IMPROVEMENTS IN WOLFF'S INEQUALITY FOR DECOMPOSITIONS OF CONE MULTIPLIERS

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Abstract. We obtain mixed norm versions $\ell^s(L^p)$ of an inequality introduced by Wolff in the context of local smoothing for the wave equation. We show that suitable modifications of the original arguments of Łaba and Wolff allow to improve in their range of p, both in the original $\ell^p(L^p)$ formulation and also in the stronger $\ell^2(L^p)$, in all dimensions $d \geq 2$. As a consequence of the latter we obtain a new L^4 bound for the cone multiplier operator in $\mathbb{R}^3$, as well as further progress in the boundedness of Bergman projections in tubes over light-cones.

1. Introduction

Let $\Gamma = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \ : \ \tau = |\xi|\,\}$ denote the forward light-cone in $\mathbb{R}^{d+1}$, $d \geq 2$. For small $\delta > 0$ and constants $c \approx 1$, we shall consider δ-neighborhoods of the truncated cone

$$\Gamma_\delta(c) = \{(\tau, \xi) \in \mathbb{R}^{d+1} \ : \ 2^{-c} \leq \tau \leq 2^c \quad \text{and} \quad \big|\tau - |\xi|\big| \leq c\delta\}. \tag{1.1}$$

We also consider the usual plate decomposition of $\Gamma_\delta(c)$ subordinated to a covering of the sphere by $\sqrt{\delta}$-caps. Namely, given a maximal sequence $\Omega = \{\omega_k\} \subset S^{d-1}$ so that $\text{dist}(\omega_k, \omega_{k'}) \geq \sqrt{\delta}$, for $k \neq k'$, and a constant $c' \approx 1$, we let

$$\Pi_k^{(\delta)} = \Pi_{\omega_k}^{(\delta)}(c, c') = \Big\{(\tau, \xi) \in \Gamma_\delta(c) \ : \ \big|\xi/|\xi| - \omega_k\big| \leq c'\sqrt{\delta}\,\Big\}. \tag{1.2}$$

By a slight abuse of language we shall call the above sets "plates", since by appropriately adjusting the constants c, c' they become comparable to $1 \times \sqrt{\delta} \times \overset{(d-1)}{\cdots} \times \sqrt{\delta} \times \delta$ rectangles with longest axis in the direction of $(1, \omega_k)$ and mid-length and short axes, respectively, tangent and normal to Γ at the point $(1, \omega_k)$.

In [16], T. Wolff proposes the validity of the following inequality related to the above plate decomposition: for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ (independent of δ and $\{\omega_k\}$) so that

$$\Big\| \sum_k f_k \Big\|_p \leq C_\varepsilon \, \delta^{-\alpha(p)-\varepsilon} \Big(\sum_k \|f_k\|_p^p \Big)^{1/p}, \quad \forall \, f_k \ : \ \text{supp } \widehat{f_k} \subset \Pi_k^{(\delta)}, \tag{1.3}$$

`This is a preliminary version from 2008. Further work is still in progress.`

First author partially supported by grant "MTM2007-60952" and *Programa Ramón y Cajal*, MCyT (Spain). Third author partially supported by NSF grant DMS 0200186.

where $\alpha(p) := d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$ is the standard Bochner-Riesz critical index in d dimensions. The inequality is conjectured to hold for all $p \geq 2 + \frac{4}{d-1}$, and for each such p, the power $\alpha(p)$ is optimal (except perhaps for $\varepsilon > 0$). (1.3) arises naturally when dealing with L^p-boundedness of cone multipliers and local smoothing of the wave equation [16].

In his fundamental work [16], Wolff developed a method to prove such inequalities for large values of p, obtaining a positive answer for $p > 74$ when $d = 2$. This method was extended to higher dimensions in the paper by Łaba and Wolff [8], establishing (1.3) for all $p > 2 + \min\{\frac{32}{3d-7}, \frac{8}{d-3}\}$. In both papers the authors announce that improvements over these indices should certainly be possible, although perhaps still far from the conjectured exponents. In fact, a slightly better range for all $d \geq 2$ was already presented by the first and third authors in [6], based on a minor modification in the original proof.

One of the purposes of this note is to observe that, with a bit more effort, the techniques of the papers [16, 8] actually lead to the validity of (1.3) in the following improved range.

Theorem 1.1. *Let $d \geq 2$. Then, for all $\varepsilon > 0$ the inequality (1.3) holds when $p > p_d$, where*

$$p_d = 2 + \tfrac{8}{d-2}\big(1 - \tfrac{1/2}{d+1}\big) \quad \textit{for} \quad d \geq 3, \quad \textit{and} \quad p_2 = 20 \quad \textit{for} \quad d = 2. \tag{1.4}$$

We shall actually do more by proving a stronger inequality than (1.3). In fact, motivated by questions on the Bergman projection for tube domains over light cones [2, 1] it is natural to consider as well mixed norm versions $\ell^s(L^p)$ of (1.3); namely

$$\Big\| \sum_k f_k \Big\|_p \leq C_\varepsilon \, \delta^{-\beta(p,s)-\varepsilon} \Big(\sum_k \|f_k\|_p^s \Big)^{1/s}, \quad \forall \, f_k \;:\; \operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}, \tag{$W_{p,s}$}$$

where

$$\beta(p,s) = \frac{d-1}{2s'} - \frac{d+1}{2p}\,. \tag{1.5}$$

Here $p \geq \frac{2(d+1)}{d-1}$ and we consider only indices $s > 1$ so that $\beta(p,s) \geq 0$, in which case the exponent $\beta(p,s)$ is best possible. Observe that for fixed p, the inequality becomes stronger as s is smaller. In fact, $(W_{p,s})$ and Hölder's inequality imply $(W_{p,\sigma})$ for all $\sigma > s$, since

$$\#\, k'\text{s} \;=\; \text{Card}\ (\Omega) \;\leq\; C\,\delta^{-\frac{d-1}{2}},$$

and $\beta(p,s) + \frac{d-1}{2}(\frac{1}{s} - \frac{1}{\sigma}) = \beta(p,\sigma)$. The hardest case should be $(p = \frac{2(d+1)}{d-1}, s = 2)$, which will imply all the other cases by interpolation with the trivial $(p = \infty, s = 1)$. One can also ask for the validity of $(W_{p,s})$ when $2 < p < \frac{2(d+1)}{d-1}$, in which case $\beta(p,s)$ must be replaced by $\frac{d-1}{2}(\frac{1}{2} - \frac{1}{s})$ (see Figure 1.1). However, no such result seems to be known, even in the simpler case of functions with spectrum in a δ-neighborhood of the unit spheres*.

*Except for $(W_{4,2})$ in the circle case, which is a trivial consequence of the classical estimate $\|\sum_k f_k\|_{L^4(\mathbb{R}^2)} \lesssim \|(\sum_k |f_k|^2)^{1/2}\|_{L^4(\mathbb{R}^2)}$.

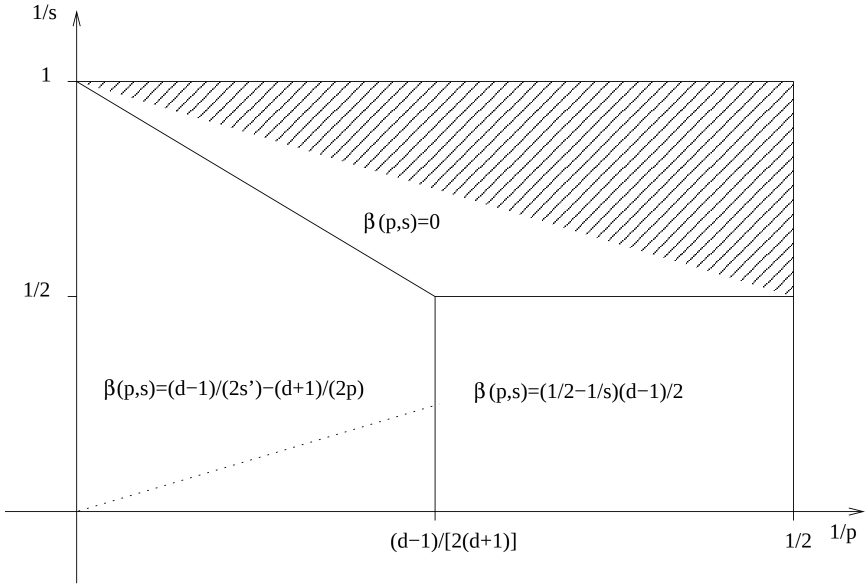

FIGURE 1.1. Conjectured region of validity for the inequality $(W_{p,s})$ for different values of $\beta(p,s)$. In the dashed region the result is trivial, while the dotted line corresponds to the inequality (1.3) conjectured by Wolff.

Our contribution to this problem is mainly restricted to the case $s = 2$, that is

$$\Big\|\sum_k f_k\Big\|_p \le C_\varepsilon\, \delta^{-\beta(p)-\varepsilon}\Big(\sum_k \|f_k\|_p^2\Big)^{1/2}, \quad \forall\, f_k \;:\; \text{supp}\ \widehat{f_k} \subset \Pi_k^{(\delta)}, \tag{1.6}$$

where now

$$\beta(p) = \beta(p,2) = \frac{d-1}{4} - \frac{d+1}{2p}. \tag{1.7}$$

Our main result can then be stated as follows.

Theorem 1.2. *Let $d \ge 3$. Then, the $(W_{p,2})$ inequality in (1.6) holds for all $\varepsilon > 0$ when*

$$p > p_d = 2 + \tfrac{8}{d-2}\big(1 - \tfrac{1/2}{d+1}\big). \tag{1.8}$$

When $d = 2$, the inequality $(W_{p,s})$ holds for all $\varepsilon > 0$ in the range

$$\begin{array}{lll} p > 20 & \text{if} & s \ge 3 - \tfrac{3}{13}, \\ p > p(s) & \text{if} & 2 \le s \le 3 - \tfrac{3}{13}, \\ p > \tfrac{70}{3} & \text{if} & s = 2, \end{array} \tag{1.9}$$

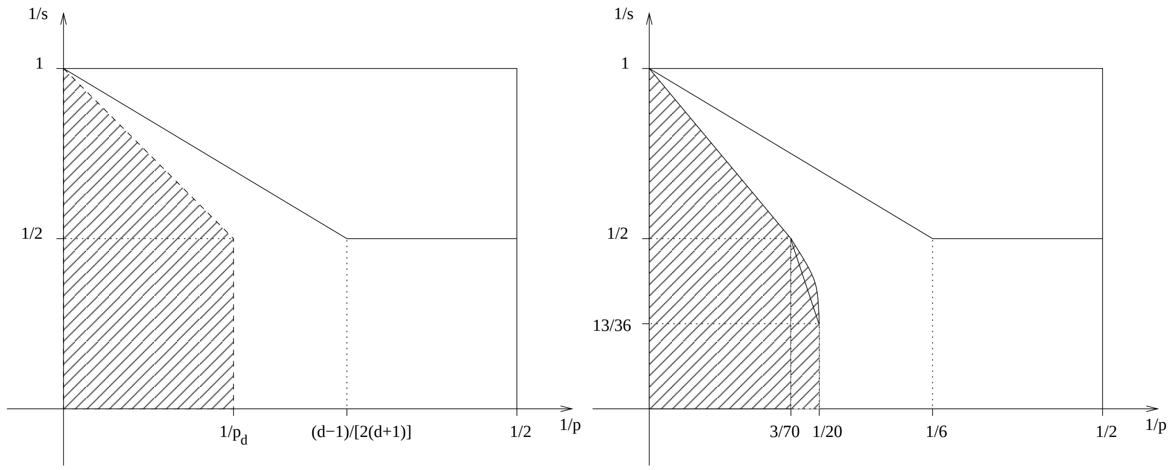

FIGURE 1.2. Region of validity of inequality $(W_{p,s})$ for $d \geq 3$ and $d = 2$, according to Theorem 1.2.

where $p(s) = 5(11s - 6 + \sqrt{65s^2 - 76s + 36})/(6(s-1))$.

Interpolating these results with the trivial $(p = \infty, s = 1)$ estimates one obtains the regions drawn in Figure 1.2.

As mentioned before, Theorem 1.2 can be proved following very closely the arguments in [16, 8]. Our contribution lies on three points: first, a new packet decomposition adapted to the $\ell^s(L^p)$ formulation of the problem. Secondly, a suitable iteration of the induction on scales method from the original proof, leading in particular to a unified exponent for all dimensions $d \geq 3$. Third, in the special (and more difficult) case $d = 2$ we additionally refine one of the combinatorial lemmas of Wolff, which in turn improves and slightly simplifies the results in [16]. These methods, together with the use of bilinear restriction estimates as described in [6], give the improved exponents announced in Theorems 1.1 and 1.2. Although most of the ideas are contained in the original papers [16, 8], we have preferred to make proofs as detailed and self-contained as possible, specially in what concerns to the more general $\ell^s(L^p)$ setting. Nevertheless, the main combinatorial arguments (and specially the very deep ones for $d = 2$ involving circle tangencies) remain untouched and have just been quoted from [16, 8].

We recall that the validity of (1.3) implies progress in various remarkable problems in harmonic analysis. We refer to [6] for a more complete discussion. In some of these problems, the stronger $(W_{p,s})$ estimates obtained in this paper imply slightly better results. For instance, when $d = 2$, the $\ell^4(L^{20})$ inequality in Theorem 1.2, when suitably combined with the bilinear methods of Tao and Vargas [14], implies the following improvement in the square function estimate (see [6, Theorem 5.1]).

Corollary 1.3. *Let $d = 2$. Then for all $\alpha > \frac{1}{9}$ and all f_k with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$, we have*

$$\Big\|\sum_k f_k\Big\|_{L^4(\mathbb{R}^3)} \leq C_\alpha\, \delta^{-\alpha} \,\Big\|\,\big(\sum_k |f_k|^2\big)^{1/2}\Big\|_{L^4(\mathbb{R}^3)}. \tag{1.10}$$

This is a slight improvement over the previously known $\alpha > 5/44$, due to Tao-Vargas and Wolff [14, 17].

As mentioned before, the $\ell^2(L^p)$ inequalities (1.6) also imply progress in the boundedness of Bergman projections in tubes over cones. We discuss this question in more detail in §7.

A reformulation. In order to prove inequalities of the form $(W_{p,s})$ we may assume, without loss of generality, that the points in $\Omega = \{\omega_k\}_k$ are *strongly separated*, meaning that $\operatorname{dist}(\omega_k, \omega_{k'}) \geq c''\sqrt{\delta}$, $k \neq k'$, for a certain constant $c'' > 0$ so that the plates $\Pi_k^{(\delta)}$ are pairwise disjoint. One can also replace the f_k's by suitable projections of Schwartz functions in $\mathbb{R}^{d+1}$.

More precisely, let $\zeta, \chi \in C_c^\infty(\mathbb{R})$ with supp $\zeta \subset [-2c, 2c]$ and supp $\chi \subset [4^{-c}, 4^c]$. For each $\omega \in S^{d-1}$, consider as well a function $\eta_\omega \in C^\infty(\mathbb{R}^d \setminus \{0\})$, homogeneous of degree 0 and supported in the open cone $\{\xi \in \mathbb{R}^d \ :\ \operatorname{dist}\,(\xi/|\xi|, \omega) \leq 2c'\sqrt{\delta}\}$. Then, define operators $P_k = P_{\omega_k}^{(\delta)}$ by

$$\widehat{P_k f}(\tau, \xi) = \zeta\big(\tfrac{\tau - |\xi|}{\delta}\big)\, \chi(\tau)\, \eta_{\omega_k}(\xi)\, \widehat{f}(\tau, \xi). \tag{1.11}$$

Note that the functions $\widehat{P_k f}$ are supported in slight expansions of the plates $\Pi_k^{(\delta)}$, and that the operators P_k are uniformly bounded on $L^p(\mathbb{R}^{d+1})$. We may choose the functions ζ, χ and η_ω so that P_k is the identity on functions with spectrum in $\Pi_k^{(\delta)}$, and there is no loss of generality if we adjust the constants so that the $\widehat{P_k f}$'s have disjoint supports. Then $(W_{p,s})$ is equivalent with the statement that

$$\Big\|\sum_k P_k h_k\Big\|_p \leq C_\varepsilon \delta^{-\beta(p,s)-\varepsilon}\Big(\sum_k \|h_k\|_p^s\Big)^{1/s}, \quad \forall\ \{h_k\} \subset L^p(\mathbb{R}^{d+1}). \tag{1.12}$$

For $1 \leq p, s \leq \infty$ and functions in $f \in L^p(\mathbb{R}^{d+1})$ we may define a semi-norm by

$$\|f\|_{p,s;\delta} = \Big(\sum_k \|P_k f\|_p^s\Big)^{1/s}. \tag{1.13}$$

Note that if $f = \sum f_k$ with supp $\widehat{f_k} \subset \Pi_k^\delta$, then

$$\|f\|_{p,s;\delta} = \Big(\sum_k \|f_k\|_p^s\Big)^{1/s} \quad \text{and} \quad \|f\|_{2,2;\delta} = \|f\|_2. \tag{1.14}$$

Also, when there is no ambiguity about the magnitude of δ we shall just write $\|f\|_{p,s}$.

2. Notation and basic definitions

Throughout we fix $p > p_d$. We also fix a positive but very small ϵ_0, which may depend on p and will be determined later. Statements involving the parameter δ are assumed to hold for all $\delta \in (0, \delta_0]$, for some fixed $\delta_0 \ll 1$. For each such δ we set

$$N = 1/\delta \qquad \text{and} \qquad t = \delta^{\epsilon_0} = N^{-\epsilon_0}.$$

The constants $C, c_0, c_1, \ldots$ appearing below may depend on $p, d, \epsilon_0, \delta_0$ and also on other constants appearing below, but will be independent of δ, f_k, $\{\omega_k\}$, and parameters such as λ or ε. Otherwise we will indicate it by C_ε, etc... By $A \lesssim B$ we will mean $A \leq C\,B$ for some C as above, and by $A \lessapprox B$ we mean $A \leq C\,(\log N)^C\,B$, for some $C > 0$. We shall write either $\#\mathcal{P}$ or $|\mathcal{P}|$ for the cardinality of a finite set $\mathcal{P}$. Recall that $\Omega = \{\omega_k\}$ has cardinality $\lesssim N^{\frac{d-1}{2}}$. Below it will be convenient to identify (perhaps with some abuse of language) each point ω_k with its subindex k; that is, we shall assume that $k = \omega_k \in S^{d-1}$.

Remark. For notational reasons, the scaling used in this paper is slightly different from that in [16, 8], where Fourier transforms are supported in cone sectors of $\{(\tau', \xi') \in \mathbb{R}^{d+1} \ : \ 2^{-c}N \leq \tau' \leq 2^c N \quad \text{and} \quad \big|\tau' - |\xi'|\big| \leq c\}$ rather than in $\Gamma_\delta(c)$. Passing from one setting to the other is straightforward, using the dilation $(\tau, \xi) \in \Gamma_\delta(c) \mapsto (\tau', \xi') = (N\tau, N\xi)$, and we shall do so without mention when quoting results from [16, 8]. Alternatively, the modified statements can also be found in [7].

Plates and plate families. We recall the basic notation concerning plates and tubes in [16, 8], which we write with the same scaling as in [7]. An *N-plate* will be a rectangular box in $\mathbb{R}^{d+1}$ of size $c_0(1 \times \sqrt{N} \times \overset{(d-1)}{\cdots} \times \sqrt{N} \times N)$, whose longest axis is a light-ray and the mid-length and short axes are respectively tangent and normal to the corresponding light-cone at the center of the plate. We typically denote plates in (t, x)-space by π and plate families by $\mathcal{P}$. We shall always assume that N-plates are essentially dual to some $\Pi_k^{(\delta)}$, in which case we use the notation

$$\pi \parallel k \tag{2.1}$$

to indicate that π is an N-plate with long side parallel to $\mathfrak{n}_k = (1, -\omega_k)$. In particular, any two plates are either parallel or point in $\sqrt{\delta}$-separated directions. We shall also assume that families $\mathcal{P}$ consist only of *separated* plates, meaning that for each $\pi \in \mathcal{P}$ at most C_1 plates from $\mathcal{P}$ can be contained in a fixed dilate $C_2\pi$, where C_1 and C_2 are fixed universal constants.

An *N-tube* τ is a rectangular box of size $c_0'(\sqrt{N} \times \overset{(d\text{ times})}{\cdots} \times \sqrt{N} \times N)$, whose longest axis points also in some $(1, -\omega_k)$ direction. Tube families $\mathcal{T}$ will also be assumed to be separated. Finally, a *σ-cube* Δ is a cube of sidelength σ centered at some point of the grid $\sigma\mathbb{Z}^{d+1}$. By $\{\Delta\}$ we denote the tiling of $\mathbb{R}^{d+1}$ formed by all such cubes. In general, given a rectangular box R (e.g., a cube, tube or plate), we denote by cR the box obtained from R by dilating it by a factor $c > 0$ about its center.

Bump functions. Given a fixed large M (which will be chosen later) we let

$$w(x) = (1 + |x|^2)^{-M/2}, \tag{2.2}$$

and given a rectangle R we denote $w_R = w \circ a_R^{-1}$, where a_R is an affine map taking the unit cube centered at 0 to the rectangle R. Thus w_R is roughly the characteristic function of R with "Schwartz tails"(with an abuse of language, as for fixed M the function w is not a Schwartz-function).

We shall also use a fixed non-negative Schwartz function ψ, strictly positive in $B_2(0)$, with Fourier transform supported in $B_{\frac{1}{100}}(0)$, and so that $\sum_{n \in \mathbb{Z}^{d+1}} \psi(\cdot + n)^2 = 1$. Again we set

$$\psi_R = \psi \circ a_R^{-1}. \tag{2.3}$$

In particular, if $\{R\}$ is a tiling of $\mathbb{R}^{d+1}$ by cubes (or plates, or tubes), then $\sum_R \psi_R^2 = 1$.

Elementary properties of the norms $\|\cdot\|_{p,s;\delta}$.

Lemma 2.1. *Let $p \geq 2$ and $f = \sum f_k$ with $\widehat{f_k}$ supported in $\Pi_k^{(\delta)}$. Then*

$$\|f\|_{\infty,2} \lesssim N^{-(d+1)/2p} \|f\|_{p,2}\ , \tag{2.4}$$

$$\|f\|_{\infty} \lesssim N^{\beta(p)} \|f\|_{p,2}\ , \tag{2.5}$$

$$\|f\|_{p,2} \lesssim \|f\|_2^{2/p} \|f\|_{\infty,2}^{1-2/p}\ . \tag{2.6}$$

Moreover, for every $s \in [1,p]$ we have

$$\|f\|_{p,s} \lesssim \|f\|_2^{2/p} \|f\|_{\infty,r}^{1-2/p}\ , \tag{2.7}$$

where $r = r(p,s)$ is defined by

$$\frac{2}{r} = (\frac{1}{s} - \frac{1}{p})/(\frac{1}{2} - \frac{1}{p}). \tag{2.8}$$

Proof. Observe that, by Young's inequality,

$$\|g\|_\infty \lesssim N^{-(d+1)/2p} \|g\|_p\ , \quad \text{when supp } \hat{g} \subset \Pi_k^{(\delta)}. \tag{2.9}$$

This yields (2.4). If $f = \sum f_k$ with $\widehat{f_k}$ supported in $\Pi_k^{(\delta)}$ then, using (2.4),

$$\|f\|_\infty \lesssim \sum_k \big\|f_k\big\|_\infty \lesssim N^{\frac{d-1}{4}} \Big(\sum_k \big\|f_k\big\|_\infty^2\Big)^{1/2} \lesssim N^{\frac{d-1}{4} - \frac{d+1}{2p}} \Big(\sum_k \big\|f_k\big\|_p^2\Big)^{1/2}$$

which is (2.5). Inequality (2.6) is a special case of a corresponding inequality for the projection operators P_k, namely for $\vartheta = 1 - 2/p$ and $\{h_k\} \subset L^p(\mathbb{R}^{d+1})$,

$$\Big(\sum_k \|P_k h_k\|_p^2\Big)^{1/2} \lesssim \Big(\sum_k \|h_k\|_2^2\Big)^{(1-\vartheta)/2} \Big(\sum_k \|h_k\|_\infty^2\Big)^{\vartheta/2}.$$

This follows by convexity from the obvious cases $p = 2$ and $p = \infty$. The more general inequality (2.7) follows similarly, after noticing that $\frac{1}{s} = \frac{1-\vartheta}{2} + \frac{\vartheta}{r}$ and $\vartheta = 1 - 2/p$ imply that $r = r(p,s)$ as in (2.8). □

We shall also use the following localization estimate.

Lemma 2.2. *Let $1 \le s \le p \le \infty$ and $f = \sum f_k$ with $\widehat{f_k}$ supported in $\Pi_k^{(\delta)}$. Let $\mathcal{Q} = \{Q\}$ be a grid of N-cubes and ψ_Q be as in (2.3) (so that $\widehat{\psi_Q}$ is supported in $B_{\delta/100}(0)$). Then*

$$\Big(\sum_{Q\in\mathcal{Q}} \|\psi_Q f\|^p_{p,s;2\delta}\Big)^{1/p} \lesssim \|f\|_{p,s;\delta}\,. \tag{2.10}$$

Proof. Note that $\widehat{\psi_Q} * \widehat{f_k}$ is supported in $\Pi_k^{(2\delta)}$. Also, $\|P_k^{(2\delta)}(\psi_Q f)\|_p \lesssim \sum_{k'\in\Omega_k} \|\psi_Q f_{k'}\|_p$, where $\Omega_k = \{k' \colon \mathrm{dist}\,(\omega_{k'} - \omega_k) \lesssim \sqrt{\delta}\}$ has cardinality bounded by a constant. Applying Minkowski's inequality (since $p \ge s$) and using the above observation we obtain

$$\begin{aligned}\Big(\sum_Q \Big[\sum_k \|P_k^{(2\delta)}(\psi_Q f)\|_p^s\Big]^{\frac{p}{s}}\Big)^{\frac{1}{p}} &\le \Big(\sum_k \Big[\sum_Q \|P_k^{(2\delta)}(\psi_Q f)\|_p^p\Big]^{\frac{s}{p}}\Big)^{\frac{1}{s}} \\ &\lesssim \Big(\sum_{k'} \Big[\sum_Q \|\psi_Q f_{k'}\|_p^p\Big]^{\frac{s}{p}}\Big)^{\frac{1}{s}} \lesssim \Big(\sum_{k'} \|f_{k'}\|_p^s\Big)^{\frac{1}{s}}.\end{aligned}$$

□

Packets.

Definition 2.3. (i) f is an called an *N-packet associated with* $\Pi_k^{(\delta)}$ if it can be written as $f = \sum_{\pi\in\mathcal{P}} f_\pi$ for some family $\mathcal{P} = \mathcal{P}_k(f)$ of separated N-plates with $\pi \parallel k$, in such a way that every function f_π, $\pi \in \mathcal{P}_k$, satisfies

$$|f_\pi| \le c_1\, w_\pi \qquad \text{and} \qquad \mathrm{supp}\, \widehat{f_\pi} \subset c_2\, \Pi_k^{(\delta)}, \tag{2.11}$$

where c_1, c_2 are suitable fixed constants.

(ii) Let E be a set of directions in Ω and let Q an N-cube. We say that f is an (N,E,Q)-*packet*[†] if it can be written as

$$f = \sum_{k\in E} \sum_{\pi\in\mathcal{P}_k} f_\pi \tag{2.12}$$

where $\mathcal{P}_k$ is a family of plates $\pi \parallel k$ so that $\pi \subset 2N^{\varepsilon_0} Q$ for all $\pi \in \mathcal{P}_k$, and f_π are functions so that (2.11) holds for all $\pi \in \mathcal{P}_k$ and all $k \in E$. We denote *the plate family of f* by $\mathcal{P}(f) = \cup_{k\in E}\mathcal{P}_k$.

[†]These packets are special cases of "N-functions", as defined in [16, 8].

(iii) We say that f is a *stable* (N,E,Q)*-packet* if it is an (N,E,Q)-packet with plate family $\mathcal{P}(f)=\cup_{k\in E}\mathcal{P}_k$ which in addition satisfies

$$|\mathcal{P}_k| \le 2|\mathcal{P}_{k'}|, \quad \forall\, k,k' \in E. \tag{2.13}$$

(iv) Let f be an (N,E,Q)-packet with plate family $\mathcal{P}=\cup_{k\in E}\mathcal{P}_k$ and with the representation (2.12). A *subpacket of* f is a function $\tilde f$ of the form

$$\tilde f = \sum_{k\in E}\sum_{\pi\in\widetilde{\mathcal{P}}_k} f_\pi, \tag{2.14}$$

where each $\widetilde{\mathcal{P}}_k$ is a subset of $\mathcal{P}_k$. Observe that every subpacket of an (N,E,Q)-packet is again an (N,E,Q)-packet. However, subpackets of stable (N,E,Q)-packets are not necessarily stable.

Condition (2.13) is crucial to deal with $\|\cdot\|_{p,s;\delta}$ norms, and implies that the cardinalities of the $\mathcal{P}_k$'s are comparable, for all $k\in E$. Elementary properties of packets are listed below.

Lemma 2.4. *Let f be an (N,E,Q)-packet. Then*

$$\|f\|_{\infty,r} \lesssim |E|^{\frac1r}, \quad 1\le r\le\infty, \tag{2.15}$$

$$\|f\|_\infty \lesssim |E| \lesssim N^{\frac{d-1}{2}}, \tag{2.16}$$

$$\|f\|_2^2 \lesssim N^{(d+1)/2}|\mathcal{P}(f)|, \tag{2.17}$$

Moreover, if $p\ge 2$ and $1\le s\le p$ we also have

$$\|f\|_{p,s} \lesssim \big(N^{\frac{d+1}{2}}|\mathcal{P}(f)|\big)^{\frac1p}\,|E|^{\frac1s-\frac1p}. \tag{2.18}$$

If in addition f is a stable (N,E,Q)-packet, then (2.18) *holds for all $p,s\ge 1$.*

Proof. (2.15) and (2.16) are immediate. We next prove (2.18) for stable packets:

$$\begin{aligned}\|f\|_{p,s} &\lesssim \Big(\sum_{k\in E}\Big\|\sum_{\pi\in\mathcal{P}_k} f_\pi\Big\|_p^s\Big)^{1/s} \lesssim \Big(\sum_{k\in E}\Big\|\sum_{\pi\in\mathcal{P}_k} w_\pi\Big\|_p^s\Big)^{1/s}\\ &\lesssim \Big(\sum_{k\in E}\big[N^{\frac{d+1}{2}}\,|\mathcal{P}_k|\big]^{s/p}\Big)^{1/s} \lesssim \big(N^{\frac{d+1}{2}}\,|\mathcal{P}(f)|\big)^{1/p}|E|^{\frac1s-\frac1p}\end{aligned}$$

where for the last inequality we use $|\mathcal{P}_k|\approx|\mathcal{P}(f)|/|E|$ by (2.13). Observe that the previous proof does not require the stability assumption when $p=s$, which in particular gives (2.17).

Finally, to obtain (2.18) for general packets when $p\ge 2$ and $1\le s\le p$, one uses the interpolation estimate in (2.7), together with the previous (2.15) and (2.17):

$$\|f\|_{p,s} \lesssim \|f\|_2^{\frac2p}\,\|f\|_{\infty,r}^{1-\frac2p} \lesssim \big(N^{\frac{d+1}{2}}\,|\mathcal{P}(f)|\big)^{1/p}|E|^{(1-\frac2p)/r},$$

where $r = r(p,s)$ is defined in (2.8). However, from the definition of $r(p,s)$ one sees that $(1-\frac{2}{p})/r = \frac{1}{s} - \frac{1}{p}$, which establishes (2.18). $\square$

The main lemma in this section concerns decompositions of functions with Fourier support in $\Gamma_\delta(c)$ into stable N-packets. The stability condition on the packets is crucial to obtain the inequality in (2.22) below, which is a sort of converse to the inequality in (2.18). This estimate will be strongly used in the proof of Proposition 3.3 and in the iteration process which starts with Lemma 5.2.

Lemma 2.5. *Let $f = \sum f_k$ with $\widehat{f_k}$ supported in $\Pi_k^{(\delta)}$ and assume that*

$$\sup_k \|f_k\|_\infty \le A. \tag{2.19}$$

Then, for every N-cube Q, we may decompose

$$f(x) \;=\; \sum_{AN^{-10d} \le 2^j \lesssim A} \; \sum_{\ell=1}^{n_j} 2^j\, f^{[j,\ell]}(x) \;+\; g(x), \quad x \in Q \tag{2.20}$$

for some constant integers $n_j \lesssim \log N$, and where

(i) for each j,ℓ, the functions $f^{[j,\ell]}$ are stable $(N, E_{j,\ell}, Q)$-packets, for certain sets of directions $E_{j,\ell} \subset \Omega$. The corresponding plate families $\mathcal{P}^{[j,\ell]}$ consist only of plates $\pi \subset 2N^{\varepsilon_0}Q$. Also, we can write

$$f^{[j,\ell]} = \sum_{k \in E_{j,\ell}} \sum_{\pi \in \mathcal{P}_k^{[j,\ell]}} f_\pi,$$

for plate families $\mathcal{P}_k^{[j,\ell]}$ consisting of plates $\parallel k$, and so that $\mathcal{P}^{[j,\ell]} = \cup_{k \in E_{j,\ell}} \mathcal{P}_k^{[j,\ell]}$.

(ii) The function $g(x)$ satisfies

$$\|g\|_{L^\infty(Q)} \lesssim N^{-8d} A. \tag{2.21}$$

(iii) For every $s, p \ge 1$ and every j, ℓ

$$2^j \left(N^{\frac{d+1}{2}} |\mathcal{P}^{[j,\ell]}|\right)^{\frac{1}{p}} |E_{j,\ell}|^{\frac{1}{s} - \frac{1}{p}} \;\lesssim\; \|f\|_{p,s;\delta}\,. \tag{2.22}$$

Proof. Fix k, and consider a tiling $\{\pi\}$ of $\mathbb{R}^{d+1}$ by plates $\pi \parallel k$. Write

$$f_k \;=\; \sum_{\pi \| k} f_k \psi_\pi^2 \;=\; \sum_{\pi \| k:\; \pi \cap (N^{\varepsilon_0} Q) \neq \emptyset} f_k \psi_\pi^2 \;+\; g_k. \tag{2.23}$$

For each $j \in \mathbb{Z}$, let

$$\mathcal{P}_k^{[j]} = \big\{ \pi \;:\; \pi \parallel k,\; \pi \cap (N^{\varepsilon_0} Q) \neq \emptyset \quad \text{and} \quad 2^j \le \|f_k \psi_\pi\|_\infty < 2^{j+1} \,\big\}.$$

Observe that $\|f_k\psi_\pi\|_\infty \lesssim \|f_k\|_\infty \le A$, and therefore these plate sets are non-empty only for $2^j \lesssim A$. Next, fix j, and for every positive integer ℓ define

$$E_{j,\ell} = \{k \ : \ 2^{\ell-1} \le \#\mathcal{P}_k^{[j]} < 2^\ell\}.$$

Since $\#\mathcal{P}_k^{[j]} \le \#\{\pi \ : \ \pi \subset 2N^{\varepsilon_0}Q\} \lesssim N^{(d+1)(1+\varepsilon_0)}/N^{\frac{d+1}{2}}$, the sets $E_{j,\ell}$ are non-empty only for $\ell \lesssim \log N$.

Call $f_\pi := 2^{-j} f_k \psi_\pi^2$, when $\pi \in \mathcal{P}_k^{[j]}$. Clearly, these are N-packets associated with $\Pi_k^{(\delta)}$. Define the functions

$$f^{[j,\ell]} = \sum_{k\in E_{j,\ell}} \sum_{\pi \in \mathcal{P}_k^{[j]}} f_\pi,$$

so that from (2.23) we see that

$$f(x) \;=\; \sum_{2^j \lesssim A} 2^j \sum_{\ell=1}^{C\log N} f^{[j,\ell]}(x) \;+\; \sum_k g_k(x). \tag{2.24}$$

By construction it is easy to see that, for each j and ℓ, the function $f^{[j,\ell]}$ is an $(N, E_{j,\ell}, Q)$-packet. The stability condition in (2.13) is immediate since

$$|\mathcal{P}_{k_0}^{[j]}| \;<\; 2^\ell \;\le\; 2\,|\mathcal{P}_{k_1}^{[j]}|, \quad \forall\, k_0, k_1 \in |E_{j,\ell}|.$$

To pass from (2.24) to the decomposition in (2.20), define the function

$$g = \sum_k g_k \;+\; \sum_{2^j < AN^{-10d}} 2^j \sum_\ell f^{[j,\ell]}.$$

Observe that, from (2.23) and the Schwartz decay of ψ,

$$\begin{aligned} \|\sum_k g_k\|_{L^\infty(Q)} \;&\le\; \sum_k \sum_{\pi\|k:\, \pi\cap(N^{\varepsilon_0}Q)=\emptyset} \|f_k\psi_\pi^2\|_{L^\infty(Q)} \\ &\lesssim\; N^{\frac{d-1}{2}} \sup_k \|f_k\|_\infty\, C_L\, N^{-\varepsilon_0 L} \;\lesssim\; N^{-9d}A, \end{aligned}$$

if we choose L sufficiently large (depending on ε_0). On the other hand, using (2.16),

$$\sum_{2^j \le AN^{-10d}} 2^j \sum_\ell \|f^{[j,\ell]}\|_\infty \;\lesssim\; (\log N) \sum_{2^j < AN^{-10d}} 2^j\, N^{\frac{d-1}{2}} \;\lesssim\; AN^{-9d}.$$

Putting the last two estimates together we obtain (2.21).

Finally, we must verify (2.22), for every j,ℓ. Fix $k_0 \in E_{j,\ell}$, and use that $|\mathcal{P}_k^{[j]}| \approx |\mathcal{P}_{k_0}^{[j]}| \approx 2^\ell$, for all $k \in E_{j,\ell}$, which implies

$$\begin{aligned} 2^{jp}\, N^{\frac{d+1}{2}}\, |\mathcal{P}^{[j,\ell]}| &\leq 2^{jp}\, N^{\frac{d+1}{2}} |E_{j,\ell}|\, 2^\ell \lesssim |E_{j,\ell}| \sum_{\pi\in\mathcal{P}_{k_0}^{[j]}} 2^{jp}\, |\pi| \\ &\lesssim |E_{j,\ell}| \sum_{\pi\in\mathcal{P}_{k_0}^{[j]}} \|f_{k_0}\psi_\pi\|_\infty^p\, |\pi| \\ &\lesssim |E_{j,\ell}| \sum_{\pi\in\mathcal{P}_{k_0}^{[j]}} \|f_{k_0}\psi_\pi\|_p^p \lesssim |E_{j,\ell}|\, \|f_{k_0}\|_p^p\,, \end{aligned}$$

where in the last two inequalities we have used (2.9) and $\sum_{n\in\mathbb{Z}^{d+1}} \psi(\cdot+n)^\rho \lesssim 1$. Thus, we have

$$\begin{aligned} \|f\|_{p,s} &\geq \Big(\sum_{k_0\in E_{j,\ell}} \|f_{k_0}\|_p^s\Big)^{\frac{1}{s}} \\ &\gtrsim \Big(\sum_{k_0\in E_{j,\ell}} \Big[\frac{2^{jp}\, N^{\frac{d+1}{2}}\, |\mathcal{P}^{[j,\ell]}|}{|E_{j,\ell}|}\Big]^{s/p}\Big)^{\frac{1}{s}} = 2^j\, (N^{\frac{d+1}{2}}\, |\mathcal{P}^{[j,\ell]}|)^{\frac{1}{p}}\, |E_{j,\ell}|^{\frac{1}{s}-\frac{1}{p}}, \end{aligned}$$

as we wished to prove. □

3. Equivalent formulations of the problem

Definition 3.1. Given $p \geq 2$, $s \in [1,p]$ and $\gamma > 0$, we say that *hypothesis* $\mathcal{H}^{str}(p,s,\gamma)$ *holds* if there exists $C_\gamma > 0$ so that for any $\delta = N^{-1} \leq \delta_0$ and any $f = \sum_k f_k$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$

$$\|f\|_p \leq C_\gamma N^{\beta(p,s)+\gamma} \Big(\sum_k \|f_k\|_p^s\Big)^{1/s}, \tag{3.1}$$

where $\beta(p,s) = \frac{d-1}{2s'} - \frac{d+1}{2p}$.

It is our objective to prove $\mathcal{H}^{str}(p,2,\gamma)$ for all $\gamma > 0$, in the asserted range of p's in (1.8) (and likewise for $\mathcal{H}^{str}(p,s,\gamma)$ when $d = 2$ in the range in (1.9)). We formulate a slightly weaker condition which can be seen as an analogue of a restricted weak type inequality.

Definition 3.2. Given $p \geq 2$, $s \in [1,p]$ and $\gamma > 0$, we say that *hypothesis* $\mathcal{H}(p,s,\gamma)$ *holds* if there exists $C_\gamma > 0$ so that for all $\delta = N^{-1} \leq \delta_0$, for all N-cubes Q, all $E \subset \Omega$ and all stable (N,E,Q)-packets f with plate family $\mathcal{P}(f)$ the following estimate holds

$$\big|\{x \in Q : |f(x)| > \lambda\}\big| \leq C_\gamma\, \lambda^{-p}\, N^{(\beta(p,s)+\gamma)p}\, N^{\frac{d+1}{2}}\, |\mathcal{P}(f)|\, |E|^{\frac{p}{s}-1}, \tag{3.2}$$

for all positive real number $\lambda > 0$.

The main result in this section is the following.

Proposition 3.3. *Let $p \geq 2$, $s \in [1,p]$ and $0 < \gamma < \gamma_1$. Then*

$$\mathcal{H}^{str}(p,s,\gamma) \implies \mathcal{H}(p,s,\gamma) \implies \mathcal{H}^{str}(p,s,\gamma_1). \tag{3.3}$$

The first implication follows by Čebyšev's inequality and the estimate (2.18) for the (p,s)-norm of stable plates. The second implication is less trivial and will be proved below. Observe that one always has the trivial bound $\mathcal{H}^{str}(p,s,\gamma = \frac{d+1}{2p})$, since $\| \sum_k f_k\|_p \leq \sum_k \|f_k\|_p \lesssim N^{\frac{d-1}{2s'}} \|f\|_{p,s} = N^{\beta(p,s)+\frac{d+1}{2p}} \|f\|_{p,s}$. Thus, assuming Proposition 3.3, Theorem 1.2 is reduced to prove the following

Theorem 3.4. *Let p and s be as in* (1.8) *and* (1.9)*. Then, there exists $\epsilon_0' = \epsilon_0'(p,s)$ so that if hypothesis $\mathcal{H}^{str}(p,s,\gamma_0)$ holds for some $\gamma_0 > 0$, then hypothesis $\mathcal{H}(p,s,\gamma)$ holds for all $\gamma > (1-\epsilon_0')\gamma_0$.*

Indeed, if Theorem 3.4 holds, then Proposition 3.3 together with an iteration gives the validity of the strong type estimate $\mathcal{H}^{str}(p,s,\epsilon)$ for all $\epsilon > 0$, thus establishing Theorem 1.2.

In the proof of Proposition 3.3 we shall also use the following localization lemma.

Lemma 3.5. *Let $1 \leq s \leq p < \infty$ and $\alpha > 0$. Assume that for all N-cubes Q and all $f = \sum_k f_k$ with $\widehat{f_k} \subset \Pi_k^{(\delta)}$ we have*

$$\|f\|_{L^p(Q)} \leq C\, N^{\alpha}\, \|f\|_{p,s;\delta}. \tag{3.4}$$

Then,

$$\|f\|_{L^p(\mathbb{R}^{d+1})} \lesssim C\, N^{\alpha}\, \|f\|_{p,s;\delta}. \tag{3.5}$$

Proof. Write $f = \sum_{Q\in\mathcal{Q}} \psi_Q^2 f$, where $\mathcal{Q}$ is a tiling of $\mathbb{R}^{d+1}$ by N-cubes and ψ_Q is as in (2.3). Then, using the Schwartz decay of ψ,

$$\begin{aligned}\|f\|^p_{L^p(\mathbb{R}^{d+1})} &= \sum_{Q'} \Big\|\sum_{Q} \psi_Q^2 f\Big\|^p_{L^p(Q')} \lesssim \sum_{Q,Q'} \big\|\psi_Q^{3/2} f\big\|^p_{L^p(Q')} \\ &\lesssim \sum_{Q,Q'} \big\|\psi_Q f\big\|^p_{L^p(Q')} \big(1+\mathrm{dist}\,(Q,Q')/N\big)^{-10d}.\end{aligned}$$

Since each $\psi_Q f_k$ has spectrum contained in $\Pi_k^{(2\delta)}$, we can apply (3.4) with f replaced by $\psi_Q f$ (and δ by 2δ) to obtain

$$\|f\|^p_{L^p(\mathbb{R}^{d+1})} \lesssim C^p\, N^{\alpha p} \sum_{Q,Q'} \big\|\psi_Q f\big\|^p_{p,s;2\delta} \big(1+\mathrm{dist}\,(Q,Q')/N\big)^{-10d},$$

which by Lemma 2.2 is controlled by $\|f\|^p_{p,s;\delta}$. □

Proof of Proposition 3.3. We show the proof of the main implication

$$\mathcal{H}(p,s,\gamma) \implies \mathcal{H}^{str}(p,s,\gamma_1), \text{ for } \gamma_1 > \gamma. \tag{3.6}$$

By the previous lemma it suffices to show

$$\|f\|_{L^p(Q)} \le C_\varepsilon\, N^{(\beta(p,s)+\gamma+\varepsilon)} \|f\|_{p,s;\delta} \tag{3.7}$$

for all $\varepsilon > 0$, all N-cubes Q and all $f = \sum_k f_k$ with $\widehat{f_k} \subset \Pi_k^{(\delta)}$. To do so we may assume

$$\|f\|_{p,s;\delta} = 1. \tag{3.8}$$

Fix an N-cube Q. Then

$$\|f\|^p_{L^p(Q)} \lesssim \sum_{m\in\mathbb{Z}} 2^{mp} \operatorname{meas}\big(\{x \in Q : |f| > 2^m\}\big). \tag{3.9}$$

By the arguments in Lemma 2.1 we have $\|f\|_\infty \lesssim N^{\beta(p,s)}$, and thus we may assume $m \lesssim \log N$ in (3.9). Also, if $2^m \le N^{-(d+1)}$, then the right hand side of (3.9) is controlled by

$$\sum_{2^m \le N^{-(d+1)}} 2^{mp}\, |Q| \lesssim N^{-(d+1)(p-1)} \le 1.$$

Thus, only a logarithmic number of m's are relevant in (3.9), so by a pigeonhole argument we can find m_* so that

$$\|f\|^p_{L^p(Q)} \lesssim (\log N)\, 2^{m_* p} \operatorname{meas}\big(\{x \in Q : |f| > 2^{m_*}\}\big) + 1. \tag{3.10}$$

Using $\sup_k \|f_k\|_\infty \lesssim \sup_k N^{-\frac{d+1}{2p}} \|f_k\|_p \le N^{-\frac{d+1}{2p}}$, we can apply to f the packet decomposition in Lemma 2.5, with $A = N^{-\frac{d+1}{2p}}$ and the N-cube Q fixed above. By (2.21), the function g in (2.20) is then $\lesssim N^{-8d}$ which in turn is $\ll 2^{m_*}$. By the pidgeonhole principle applied to the $O((\log N)^2)$ terms in the sum in (2.20), there are integers j_* and ℓ_*, so that the set of directions $E^* = E_{j_*,\ell_*}$ and the stable (N, E^*, Q)-packet $f^* = f^{[j_*,\ell_*]}$ satisfy

$$\operatorname{meas}\{x \in Q : |f| > 2^{m_*}\} \lesssim (\log N)^2 \operatorname{meas}\Big(\Big\{x \in Q \;:\; 2^{j^*}\, |f_*| > \tfrac{2^{m_*}}{C(\log N)^2}\Big\}\Big).$$

By Hypothesis $\mathcal{H}(p, s, \gamma)$ the right hand side of (3.10) is then estimated by

$$C_\gamma\, (\log N)^{3+2p}\, N^{(\beta(p,s)+\gamma)p}\, 2^{j_* p}\, N^{\frac{d+1}{2}} |\mathcal{P}(f_*)|\, |E_*|^{\frac{p}{s}-1} \lessapprox C_\gamma\, N^{(\beta(p,s)+\gamma)p},$$

where the last inequality follows from the crucial estimate (2.22) and the assumption $\|f\|_{p,s;\delta} = 1$. Since the powers of $\log N$ are controlled by $C_\varepsilon N^\varepsilon$, for any $\varepsilon > 0$, this finishes the proof of (3.7) and thus the proposition. □

There are some situations in which the inequality in (3.2) is trivial to verify, namely when either $|E|$ or λ are sufficiently small.

Lemma 3.6. *Let $p > 2$ and $1 \le s \le p$. Then the inequality* (3.2) *is true for every $\gamma > 0$ and every (N, E, Q)-packet when either*

$$\lambda \le N^{\frac{\beta(p,s)\,p}{p-2}}\, |E|^{(\frac{p}{s}-1)/(p-2)}, \tag{3.11}$$

or when

$$|E|^{\frac{p}{s'}-1} \le N^{\beta(p,s)\,p}. \tag{3.12}$$

Proof. By Čebyšev's inequality and Lemma 2.4

$$\text{meas}\,\big(\{x : |f(x)| > \lambda\}\big) \le \lambda^{-2}\|f\|_2^2 \lesssim \lambda^{-2}\, N^{(d+1)/2}\, |\mathcal{P}(f)|\,,$$

and therefore

$$\text{meas}\,\big(\{x : |f(x)| > \lambda\}\big) \le \lambda^{-p}\, N^{\beta(p,s)\,p}\, N^{\frac{d+1}{2}}\, |\mathcal{P}(f)||E|^{\frac{p}{s}-1}, \tag{3.13}$$

if $\lambda^{-2} \le \lambda^{-p}\, N^{\beta(p,s)\,p}|E|^{\frac{p}{s}-1}$, which is easily seen to the same as (3.11). On the other hand, for packets f we have $\|f\|_\infty \lesssim |E|$ (by (2.16)), so that (3.2) only needs to be verified when $\lambda \lesssim |E|$. But in this range (3.11) always holds if $|E| \lesssim N^{\beta(p,s)p/(p-2)}|E|^{(\frac{p}{s}-1)/(p-2)}$, which is the same as (3.12). □

The previous lemma can be slightly improved using the following known (although probably non optimal) square function estimate: *for all* $f = \sum f_k$ *with supp* $\hat{f}_k \subset \Pi_k^{(\delta)}$ *and for all* $\varepsilon > 0$ *it holds*

$$\|\sum_k f_k\|_q \le C_\varepsilon\, N^{\frac{d-1}{4(d+3)}+\varepsilon}\, \big\|\big(\sum_k |f_k|^2\big)^{\frac{1}{2}}\big\|_q, \qquad \text{where} \quad q = \tfrac{2(d+3)}{d+1}. \tag{3.14}$$

This inequality follows from the bilinear methods of Tao and Vargas [14], combined with Wolff's bilinear restriction theorem for the cone [17]. See e.g. [6, Prop. 2.3] for a detailed proof.

Lemma 3.7. *Let* $q = 2(d+3)/(d+1)$, $p > q$ *and* $1 \le s \le p$. *Then the inequality* (3.2) *is true for every* $\gamma > 0$ *and every* (N, E, Q)*-packet when either*

$$\lambda \lessapprox N^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)}\, |E|^{(\frac{p}{s}-\frac{q}{2})/(p-q)}, \tag{3.15}$$

or when

$$|E|^{\frac{p}{s'}-\frac{q}{2}} \lessapprox N^{\beta(p,s)\,p-\frac{d-1}{2(d+1)}}. \tag{3.16}$$

Proof. Using Chebichev's inequality and (3.14) we see that

$$\text{meas}\,\big(\{x : |f(x)| > \lambda\}\big) \le \lambda^{-q}\|f\|_q^q \le C_\varepsilon\, \lambda^{-q}\, N^{(\frac{d-1}{4(d+3)}+\varepsilon)q}\, \big\|\big(\sum_k |f_k|^2\big)^{\frac{1}{2}}\big\|_q^q.$$

Since $q > 2$, Minkowski's inequality gives $\big\|\big(\sum_k |f_k|^2\big)^{\frac{1}{2}}\big\|_q \le \|f\|_{q,2;\delta}$, while for (N, E, Q)-packets we have $\|f\|_{q,2;\delta}^q \lesssim N^{\frac{d+1}{2}}\, |\mathcal{P}(f)|\, |E|^{\frac{q}{2}-1}$, by Lemma 2.4. Thus, choosing $\varepsilon < \gamma$, (3.2) will hold for all λ so that

$$\lambda^{-q}\, N^{\frac{d-1}{4(d+3)}q}\, |E|^{\frac{q}{2}} \lessapprox \lambda^{-p}\, N^{\beta(p,s)\,p}\, |E|^{\frac{p}{s}},$$

or equivalently when (3.15) holds. On the other hand, since we only consider $\lambda \le \|f\|_\infty \lesssim |E|$, we see that (3.15) is always true when $|E| \lessapprox \big(N^{\beta(p,s)p-\frac{d-1}{2(d+1)}}|E|^{\frac{p}{s}-\frac{q}{2}}\big)^{1/(p-q)}$, which after easy arithmetics gives the condition in (3.16). □

Remark 3.8. Thus, in the proof of Theorem 3.4 below we only need to consider the validity of (3.2) for (N, E, Q)-packets f whose associated direction sets E have cardinality

$$N^{(\beta(p,s)\,p-\frac{d-1}{2(d+1)})/(\frac{p}{s'}-\frac{q}{2})}\,(\log N)^C \;\le\; |E| \lesssim N^{\frac{d-1}{2}}, \tag{3.17}$$

and for real numbers λ in the range

$$N^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)}\,|E|^{(\frac{p}{s}-\frac{q}{2})/(p-q)}\,(\log N)^C \;\le \lambda \lesssim |E|, \tag{3.18}$$

where C can be a suitably large constant.

4. Sufficient conditions for Theorem 3.4

The purpose of this section is identify properties of packets so that the improvement in Theorem 3.4 holds. As in [16, 8] these can be phrased via localization of the level sets $\{|f| > \lambda\}$ using grids of slightly smaller cubes. Also, such localization assumptions will hold when the cardinality of the involved plate families is suitably controlled in terms of λ.

4.1. **Localization.** We begin with an easy (but crucial) localization estimate.

Lemma 4.1. *Let $\widehat{f}$ be supported in $\Gamma_\delta(c)$, let R be a cube of diameter tN, where $t \le 1$. Then*

$$\|\psi_R f\|_2 \lesssim t^{1/2}\|f\|_2 \tag{4.1}$$

Proof. By Plancherel this is equivalent with a statement about the integral operator $TF(\xi) = \int K_\delta(\xi,\eta)F(\eta)d\eta$ with kernel

$$K_\delta(\xi,\eta) = \widehat{\psi_R}(\xi-\eta)\chi_{\Gamma_\delta(c)}(\eta).$$

The L^2 operator norm is $\le \sqrt{A_1A_2}$ where

$$A_1 = \sup_\xi \int |K_\delta(\xi,\eta)|d\eta$$

$$A_2 = \sup_\eta \int |K_\delta(\xi,\eta)|d\xi.$$

Now clearly $A_2 = O(1)$ while the smaller η-support yields $A_1 = O(t)$. This implies the assertion. □

We shall also use the following result.

Lemma 4.2. *Let* $f = \sum_k f_k$ *with supp* $\hat{f}_k \subset \Pi_k^{(\delta)}$, $t \in [\sqrt{\delta}, 1]$ *and* R *a* tN*-cube. Then the function* $f\psi_R$ *has Fourier transform supported in* $\Gamma_{\delta/t}(C)$ *and*

$$\big\|f\psi_R\big\|_{\infty,r;\delta/t} \lesssim t^{-\frac{d-1}{2r'}} \|f\|_{\infty,r;\delta}, \quad \forall\, r \geq 1. \tag{4.2}$$

Proof. Since $\widehat{\psi_R}$ is supported in $B_{\delta/(100t)}(0)$, it follows immediately that $\widehat{f\psi_R} = \hat{f} * \widehat{\psi_R}$ is supported in $\Gamma_{\delta/t}(C)$, for a sufficiently large constant $C > 0$. Next, denote by $P_{k'}^{(\delta/t)}$ the projections adapted to the plates $\Pi_{k'}^{(\delta/t)}$ as in (1.11), and for each k' let $\Omega_{k'} = \{k \;:\; \big(\Pi_k^{(\delta)} + B_{\delta/(100t)}(0)\big) \cap \Pi_{k'}^{(2\delta/t)} \neq \emptyset\}$. Observe that $t \in [\sqrt{\delta}, 1]$ implies $\#\Omega_{k'} \lesssim t^{-(d-1)/2}$, $\forall\, k'$, and $\#\{k' \;:\; k \in \Omega_{k'}\} \lesssim 1$, $\forall\, k$. Also we have

$$\big\|P_{k'}^{(\delta/t)}(f\psi_R)\big\|_\infty = \big\|P_{k'}^{(\delta/t)}\big(\sum_{k\in\Omega_{k'}} f_k\psi_R\big)\big\|_\infty \lesssim \sum_{k\in\Omega_{k'}} \|f_k\|_\infty.$$

Then, (4.2) follows from the above observations and Hölder's inequality. □

We now state a definition of λ-localization using tN-cubes. Below, $\mathcal{Q}(t) = \{B\}$ denotes a fixed partition of $\mathbb{R}^{d+1}$ by tN-cubes[‡].

Definition 4.3. *Let* f *be an* (N, E, Q)*-packet, let* $\lambda > 0$ *and as before* $t = \delta^{\epsilon_0} = N^{-\epsilon_0}$*. We say that* f ***localizes at height*** λ *if there are subpackets* f^B *of* f*, where* B *runs over* tN*-cubes in a grid* $\mathcal{Q}(t)$*, such that*

$$\sum_B \#\mathcal{P}(f^B) \lessapprox \#\mathcal{P}(f) \tag{4.3}$$

and

$$meas\ \big(\{x : |f(x)| > \lambda\}\big) \lessapprox \sum_B meas\ \big(B \cap \{x : |f^B| \gtrapprox \lambda\}\big). \tag{4.4}$$

The next lemma gives, under the localization assumption, the crucial gain in the exponent γ asserted in Theorem 3.4. The statement is just a straightforward modification of [8, Lemma 6.2], but we sketch the proof below for completeness.

Lemma 4.4. *Let* $p \geq 2$, $s \in [1, p]$ *and suppose that* $\mathcal{H}^{str}(p, s, \gamma_0)$ *holds for a fixed* $\gamma_0 > 0$*. Let* $\lambda > 0$ *and suppose that* f *is an* (N, E, Q)*-packet which localizes at height* λ *(with respect to* tN*-cubes). Then, the estimate* (3.2)*, i.e.*

$$\big|\{x \in Q : |f(x)| > \lambda\}\big| \leq C_\gamma\, \lambda^{-p}\, N^{(\beta(p,s)+\gamma)p}\, N^{(d+1)/2} |\mathcal{P}(f)|\, |E|^{\frac{p}{s}-1}$$

holds for such f*,* Q *and* λ*, and for all* $\gamma > \gamma_0(1 - \epsilon_0/2)$*.*

[‡]Below B will always denote a tN-cube, while we keep the notation Q for N-cubes, and Δ for $\sqrt{N}$-cubes.

Proof. For each tN-cube $B \in \mathcal{Q}(t)$, the function $f^B\psi_B$ has Fourier transform supported in $\Gamma_{\delta/t}(C)$. We claim that

$$\big\|f^B\psi_B\big\|_{p,s;\delta/t} \lesssim t^{-\beta(p,s)}\,\|f^B\|_2^{2/p}\,|E|^{\frac{1}{s}-\frac{1}{p}}\,. \tag{4.5}$$

Indeed, using the convexity inequality in (2.7) (with $r = r(s,p)$ as in (2.8)), followed by Lemmas 4.1 and 4.2, we have

$$\begin{aligned} \big\|f^B\psi_B\big\|_{p,s;\delta/t}^p &\lesssim \big\|f^B\psi_B\big\|_2^2\,\big\|f^B\psi_B\big\|_{\infty,r;\delta/t}^{p-2} \\ &\lesssim t\,\big\|f^B\big\|_2^2\,t^{-\frac{d-1}{2r'}(p-2)}\,\big\|f^B\big\|_{\infty,r;\delta}^{p-2}\,. \end{aligned} \tag{4.6}$$

Now, $\|f^B\|_{\infty,r} \lesssim |E|^{1/r}$, while by the definition of $r = r(s,p)$ in (2.8) we can write $\frac{p-2}{r'} = \frac{p}{s'} - 1$ and $\frac{p-2}{r} = \frac{p}{s} - 1$. Inserting these estimates in the right hand side of (4.6), the claimed inequality (4.5) follows easily.

Thus, using the localization condition and the hypothesis $\mathcal{H}^{str}(p,s,\gamma_0)$ (with δ replaced by δ/t) we obtain

$$\begin{aligned} \big|\{|f|>\lambda\}\big| &\lessapprox \sum_B \big|\,\{\,|f^B\psi_B| \gtrapprox \lambda\,\}\,\big| \qquad \text{(by (4.4))} \\ &\lessapprox \lambda^{-p}\sum_B (tN)^{(\beta(p,s)+\gamma_0)p}\,\|f^B\psi_B\|_{p,s;\delta/t}^p \\ &\lesssim \lambda^{-p}\sum_B N^{(\beta(p,s)+\gamma_0)p}\,t^{\gamma_0 p}\,\|f^B\|_2^2\,|E|^{\frac{p}{s}-1}, \end{aligned}$$

where in the last step we have used (4.5). Since by (4.3)

$$\sum_B \|f^B\|_2^2 \lesssim N^{\frac{d+1}{2}}\sum_B \#\mathcal{P}(f^B) \lessapprox N^{\frac{d+1}{2}}\,\#\mathcal{P}(f),$$

the lemma follows. □

4.2. **Sufficient conditions for λ-localization.** It is now important to identify situations in which the localization conditions of Definition 4.3 apply and thus the improvement of Lemma 4.4 holds. In [16, 8] a number of sufficient conditions are given, when the cardinality of $\mathcal{P}(f)$ is controlled by a power of λ. The simplest one is the following.

Proposition 4.5. [8, Lemma 5.2]**.** *Let f be an (N,E,Q)-packet and $\lambda > 0$ such that*

$$\#\mathcal{P}(f) \le t^{14d}\,\lambda^2\,. \tag{4.7}$$

Then f localizes at height λ with tN-cubes. In particular, if $p \ge 2$, $s \in [1,p]$ and we assume $\mathcal{H}^{str}(p,s,\gamma_0)$ for some $\gamma_0 > 0$, then the inequality (3.2), i.e.

$$\big|\{x \in Q : |f(x)| > \lambda\}\big| \le C\,\lambda^{-p}\,N^{(\beta(p,s)+\gamma)p}\,N^{\frac{d+1}{2}}\,|\mathcal{P}(f)|\,|E|^{\frac{p}{s}-1}$$

holds for such f, λ and Q, and for all $\gamma > (1-\epsilon_0/2)\,\gamma_0$.

We refer to [8] for details about the proof, which involves only simple combinatorial arguments.

For completeness, the proof is included in small print.

The main geometrical argument behind Proposition 4.5 is in the following result which (in a slightly more complicated version) will be applied to $W = \{|f| > \lambda\}$. For a proof we refer to [8, Lemma 4.2]. Below, $\mathcal{Q}(t) = \{B\}$ denotes a grid of tN-cubes, and for $x \in \mathbb{R}^{d+1}$ we define $B(x)$ as the cube B in the grid containing x (which is well defined apart from a null set).

Lemma 4.6. *Let W be a measurable subset of $\mathbb{R}^{d+1}$ and let $\mathcal{P}$ be a plate family, whose elements are contained in a fixed cube of diameter $CN^{1+\epsilon_0}$. As before, let $t = \delta^{\epsilon_0} = N^{-\epsilon_0}$. Consider the following relation "$\sim$" between plates $\pi \in \mathcal{P}$ and cubes $B \in \mathcal{Q}(t)$: we say that $\pi \sim B$ if B intersects the 9-fold dilate of B_π, where B_π is a tN-cube in $\mathcal{Q}(t)$ for which the quantity $|W \cap \pi \cap B_\pi|$ is maximal. . Then*

$$\#\{B : \pi \sim B\} \le 10^d, \quad \text{for every } \pi \in \mathcal{P} \tag{4.8}$$

and

$$\mathcal{I} := \int_W \sum_{\pi \in \mathcal{P}, \pi \not\sim B(x)} \chi_\pi(x) dx \lessapprox t^{-5d} |W| \sqrt{\#\mathcal{P}}. \tag{4.9}$$

Proof. The condition that all plates in $\mathcal{P}$ are contained in a fixed $CN^{1+\epsilon_0}$-cube, and the separation property of the plates implies $\#\mathcal{P} = O(t^{-d-1}N^d)$. Note that (4.8) is trivial from the definition of the relation. To prove (4.9) we first note that

$$\mathcal{I} = \sum_\pi \nu(\pi)$$

where

$$\nu(\pi) = \Big|\{x \in W \cap \pi : B(x) \not\sim \pi\}\Big|.$$

There is the trivial estimate

$$\int_W \sum_{\substack{\pi \in \mathcal{P}: \pi \not\sim B(x) \\ \nu(\pi) \le |W| N^{-d}}} \chi_\pi(x) dx \lesssim \#\mathcal{P}|W|N^{-d} \lesssim t^{-d-1}\,|W|.$$

Thus we only need to bound

$$\widetilde{\mathcal{I}} = \sum_{\substack{\pi \in \mathcal{P}: \\ N^{-d}|W| \le \nu(\pi) \le |W|}} \nu(\pi)$$

As there are $O(\log N)$ dyadic intervals between $N^{-d}|W|$ and $|W|$ we can use a pidgeonhole argument to get a subfamily $\mathcal{P}' \subset \mathcal{P}$ and a value of ν between $N^{-d}|W|$ and $|W|$ so that

$$|\widetilde{\mathcal{I}}| \lessapprox \nu \,\mathrm{card}(\mathcal{P}') \tag{4.10}$$

and

$$\nu \le \nu(\pi) \le 2\nu \quad \text{ for each } \pi \in \mathcal{P}'.$$

Since every plate can be covered with $O(t^{-1})$ cubes, for each $\pi \in \mathcal{P}'$ there must be a cube $B'(\pi)$ *not related to* π so that

$$|W \cap B'(\pi) \cap \pi| \gtrsim t\nu.$$

By the maximality condition in the definition of B_π we must then also have

$$|W \cap B_\pi \cap \pi| \gtrsim t\nu \text{ for each } \pi \in \mathcal{P}'.$$

Clearly the number of all possible pairs of tN cubes is $O(t^{-4(d+1)})$. This means that we can find two tN-cubes B, B' in $\mathcal{Q}(t)$ and a subfamily $\mathcal{P}''$ of $\mathcal{P}'$ which has cardinality $\gtrsim t^{4(d+1)}\#\mathcal{P}'$ so that for all $\pi \in \mathcal{P}''$ we have $B_\pi = B$ and $B'(\pi) = B'$.

We now fix these two tN-cubes B and B' and consider the auxiliary expression

$$\mathcal{A} = \sum_{\pi\in\mathcal{P}''} |W \cap B \cap \pi||W \cap B' \cap \pi|.$$

Then we have the lower bound

$$\mathcal{A} \gtrsim (t\nu)^2 \mathrm{card}(\mathcal{P}'') \gtrsim t^{4d+6}\mathrm{card}(\mathcal{P}')\nu^2.$$

We can also derive an upper bound by rewriting

$$\mathcal{A} = \int_{W\cap B}\int_{W\cap B'} \sum_{\pi\in\mathcal{P}''} \chi_\pi(x)\chi_\pi(x')dxdx'$$

If $\pi \cap B \neq \emptyset$ and $\pi \cap B' \neq \emptyset$ for some $\pi \in \mathcal{P}''$ then π is related to B but not to B', thus the distance of B to B' is at least tN. This means that for each pair of points $(x, x') \in B \times B'$ there are $\lesssim t^{-d+1}$ separated plates which go to both x and x'. This means that the integrand $\sum_{\pi\in\mathcal{P}''}\chi_\pi(x)\chi_\pi(x)$ is $O(t^{-d+1})$ and hence we get the upper bound

$$\mathcal{A} \lesssim t^{-d+1}|W \cap B||W \cap B'| \lesssim t^{-d+1}|W|^2.$$

Comparing the upper and the lower bounds for $\mathcal{A}$ we find that

$$\nu \le t^{-d-1}(\#\mathcal{P}')^{-1/2}\sqrt{\mathcal{A}} \le t^{-5(d+1)/2}|W|(\#\mathcal{P}')^{-1/2}$$

and thus using (4.10) (i.e. $\widetilde{\mathcal{I}} \lessapprox \nu\, \mathrm{card}(\mathcal{P}')$) and we obtain

$$\widetilde{\mathcal{I}} \lessapprox t^{-5(d+1)/2}|W|\sqrt{\#\mathcal{P}'}.$$

□

For technical reasons Lemma 4.6 is not quite enough for us since we wish to replace the characteristic functions χ_π by the similar weights w_π with "Schwartz-tails". This is fairly straightforward and requires adjustments in the definition of the relation $\sim$ between plates and tN-cubes and some additional pidgeon-holing. We state the required estimate and refer to Lemma 4.3 in the paper by Łaba and Wolff [8] for details of the proof.

Lemma 4.7. *Let W be a measurable subset of $\mathbb{R}^{d+1}$ and let $\mathcal{P}$ be a plate family, whose elements are contained in a fixed cube of diameter $CN^{1+\epsilon_0}$. Let M_0 be a large constant, and assume the constant M in the definition of w (see (2.2)) is so large that $M \ge 10M_0 d$. Let $t = N^{-\varepsilon_0}$ and $\mathcal{Q}(t) = \{B\}$ be a grid of tN cubes as before. Then, there is a relation "$\sim$" between plates in $\mathcal{P}$ and tN-cubes in $\mathcal{Q}(t)$ so that*

$$\#\{B : \pi \sim B\} \lessapprox 1, \quad \textit{for every } \pi \in \mathcal{P} \tag{4.11}$$

and if

$$\mathfrak{W}_{\mathcal{P}}(x) = \sum_{\substack{\pi\in\mathcal{P}\\ \pi\not\sim B(x)}} w_\pi(x)$$

then

$$\int_W \mathfrak{W}_{\mathcal{P}}(x)\, dx \lessapprox t^{-5d}\,|W|\,\sqrt{\#\mathcal{P}} + N^{-M_0}|W|.$$

Proof of Proposition 4.5. We wish to apply Lemma 4.4 and therefore have to show that with $\mathcal{P} \equiv \mathcal{P}(f)$ under the assumption $\#\mathcal{P} \leq t^{14d}\lambda^2$ the localization condition in Definition 4.3 holds.

We proceed applying Lemma 4.7 to $W = \{x : |f| \geq \lambda\}$ and $\mathcal{P}$, and let $\sim$ be the relation between N-plates and tN-cubes from Lemma 4.7. Recall that

$$f(x) = \sum_{\pi \in \mathcal{P}} f_\pi$$

with $|f_\pi| \lesssim w_\pi$. For every tN-cube $B \in \mathcal{Q}(t)$ define

$$f^B(x) = \sum_{\pi \sim B} f_\pi .$$

By condition (4.11) we have $\sum_B |\mathcal{P}(f^B)| \lessapprox |\mathcal{P}(f)|$, i.e. (4.3). Moreover with $\mathcal{P} \equiv \mathcal{P}(f)$

$$\int_W \mathfrak{W}_{\mathcal{P}}(x)dx \lessapprox t^{-5d}|W|\sqrt{\#\mathcal{P}} \lesssim t^{-5d}|W|\sqrt{t^{14d}\,\lambda^2} \;\lesssim t^{2d}\,|W|\,\lambda\,.$$

This means that there is a subset W^* of W so that $|W^*| \geq |W|/2$ on which we have the pointwise bound

$$\mathfrak{W}_{\mathcal{P}}(x) \lesssim t\lambda, \quad x \in W^*$$

Also if $x \in W^* \cap B$ we have

$$|f(x) - f^B(x)| = \Big|\sum_{\pi : \pi \not\sim B} f_\pi(x)\Big| \lesssim \mathfrak{W}_{\mathcal{P}}(x) \lesssim t\lambda$$

and hence

$$|f^B(x)| \geq \lambda/2, \qquad x \in W^* \cup B.$$

This implies the localization condition (4.4). □

A second sufficient condition, which also appears in [8] can be described as follows. Following [8, §4], to every plate family $\mathcal{P}$ we can associate a (separated) N-tube family $\mathcal{T} = \mathcal{T}(\mathcal{P})$ of minimal cardinality so that each $\pi \in \mathcal{P}$ is contained in a 10-fold dilate of some $\tau \in \mathcal{T}$. For each $\tau \in \mathcal{T}(\mathcal{P})$ we call

$$\mathcal{P}(\tau) = \{\pi \in \mathcal{P} \;:\; \pi \subset 10\tau\}$$

and for every positive integer $\mu \in \mathbb{N}$ we define a subfamily of $\mathcal{P}$ by

$$\mathcal{P}^{(\mu)} = \bigcup_{\tau \in \mathcal{T}(\mathcal{P})} \big\{\mathcal{P}(\tau) \;:\; 2^{\mu-1} \leq \#\mathcal{P}(\tau) < 2^\mu\big\}, \tag{4.12}$$

and a corresponding subfamily of tubes

$$\mathcal{T}(\mathcal{P}^{(\mu)}) = \big\{\tau \in \mathcal{T}(\mathcal{P}) \;:\; 2^{\mu-1} \leq \#\mathcal{P}(\tau) < 2^\mu\big\}. \tag{4.13}$$

Observe that the families $\mathcal{P}^{(\mu)}$ are nonempty only for $\mu \lesssim \log N$, since we always have

$$\#\{\pi \;:\; \pi \subset 10\tau\} \lesssim \sqrt{N}, \quad \forall\, \tau.$$

It is also clear that

$$\#\mathcal{T}(\mathcal{P}^{(\mu)}) \lesssim \frac{\#\mathcal{P}}{2^\mu}. \tag{4.14}$$

Definition 4.8. Given an (N,E,Q)-packet $f=\sum_{\pi\in\mathcal{P}} f_\pi$ and a real number $\lambda>0$, we define $\mu_*=\mu_*(f,\lambda)$ as a positive integer at random among those for which the subpacket $f^*=\sum_{\pi\in\mathcal{P}^*} f_\pi$ with plate family $\mathcal{P}^*=[\mathcal{P}(f)]^{(\mu_*)}$ (defined as in (4.12)), satisfies

$$\big|\{|f|>\lambda\}\big| \le C_0 \log N\,\big|\{|f^*|>\frac{\lambda}{C_0\log N}\}\big|, \tag{4.15}$$

for a fixed constant $C_0>0$. Observe that by an elementary pigeonhole argument at least one such μ_* exists provided C_0 is chosen large enough.

The second sufficient condition for λ-localization can now be written as follows (see [8, Lemma 5.3]).

Proposition 4.9. *Let f be an (N,E,Q)-packet and $\lambda>0$, and assume that for $\mu_*=\mu_*(f,\lambda)$ defined as above we have*

$$\frac{|\mathcal{P}(f)|}{2^{\mu_*}} \le t^{14d}\,\lambda^2\,. \tag{4.16}$$

Then, f localizes at height λ with tN-cubes. In particular, if $p\ge 2$, $s\in[1,p]$ and we assume $\mathcal{H}^{str}(p,s,\gamma_0)$ for some $\gamma_0>0$, then the inequality (3.2), i.e.

$$\big|\{x\in Q: |f(x)|>\lambda\}\big| \le C\,\lambda^{-p}\,N^{(\beta(p,s)+\gamma)p}\,N^{\frac{d+1}{2}}\,|\mathcal{P}(f)|\,|E|^{\frac{p}{s}-1}$$

holds for such f, Q and λ, and for all $\gamma>\gamma_0(1-\epsilon_0/2)$.

Moreover, if (4.16) does not hold, then for every $\sqrt{N}$-cube Δ the subpacket f^* in Definition 4.8 satisfies

$$\big\|f^*\psi_\Delta\big\|_2^2 \lesssim t^{-14d}\,\frac{|\mathcal{P}(f)|}{\lambda^2\sqrt{N}}\,|\Delta||E|. \tag{4.17}$$

Proof. The first part of Proposition 4.9 is precisely the statement of [8, Lemma 5.3, Case 1], so we refer to this paper for a detailed proof.

We now establish the second part of the proposition, that is the inequality (4.17). Write $f^*=\sum_k f_k^*$ with supp $\widehat{f_k^*}\subset\Pi_k^{(\delta)}$. Since for each $\sqrt{N}$-cube Δ the functions in $\{f_k^*\psi_\Delta\}_k$ are essentially orthogonal, by Plancherel we have

$$\big\|f^*\psi_\Delta\big\|_2^2 \lesssim \sum_k \int |f_k^*\,\psi_\Delta|^2 \lesssim \int \sum_{\pi\in\mathcal{P}^*} w_\pi\, w_\Delta.$$

By Lemma 4.1 in [8], we can estimate

$$\int \sum_{\pi\in\mathcal{P}^*} w_\pi\, w_\Delta \lesssim \frac{2^{\mu_*}}{\sqrt{N}} \int \sum_{\tau\in\mathcal{T}(\mathcal{P}^*)} w_\tau\, w_\Delta \lesssim \frac{2^{\mu_*}\,|E|\,|\Delta|}{\sqrt{N}}.$$

Then, (4.17) follows using the upper bound for 2^{μ_*} obtained when (4.16) does not hold. □

4.3. **Sufficient conditions for** $d = 2$**.** One cannot expect the sufficient conditions (4.7) or (4.16) to hold for general packets f, since $|\mathcal{P}(f)|$ can be as large as N^d while λ^2 is at most N^{d-1}. As explained below, one can go over this difficulty localizing the problem with $\sqrt{N}$-cubes, and reconsidering the above sufficient conditions at scale $\sqrt{N}$. As noticed in [8], this idea turns out to work well for dimensions $d \geq 3$, but does not give anything when $d = 2$. Fortunately in the latter case much better sufficient conditions hold, as was proved by Wolff in [16].

Proposition 4.10. : [16, Lemma 3.1]**.** *Let f be an (N, E, Q)-packet in $\mathbb{R}^{2+1}$ and $\lambda \geq 1$ so that*

$$|\mathcal{P}(f)| \leq t^{300} \lambda^3 . \tag{4.18}$$

Then f localizes at λ with tN-cubes. In particular, if $p \geq 2$, $s \in [1, p]$ and we assume $\mathcal{H}^{str}(p, s, \gamma_0)$ for some $\gamma_0 > 0$, then the inequality (3.2) *holds for such f, Q and λ, and for all $\gamma > \gamma_0(1 - \epsilon_0/2)$.*

Proposition 4.11. *Let f be an (N, E, Q)-packet in $\mathbb{R}^{2+1}$ and $\lambda \geq 1$, and assume that for $\mu_* = \mu_*(f, \lambda)$ as in Definition 4.8 we have*

$$\frac{|\mathcal{P}(f)|}{2^{\mu_*}} \leq t^{3000} \delta^{1/4} \lambda^3 , \tag{4.19}$$

then f localizes at height λ. In particular, if $p \geq 2$, $s \in [1, p]$ and we assume $\mathcal{H}^{str}(p, s, \gamma_0)$ for some $\gamma_0 > 0$, then the inequality (3.2) *holds for all $\gamma > \gamma_0(1 - \epsilon_0/2)$.*

Moreover, if (4.19) *does not hold, then for every $\sqrt{N}$-cube Δ we have*

$$\left\| f^* \psi_\Delta \right\|_2^2 \lesssim t^{-C} \frac{N^{5/4}}{\lambda^3} |\mathcal{P}(f)||E|. \tag{4.20}$$

We refer to [16] for the deep proof of Proposition 4.10, which among other things relies on combinatorial methods of Clarkson et al [4] for counting tangencies in arrangements of circles.

Proposition 4.11 is new, and improves over Lemma 3.2 in [16], which (essentially) requires the stronger sufficient condition

$$\frac{\#\mathcal{P}(f)}{2^{r_*}} \leq t^C \delta^{7/4} \lambda^6. \tag{4.21}$$

It is straightforward to verify that (4.21) implies (4.19) since $\lambda \leq N^{\frac{1}{2}}$. We will give a complete proof of Proposition 4.11 in §6.

5. The proof of theorem 3.4

5.1. **A parabolic rescaling.** The next lemma is an analogue and consequence of Wolff's inequality for Fourier plates contained in an angular sector of length $\sqrt{\sigma} \gg \sqrt{\delta}$.

Lemma 5.1. *Let $\delta < \sigma < 1$ and consider a fixed σ-plate $\Pi^{(\sigma)}$ contained in $\Gamma_\sigma(C)$. Suppose that Hypothesis $\mathcal{H}^{str}(p, s, \gamma)$ holds for some $p, s \geq 1$ and $\gamma > 0$. Then*

$$\Big\| \sum_{\substack{k: \\ \Pi_k^{(\delta)} \subset \Pi^{(\sigma)}}} P_k^{(\delta)}(h_k) \Big\|_p \lesssim (\delta/\sigma)^{-\beta(p,s)-\gamma} \Big(\sum_k \|h_k\|_p^s \Big)^{1/s}, \qquad \forall \, \{h_k\} \subset L^p(\mathbb{R}^{d+1}). \tag{5.1}$$

Proof. The lemma follows by rescaling the problem with a suitable Lorentz transformation and using hypothesis $\mathcal{H}^{str}(p, s, \gamma)$ (see e.g. [8, p. 167]). For completeness, we describe the argument here.

Let $\{\eta_1, \ldots, \eta_d\}$ be an orthonormal basis of $\mathbb{R}^d$, where η_1 is chosen so that $(1, \eta_1)$ is the center of the plate $\Pi^{(\sigma)}$. Then $\{(1, \eta_1), (-1, \eta_1), (0, \eta_2), \ldots, (0, \eta_d)\}$ is a basis of $\mathbb{R}^{d+1}$. Define a linear operator $L \in Gl_{d+1}(\mathbb{R})$ preserving the cone and acting on this basis by

$$L(1, \eta_1) = (1, \eta_1), \quad L(-1, \eta_1) = \tfrac{1}{\sigma} (-1, \eta_1) \quad \text{and} \quad L(0, \eta_\ell) = \tfrac{1}{\sqrt{\sigma}} (0, \eta_\ell), \ \ell = 2, ..., d.$$

Set $f_k = P_k^{(\delta)}(h_k)$, so that the functions $f_k \circ L$ have now spectrum in (perhaps a multiple) of the plates $\Pi_k^{(\delta/\sigma)}$ corresponding to the $\sqrt{\delta/\sigma}$-separated centers $\{L(1, \omega_k)\}$. Thus, hypothesis $\mathcal{H}^{str}(p, s, \gamma)$ can be applied at scale δ/σ giving

$$\big\| \sum_k f_k \circ L \big\|_p \lesssim (\delta/\sigma)^{-\beta(p,s)-\gamma} \Big(\sum_k \big\| f_k \circ L \big\|_p^s \Big)^{\frac{1}{s}},$$

which after a change of variables yields (5.1). $\square$

5.2. **The two main lemmas.** To prove Theorem 3.4 we must show that for every (N, E, Q)-packet f and λ as in (3.18) the inequality (3.2) holds in the improved range $\gamma > \gamma_0(1 - \epsilon_0')$, under the assumption $\mathcal{H}^{str}(p, s, \gamma_0)$. This will be done by repeatedly localizing at smaller scales, and then using the induction hypothesis at the lowest scale. In this section we prove the main two lemmas which show how this process works at each step. Proofs are similar to [8, Lemma 6.1].

Below, we let $N_1 = \sqrt{N}$ (hence $\delta_1 = \sqrt{\delta}$) and denote by $\mathcal{Q}_1 = \{\Delta\}$ a tiling of $\mathbb{R}^{d+1}$ by N_1-cubes. Then, for every (N, E, Q_0)-packet f we can write

$$\big| \{ x \in Q_0 \ : \ |f(x)| > \lambda \} \big| \leq \sum_{\Delta \subset Q_0} \big| \{ x \in \Delta \ : \ |f \psi_\Delta(x)| > c\lambda \} \big| \tag{5.2}$$

for some constant $c > 0$. Observe that, for a fixed Δ, the function $f\psi_\Delta$ has Fourier transform supported in $\Gamma_{\delta_1}(C)$, but in general is not a packet. However, by Lemma 2.5, $f\psi_\Delta$ can be decomposed on Δ in terms of (N_1, E_1, Δ) packets. Below we denote by Ω_1 a $\sqrt{\delta_1}$-separated set in S^{d-1}, and by $\{\Pi_k^{(\delta_1)}\}_{k \in \Omega_1}$ the corresponding plate decomposition of $\Gamma_{\delta_1}(C)$.

Lemma 5.2. *Let Q_0 be an N-cube, f be an (N, E, Q_0)-packet, and let $\lambda \geq 1$. Then there exists $\lambda_1 > 0$ so that for every N_1-cube $\Delta \subset Q_0$ there is a plate family $\mathcal{P}_{(1,\Delta)}$, a set*

$E_{(1,\Delta)} \subset \Omega_1$, *and a stable* $(N_1, E_{(1,\Delta)}, \Delta)$*-packet* $f_{(1,\Delta)}$ *with plate set* $\mathcal{P}_{(1,\Delta)}$ *so that*

$$\big|\{\, x \in Q_0 \ : \ |f(x)| > \lambda \,\}\big| \lessapprox \sum_{\Delta \subset Q_0} \big|\{\, x \in \Delta \ : \ |f_{(1,\Delta)}(x)| \geq \lambda_1 \,\}\big| \tag{5.3}$$

and

$$|\mathcal{P}_{(1,\Delta)}| \lessapprox \frac{\lambda_1^2}{\lambda^2} \frac{\|f\psi_\Delta\|_2^2}{N_1^{\frac{d+1}{2}}} \lessapprox \frac{\lambda_1^2}{\lambda^2} N_1^{\frac{d+1}{2}} |E|. \tag{5.4}$$

Moreover, for all $p, s \geq 1$ *we have*

$$|\mathcal{P}_{(1,\Delta)}|\, |E_{(1,\Delta)}|^{\frac{p}{s}-1} \lessapprox \frac{\lambda_1^p}{\lambda^p} \frac{\|f\psi_\Delta\|_{p,s;\delta_1}^p}{N_1^{\frac{d+1}{2}}}. \tag{5.5}$$

Proof. Fix $\Delta \subset Q_0$ and let $g^\Delta \equiv f\psi_\Delta$, which has Fourier transform supported in $\Gamma^{\delta_1}(C)$ and satisfies

$$\|g^\Delta\|_{\infty,\infty;\delta_1} \lesssim (N/N_1)^{(d-1)/2} = N_1^{\frac{d-1}{2}}$$

(by Lemma 4.2). Applying Lemma 2.5 with $A = N_1^{\frac{d-1}{2}}$ and $Q = \Delta$, we can write

$$g^\Delta(x) = \sum_{N_1^{-10d} A \lesssim 2^j \lesssim A N_1^d} 2^j \sum_{\ell=1}^{n_{j,\Delta}} g^\Delta_{[j,\ell]}(x) + h^\Delta(x), \quad x \in \Delta, \tag{5.6}$$

where

$$n_{j,\Delta} \lesssim \log N_1, \tag{5.7}$$

$$\sup_{x \in \Delta} |h_\Delta(x)| \lesssim N_1^{-8d} A \leq N_1^{-7d}; \tag{5.8}$$

moreover, for each (j, ℓ, Δ) there is a subset $E^\Delta_{j,\ell}$ of Ω_1 so that $g^\Delta_{[j,\ell]}$ is a stable $(N_1, E^\Delta_{j,\ell}, \Delta)$-packet, with associated plate family $\mathcal{P}^\Delta_{j,\ell}$, consisting only of N_1-plates π contained in $2N_1^{1+\epsilon_0}\Delta$, and more importantly satisfying

$$2^{jp} N_1^{\frac{d+1}{2}} |\mathcal{P}^\Delta_{j,\ell}|\, |E^\Delta_{j,\ell}|^{\frac{p}{s}-1} \lesssim \|f\psi_\Delta\|^p_{p,s;\delta_1}, \qquad \forall\, p, s \geq 1. \tag{5.9}$$

As there are only $O(\log N)$ values of j and $O(\log N)$ values of ℓ a simple pidgeonhole argument and (5.8) show that, for $\lambda \geq 1$,

$$\begin{aligned}\Big|\big\{x \in \Delta \ : \ |g^\Delta| > c\lambda\big\}\Big| &\leq \Big|\big\{x \in \Delta \ : \ \big|\textstyle\sum_{N_1^{-10d} A \lesssim 2^j \lesssim N_1^d A} 2^j \sum_{\ell=1}^{n_{j,\Delta}} g^\Delta_{[j,\ell]}(x)\big| > \frac{c\lambda}{2}\big\}\Big| \\ &\leq C\,(\log N)^2 \Big|\big\{x \in \Delta \ : \ \big|2^{j_\Delta} g^\Delta_{[j_\Delta,\ell_\Delta]}(x)\big| > \tfrac{\lambda}{C(\log N)^2}\big\}\Big|\end{aligned}$$

for some fixed j_Δ, ℓ_Δ. Pigeonholing once again we can find, among the (j_Δ, ℓ_Δ)'s, a fixed pair $j_*, \ell_* \in \mathbb{Z}$ (independent of Δ) so that

$$\sum_\Delta \Big|\big\{x \in \Delta \ : \ |g^\Delta| > c\lambda\big\}\Big| \lessapprox \sum_\Delta \Big|\big\{x \in \Delta \ : \ \big|2^{j_*} g^\Delta_{[j_*,\ell_*]}(x)\big| > \tfrac{\lambda}{C(\log N)^2}\big\}\Big|.$$

Using (5.2) this means that (5.3) holds with $\lambda_1 = 2^{-j_*}\lambda/(C\log N)^2$ and $f_{(1,\Delta)} = g^{\Delta}_{[j_*,\ell_*]}$, and hence that $E_{(1,\Delta)} = E^{\Delta}_{j_*,\ell_*}$ and $\mathcal{P}_{(1,\Delta)} = \mathcal{P}(g^{\Delta}_{[j_*,\ell_*]})$. Observe also that (5.5) follows immediately from (5.9) and the definition of λ_1.

The first inequality in (5.4) follows from the case $p = s = 2$ of (5.9) in the same fashion. For the second inequality in (5.4) we observe that if $f = \sum_k f_k$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$ then the Fourier transforms $\widehat{f_k\psi_\Delta}$ are supported in essentially disjoint sets. Thus we have the crucial orthogonality estimate

$$\|f\psi_\Delta\|_2^2 \lesssim \sum_k \|f_k\psi_\Delta\|_2^2 \lesssim |\Delta| \sum_{k\in E} \|f_k\|_\infty^2 \lesssim N_1^{d+1}\,|E|, \tag{5.10}$$

the last step following from the fact that $\|f_k\|_\infty \lesssim 1$ for N-packets. This establishes the second inequality in (5.4) and hence the lemma. □

Below we shall use the bound in (5.4) to argue that at least one of the sufficient conditions, Proposition 4.5 or Proposition 4.9, can be applied to the triplet $(f_{(1,\Delta)}, \lambda_1, \Delta)$. The next lemma, shows how to conclude the theorem for the original packet f in such case.

Lemma 5.3. *Let $p \geq 2$, $s \in [1,p]$ and assume that $\mathcal{H}^{str}(p,s,\gamma_0)$ holds for some $\gamma_0 > 0$. Consider an (N,E,Q_0)-packet f and a real number $\lambda \geq 1$. Suppose we are given a number $\lambda_1 > 0$ and a collection $\{f_{(1,\Delta)}\}_\Delta$, where Δ runs over a grid of N_1-cubes contained in Q_0, where each $f_{(1,\Delta)}$ is an $(N_1, E_{(1,\Delta)}, \Delta)$-packet with plate family $\mathcal{P}_{(1,\Delta)}$ satisfying* (5.3), (5.4) *and* (5.5) *(e.g., when $f_{(1,\Delta)}$ are generated as in Lemma 5.2). Assume in addition that there is a real number $\alpha > 0$ so that, for every $\Delta \subset Q_0$, the pairs $(f_{(1,\Delta)}, \lambda_1)$ satisfy the inequality:*

$$\big|\{\, x \in \Delta \;:\; |f_{(1,\Delta)}(x)| > \lambda_1 \,\}\big| \lessapprox \frac{N_1^{(\beta(p,s)+\alpha)p}}{\lambda_1^p}\, N_1^{\frac{d+1}{2}}\, |\mathcal{P}_{(1,\Delta)}|\, |E_{(1,\Delta)}|^{\frac{p}{s}-1}, \tag{5.11}$$

Then, we also have

$$\big|\{\, x \in Q_0 \;:\; |f(x)| > \lambda \,\}\big| \lessapprox \lambda^{-p}\, N^{(\beta(p,s)+\gamma)p}\, N^{\frac{d+1}{2}}\, |\mathcal{P}(f)|\, |E|^{\frac{p}{s}-1}, \tag{5.12}$$

with $\gamma = (\gamma_0 + \alpha)/2$.

In particular, (5.12) *holds with $\gamma = \gamma_0(1 - \epsilon_0/4)$ at least when one of the following conditions is satisfied for every N_1-cube $\Delta \subset Q_0$:*

(i) $\lambda_1 \lessapprox N_1^{(\beta(p,s)p - \frac{d-1}{2(d+1)})/(p-q)}\, |E_{(1,\Delta)}|^{(\frac{p}{s}-\frac{q}{2})/(p-q)}$;

(ii) $|E_{(1,\Delta)}|^{\frac{p}{s'}-\frac{q}{2}} \lessapprox N_1^{\beta(p,s)\,p - \frac{d-1}{2(d+1)}}$;

(iii) $(f_{(1,\Delta)}, \lambda_1)$ *satisfies any of the sufficient conditions* (4.7) *or* (4.16)*;*

(iv) when $d = 2$, $(f_{(1,\Delta)}, \lambda_1)$ satisfies any of the sufficient conditions (4.18) *or* (4.19)*.*

Remark 5.4. We observe that the previous two lemmas already give Theorem 3.4 for p and d sufficiently large. For instance, to verify that (5.11) holds, say with $s = 2$, by Proposition 4.5 we only need to check that the plate families $\mathcal{P}_{(1,\Delta)}$ satisfy

$$|\mathcal{P}_{(1,\Delta)}| \lesssim t^{7d}\lambda_1^2.$$

By the inequality (5.4) and the fact that we only consider $\lambda \geq N^{\frac{\beta(p,2)\,p}{p-2}}|E|^{1/2}$ (by Lemma 3.6 with $s = 2$), we obtain (after some arithmetics)

$$\lambda_1^{-2}\,|\mathcal{P}_{(1,\Delta)}| \lessapprox \frac{N_1^{(d+1)/2}\,|E|}{\lambda^2} \leq N^{\frac{d+1}{4}-\frac{2p\,\beta(p,2)}{p-2}} = N^{\frac{2}{p-2}-\frac{d-3}{4}}, \tag{5.13}$$

which is $\lesssim t^{7d} = N^{-7d\varepsilon_0}$ if $d > 3$ and $p > 2 + \frac{8}{d-3-4d\varepsilon_0}$. Thus, choosing $\varepsilon_0 = \varepsilon_0(p)$ small we can exhaust the range $p > 2 + 8/(d-3)$, which is one of the indices obtained in [8] for the validity of (1.3). To improve over this index one must iterate the process with successive $N^{1/4}, N^{1/8}$,... localizations, as described in the next subsection.

Proof of Lemma 5.3. By (5.3) and (5.11) we have

$$\begin{aligned}\big|\{\, x \in Q_0 \;:\; |f| > \lambda\,\}\big| &\lessapprox \sum_{\Delta\subset Q_0} \big|\{\, x \in \Delta \;:\; |f_{(1,\Delta)}| > \lambda_1 \,\}\big| \\ &\lessapprox \sum_{\Delta\subset Q_0} \lambda_1^{-p}\, N_1^{(\beta(p,s)+\alpha)p}\, N_1^{\frac{d+1}{2}}\, |\mathcal{P}_{(1,\Delta)}|\,|E_{(1,\Delta)}|^{\frac{p}{s}-1}.\end{aligned}$$

Thus, the result will be established if we can show

$$\sum_{\Delta\subset Q_0} N_1^{\frac{d+1}{2}}\, |\mathcal{P}_{(1,\Delta)}|\,|E_{(1,\Delta)}|^{\frac{p}{s}-1} \lesssim \frac{{\lambda_1}^p}{\lambda^p}\, N_1^{(\beta(p,s)+\gamma_0)p}\, N^{\frac{d+1}{2}}\, |\mathcal{P}(f)|\,|E|^{\frac{p}{s}-1}. \tag{5.14}$$

To do so, recall that $f\psi_\Delta$ are functions with spectrum in $\Gamma_{\delta_1}(C)$, and denote by $P_\ell^{(\delta_1)}$ the projections as in (1.11) associated with the usual partition of $\Gamma_{\delta_1}(C)$ by $1\times\delta_1\times\sqrt{\delta_1}\times\ldots\times\sqrt{\delta_1}$ plates: $\{\Pi_\ell^{(\delta_1)}\}_\ell$. Then by (5.5) we have for each Δ,

$$\begin{aligned} N_1^{\frac{d+1}{2}}\, |\mathcal{P}_{(1,\Delta)}|\,|E_{(1,\Delta)}|^{\frac{p}{s}-1} &\lessapprox \frac{{\lambda_1}^p}{\lambda^p}\, \|f\psi_\Delta\|_{p,s;\delta_1}^p \\ &\lesssim \frac{{\lambda_1}^p}{\lambda^p}\Big(\sum_\ell \big\|P_\ell^{(\delta_1)}(f\psi_\Delta)\big\|_p^s\Big)^{p/s} \\ &= \frac{{\lambda_1}^p}{\lambda^p}\Big(\sum_\ell \Big\| P_\ell^{(\delta_1)}\big[\,\psi_\Delta\big(\sum_{k\;:\;\Pi_k^{(\delta)}\subset C\Pi_\ell^{(\delta_1)}} f_k\big)\big]\Big\|_p^s\Big)^{p/s} \\ &\lesssim \frac{{\lambda_1}^p}{\lambda^p}\Big(\sum_\ell \Big\|\psi_\Delta\big(\sum_{\Pi_k^{(\delta)}\subset C\Pi_\ell^{(\delta_1)}} f_k\big)\Big\|_p^s\Big)^{p/s}. \end{aligned}$$

We sum in Δ and apply Minkowski's inequality (since $p \geq s$) to obtain

$$\begin{aligned}\sum_{\Delta} N_1^{\frac{d+1}{2}} |\mathcal{P}_{(1,\Delta)}|\,|E_{(1,\Delta)}|^{\frac{p}{s}-1} &\lessapprox \frac{\lambda_1^p}{\lambda^p}\sum_{\Delta}\Big(\sum_{\ell}\Big\|\psi_\Delta\big(\sum_{\Pi_k^{(\delta)}\subset C\Pi_\ell^{(\delta_1)}} f_k\big)\Big\|_p^s\Big)^{p/s}\\ &\lesssim \frac{\lambda_1^p}{\lambda^p}\Big(\sum_{\ell}\Big[\sum_{\Delta}\Big\|\psi_\Delta\big(\sum_{\Pi_k^{(\delta)}\subset C\Pi_\ell^{(\delta_1)}} f_k\big)\Big\|_p^p\Big]^{s/p}\Big)^{p/s}\\ &\lesssim \frac{\lambda_1^p}{\lambda^p}\Big(\sum_{\ell}\Big\|\sum_{\Pi_k^{(\delta)}\subset C\Pi_\ell^{(\delta_1)}} f_k\Big\|_p^s\Big)^{p/s}.\end{aligned}$$

Now, we apply Hypothesis $\mathcal{H}^{str}(p,s,\gamma_0)$ in the rescaled version of Lemma 5.1 and bound for each ℓ

$$\Big\|\sum_{k:\Pi_k^{(\delta)}\subset C\Pi_\ell^{(\delta_1)}} f_k\Big\|_p \lesssim (N/N_1)^{\beta(p,s)+\gamma_0}\Big(\sum_{k:\Pi_k^{(\delta)}\subset C\Pi_\ell^{(\delta_1)}}\|f_k\|_p^s\Big)^{1/s}.$$

This yields

$$\begin{aligned}\Big(\sum_{\ell}\Big\|\sum_{\Pi_k^{(\delta)}\subset c\Pi_\ell^{(\delta_1)}} f_k\Big\|_p^s\Big)^{p/s} &\lessapprox N_1^{(\beta(p,s)+\gamma_0)p}\Big(\sum_{\ell}\sum_{\Pi_k^{(\delta)}\subset c\Pi_\ell^{(\delta_1)}}\|f_k\|_p^s\Big)^{p/s}\\ &\lesssim N_1^{(\beta(p,s)+\gamma_0)p}\,\|f\|_{p,s}^p\\ &\lesssim N_1^{(\beta(p,s)+\gamma_0)p}\,N^{\frac{d+1}{2}}|\mathcal{P}(f)|\,|E|^{\frac{p}{s}-1},\end{aligned}$$

where the last inequality follows from (2.18). This proves (5.14) and establishes the lemma. □

5.3. **Iteration.** We are now ready to describe the iteration. Here we fix $p > p_d$ as in (1.4) and $s \in [1,p]$. We let $N_j = N^{1/2^j}$ for $j = 0,1,2,\dots$ Starting with an (N,E,Q_0)-packet $f = f_0$ and $\lambda = \lambda_0$ as in (3.18), at step j we shall define, for each N_{j-1}-cube Δ_{j-1}, a real number $\lambda_j > 0$ and a collection of functions $\{f_{(j,\Delta)}\}_{\Delta\subset\Delta_{j-1}}$, where Δ runs in a grid $\mathcal{Q}_j$ of N_j-cubes and each $f_{(j,\Delta)}$ is an $(N_j, E_{(j,\Delta)}, \Delta)$-packet with plate family $\mathcal{P}_{(j,\Delta)}$, and so that the pair $(f_{(j,\Delta)}, \lambda_j)$ satisfies

$$(a)\qquad \big|\{\, x\in\Delta_{j-1} \;:\; |f_{(j-1,\Delta_{j-1})}| > \lambda_{j-1}\,\}\big| \lessapprox \sum_{\substack{\Delta\in\mathcal{Q}_j\\ \Delta\subset\Delta_{j-1}}}\big|\{\, x\in\Delta \;:\; |f_{(j,\Delta)}| > \lambda_j\,\}\big|$$

$$(b)\qquad |\mathcal{P}_{(j,\Delta)}| \lessapprox \frac{\lambda_j^2}{\lambda_{j-1}^2}\,\frac{\|f_{(j-1,\Delta_{j-1})}\psi_\Delta\|_2^2}{N_j^{\frac{d+1}{2}}};$$

$$(c)\qquad |\mathcal{P}_{(j,\Delta)}| \lessapprox \frac{\lambda_j^p}{\lambda_{j-1}^p}\,\frac{\|f_{(j-1,\Delta_{j-1})}\psi_\Delta\|_{p,s;\delta_j}}{N_j^{\frac{d+1}{2}}}.$$

It is clear from Lemma 5.2 that this is possible for $j=1$. We next show how to pass from step j to step $j+1$.

Suppose we are at step j. Then we stop the process for the N_j-cubes $\Delta \in \mathcal{Q}_j$ for which the pair $(f_{(j,\Delta)}, \lambda_j)$ already satisfies the improved inequality in (5.11); in particular when at least one of the conditions (i)-(iv) in Lemma 5.3 holds (with the subindex "1" replaced by "j"). Observe that when for *all* cubes $\Delta \subset \Delta_{j-1}$ the inequality (5.11) is satisfied, then a direct application of Lemma 5.3 gives the improved estimate at the next scale, i.e.

$$\big|\{\, x \in \Delta_{j-1} \;:\; |f_{(j-1,\Delta_{j-1})}| > \lambda_{j-1} \,\}\big| \lessapprox \lambda_{j-1}^{-p}\, N_{j-1}^{(\alpha+\gamma_0(1-\frac{\epsilon_0}{4}))p}\, N_{j-1}^{\frac{d+1}{2}}\, |\mathcal{P}_{(j-1,\Delta_{j-1})}|^{\frac{p}{s}-1},$$

which after $j-1$ more applications of the lemma leads to (5.12) with $\gamma = \gamma_0(1-\epsilon_0/2^{j+1})$, hence establishing Theorem 3.4 with $\epsilon_0' = \epsilon_0/2^{j+1}$.

Assume therefore that we are dealing with cubes $\Delta \in \mathcal{Q}_j$ for which $(f_{(j,\Delta)}, \lambda_j)$ does not satisfy any of the conditions (i)-(iv) in Lemma 5.3. That is, we are only considering

$$\lambda_j \;\geq\; (\log N)^C N^{(\beta(p,s)p - \frac{d-1}{2(d+1)})/(p-q)}\, |E_{(j,\Delta)}|^{(\frac{p}{s}-\frac{q}{2})/(p-q)} \tag{5.15}$$

and

$$|E_{(j,\Delta)}| \geq\; N^{(\beta(p,s)\,p - \frac{d-1}{2(d+1)})/(\frac{p}{s'}-\frac{q}{2})}. \tag{5.16}$$

Also, since (iii) fails, by Proposition 4.9, there must exist a subpacket $f^*_{(j,\Delta)}$ of $f_{(j,\Delta)}$ so that

$$\big|\{|f_{(j,\Delta)}| > \lambda_j\}\big| \lessapprox \big|\{|f^*_{(j,\Delta)}| > c\lambda_j/\log N\}\big| \tag{5.17}$$

and moreover, for every N_{j+1}-cube Δ_{j+1}

$$\big\|f^*_{(j,\Delta)}\psi_{\Delta_{j+1}}\big\|_2^2 \lesssim t^{-14d}\,\frac{N_j^{d/2}}{\lambda_j^2}\,|\mathcal{P}_{(j,\Delta)}|\,|E_{(j,\Delta)}|. \tag{5.18}$$

Then, we can replace the original $(f_{(j,\Delta)}, \mathcal{P}_{(j,\Delta)}, \lambda_j)$ by $(f^*_{(j,\Delta)}, \mathcal{P}^*_{(j,\Delta)}, \lambda^*_j = c\lambda_j/\log N)$, which also satisfies (a), (b), (c) and (5.15), (5.16). Next, we apply Lemma 5.2 to each pair $(f^*_{(j,\Delta)}, \lambda^*_j)$ to obtain new quadruplets $(f_{(j+1,\Delta_{j+1})}, \mathcal{P}_{(j+1,\Delta_{j+1})}, E_{(j+1,\Delta_{j+1})}, \lambda_{j+1})$ with the required conditions, i.e.

$$\big|\{\, x \in \Delta \;:\; |f^*_{(j,\Delta)}| > \lambda^*_j \,\}\big| \lessapprox \sum_{\substack{\Delta_{j+1}\in\mathcal{Q}_{j+1}\\ \Delta_{j+1}\subset\Delta}} \big|\{\, x \in \Delta_{j+1} \;:\; |f_{(j+1,\Delta_{j+1})}| > \lambda_{j+1} \,\}\big|$$

$$|\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox \frac{\lambda_{j+1}^2}{\lambda_j^2}\,\frac{\|f^*_{(j,\Delta_j)}\psi_{\Delta_{j+1}}\|_2^2}{N_{j+1}^{(d+1)/2}}; \tag{5.19}$$

$$|\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox \frac{\lambda_{j+1}^p}{\lambda_j^p}\,\frac{\|f^*_{(j,\Delta_j)}\psi_{\Delta_{j+1}}\|_{p,s;\delta_{j+1}}^p}{N_{j+1}^{(d+1)/2}}.$$

Observe that, with this construction, if we combine (5.19) and (5.18) we obtain in addition the inequality

$$\text{(d)}\qquad |\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox t^{-14d}\,\frac{\lambda_{j+1}^2}{\lambda_j^4}\,N_j^{\frac{d-1}{4}}\,|\mathcal{P}_{(j,\Delta_j)}|\,|E_{(j,\Delta_j)}|,\quad j=1,2,\dots$$

The case $d\geq 3$ and $s=2$.

Claim. *If $d\geq 3$, $p>p_d$ and $s=2$, then the above process will stop after a finite number of iterations. More precisely, there exists $\ell=\ell(p)\in\mathbb{N}$ so that the quadruplets $(f_{(\ell,\Delta)},\mathcal{P}_{(\ell,\Delta)},E_{(\ell,\Delta)},\lambda_\ell)$ satisfy the sufficient condition (4.7) in Lemma 4.5 for all $\Delta\in\mathcal{Q}_\ell$.*

For simplicity, denote $A_j=|\mathcal{P}_{(j,\Delta_j)}|$ and $E_j=|E_{(j,\Delta_j)}|$. Then, from (d) above one obtains

$$\lambda_\ell^{-2}\,A_\ell \lessapprox t^{-14d(\ell-1)}\,\frac{E_{\ell-1}\cdots E_1}{\lambda_{\ell-1}^2\cdots\lambda_2^2\lambda_1^4}\,(N_{\ell-1}\dots N_1)^{\frac{d-1}{4}}\,A_1. \tag{5.20}$$

Now, to estimate A_1 we use (5.4), that is

$$A_1 \lessapprox \frac{\lambda_1^2}{\lambda^2}\,N^{\frac{d+1}{4}}\,E_0. \tag{5.21}$$

Inserting this into (5.20) leads to

$$\lambda_\ell^{-2}\,A_\ell \lessapprox t^{-14d(\ell-1)}\,\frac{E_{\ell-1}\cdots E_1E_0}{\lambda_{\ell-1}^2\cdots\lambda_1^2\lambda^2}\,(N_{\ell-1}\dots N_1)^{\frac{d-1}{4}}\,N^{\frac{d+1}{4}}. \tag{5.22}$$

We need to show that the right hand side of this expression is smaller than t^{14d}. Observe that we can replace the symbol "$\lessapprox$" in (5.22) by "$\leq t^{-1}$", provided $N\geq N_0(\epsilon_0)$. Thus it will suffice to prove the inequality

$$\big(N^{1-\frac{1}{2^{\ell-1}}}\big)^{\frac{d-1}{4}}\,N^{\frac{d+1}{4}} \leq t^{30d\ell}\,\frac{\lambda_{\ell-1}^2}{E_{\ell-1}}\cdots\frac{\lambda_1^2}{E_1}\,\frac{\lambda^2}{E_0}\,. \tag{5.23}$$

By (5.15) (with $s=2$) we know that

$$\lambda_j \geq N^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)}\,E_j^{1/2} = N_j^{\frac{d-1}{4}-\frac{q}{4(p-q)}}\,E_j^{1/2},$$

and therefore it is enough to show that

$$N^{\frac{d-1}{4}}\,N^{\frac{d+1}{4}} \leq t^{30d\ell}\,\big(N^{\frac{d-1}{2}-\frac{q}{2(p-q)}}\big)^{2-\frac{1}{2^{\ell-1}}}\,. \tag{5.24}$$

Since $t=N^{-\epsilon_0}$, the previous is equivalent to

$$2(1-\tfrac{1}{2^\ell})(\tfrac{d-1}{2}-\tfrac{q}{2(p-q)})-\tfrac{d}{2} \geq 30d\ell\epsilon_0. \tag{5.25}$$

It is now easy to verify that this holds when $p>p_d$, for a sufficiently large integer $\ell=\ell(p)$, and a suitable choice of $\epsilon_0=\epsilon_0(p)$. More precisely, condition $p>p_d$ (for $d\geq 3$) can be read as $\frac{q}{4(p-q)}<\frac{d-2}{8}$, which is equivalent to

$$\epsilon_p := 2(\tfrac{d-1}{2}-\tfrac{q}{2(p-q)})-\tfrac{d}{2}>0.$$

Thus, we only need to choose $\ell = \ell(p)$ so that $2^{-\ell+1}(\frac{d-1}{2} - \frac{q}{2(p-q)}) < \epsilon_p/2$, and next choose $\epsilon_0 = \epsilon_0(\ell, p) = \epsilon_0(p)$ so that $30d\ell\epsilon_0 < \epsilon_p/2$. This will satisfy (5.25) and establish the claim. Thus letting $\epsilon_0' = \epsilon_0/2^{\ell+1}$, one obtains Theorem 3.4 for $d \geq 3$.

The case $d = 2$. In this case the previous scheme does not give anything. One must use in (5.18) above Proposition 4.11, rather than the weaker Proposition 4.9. In such case the inequality in (5.18) can be replaced with the improved version

$$\big\|f^*_{(j,\Delta)}\psi_{\Delta_{j+1}}\big\|_2^2 \lessapprox t^{-C}\,\frac{N_j^{5/4}}{\lambda_j^3}\,|\mathcal{P}_{(j,\Delta)}|\,|E_{(j,\Delta_j)}| \tag{5.26}$$

(which follows from (4.20)). Thus (d) will take the form

$$|\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox t^{-C}\,\frac{\lambda_{j+1}^2}{\lambda_j^5}\,N_{j+1}\,|\mathcal{P}_{(j,\Delta)}|\,|E_{(j,\Delta_j)}|, \quad j = 1, 2, \dots \tag{5.27}$$

Then calling $A_j = |\mathcal{P}_{(j,\Delta_j)}|$, $E_j = |E_{(j,\Delta_j)}|$ and iterating as in the proof of the claim we are led to

$$\begin{aligned} \lambda_\ell^{-2}\,A_\ell \;&\leq\; t^{-C}\,\frac{E_{\ell-1}\dots E_1}{\lambda_{\ell-1}^3\dots\lambda_2^3\lambda_1^5}\,\big(N_\ell\dots N_2\big)\,A_1 \\ \text{(by (5.21))}\qquad &\leq\; t^{-C}\,\frac{E_{\ell-1}}{\lambda_{\ell-1}^3}\cdots\frac{E_1}{\lambda_1^3}\,\frac{E_0}{\lambda^2}\,\big(N_\ell\dots N_2\big)\,N^{\frac34}. \end{aligned} \tag{5.28}$$

The lower bound for λ_j from (5.15) gives

$$\begin{aligned} \frac{|E|}{\lambda^2} \;&\leq\; \frac{|E|^{1-2(\frac{p}{s}-\frac{q}{2})/(p-q)}}{N^{2(\beta(p,s)-\frac16)/(p-q)}} \;=\; \frac{|E|^{\frac{2p}{p-q}\left(\frac12-\frac1s\right)}}{N^{\frac{p}{p-q}\left(\frac{1}{s'}-\frac{q}{p}\right)}} \\ &\leq\; \frac{N^{\frac{p}{p-q}\left(\frac12-\frac1s\right)}}{N^{\frac{p}{p-q}\left(\frac{1}{s'}-\frac{q}{p}\right)}} \;=\; N^{-\frac{p-2q}{2(p-q)}}, \end{aligned} \tag{5.29}$$

where in the last inequality we have used that $|E| \lesssim N^{1/2}$ (since we only consider $s \geq 2$). On the other hand the same bound for λ_j from (5.15) gives

$$\frac{E_j}{\lambda_j^3} \leq \frac{E_j^{1-3(\frac{p}{s}-\frac{q}{2})/(p-q)}}{N_j^{3(\beta(p,s)-\frac16)/(p-q)}} \;=\; \frac{E_j^{\frac{3p}{p-q}\left(\frac{2p+q}{6p}-\frac1s\right)}}{N_j^{\frac{3p}{2(p-q)}\left(\frac{1}{s'}-\frac{q}{p}\right)}} \tag{5.30}$$

To estimate further this quantity we must distinguish cases.

Case 1: $s \geq \frac{6p}{2p+q} = 3 - \frac{15}{3p+5}$. Then the exponent of E_j in (5.30) is positive and we can use again the trivial bound $E_j \lesssim N_j^{1/2}$, which leads to

$$\frac{E_j}{\lambda_j^3} \leq \frac{N_j^{\frac{3p}{2(p-q)}\left(\frac{2p+q}{6p}-\frac1s\right)}}{N_j^{\frac{3p}{2(p-q)}\left(\frac{1}{s'}-\frac{q}{p}\right)}} \;=\; N_j^{-\frac{4p-7q}{4(p-q)}}. \tag{5.31}$$

Inserting (5.29) and (5.31) into (5.28) we obtain

$$\lambda_\ell^{-2} A_\ell \,\le\, t^{-C} N^{-\frac{4p-7q}{4(p-q)}} N^{-\frac{p-2q}{2(p-q)}} N^{1/2} N^{3/4}.$$

Since by Lemma 4.5 it suffices to show[§] that $\lambda_\ell^{-2} A_\ell \le t^{28}$, we will be done when

$$\tfrac{5}{4} \,<\, \tfrac{4p-7q}{4(p-q)} \,+\, \tfrac{p-2q}{2(p-q)} \,=\, \tfrac{6p-11q}{4(p-q)},$$

or equivalently when

$$p > 6q = 20.$$

This establishes Theorem 1.2 in this case.

Case 2: $2 \le s \le \frac{6p}{2p+q} = 3 - \frac{15}{3p+5}$. Then the exponent of E_j in (5.30) is negative and we must use instead the lower bound $E_j \gtrsim N_j^{(\beta(p,s)\,p-\frac{1}{6})/(\frac{p}{s'}-\frac{q}{2})}$ in (5.16), which after a simple but tedious computation leads to

$$\frac{E_j}{\lambda_j^3} \,\le\, N_j^{-(\frac{p}{s'}-q)/(\frac{p}{s'}-\frac{q}{2})}. \tag{5.32}$$

Inserting this expression together with (5.29) in (5.28) we obtain

$$\lambda_\ell^{-2} A_\ell \,\le\, t^{-C} N^{-(\frac{p}{s'}-q)/(\frac{p}{s'}-\frac{q}{2})} N^{-\frac{p-2q}{2(p-q)}} N^{1/2} N^{3/4},$$

so that we will have $\lambda_\ell^{-2} A_\ell \le t^{28}$ at least when

$$\tfrac{5}{4} \,<\, \tfrac{2p-2s'q}{2p-s'q} \,+\, \tfrac{p-2q}{2(p-q)}\,. \tag{5.33}$$

When $s = 2$ this is easily seen to be equivalent to

$$p > 7q = \tfrac{70}{3} \,=\, 23.333...$$

as asserted in Theorem 1.2. When $2 < s < 3 - \frac{15}{3p+5}$, then solving for p in (5.33) leads to the range

$$p > p(s) = \big(11s - 6 + \sqrt{65s^2 - 76s + 36}\big)\, q/(4(s-1)), \tag{5.34}$$

which therefore completes the proof of Theorem 1.2.

Remark 5.5. We point out that the range of p obtained in (5.34) when $2 < s < 3 - \frac{15}{3p+5}$ and $d = 2$ is slightly better than the interpolated line between $(\frac{1}{p} = \frac{3}{70}, \frac{1}{s} = \frac{1}{2})$ and $(\frac{1}{p} = \frac{1}{20}, \frac{1}{s} = \frac{13}{36})$ (see Figure 1.2).

Remark 5.6. One can do similar computations to establish a range for $s < 2$, however the region that comes out corresponds precisely to interpolating the case $s = 2$ with the trivial $p = \infty, s = 1$, and therefore no new result appears in this case (again, see Figure 1.2).

[§]We could also require the weaker estimate $A_\ell \le t^{C'} \lambda_\ell^3 |E_\ell|^{\frac{1}{2}}$ (by Lemma 4.10), but this makes no difference at this point.

6. Proof of Proposition 4.11

The main result in this section is Lemma 6.2, which gives an improvement over Lemma 2.5 in [16]. The rest of the proof of Proposition 4.11 follows from exactly the same reasoning as in [16], replacing at each occurrence Wolff's Lemma 2.5 by its improved version; we sketch the argument in §6.2. Recall that throughout this section $d = 2$.

6.1. **The combinatorial lemma.** In this subsection it will be convenient to follow the notation in [16, §2]. Namely, $\mathcal{P} = \{\pi\}$ will denote a collection of $1 \times \sqrt{\delta} \times \delta$ plates, and $\mathcal{T} = \{\tau\}$ a collection of $1 \times \sqrt{\delta} \times \sqrt{\delta}$ tubes. As usual the longest axes of π and τ point in $\sqrt{\delta}$-separated light rays. We shall also use a collection $\mathbb{P} = \{\Pi\}$ of much larger plates with dimensions $1 \times \delta^{\frac{1}{4}} \times \sqrt{\delta}$, and longest axes pointing in $\delta^{\frac{1}{4}}$-separated directions. All such families are assumed to consist of *separated* plates or tubes, meaning that $C_1\pi$ contains less than C_2 plates from $\mathcal{P}$, and similarly with $\mathcal{T}$ and $\mathbb{P}$.

Fix $t = \delta^{\epsilon_0}$ and consider a tiling $\{B\}$ of $\mathbb{R}^3$ by t-cubes. If $w \in \mathbb{R}^3$, we denote by $B(w)$ the t-cube containing w. Given a finite set $\mathcal{W}$ consisting of $\sqrt{\delta}$-separated points in $\mathbb{R}^3$ and a tube family $\mathcal{T}$, we wish to define relations "$\sim$" between tubes and t-cubes which keep as small as possible the cardinality of the *bad incidence set*

$$I_b(\mathcal{W}, \mathcal{T}) \;=\; \big\{(w,\tau) \in \mathcal{W} \times \mathcal{T} \;:\; w \in \tau,\; \tau \not\sim B(w)\big\}.$$

These relations will be *admissible* if they satisfy the property

$$\text{for every } \tau \in \mathcal{T} \qquad \text{Card}\,\{B \;:\; \tau \sim B\} \lessapprox 1. \tag{6.1}$$

One defines likewise the concept of admissible relation between t-cubes and $\mathbb{P}$-plates, as well as the bad incidence set $I_b(\mathcal{W}, \mathbb{P})$.

As a special example consider the relation $\tau \sim B$ if B is equal or adjacent to a fixed cube maximizing $|\mathcal{W} \cap \tau \cap B|$, and likewise for $\mathbb{P}$-plates. Using this relation, Wolff proves the following result.

Lemma 6.1. : (see [16, Lemma 2.3]**).** *Let $\mathcal{W}$ be a $\sqrt{\delta}$-separated set in $\mathbb{R}^3$.*
(i) *Given a plate family $\mathbb{P}$, there exists an admissible relation $\sim$ so that, for every $\varepsilon > 0$*

$$\textit{Card}\; I_b(\mathcal{W}, \mathbb{P}) \le C_\varepsilon\, \delta^{-\varepsilon}\, t^{-6}\, |\mathbb{P}|^{1/3}\, |\mathcal{W}|. \tag{6.2}$$

(ii) *Given a tube family $\mathcal{T}$, there exists an admissible relation $\sim$ so that*

$$\textit{Card}\; I_b(\mathcal{W}, \mathcal{T}) \lessapprox t^{-5}\, |\mathcal{T}|^{1/2}\, |\mathcal{W}|. \tag{6.3}$$

The statement in (i) is by far much deeper than its counterpart in (ii), relying on highly non trivial bounds for circle tangencies. In his paper, Wolff improves the bound in (6.3) by combining it with (6.2) (see [16, Lemma 2.5]). It seems, though, that both his statement and proof can be simplified. Below, given a set $\mathcal{T}$, we denote by $\mathbb{P}(\mathcal{T})$ a plate family of minimal cardinality so that each $\tau \in \mathcal{T}$ is contained in some $\Pi \in \mathbb{P}(\mathcal{T})$ (as in [16, p. 1255]).

Lemma 6.2. *Let $\mathcal{W}$ be a $\sqrt{\delta}$-separated set in $\mathbb{R}^3$, and $\mathcal{T}$ a tube family so that every $\Pi \in \mathbb{P}(\mathcal{T})$ contains at most m tubes. Then, there exists an admissible relation $\sim$ so that, for every $\varepsilon > 0$*

$$\text{Card } I_b(\mathcal{W}, \mathcal{T}) \le C_\varepsilon\, \delta^{-\varepsilon}\, t^{-11}\, m^{1/6}\, |\mathcal{T}|^{1/3}\, |\mathcal{W}|. \tag{6.4}$$

Proof. Assume first that every plate $\Pi \in \mathbb{P}(\mathcal{T})$ contains between $m/2$ and m tubes from $\mathcal{T}$. Given $\tau \in \mathcal{T}$, let Π be the plate in $\mathbb{P}(\mathcal{T})$ containing τ, and $\mathcal{T}_\Pi$ the subset of all tubes from $\mathcal{T}$ contained in Π. Define the relation $\tau \sim B$ when one of the following holds:

(a) $\Pi \sim B$, as in (i) of Lemma 6.1, with respect to the set $\mathcal{W}$ and the plate family $\mathbb{P}(\mathcal{T})$;

(b) $\tau \sim B$, as in (ii) of Lemma 6.1, with respect to the set $\mathcal{W} \cap \Pi \cap [\cup_{B \not\sim \Pi} B]$ and the tube family $\mathcal{T}_\Pi$.

More precisely, if we denote by B_Π the union of the t-cubes B which are equal or adjacent to the cube maximizing $|\mathcal{W} \cap \Pi \cap B|$, and denote by B_τ the union of the t-cubes B which are equal or adjacent to the cube maximizing $|[\mathcal{W} \cap B_\Pi^c] \cap \tau \cap B|$, then

$$\tau \sim B \quad \text{iff} \quad B \subset B_\Pi \cup B_\tau.$$

Clearly $\sim$ is an admissible relation. Moreover,

$$\begin{aligned}
\text{Card } I_b(\mathcal{W}, \mathcal{T}) &= \sum_{\tau \in \mathcal{T}} \big|\mathcal{W} \cap \tau \cap [\cup_{B \not\sim \tau} B]\big| = \sum_{\Pi \in \Pi(\mathcal{T})} \sum_{\substack{\tau \in \mathcal{T} \\ \tau \subset \Pi}} \big|\mathcal{W} \cap \tau \cap B_\Pi^c \cap B_\tau^c\big| \\
\text{(by (6.3))} \quad &\lessapprox \sum_{\Pi \in \Pi(\mathcal{T})} t^{-5}\, m^{1/2}\, \big|\mathcal{W} \cap \Pi \cap B_\Pi^c\big| \\
\text{(by (6.2))} \quad &\le C_\varepsilon\, \delta^{-\varepsilon}\, t^{-11}\, m^{1/2}\, \big|\mathbb{P}(\mathcal{T})\big|^{1/3}\, \big|\mathcal{W}\big| \\
&\lesssim C_\varepsilon\, \delta^{-\varepsilon}\, t^{-11}\, m^{\frac{1}{2}-\frac{1}{3}}\, \big|\mathcal{T}\big|^{1/3}\, \big|\mathcal{W}\big|,
\end{aligned}$$

since by assumption $|\mathbb{P}(\mathcal{T})| \approx |\mathcal{T}|/m$. Finally, to remove the condition that each Π contains at least $m/2$ tubes, simply partition $\mathcal{T}$ into the subfamilies $\mathcal{T}_j = \cup\{\mathcal{T}_\Pi \ : \ 2^{j-1} \le |\mathcal{T}_\Pi| < 2^j\}$, and apply the above reasoning to each $\mathcal{T}_j$. □

Remark 6.3. Observe that m in the statement of the lemma is always $m \lesssim N^{\frac{1}{2}}$, since each Π may contain at most $N^{\frac{1}{4}}$ parallel tubes pointing in each of $N^{\frac{1}{4}}$ different directions. In fact, below we shall only use (6.4) with $m = N^{\frac{1}{2}}$.

Remark 6.4. From Lemma 6.2 it is easy to derive a version with "Schwartz tails" as in [16, Lemma 2.7]. Namely, letting

$$\mathcal{I}_b(\mathcal{T},\mathcal{W}) = \sum_{w\in\mathcal{W}} \sum_{\substack{\tau\in\mathcal{T} \\ \tau\not\sim B(w)}} w_\tau(w),$$

then with the same conditions as in Lemma 6.2 there is an admissible relation $\sim$ so that for all $\varepsilon > 0$

$$\mathcal{I}_b(\mathcal{T},\mathcal{W}) \le C_\varepsilon\,\delta^{-\varepsilon}\,t^{-11}\,N^{1/12}\,|\mathcal{T}|^{1/3}\,|\mathcal{W}| + \delta^{100}|\mathcal{W}|. \tag{6.5}$$

Remark 6.5. We point out that, according to the scaling we have adopted in the paper, we will use the results in this subsection with families $\mathcal{T}$ of $N\times\sqrt{N}\times\sqrt{N}$-tubes and sets $\mathcal{W}$ of $\sqrt{N}$-separated points. Of course, all the results remain valid with this scaling, by a simple change of variables.

6.2. **Proof of Proposition 4.11.** We only sketch the proof of Proposition 4.11, since it is essentially the same as in [16, Lemma 3.2] or [8, Lemma 5.3].

We are given an (N,E,Q)-packet f, and consider the subpacket $f^* = \sum_{\pi\in\mathcal{P}^*} f_\pi$ in Definition 4.8 and $\lambda\ge 1$ so that (4.19) holds. Reasoning as in [16, p. 1267] one can find a finite set of $N^{\frac12}$-separated points $\mathcal{W}\subset\{|f^*|>c\lambda/\log N\}$ and a real number $a=a(N)>0$ so that the set

$$\widetilde{W} := \bigcup_{w\in\mathcal{W}} \Delta(w)\cap\{|f^*|>\tfrac{c\lambda}{\log N}\}$$

(with $\Delta(w)$ denoting the $\sqrt{N}$-cube containing w) satisfies

$$\text{meas}\,\big\{|f^*|>c\lambda/\log N\big\} \lessapprox \text{meas}\,(\widetilde{W})$$

and

$$\text{meas}\,\big(\Delta(w)\cap\{|f^*|>\tfrac{c\lambda}{\log N}\}\big) \approx a\,N^{\frac32}, \quad \forall\, w\in\mathcal{W}. \tag{6.6}$$

Let $\sim$ denote the equivalence relation relative to $(\mathcal{W},\mathcal{T}(\mathcal{P}^*))$ obtained in Remark 6.4, and given $\pi\in\mathcal{P}^*$, define $\pi\sim B$ when the tube $\tau\in\mathcal{T}(\mathcal{P}^*)$ whose 10-fold dilate contains π satisfies $\tau\sim B$. Define the plate families $\mathcal{P}_B=\{\pi\in\mathcal{P}^* \ : \ \pi\sim B\}$, which satisfy

$$\sum_B |\mathcal{P}_B| \lessapprox |\mathcal{P}^*|$$

by property (6.1) from the previous subsection. By (4.15), to obtain the λ-localization of f as in Definition 4.3 it suffices to show that

$$\big|\{|f^*|>c\lambda/\log N\}\big| \lessapprox \sum_B \big|B\cap\{|f^B|\gtrapprox\lambda\}\big|, \tag{6.7}$$

where $f^B = \sum_{\pi\in\mathcal{P}_B} f_\pi$. To prove (6.7) we use the crude estimate

$$|f^*(x) - f^B(x)| \lesssim \sum_{\tau\not\sim B} w_\tau(x)$$

and show that the right hand side is $\ll \lambda/\log N$ when $x \in B \cap \widetilde{W}$. Indeed, by Lemma 6.2 (in its version with Schwartz tails; see Remark 6.4) and the fact that w_τ is essentially constant in $\sqrt{N}$-cubes we have

$$\begin{aligned}\int_{\widetilde{W}} \sum_{\tau\not\sim B(x)} w_\tau(x)\,dx \;&\lesssim\; a\,N^{\frac{3}{2}} \sum_{w\in\mathcal{W}} \sum_{\tau\not\sim B(w)} w_\tau(w) \;=\; a\,N^{\frac{3}{2}}\,\mathcal{I}_b(\mathcal{T}(\mathcal{P}^*),\mathcal{W})\\ &\le\; C_\varepsilon\,N^\varepsilon\,t^{-11}\,N^{\frac{1}{12}}\,\big|\mathcal{T}(\mathcal{P}_r)\big|^{\frac{1}{3}}\,\#(\mathcal{W})\,a\,N^{\frac{3}{2}}\\ &\lesssim\; C_\varepsilon\,N^\varepsilon\,t^{-11}\,N^{\frac{1}{12}}\left[\tfrac{|\mathcal{P}_f|}{2^{\mu_*}}\right]^{\frac{1}{3}}\big|\widetilde{W}\big|,\end{aligned}$$

which is smaller than $c|\widetilde{W}|\lambda/(4\log N)$ if the sufficient condition (4.19) holds (choosing $\varepsilon \ll \epsilon_0$ and $N \ge N_0(\epsilon_0)$). Thus, there exists a subset W^* of $\widetilde{W}$ with proportional measure so that

$$\sum_{\tau\not\sim B(x)} w_\tau(x) < c\lambda/(4\log N), \quad x \in W^*.$$

Therefore, if $x \in B\cap W^*$ we have $|f^*(x) - f^B(x)| \le c\lambda/(4\log N)$, which implies $|f^B(x)| > c\lambda/(4\log N)$. Thus,

$$\big|\{|f^*| > c\lambda/\log N\}\big| \lessapprox \big|\widetilde{W}\big| \lesssim \big|W^*\big| \le \sum_B \big|B\cap\{|f^B| \gtrapprox \lambda\}\big|,$$

as we wished to prove. Finally, to obtain (4.20) when the condition (4.19) does not hold, one repeats the same argument as at the end of the proof of Proposition 4.9. We leave details to the reader.

7. Boundedness of Bergman projections

As mentioned in the introduction, Wolff's inequality, and their variants with $\ell^2(L^p)$-norms, have also played a role in a complex analysis problem, namely the boundedness of Bergman projections in tube domains over full light cones, see e.g. [2, 1]. Denote by $Q(Y) = y_0^2 - |y'|^2$ the Lorentz form in $\mathbb{R}^{d+1}$ and consider the forward light cone on which Q is positive;

$$\Lambda^{d+1} = \{Y = (y_0, y') \in \mathbb{R}\times\mathbb{R}^d : y_0^2 - |y'|^2 > 0, y_0 > 0\}.$$

Let $\mathcal{T}^{d+1} \subset \mathbb{C}^{d+1}$ be the tube domain over Λ^{d+1}, i.e.

$$\mathcal{T}^{d+1} = \mathbb{R}^{d+1} + i\Lambda^{d+1}.$$

Let $w_\gamma(Y) = Q(Y)^\gamma$ and consider the weighted space $L^p(\mathcal{T}^{d+1}, w_\gamma)$ with norm

$$\|F\|_{L^p(w_\gamma)} = \Big(\iint_{\mathcal{T}^{d+1}} |F(X+iY)|^p \Delta^\gamma(Y)\, dY dX\Big)^{1/p}.$$

Let $\mathcal{P}_\gamma$ be the orthogonal projection mapping the weighted space $L^2(\mathcal{T}^{d+1}, w_\gamma)$ to its subspace $\mathcal{A}^p_\gamma$ consisting of the holomorphic functions. Only the case $\gamma > -1$ is relevant since $\mathcal{A}^p_\gamma = \{0\}$ for $\gamma \le -1$. One is interested in the boundedness of $\mathcal{P}_\gamma$ in $L^p(\mathcal{T}^{d+1}, w_\gamma)$. A known and trivial necessary condition is

$$1 + \frac{d-1}{2(\gamma+d+1)} < p < 1 + \frac{2(\gamma+d+1)}{d-1} \tag{7.1}$$

(see e.g. [2]). In fact it has been conjectured that boundedness should hold in this range(7.1)¶.

Corollary 7.1. *Let $d \ge 2$ and p_d as in* (1.4)*. Then for all*

$$\gamma \ge \max\big\{-1 + \tfrac{d-1}{4}(p_d - \tfrac{2(d+1)}{d-1}), \tfrac{d-1}{2}(p_d - \tfrac{2(d+1)}{d-1} - 1)\big\}, \tag{7.2}$$

the Bergman projection $\mathcal{P}_\gamma$ is a bounded operator in $L^p(\mathcal{T}^{d+1}, w_\gamma)$ in the sharp range (7.1).

Remark 7.2. We point out that the range in Corollary 7.1 is a consequence of the stronger $\ell^2(L^p)$ inequalities in Theorem 1.2. The weaker $\ell^p(L^p)$ estimates in Theorem 1.1 only imply a solution to the problem in the smaller range $\gamma \ge \frac{d-1}{2}(p_d - \frac{2(d+1)}{d-1})$ (see [6, Corollary 1.4]).

The corollary follows from Theorem 1.2 and the arguments in [1, §5]. To be more precise, one has the following (stronger) result:

Proposition 7.3. *Let $2 \le s \le w < \infty$ and suppose that $\mathcal{H}^{str}(w, s, \varepsilon)$ holds for all $\varepsilon > 0$. Let*

$$\gamma(w, s) = -1 + 2s\beta(w, s). \tag{7.3}$$

Then, for every $\gamma > -1$, the Bergman projection P_γ is bounded in the mixed-norm space $L^{p,u}_\gamma(\mathcal{T}^{d+1}) = L^u(\Delta^\gamma(Y)dY; L^p(dX))$ in the optimal range $2 \le u < \tilde{u}_{\gamma,p} = (\gamma+d)/(\frac{d+1}{2p'} - 1)$ whenever

$$p \ge p_{w,s,\gamma} := w + \tfrac{w}{s}\,\frac{(\gamma_{w,s} - \gamma)_+}{\gamma+1}. \tag{7.4}$$

Proof. The result follows from [1, Prop. 5.5] by using a similar reasoning as in [1, Corol. 5.11]. Namely, assuming first $\gamma \ge \gamma(w, s)$, then $\mathcal{H}^{str}(w, s)$ implies [1, (5.6)] for all $p \ge w$ and all $\mu > s\beta(p, s)$, which in turn by [1, Prop. 5.5] implies (after some arithmetics) the boundedness of P_γ in $L^{p,u}_\gamma(\mathcal{T}^{d+1})$ in the optimal range $2 \le u < \tilde{u}_{\gamma,p}$.

¶Except for $d = 2$ and $\gamma \in (-1, -1/2)$, in which case there are additional counterexamples for $p \ge 8 + 4\gamma$ (see [1]).

When $\gamma < \gamma(w,s)$ one must find (ρ,σ) so that $\mathcal{H}^{str}(\rho,\sigma)$ holds and $\gamma = \gamma(\rho,\sigma)$. By interpolation with the trivial $(\infty,1)$-estimate, $\mathcal{H}^{str}(\rho,\sigma)$ holds when $\rho \geq w$ and $\sigma' = s'\rho/w$. Since with this choice $\gamma(\rho,\sigma) \searrow -1$ as $\rho \to \infty$, one can always find a (unique) ρ so that $\gamma = \gamma(\rho,\sigma)$. In fact, a simple computation shows that $\rho = p_{w,s,\gamma}$ as in (7.4). Thus, by the first part of the proof P_γ is bounded in $L^{p,u}_\gamma(\mathcal{T}^{d+1})$ in the optimal range $2 \leq u < \tilde{u}_{\gamma,p}$, for all $p \geq p_{w,s,\gamma}$. □

To obtain Corollary 7.1 from Proposition 7.3 one must specialize to the diagonal case $p = u$. First, an easy computation shows that $2 \leq p < \tilde{u}_{\gamma,p}$ is equivalent to $2 \leq p < 1 + 2(\gamma + d + 1)/(d-1)$, which gives the conjectured range of L^p_γ-boundedness for P_γ in (7.1) (by duality); thus, it suffices to find all γ's so that the endpoint $p = \tilde{u}_{\gamma,p}$ is $\geq p_{w,s,\gamma}$ as in (7.4). Straightforward arithmetics show that this is the case for

$$\gamma \geq \frac{d-1}{2}\Big(w - \tfrac{2(d+1)}{d-1} - 1 + \tfrac{w}{s}\,\frac{(\gamma(w,s)-\gamma)_+}{\gamma+1}\Big).$$

When $d \geq 3$, we let $w = p_d$ and $s = 2$, so that considering the two cases $\gamma > \gamma(w,s)$ and $\gamma \leq \gamma(w,s)$, one obtains the conditions in (7.2). When $d = 2$, one may use $w = p_2 = 20$ and $s = 3$, which leads to the same conditions on γ (namely to $\gamma \geq (w-7)/2 = 6.5$). This establishes Corollary 7.1.

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