}$$

which again implies the assertion using $\frac{1}{s}+s \ge \frac{10}{3}$ and choosing η sufficiently small.

Thus (5.27) is proven, and we may change variables $\gamma[t^2+(x-y)^2]/s = u$ to obtain

$$A2 \lesssim \frac{t^{2\nu} e^{-\left(\frac{1}{2}+\gamma\right)y^2}}{[t^2+(x-y)^2]^{\nu+\frac{1}{2}}} \int_{3\gamma[t^2+(x-y)^2]\max\left\{1, \frac{1}{xy}\right\}}^{\infty} u^{\nu+\frac{1}{2}} e^{-u} \frac{du}{u} \tag{5.28}$$

$$\lesssim \frac{t^{2\nu} e^{-\left(\frac{1}{2}+\gamma\right)y^2}}{(t+x+y)^{2\nu+1}} \int_{\gamma'[x^2+y^2]\max\left\{1, \frac{1}{xy}\right\}}^{\infty} u^{\nu+\frac{1}{2}} e^{-u} \frac{du}{u}, \tag{5.29}$$

since in the selected range of x, y we have $|x-y| \gtrsim x+y$. To finish the proof, we must distinguish some cases.

Case $y \ge 1$: then bounding the denominator and the integral in (5.29) by a constant we immediately see that

$$A2 \lesssim t^{2\nu} e^{-\left(\frac{1}{2}+\gamma\right)y^2} \lesssim t^{2\nu} \Phi(y)$$

since the exponential has a better decay.

Case $y \le 1$ and $y \ge Mx$: we again bound the integral by a constant and estimate the fraction in (5.29) as follows

$$A2 \lesssim \frac{t^{2\nu}}{t^{2\nu} y} = \frac{\langle y\rangle^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}} \le c_M \frac{\langle y\rangle^{\alpha+\frac{1}{2}}}{\langle x\rangle^{\alpha+\frac{3}{2}}}. \tag{5.30}$$

Case $y \le 1$ and $y \le \frac{x}{2}$: this is a relevant case, since the integral in (5.29) plays actually a role. To compute the integral, we must distinguish the two subcases

(1) If $xy \leq 1$, then since also $\frac{x}{y} \geq 2$,

$$A2 \lesssim \frac{t^{2\nu}}{t^{2\nu}x} \int_{\gamma' \frac{x}{y}}^{\infty} u^{\nu-\frac{1}{2}}\, e^{-u}\, \mathrm{d}u \approx \frac{1}{x} \left(\frac{x}{y}\right)^{\nu-\frac{1}{2}} e^{-c\frac{x}{y}} \tag{5.31}$$

$$\lesssim \frac{1}{x} \left(\frac{y}{x}\right)^{\alpha+\frac{1}{2}} \leq \frac{\langle y\rangle^{\alpha+\frac{1}{2}}}{\langle x\rangle^{\alpha+\frac{3}{2}}}.$$

(2) If $xy \geq 1$, then we have $x \geq \frac{1}{y} \geq 1$ and

$$A2 \lesssim \frac{t^{2\nu}}{x^{1+2\nu}} \int_{\gamma' x^2}^{\infty} u^{\nu-\frac{1}{2}}\, e^{-u}\, \mathrm{d}u \lesssim t^{2\nu}\, x^{-2}\, e^{-cx^2}.$$

Now, since $\frac{1}{x} \leq y \leq 1$ we can insert the estimate

$$1 \lesssim \langle y\rangle^{\alpha+\frac{1}{2}} \max\left\{x^{\alpha+\frac{1}{2}}, 1\right\},$$

to obtain $A2 \lesssim t^{2\nu}\, \langle y\rangle^{\alpha+\frac{1}{2}}$. □

Remark 5.9 As mentioned earlier in Remark 5.6, here it is also possible to obtain a bound

$$A2 \lesssim c'(x)\, t^{2\nu}\, \Phi(y), \tag{5.32}$$

with $c'(x) = 1/\langle x\rangle^{\alpha+\frac{3}{2}+2\nu}$. The loss produced by $\max\{1, t^{2\nu}\}$ can be corrected in (5.30) and (5.31) by replacing the factor $t^{2\nu}$ in the denominator by $x^{2\nu}$, as one readily notices from (5.29). As mentioned before, this estimate will play a role later.

5.5 Upper estimates for $A2$ in the local part $\frac{x}{2} < y < Mx$

As in the previous subsection, our starting point is the formula (5.26), which we must estimate in the local region $\frac{x}{2} < y < Mx$. A sufficient bound for us is stated in the next lemma.

Lemma 5.10 *If $\frac{x}{2} < y < Mx$, then*

$$A2 \lesssim t^{2\nu} \int_0^{\min\left\{\frac{1}{3}, \frac{xy}{3}\right\}} \frac{e^{-\frac{ct^2}{s} - \frac{1}{4}\left[\frac{(x-y)^2}{s} + s(x+y)^2\right]}}{s^{\nu+\frac{3}{2}}}\, ds \lesssim C(x)\, \frac{t^{2\nu}\, e^{-\frac{y^2}{2}}}{\big(t + |x-y|\big)^{1+2\nu}}, \tag{5.33}$$

where $C(x) = (1+x)^{2\nu} e^{\frac{x^2}{2}}$.

Proof We shall crudely enlarge the integral in (5.33) to the range $\int_0^{1/3}$. This last integral was already estimated in [6] and [5], by a similar procedure to the one used in the last subsection. More precisely, from the estimates in [5, Lemma 3.2], formula (3.16), it follows that

$$t^{2\nu} \int_0^{\frac{1}{2}} \frac{e^{-\frac{ct^2}{s} - \frac{1}{4}\left[\frac{(x-y)^2}{s} + s(x+y)^2\right]}}{s^{\nu+\frac{3}{2}}}\, ds \lesssim \frac{t^{2\nu}\, (1+x)^{2\nu}\, e^{\frac{x^2-y^2}{2}}}{\big(t + |x-y|\big)^{1+2\nu}},$$

which agrees with (5.33). □

5.6 Proof of Propositions 5.1 and 5.2

Proposition 5.2 follows by putting together Lemmas 5.5, 5.7, 5.8 and 5.10. Concerning Proposition 5.1, the lower bound was shown in Lemma 5.4, while the upper bound also follows from Lemmas 5.5, 5.7 and 5.8, at least when $y < \frac{x}{2}$ or $y > Mx$. This actually implies the asserted result for all x and y, since when y belongs to the compact set $[\frac{x}{2}, Mx]$, the continuous function $y \to P_t(x, y)/\Phi(y)$ is bounded above by a constant $c_2(t, x)$. □

5.7 Proof of Corollary 5.3

By Proposition 5.2, observe that

$$P_t f(x) \lesssim \frac{C_1(x)}{t} \int_{\mathbb{R}_+} \frac{g(y)\,\mathrm{d}y}{\left(1 + \frac{|x-y|}{t}\right)^{1+2\nu}} + C_2(x)(1 \vee t_0)^{2\nu} \left\|f\right\|_{L^1(\Phi)}, \tag{5.34}$$

where $g(y) = f(y)e^{-\frac{y^2}{2}} \chi_{\{\frac{x}{2}<y<Mx\}}$. The first term is then controlled by the Hardy–Littlewood maximal function by a standard slicing argument. □

Remark 5.11 We wrote in (1.9) a different version of (5.9) with $\mathcal{M}^{\mathrm{loc}}(f\Phi)$ in place of $\mathcal{M}^{\mathrm{loc}}\left(fe^{-\frac{y^2}{2}}\right)$. Since $x \approx y$,

$$\mathcal{M}^{\mathrm{loc}}(f\Phi)(x) \approx \frac{\langle x\rangle^{\alpha+\frac{1}{2}}}{[\ln(e+x)]^{1+\nu}\,(1+x)^{\mu+\frac{1}{2}}}\,\mathcal{M}^{\mathrm{loc}}\left(fe^{-\frac{y^2}{2}}\right)(x),$$

so they are actually equivalent modulo x-constants. The write-up in (1.9) has the advantage of remaining valid for other Laguerre systems; see Sect. 7 below.

6 Proofs

As indicated in Sect. 1, we present the proof of Theorems 1.1 and 1.2 for the differential operator L in (1.1) and the function Φ in (1.4). We postpone to Sect. 7 the proof of the results for the other systems mentioned in Table 1.

6.1 Proof of Theorem 1.1

First of all, it is an immediate consequence of Proposition 5.1 that $P_t|f|(x) < \infty$ for some (or all) $t, x > 0$ if and only if $f \in L^1(\Phi)$. This justifies that $f \in L^1(\Phi)$ is the right setting for this problem. Notice also that taking derivatives of the kernel $P_t(x, y)$ in (5.5) with respect to t does not worsen its decay in y, so $P_t f(x)$ automatically becomes infinitely differentiable in the t-variable when $f \in L^1(\Phi)$. Notice, moreover, that the kernel satisfies[3]

$$\left[\partial_{xx} - x^2 - \frac{\alpha^2-\frac{1}{4}}{x^2} - 2\mu + \partial_{tt} + \frac{1-2\nu}{t}\,\partial_t\right] P_t(x, y) = 0.$$

Thus, the (distributional) derivatives $\partial_x^{2m}[P_t f]$ are transformed into t-derivatives, and hence are continuous functions. Since this is valid for all m, it implies that $(t, x) \mapsto P_t f(x)$ is a

[3] See, e.g., [5, §2] for an explicit computation.

C^∞ function. We have thus completed the proof of paragraph (i) and the last statement in Theorem 1.1. We shall now prove the statement in (ii), namely that for $f \in L^1(\Phi)$

$$\lim_{t\to 0^+} P_t f(x) = f(x), \quad \forall\, x \in \mathcal{L}_f \tag{6.1}$$

where $\mathcal{L}_f$ denotes the set of Lebesgue points of f. When $f(x) = 0$, this is easily obtained from the kernel estimates in Proposition 5.2. Indeed,

$$P_t f(x) \lesssim C_1(x) \int_{\frac{x}{2}}^{Mx} \frac{t^{2\nu} |f(y)|\, \mathrm{d}y}{(t + |x-y|)^{1+2\nu}} + C_2'(x)\, t^{2\nu} \int_{\mathbb{R}_+} |f|\Phi,$$

where in the second term we have replaced $(t \vee 1)^{2\nu}$ by $t^{2\nu}$ in view of Remarks 5.6 and 5.9. Thus, this second term vanishes as $t \to 0$ (actually for all $x \in \mathbb{R}_+$). Concerning the first term, it is given by convolution of $|f(y)|\chi_{\{\frac{x}{2}<y<Mx\}} \in L^1_c(\mathbb{R}_+)$ with a radially decreasing approximate identity, so from well-known results (see, e.g., [14, p. 112]), it must vanish as $t \to 0$ at every Lebesgue point x of f with $f(x) = 0$.

It remains to prove (6.1) when $f(x)$ is not necessarily 0. To show this, we first notice that the first eigenfunction $\varphi = \varphi_0^\alpha$ (with eigenvalue $\lambda = 2(\mu + \alpha + 1)$) satisfies

$$P_t \varphi = F_t(\lambda)\varphi, \quad \text{with} \quad \lim_{t\to 0} F_t(\lambda) = 1.$$

Indeed, setting $u = t^2/(4v)$ in (1.3), gives

$$F_t(\lambda) = \tfrac{(t/2)^{2\nu}}{\Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4u} - \lambda u} \frac{\mathrm{d}u}{u^{1+\nu}} = \tfrac{1}{\Gamma(\nu)} \int_0^\infty e^{-v - \frac{t^2\lambda}{4v}}\, v^{\nu-1}\, dv \longrightarrow 1, \quad \text{as } t \to 0.$$

Therefore, we can write

$$P_t f(x) - f(x) = P_t f(x) - F_t(\lambda) f(x) + f(x)[F_t(\lambda) - 1], \tag{6.2}$$

with the last term vanishing as $t \to 0$. Since $\varphi > 0$, the first term can be rewritten as

$$P_t f(x) - \tfrac{f(x)}{\varphi(x)}\, P_t\varphi(x) = P_t \left(f - \tfrac{f(x)}{\varphi(x)} \varphi \right)(x).$$

Setting $g = f - \frac{f(x)}{\varphi(x)}\varphi$, it is easily seen that $g \in L^1(\Phi)$, $g(x) = 0$ and x is a Lebesgue point of g. This last assertion follows from

$$⨍_{I(x,r)} |g(y)| \mathrm{d}y \le ⨍_{I(x,r)} |f(y) - f(x)| \mathrm{d}y + \frac{|f(x)|}{\varphi(x)} ⨍_{I(x,r)} |\varphi(y) - \varphi(x)| \mathrm{d}y,$$

which vanishes as $r \to 0$. Thus we can apply our earlier case to g and conclude that $\lim_{t\to 0} P_t g(x) = 0$. So the left-hand side of (6.2) goes to 0 as $t \to 0$, establishing (6.1) and completing the proof of Theorem 1.1. □

Remark 6.1 A close look at the last part of the proof shows that when $f \in C([a, b])$ with $[a, b] \Subset \mathbb{R}_+$, the convergence of $P_t f(x) \to f(x)$ is uniform in $x \in [a, b]$.

6.2 Proof of Theorem 1.2

We have to show that $P^*_{t_0}$ maps $L^p(w) \to L^p(v)$, for some weight $v(x) > 0$, under the assumption that

$$\|w\|_{D_p(\Phi)} := \left[\int_{\mathbb{R}_+} w^{-\frac{p'}{p}}(x) \Phi(x)^{p'}\, \mathrm{d}x \right]^{1/p'} < \infty,$$

with Φ as in (5.6). We shall use the bound for $P^*_{t_0}$ in (5.9), namely

$$P^*_{t_0} f(x) \lesssim C_1(x)\, \mathcal{M}^{\mathrm{loc}}_M \Big(f e^{-\frac{y^2}{2}} \Big)(x) \,+\, C_2(x) \int_{\mathbb{R}_+} |f(y)|\, \Phi(y)\, \mathrm{d}y$$
$$= I(x) \,+\, II(x), \tag{6.3}$$

for a suitable $M > 1$, and $C_1(x)$, $C_2(x)$ given explicitly in Proposition 5.2. We first treat the last term, which by Hölder's inequality is bounded by

$$II(x) \,\le\, C_2(x)\, \|f\|_{L^p(w)}\, \|w\|_{D_p(\Phi)}\,.$$

Thus, it suffices to choose a weight v such that $C_2(x) = [\ln(e+x)]^{1+\nu}\,(1+x)^{|\mu+\frac{1}{2}|}\, e^{\frac{x^2}{2}} / \langle x\rangle^{\alpha+\frac{3}{2}}$ belongs to $L^p(v)$ to conclude that

$$\|II\|_{L^p(v)} \,\le\, \|C_2\|_{L^p(v)}\, \|w\|_{D_p(\Phi)}\, \|f\|_{L^p(w)}\,. \tag{6.4}$$

For instance, we may take any $v(x) \le v_2(x)$ with

$$v_2(x) := \frac{\langle x\rangle^{(\alpha+\frac{3}{2})p-1}}{[\ln(e/\langle x\rangle)]^2}\, \frac{e^{-\frac{p}{2}x^2}}{(1+x)^N} \tag{6.5}$$

for any $N > 1 + p\,|\mu + \frac{1}{2}|$. We remark that $v_2 \in D_q(\Phi)$ for all $q > p$. Indeed, the local condition was already established in (4.3). For the global condition, notice that

$$\int_1^\infty v_2(x)^{-\frac{q'}{q}}\, \Phi(x)^{q'}\, \mathrm{d}x \,\lesssim\, \int_1^\infty e^{-\left(1-\frac{p}{q}\right)\frac{q'}{2}x^2}\, (1+x)^{\frac{Nq'}{q}}\, \mathrm{d}x \,<\, \infty.$$

We now consider the term $I(x)$ in (6.3). We define a new weight $W(x) = w(x)e^{\frac{p}{2}x^2}$ and observe that

$$w \in D_p(\Phi) \quad\Longrightarrow\quad W \in D^0_p\left(\alpha + \tfrac{1}{2}\right) \cap D^{\exp_2}_p(a), \quad \forall\, a > 0, \tag{6.6}$$

for the weight classes defined in Sect. 3.2. Indeed, the local estimate follows from $\Phi(x) \approx \langle x\rangle^{\alpha+\frac{1}{2}}$ when $x \in (0,1)$, and the global estimate is a consequence of

$$\big\|W\big\|_{D^{\exp_2}_p(a)} \,=\, \int_1^\infty w^{-\frac{p'}{p}}(x)\, e^{-\frac{p'}{2}x^2}\, e^{-ap'x^2}\, \mathrm{d}x \,\le\, C_a^{p'} \int_1^\infty w^{-\frac{p'}{p}}(x)\Phi(x)^{p'}\, \mathrm{d}x\,, \tag{6.7}$$

with $C_a = \max_{x\ge1} |\ln(e+x)|^{1+\nu}|1+x|^{\mu+\frac{1}{2}} e^{-ax^2}$.

We shall now set

$$v_{1,\varepsilon}(x) = \frac{e^{-\frac{px^2}{2}}}{(1+x)^{2p\nu}}\, \mathscr{V}(x)\rho_\varepsilon\Big(\mathscr{V}(x)\Big), \quad \text{where } \mathscr{V}(x) = \left[\mathcal{M}^{\mathrm{loc}}_M \Big(w^{-\frac{p'}{p}}\, e^{-\frac{p'y^2}{2}} \Big)(x)\right]^{-\frac{p}{p'}} \tag{6.8}$$

(or $v_{1,\varepsilon}(x) = (1+x)^{-2p\nu} e^{-\frac{px^2}{2}}\, V_\varepsilon(x)$ in the notation of (3.1)). Given $f \in L^p(w)$, we denote $\tilde{f}(y) = f(y)e^{-\frac{y^2}{2}}$ which is a function in $L^p(W)$. Then, using the two-weight inequality for $\mathcal{M}^{\mathrm{loc}}_M$ in Theorem 3.1, and the expression for $C_1(x) = (1+x)^{2\nu} e^{\frac{x^2}{2}}$, we see that, for any $v \le v_{1,\varepsilon}$, the term $I(x)$ in (6.3) is controlled by

$$\|I(x)\|^p_{L^p(v)} \le \int_{\mathbb{R}_+} \frac{C_1(x)^p\, e^{-\frac{p}{2}x^2}}{(1+|x|)^{2p\nu}}\, \big|\mathcal{M}^{\mathrm{loc}}_M \tilde{f}(x)\big|^p\, V_\varepsilon(x)\, \mathrm{d}x$$
$$\lesssim \|\tilde{f}\|^p_{L^p(W)} \,=\, \|f\|^p_{L^p(w)}. \tag{6.9}$$

So, combining (6.3), (6.4) and (6.9), we have shown that $\|P^*_{t_0} f\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$, provided

$$v(x) = \min\{v_{1,\varepsilon}(x), v_2(x)\}, \tag{6.10}$$

with $v_{1,\varepsilon}(x)$ and $v_2(x)$ defined in (6.8) and (6.5).

We only have to verify that if $q > p$, then we can choose ε sufficiently small so that $v_{1,\varepsilon} \in D_q(\Phi)$ (which implies $v \in D_q(\Phi)$). This actually follows from (6.6) and Proposition 3.3. Indeed, on the one hand, since $W \in D^0_p(\alpha + \frac{1}{2})$

$$\int_0^1 v_{1,\varepsilon}(x)^{-\frac{q'}{q}} \,\Phi(x)^{q'}\, dx \lesssim \int_0^1 V_\varepsilon(x)^{-\frac{q'}{q}} \,\langle x\rangle^{(\alpha+\frac{1}{2})q'}\, dx, \tag{6.11}$$

which is finite by (i) in the proposition (choosing ε sufficiently small). On the other hand,

$$\int_1^\infty v_{1,\varepsilon}(x)^{-\frac{q'}{q}} \,\Phi(x)^{q'}\, dx \lesssim \int_1^\infty V_\varepsilon(x)^{-\frac{q'}{q}} \, e^{-q'\left(1-\frac{p}{q}\right)\frac{x^2}{2}} \,(1+x)^{2\nu p q'/q}\, dx\,, \tag{6.12}$$

and since $W \in D^{\exp_2}_p(a)$ for all $a > 0$, we can apply part (ii) of Proposition 3.3 (for a sufficiently small ε) to conclude that this is also a finite quantity. □

Remark 6.2 **Alternative expression for the second weight.** A slight modification of the above construction suggests to define a new weight by

$$v^{\Phi,w}_\varepsilon(x) := \min\left\{\Phi(x)^p \left[\mathcal{M}^{\rm loc}\left(w^{-\frac{p'}{p}}\Phi^{p'}\right)(x)\right]^{-\frac{p}{p'}} \Upsilon_\varepsilon(x),\ \frac{\langle x\rangle^{p-1}}{[\ln(e/\langle x\rangle)]^2}\,\frac{\Phi(x)^p}{(1+x)^{N_0}}\right\} \tag{6.13}$$

with

$$\Upsilon_\varepsilon(x) = \frac{\langle x\rangle^{\varepsilon N_1}}{(1+x)^{N_2}}\,\rho_\varepsilon\left(\left[\mathcal{M}^{\rm loc}\left(w^{-\frac{p'}{p}}\Phi^{p'}\right)(x)\right]^{-\frac{p}{p'}}\right), \quad \varepsilon \in (0,1).$$

If N_0, N_1, N_2 are sufficiently large, then similar arguments as above lead to the boundedness of $P^*_{t_0} : L^p(w) \to L^p(v^{\Phi,w}_\varepsilon)$ when $w \in D_p(\Phi)$ and $\varepsilon \in (0,1)$, and give also the property that $v^{\Phi,w}_\varepsilon \in D_q(\Phi)$ if ε is sufficiently small.

We only sketch the proof for the first weight inside (6.13), namely

$$\tilde v_{1,\varepsilon}(x) := \Phi(x)^p \left[\mathcal{M}^{\rm loc}\left(w^{-\frac{p'}{p}}\Phi^{p'}\right)(x)\right]^{-\frac{p}{p'}} \Upsilon_\varepsilon(x). \tag{6.14}$$

We claim that if N_1, N_2 are sufficiently large, then

$$v_{1,\varepsilon}(x)\,\frac{\langle x\rangle^{\varepsilon M_1}}{(1+x)^{M_2}} \lesssim \tilde v_{1,\varepsilon}(x) \lesssim v_{1,\varepsilon}(x), \quad \forall\,\varepsilon \in (0,1), \tag{6.15}$$

for suitable $M_1, M_2 > 0$. Assuming this claim, it is not difficult to deduce that

(i) $\|I(x)\|_{L^p(\tilde v_{1,\varepsilon})} \lesssim \|w\|_{D_p(\Phi)}\|f\|_{L^p(w)}$.
(ii) Given $q > p$, then $\tilde v_{1,\varepsilon} \in D_q(\Phi)$, provided ε is sufficiently small.

Indeed, (i) is immediate from (6.9), while (ii) is not hard to obtain from (6.12) and (6.11) (here using Proposition 3.3 (i) with some $\beta_1 < \beta$). Finally, to justify (6.15), first notice that, by the local restriction in the maximal function,

$$\Phi(x)^p \left[\mathcal{M}^{\rm loc}\left(w^{-\frac{p'}{p}}\Phi^{p'}\right)(x)\right]^{-\frac{p}{p'}} \approx e^{-\frac{px^2}{2}}\mathcal{V}(x),$$

with the notation in (6.8). Moreover, using the trivial inequality

$$\left(\lambda \wedge \tfrac{1}{\lambda}\right)^{\varepsilon} \rho_\varepsilon(x) \;\le\; \rho_\varepsilon(\lambda x) \;\le\; \left(\lambda \vee \tfrac{1}{\lambda}\right)^{\varepsilon} \rho_\varepsilon(x),$$

we obtain that

$$\Upsilon_\varepsilon(x) \le \frac{\langle x\rangle^{\varepsilon N_1}}{(1+x)^{N_2}} \left[\frac{(1+x)^{|\mu+\frac12|}[\ln(e+x)]^{1+\nu}}{\langle x\rangle^{|\alpha+\frac12|}}\right]^{p\varepsilon} \rho_\varepsilon\Big(\mathscr{V}(x)\Big) \lesssim \frac{\rho_\varepsilon(\mathscr{V}(x))}{(1+x)^{2\nu p}},$$

provided we choose $N_1 \ge p|\alpha+\frac12|$ and $N_2 > 2\nu p + \varepsilon p|\mu+\frac12|$. Thus $\tilde v_{1,\varepsilon}(x) \lesssim v_{1,\varepsilon}(x)$. On the other hand, the reverse estimate

$$\Upsilon_\varepsilon(x) \ge \frac{\langle x\rangle^{\varepsilon N_1}}{(1+x)^{N_2}} \left[\frac{\langle x\rangle^{|\alpha+\frac12|}}{(1+x)^{|\mu+\frac12|}[\ln(e+x)]^{1+\nu}}\right]^{p\varepsilon} \rho_\varepsilon\Big(\mathscr{V}(x)\Big),$$

implies the lower bound in (6.15) with $M_1 = N_1 + p|\alpha+\frac12|$ and any $M_2 > N_2 + \varepsilon p|\mu+\frac12|$.

The new weight expression in (6.13) will have the advantage of being jointly valid for all the Laguerre systems in Table 1 (with the corresponding Φ functions).

7 Transference to other Laguerre-type systems

In this section, we show how to transfer the results already proved for the system $\{\varphi_n^\alpha\}$ and the operator L to the other Laguerre systems and operators in Table 1. This procedure is completely general and has been used before in other instances (see, e.g., [1]).

7.1 Results for the system ψ_n^α

The starting point is the identity defining ψ_n^α, namely

$$\psi_n^\alpha(y) = a(y)\,\varphi_n^\alpha(y), \quad \text{with } a(y) = y^{-\alpha-\frac12}. \tag{7.1}$$

Clearly, φ_n^α is an eigenvector of L if and only if ψ_n^α is an eigenvector of the operator

$$f \longmapsto \Lambda f(x) = a(x) L[a^{-1} f\,](x)$$

(with the same eigenvalue $\lambda_n = 4n + 2(\alpha+1+\mu)$). An elementary computation shows that the differential operator Λ obtained in this fashion is exactly the one listed in Table 1. Remark also that $\{\psi_n^\alpha\}$ becomes an orthonormal basis in L^2 with the measure $a^{-2}(y)\mathrm{d}y = y^{2\alpha+1}\mathrm{d}y$.

The identity in (7.1) leads to a pointwise relation of the corresponding heat kernels

$$e^{-t\Lambda}(x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\psi_n^\alpha(x)\psi_n^\alpha(y) = a(x)\,a(y)\,e^{-tL}(x,y),$$

and by the subordination formula, also of the corresponding Poisson kernels

$$P_t^\Lambda(x,y) = a(x)\,a(y)\,P_t^L(x,y).$$

In particular,

$$P_t^\Lambda f(x) = \int_{\mathbb{R}_+} P_t^\Lambda(x,y)\,f(y)\,a^{-2}(y)\mathrm{d}y = a(x)\,P_t^L[a^{-1}f](x). \tag{7.2}$$

From this relation, it is clear that Theorem 1.1 becomes true for the operator Λ with

$$\Phi^{\Lambda}(y) = a(y)^{-1}\Phi^{L}(y),$$

as listed in Table 1. From (7.2), it also follows that $P_{t_0}^{*,\Lambda}$ maps $L^p(w) \to L^p(v)$ if and only if $P_{t_0}^{*,L}$ maps $L^p(a^p w) \to L^p(a^p v)$, and hence the necessary and sufficient condition becomes

$$a^p w \in D_p(\Phi^L) \Longleftrightarrow \|a^{-1} w^{-\frac{1}{p}} \Phi^L\|_{p'} < \infty \Longleftrightarrow w \in D_p(\Phi^{\Lambda}),$$

as was claimed in Theorem 1.2. For the assertions about the weight v, one may argue directly as follows. Observe from (7.2) and Corollary 5.3 that we can write

$$P_{t_0}^{*,\Lambda} f(x) \lesssim C_1(x)\, a(x) \mathcal{M}_M^{\mathrm{loc}} \left(f a^{-1} e^{-\frac{y^2}{2}} \right)(x) \,+\, C_2(x) a(x) \int_{\mathbb{R}_+} |f| \Phi^{\Lambda},$$

with a cancelation in the first term due to $a(x)a(y)^{-1} \approx 1$ when $\frac{x}{2} < y < Mx$. At this point, we can apply the same arguments as in Sect. 6.2. Namely, we construct $v = \min\{v_{1,\varepsilon}, v_2\}$ with the same choice of $v_{1,\varepsilon}$, and with v_2 in (6.5) now replaced by

$$v_2(x) := \frac{\langle x \rangle^{(2\alpha+2)p-1}}{[\ln{(e/\langle x \rangle)}]^2}\, \frac{e^{-\frac{p}{2}x^2}}{(1+x)^N}\,.$$

The same proof will give that, for any $q > p$, there is a sufficiently small ε so that $v \in D_q(\Phi^{\Lambda})$ (the only difference being that, locally, this condition now becomes $v \in D_q^0(2\alpha+1)$). We remark that this part will work as well with the choice

$$v(x) \,=\, a^{-p}(x)\, v^{\Phi^L, a^p w}(x) \,=\, v^{\Phi^{\Lambda}, w}(x),$$

as defined in (6.13).

7.2 Results for the system $\mathfrak{L}_n^{\alpha}$

Consider the following isometry of $L^2(\mathbb{R}_+, \mathrm{d}y)$

$$f \longmapsto Af(x) \,=\, \sqrt{2x}\, f(x^2).$$

The systems $\mathfrak{L}_n^{\alpha}$ and φ_n^{α} are related by $\varphi_n^{\alpha} = A\mathfrak{L}_n^{\alpha}$, or equivalently

$$\mathfrak{L}_n^{\alpha}(y) \,=\, [A^{-1}\varphi_n^{\alpha}](y) \,=\, (4y)^{-\frac{1}{4}}\, \varphi_n^{\alpha}\left(\sqrt{y}\right). \tag{7.3}$$

In particular, $\mathfrak{L}_n^{\alpha}$ is an eigenvector of the operator

$$\mathfrak{L} \,=\, \frac{1}{4}\, A^{-1} \circ L \circ A,$$

this time with eigenvalue $\lambda_n/4 = n + (\alpha+1+\mu)/2$. The factor $\frac{1}{4}$ has been inserted so that $\mathfrak{L}$ coincides with the operator listed in Table 1.

The heat kernels are now related by

$$e^{-t\mathfrak{L}}(x,y) \,=\, \sum_{n=0}^{\infty} e^{-t\lambda_n/4} \mathfrak{L}_n^{\alpha}(x)\mathfrak{L}_n^{\alpha}(y) \,=\, \frac{1}{2(xy)^{\frac{1}{4}}}\, e^{-\frac{t}{4}L}\left(\sqrt{x}, \sqrt{y}\right),$$

and therefore, a substitution in the subordinated integral in (5.5) gives

$$P_t^{\mathfrak{L}}(x,y) \,=\, (16xy)^{-\frac{1}{4}}\, P_{t/2}^{L}\left(\sqrt{x}, \sqrt{y}\right).$$

Thus, we obtain the formula

$$P_t^{\mathfrak{L}} f(x) = \int_{\mathbb{R}_+} P_t^{\mathfrak{L}}(x, y)\, f(y)\, \mathrm{d}y = (4x)^{-\frac{1}{4}}\, P_{t/2}^L[Af]\left(\sqrt{x}\right). \tag{7.4}$$

From this relation, one easily deduces Theorem 1.1 for the operator $\mathfrak{L}$, provided that

$$\Phi^{\mathfrak{L}}(y) = [A^{-1}\Phi^L](y), \tag{7.5}$$

which is comparable to the function in Table 1.

To establish the second theorem, first observe from (7.4) and Corollary 5.3 that

$$P_{t_0}^{*,\mathfrak{L}} f(x) \lesssim \frac{C_1(\sqrt{x})}{x^{1/4}}\, \mathcal{M}_M^{\mathrm{loc}}\left(\sqrt{y} f(y^2) e^{-\frac{y^2}{2}}\right)(\sqrt{x}) + \frac{C_2\left(\sqrt{x}\right)}{x^{1/4}} \int_{\mathbb{R}_+} \sqrt{2y} |f(y^2)| \Phi^L(y).$$

We claim that this inequality can be rewritten as

$$P_{t_0}^{*,\mathfrak{L}} f(x) \lesssim C_1\left(\sqrt{x}\right) \mathcal{M}_{\frac{1}{4},M^2}^{\mathrm{loc}}\left(f e^{-\frac{y}{2}}\right)(x) + \frac{C_2\left(\sqrt{x}\right)}{x^{1/4}} \int_{\mathbb{R}_+} |f(u)| \Phi^{\mathfrak{L}}(u), \tag{7.6}$$

where $\mathcal{M}_{\frac{1}{4},M^2}^{\mathrm{loc}}$ stands for the local maximal function in (1.10) with the cutoff replaced by $\chi_{\{\frac{x}{4} \le y \le M^2 x\}}$. The expression for the second term is clear from (7.5) (after a change of variables $y^2 = u$). To handle the first term, notice that the local region now becomes $\frac{\sqrt{x}}{2} < y < M\sqrt{x}$, which in particular gives $x^{-\frac{1}{4}}\sqrt{y} \approx 1$. We also need the following lemma for the maximal function.

Lemma 7.1 *For every function $g : \mathbb{R}_+ \to \mathbb{C}$ and every $x \in \mathbb{R}_+$, we have*

$$\mathcal{M}\left(g(y^2)\chi_{\{\frac{\sqrt{x}}{2} < y < M\sqrt{x}\}}\right)\left(\sqrt{x}\right) \lesssim \mathcal{M}\left(g(u)\chi_{\{\frac{x}{4} < u < M^2 x\}}\right)(x).$$

Moreover, x is a Lebesgue point of g if and only if $\sqrt{x}$ is a Lebesgue point of $g(y^2)$.

Proof The first assertion follows essentially from the change of variables $y^2 = u$,

$$\begin{aligned} LHS &\le \sup_{r>0} \frac{1}{r} \int_{|y-\sqrt{x}|<r} |g(y^2)|\, \chi_{\{\frac{x}{4} < y^2 < M^2 x\}}\, \mathrm{d}y \\ &= \sup_{r>0} \frac{1}{r} \int_{|\sqrt{u}-\sqrt{x}|<r} |g(u)|\, \chi_{\{\frac{x}{4} < u < M^2 x\}}\, \frac{\mathrm{d}u}{2\sqrt{u}}. \end{aligned}$$

In this local range, we have $\sqrt{u} \approx \sqrt{x}$, so may take the denominator outside the integral. The local behavior also implies that

$$|\sqrt{u} - \sqrt{x}| = \left|\int_x^u \frac{\mathrm{d}s}{2\sqrt{s}}\right| \approx \frac{|u - x|}{\sqrt{x}}. \tag{7.7}$$

Therefore, we conclude that

$$LHS \lesssim \sup_{r>0} \frac{1}{r\sqrt{x}} \int_{|u-x|<r\sqrt{x}} |g(u)|\, \chi_{\{\frac{x}{4} < u < M^2 x\}}\, \mathrm{d}u \lesssim RHS.$$

For the last assertion, we only show that $x \in \mathcal{L}_g$ implies $\sqrt{x} \in \mathcal{L}_{\tilde{g}}$ with $\tilde{g}(y) = g(y^2)$ (the converse is similar). As before, the change in variables $y = \sqrt{u}$ gives

$$\mathcal{I}_r(x) := \frac{1}{r} \int_{|y-\sqrt{x}|<r} |\tilde{g}(y) - \tilde{g}(\sqrt{x})|\, \mathrm{d}y = \frac{1}{r} \int_{|\sqrt{u}-\sqrt{x}|<r} |g(u) - g(x)|\, \frac{\mathrm{d}u}{2\sqrt{u}}.$$

If r is sufficiently small (e.g., $r < \sqrt{x}/2$), we have $\sqrt{u} \approx \sqrt{x}$ and also $|\sqrt{u} - \sqrt{x}| \approx |u - x|/\sqrt{x}$, by (7.7). Thus,

$$\mathcal{I}_r(x) \lesssim \frac{1}{r\sqrt{x}} \int_{|u-x| \le c\, r\sqrt{x}} |g(u) - g(x)|\, \mathrm{d}u$$

which vanishes as $r \to 0$ since $x \in \mathcal{L}_g$. □

We have thus shown (7.6). From here, one proves Theorem 1.2 (for the operator $\mathfrak{L}$) arguing once again as in Sect. 6.2. Remark that, in view of the new constants C_1 and C_2 in (7.6), the weight $v = \min\{v_{1,\varepsilon}, v_2\}$ must be defined with

$$v_{1,\varepsilon}(x) = \frac{e^{-\frac{px}{2}}}{(1+x)^{p\nu}} \mathscr{V}(x) \rho_\varepsilon\Big(\mathscr{V}(x)\Big) \quad \text{and} \quad v_2(x) = \frac{\langle x \rangle^{(\frac{\alpha}{2}+1)p-1}}{[\ln(e/\langle x \rangle)]^2} \frac{e^{-\frac{p}{2}x}}{(1+x)^N},$$

where $\mathscr{V}(x) = \left[\mathcal{M}^{\mathrm{loc}}_{\frac{1}{4}, M^2}\left(w^{-\frac{p'}{p}} e^{-\frac{p'x}{2}}\right)(x)\right]^{1-p}$ and $N > 1 + \frac{p}{2}\left(|\mu + \frac{1}{2}| - \frac{1}{2}\right)$. Then, the same proof as before gives that $P^{*,\mathfrak{L}}_{t_0} : L^p(w) \to L^p(v)$ if $w \in D_p(\Phi^{\mathfrak{L}})$. One can also establish (with a few obvious modifications, such as using the class $D_p^{\exp_1}$ in Proposition 3.3 ii) that for every given $q > p$, there is a sufficiently small ε so that $v \in D_q(\Phi^{\mathfrak{L}})$. Once again, we may also replace this weight by $v^{\Phi^{\mathfrak{L}}, w}(x)$, as defined in (6.13).

7.3 Results for the system ℓ_n^α

Remember that these functions satisfy

$$\ell_n^\alpha(y) = a(y)\, \mathfrak{L}_n^\alpha(y), \quad \text{with } a(y) = y^{-\frac{\alpha}{2}}. \tag{7.8}$$

Thus, they are eigenvectors of the differential operator

$$f \mapsto \mathscr{L} f(x) = a(x) \mathfrak{L}[a^{-1} f](x)$$

(with the same eigenvalues as $\mathfrak{L}_n^\alpha$) and constitute an orthonormal system in $L^2(a(y)^{-2}\mathrm{d}y)$. One then derives Theorems 1.1 and 1.2 for the operator $\mathscr{L}$, from the known results about $\mathfrak{L}$, by repeating exactly the same arguments that we gave in Sect. 7.1. We leave the details to the reader.

7.4 Results for the Laguerre polynomials L_n^α

Since $\mathbb{L}$ is the most classical Laguerre operator, we shall give a few more details here. First of all, recall that L_n^α and $\mathfrak{L}_n^\alpha$ are linked by

$$L_n^\alpha(y) = a(y)\, \mathfrak{L}_n^\alpha(y), \quad \text{with } a(y) = y^{-\frac{\alpha}{2}} e^{y/2}. \tag{7.9}$$

Thus, the functions L_n^α are orthonormal in $L^2(a(y)^{-2}\mathrm{d}y) = L^2(y^\alpha e^{-y}\mathrm{d}y)$ and are also eigenvectors of the differential operator

$$f \mapsto \mathbb{L} f(x) = a(x) \mathfrak{L}[a^{-1} f](x),$$

with the same eigenvalues as $\mathfrak{L}_n^\alpha$, namely $n + (\alpha + 1 + \mu)/2$. We remark that $\mathbb{L}$ coincides with the operator in Table 1. Thus, the heat and Poisson kernels of these two operators are related by

$$e^{-t\mathbb{L}}(x, y) = \sum_{n=0}^\infty e^{-\left(n + \frac{\alpha+\mu+1}{2}\right)t} L_n^\alpha(x) L_n^\alpha(y) = a(x)\, a(y)\, e^{-t\mathfrak{L}}(x, y),$$

and

$$P_t^{\mathbb{L}}(x,y) \,=\, a(x)\,a(y)\,P_t^{\mathfrak{L}}(x,y).$$

This implies the identity

$$P_t^{\mathbb{L}} f(x) \,=\, \int_{\mathbb{R}_+} P_t^{\mathbb{L}}(x,y)\,f(y)\,y^\alpha e^{-y}\,\mathrm{d}y \,=\, a(x)\,P_t^{\mathfrak{L}}\Big[f y^{\frac{\alpha}{2}} e^{-\frac{y}{2}}\Big](x), \tag{7.10}$$

from which one deduces the validity of Theorem 1.1 for the operator $\mathbb{L}$, provided

$$\Phi^{\mathbb{L}}(x) \,=\, y^{\frac{\alpha}{2}} e^{-\frac{y}{2}}\,\Phi^{\mathfrak{L}}(y) \,=\, \frac{y^\alpha e^{-y}}{(1+y)^{\frac{1+\alpha+\mu}{2}}\,[\ln(e+x)]^{1+\nu}}.$$

Note that this coincides with the function in Table 1. Moreover, (7.10) combined with (7.6) implies the estimate

$$P_{t_0}^{*,\mathbb{L}} f(x) \,\lesssim\, C_1^{\mathbb{L}}(x)\,\mathcal{M}^{\mathrm{loc}}_{\frac14,M^2}\big(fe^{-y}\big)(x) \,+\, C_2^{\mathbb{L}}(x)\int_{\mathbb{R}_+} |f(u)|\Phi^{\mathbb{L}}(u) = I(x) + II(x), \tag{7.11}$$

with the new constants

$$C_1^{\mathbb{L}}(x) = (1+x)^\nu e^x \quad \text{and} \quad C_2^{\mathbb{L}}(x) = [\ln(1+x)]^{1+\nu}\,(1+x)^{\left(|\mu+\frac12|-\alpha-\frac12\right)/2} e^x/\langle x\rangle^{\alpha+1}.$$

We now apply the same arguments as in Sect. 6.2 to show that, for a suitable weight v, we have $\|P_{t_0}^{*,\mathbb{L}} f\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$, under the assumption $w \in D_p(\Phi^{\mathbb{L}})$. Indeed, to control the second term $II(x)$ we choose a weight v_2 such that $C_2^{\mathbb{L}} \in L^p(v_2)$, namely

$$v_2(x) \,=\, \frac{\langle x\rangle^{(\alpha+1)p-1}}{[\ln(e/\langle x\rangle)]^2}\,\frac{e^{-px}}{(1+x)^N},$$

with say $N > 1 + p(\frac{1+\alpha+\mu}{2} + |\alpha+\frac12|)$. It is not difficult to verify that $v_2 \in D_q(\Phi^{\mathbb{L}})$ for all $q > p$. To control the first term, we set

$$v_{1,\varepsilon}(x) = \frac{e^{-px}}{(1+x)^{p\nu}}\,\mathscr{V}(x)\rho_\varepsilon\Big(\mathscr{V}(x)\Big) \text{ with } \mathscr{V}(x) = \Big[\mathcal{M}^{\mathrm{loc}}_{\frac14,M^2}\Big(w^{-\frac{p'}{p}} e^{-p'x}\Big)(x)\Big]^{1-p}.$$

That is, if $W(x) = w(x)e^{px}$, then $v_{1,\varepsilon}(x) = (1+x)^{-p\nu} e^{-px}\,V_\varepsilon(x)$ with the notation in (3.1). So we may use Theorem 3.1 to obtain

$$\|I\|_{L^p(v_{1,\varepsilon})} \,\lesssim\, \left[\int_{\mathbb{R}_+} \left|\mathcal{M}^{\mathrm{loc}}_{\frac14,M^2}\big(fe^{-y}\big)(x)\right|^p\,V_\varepsilon(x)\mathrm{d}x\right]^{\frac1p} \,\lesssim\, \|fe^{-y}\|_{L^p(W)} = \|f\|_{L^p(w)}.$$

Again, it is not difficult to verify that for a sufficiently small ε one has $v_{1,\varepsilon} \in D_q(\Phi^{\mathbb{L}})$, arguing as in the last part[4] of Sect. 6.2. Thus, Theorem 1.2 holds with $v = \min\{v_{1,\varepsilon}, v_2\}$. Alternatively, with the notation in (6.13), one may as well choose the weight $v^{\Phi^{\mathbb{L}},w}(x)$.

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[4] With the quadratic exponentials $e^{x^2/2}$ in (6.7) and (6.12) replaced by linear exponentials e^x.

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