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ON UNIFORM BOUNDEDNESS OF DYADIC AVERAGING OPERATORS IN SPACES OF HARDY–SOBOLEV TYPE

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Abstract. We give an alternative proof of recent results by the authors on uniform boundedness of dyadic averaging operators in (quasi-)Banach spaces of Hardy–Sobolev and Triebel–Lizorkin type. This result served as the main tool to establish Schauder basis properties of suitable enumerations of the univariate Haar system in the mentioned spaces. The rather elementary proof here is based on characterizations of the respective spaces in terms of orthogonal compactly supported Daubechies wavelets.

1. Introduction

Consider the dyadic averaging operators $\mathbb{E}_N$ on the real line given by

$$\mathbb{E}_N f(x) = \sum_{\mu\in\mathbb{Z}} \mathbb{1}_{I_{N,\mu}}(x)\, 2^N \int_{I_{N,\mu}} f(t)dt \tag{1}$$

with $I_{N,\mu} = [2^{-N}\mu, 2^{-N}(\mu+1))$. $\mathbb{E}_N f$ is the conditional expectation of f with respect to the σ-algebra generated by the dyadic intervals of length 2^{-N}.

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The following theorem on uniform boundedness in Triebel–Lizorkin spaces $F^s_{p,q}$ was proved by the authors in [2]. Since the uniform boundedness result is interesting in itself, we give an alternative and more elementary proof based on wavelet theory to make it accessible for a broader readership.

THEOREM 1.1 [2]. *Let* $1/2 < p < \infty$, $0 < q \le \infty$, *and* $1/p - 1 < s < \min\{1/p, 1\}$. *Then there is a constant* $C := C(p, q, s) > 0$ *such that for all* $f \in F^s_{p,q}$

$$\sup_{N\in\mathbb{N}} \|\mathbb{E}_N f\|_{F^s_{p,q}} \le C\|f\|_{F^s_{p,q}}. \tag{2}$$

In [2], this result served as the main tool to establish that suitably regular enumerations of the Haar system form a Schauder basis for the spaces $F^s_{p,q}$ in the parameter ranges of the theorem, see Section 3. The connection with the Haar system is given via the martingale difference operators

$$\mathbb{D}_N = \mathbb{E}_{N+1} - \mathbb{E}_N$$

which are the orthogonal projections to the spaces generated by Haar functions with fixed Haar frequency 2^N.

In previous works stronger notions of convergence have been examined, such as unconditional convergence for the martingale difference series. This is equivalent with the inequality

$$\Big\| \sum_n b_n \mathbb{D}_n f \Big\|_{F^s_{p,q}} \lesssim \|b\|_{\ell^\infty(\mathbb{N})} \|f\|_{F^s_{p,q}}. \tag{3}$$

It follows from the results in Triebel [6] that (3) holds if we add the condition $1/q - 1 < s < 1/q$ to the hypotheses in the theorem. For the case $q = 2$ this corresponds to the shaded region in Fig. 1.

It was shown in [3], [4] that the additional restriction on the q-parameter is necessary for (3) to hold. If we drop it then Theorem 1.1 and a summation by parts argument imply that (3) holds with the larger norm $\|b\|_\infty + \|b\|_{BV}$. It should be interesting to establish sharp results involving sequence spaces that are intermediate between $\ell^\infty(\mathbb{N})$ and $BV(\mathbb{N})$. We remark that these problems are interesting only for the $F^s_{p,q}$ spaces since inequality (3) with $B^s_{p,q}$ in place of $F^s_{p,q}$ holds in the full parameter range of Theorem 1.1, see [6] for further discussion and historical comments. In Section 2 we give a proof of Theorem 1.1 using characterizations of Triebel–Lizorkin spaces based on Daubechies wavelets. Relying on this, the proof is rather elementary due to the orthogonality and locality properties of the wavelet system. In addition, a "wavelet analog" of [2, Theorem 1.2] is provided in Proposition 2.1 below. In Section 3 we apply the methods to get an additional result needed to obtain the Schauder basis property of the Haar system.

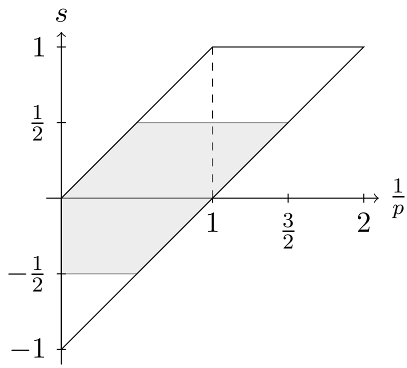

Fig. 1: Unconditional convergence in Hardy–Sobolev spaces

2. Proof of Theorem 1.1

We will exclusively use a characterization of Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R})$ and Besov spaces $B^s_{p,q}$ via compactly supported Daubechies wavelets (see [1], [7, Section 4]). Let ψ_0 and ψ be the orthogonal scaling function and corresponding wavelet of Daubechies type such that ψ_0, ψ being sufficiently smooth (C^K) and ψ having sufficiently many vanishing moments (L). We denote

$$\psi_{j,\nu}(x) := \frac{1}{\sqrt{2}}\psi(2^{j-1}x - \nu), \quad j \in \mathbb{N},\ \nu \in \mathbb{Z},$$

and $\psi_{0,\nu}(x) := \psi_0(x - \nu)$ for $\nu \in \mathbb{Z}$. Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. If K and L are large enough (depending on p, q and s) then we have an equivalent characterization (usual modification in case $q = \infty$),

$$\|f\|_{F^s_{p,q}} \asymp \Big\| \Big(\sum_{j=0}^{\infty} \Big| 2^{js} \sum_{\nu\in\mathbb{Z}} \lambda_{j,\nu}(f) \mathbb{1}_{j,\nu} \Big|^q \Big)^{1/q} \Big\|_p, \tag{4}$$

$$\|f\|_{B^s_{p,q}} \asymp \Big(\sum_{j=0}^{\infty} \Big\| 2^{js} \sum_{\nu\in\mathbb{Z}} \lambda_{j,\nu}(f) \mathbb{1}_{j,\nu} \Big\|_p^q \Big)^{1/q}, \tag{5}$$

where $\lambda_{j,\nu}(f) := 2^j \langle f, \psi_{j,\nu} \rangle$ and $\mathbb{1}_{j,\nu}$ denotes the characteristic function of the interval $I_{j,\nu} := [2^{-j}\nu, 2^{-j}(\nu+1)]$. See Triebel [5, Theorem 1.64] and the references therein. A corresponding characterization also holds true for Besov spaces $B^s_{p,q}$. Since we also deal with distributions which are not locally integrable, the inner product $\langle f, \psi_{j,\nu} \rangle$ has to be interpreted in the usual way.

Clearly, f can be decomposed into wavelet building blocks, i.e.

$$f = \sum_{j\in\mathbb{Z}} f_j \quad \text{with } f_j = \begin{cases} \sum\limits_{\nu\in\mathbb{Z}} \lambda_{j,\nu}(f)\psi_{j,\nu} & \text{if } j \geq 0, \\ 0 & \text{if } j < 0. \end{cases} \tag{6}$$

Let us denote the Nth partial sum of this representation by

$$P_N f = \sum_{j\leq N} f_j, \quad N \in \mathbb{N}\,. \tag{7}$$

Note, that the functions f_j and $P_N f$ represent K times continuously differentiable functions due to the regularity assumption on the wavelet.

In the sequel we will prove the following proposition.

Proposition 2.1. *Let* $1/2 < p \leq \infty$, $0 < r \leq \infty$, *and* $1/p - 1 < s < \min\{1/p, 1\}$. *Let* $\{\psi_{j,\nu}\}_{j,\nu}$ *represent a Daubechies wavelet system such that* (5) *holds for all* $0 < q \leq \infty$ *and let* P_N *be given by* (7). *Then there is a constant* $C := C(p, r, s) > 0$ *such that for all* $f \in B^s_{p,\infty}$

$$\sup_{N\in\mathbb{N}} \big\|\mathbb{E}_N f - P_N f\big\|_{B^s_{p,r}} \leq C\|f\|_{B^s_{p,\infty}}\,. \tag{8}$$

Note, that for a fixed wavelet system satisfying (4) we clearly have

$$\sup_{N\in\mathbb{N}} \|P_N f\|_{F^s_{p,q}} \leq C\|f\|_{F^s_{p,q}}\,. \tag{9}$$

If this wavelet system in addition satisfies (5) for all $0 < q \leq \infty$, then Proposition 2.1 together with (9) implies Theorem 1.1.

Proof of Proposition 2.1 in the case $1/2 < p \leq 1$. Let $1/p - 1 < s < 1$. Using the decomposition (6) we can write with $\theta := \min\{1, p\} = p$

$$\|\mathbb{E}_N f - P_N f\|_{B^s_{p,r}} \asymp \Big(\sum_{j=0}^{\infty} \Big\|2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N f - P_N f, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p^r\Big)^{1/r}$$

$$\lesssim \Big(\sum_{j=0}^{\infty} \Big\|2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N(P_N f) - P_N f, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p^r\Big)^{1/r} \tag{10}$$

$$+ \Big(\sum_{j=0}^{\infty} \Big\|2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N(f - P_N f), \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p^r\Big)^{1/r}. \tag{11}$$

We split the proof into several steps according to the cases we have to distinguish in the estimation of the quantities in (10) and (11).

Step 1. We deal with (11) and use that $f - P_N f = \sum_{j+\ell>N} f_{j+\ell}$. Clearly,

$$(11) \lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{j+\ell\geq N}\Big\|2^{js}\sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p^{\theta}\Big)^{r/\theta}\Big]^{1/r} \tag{12}$$

We continue estimating $\big\|2^{js}\sum_\eta 2^j \langle \mathbb{E}_N f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\big\|_p$. Note first that due to $p \leq 1$,

$$\begin{aligned} &\Big\|2^{js}\sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p \\ &\leq \Big(\sum_{\nu\in\mathbb{Z}} |\lambda_{j+\ell,\nu}(f)|^p \Big\|2^{js}\sum_{\eta} 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p^p\Big)^{1/p}. \end{aligned} \tag{13}$$

So it remains to deal with $\big\|2^{js}\sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\big\|_p$. Note, that due to $j+\ell > N$ the function $\mathbb{E}_N\psi_{j+\ell,\nu}$ is a step function consisting of $O(1)$ non-vanishing steps. These steps have length 2^{-N} and magnitude bounded by $O(2^{N-(j+\ell)})$.

Case 1.1. Assume $j \geq N$. Due to the cancellation of $\psi_{j,\eta}$ and $j \geq N$ we have that the function $\sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}$ is supported on a union of intervals of total measure $O(2^{-j})$ and bounded from above by $O(2^{N-(j+\ell)})$. This gives

$$\Big\|2^{js}\sum_{\eta} 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p \lesssim 2^{js}2^{-j/p}2^{N-j-\ell}. \tag{14}$$

Case 1.2. Assume $j \leq N$. Clearly, we have $\ell > 0$ since $j+\ell > N$. Now $\sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}$ is supported on an interval of size $O(2^{-j})$. As $\mathbb{E}_N\psi_{j+\ell,\nu}$ consists of $O(1)$ steps of length 2^{-N} each and $N \geq j$ we get by straightforward size estimates $2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle = O(2^{-\ell})$. Hence

$$\Big\|2^{js}\sum_{\eta} 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p \lesssim 2^{js}2^{-j/p}2^{-\ell}. \tag{15}$$

Step 2. We consider (10) and observe first

$$(10) \lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{j+\ell\leq N}\Big\|2^{js}\sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell} - f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p^{\theta}\Big)^{r/\theta}\Big)^{1/r}. \tag{16}$$

Analogously to (13) the matter reduces to estimate the L_p (quasi-)norm of the functions

$$2^{js}\sum_{\eta} 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu} - \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \tag{17}$$

for the different cases resulting from $j+\ell \le N$.

Case 2.1. We first deal with the case $j \le N$. Using the mean value theorem together with (1) we see for all $x \in \mathbb{R}$ that

$$\big|\mathbb{E}_N \psi_{j+\ell,\nu}(x) - \psi_{j+\ell,\nu}(x)\big| \le 2^{j+\ell-N}.$$

Due to $j+\ell \le N$, its support has length $O(2^{-(j+\ell)})$ around $\nu 2^{-(j+\ell)}$. We continue distinguishing the cases $\ell \ge 0$ and $\ell < 0$.

Case 2.1.1. Let $\ell \ge 0$. Since $j+\ell \ge j$ the inner product with $2^j \psi_{j,\eta}$ gives an additional factor $2^{-\ell}$. In addition, the support of (17) is contained in an interval of size $O(2^{-j})$. Hence, we get

$$\Big\| 2^{js} \sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu} - \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \Big\|_p \lesssim 2^{js} 2^{j+\ell-N} 2^{-\ell} 2^{-j/p}. \tag{18}$$

Case 2.1.2. Assume $\ell \le 0$. This time the inner product with $2^j \psi_{j,\eta}$ does not give an extra factor and the support has length $2^{-(j+\ell)}$. Thus, we have in this case

$$\Big\| 2^{js} \sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu} - \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \Big\|_p \lesssim 2^{js} 2^{j+\ell-N} 2^{-(j+\ell)/p}. \tag{19}$$

Case 2.2. Assume $j > N \ge j+\ell$ which implies $\ell < 0$. Due to the orthogonality of the wavelets ($\ell < 0$) we can estimate as follows:

$$\begin{aligned} &\Big\| 2^{js} \sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu} - \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \Big\|_p \\ &\lesssim 2^{js} \Big(\sum_{\mu \in \mathbb{Z}} \int_{|x-2^{-N}\mu| \lesssim 2^{-j}} \Big| \sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}(x) \Big|^p dx \Big)^{1/p} \\ &\lesssim 2^{js} \Big(\sum_{\mu \in \mathbb{Z}} \int_{|x-2^{-N}\mu| \lesssim 2^{-j}} \Big| \sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu} - \psi_{j+\ell,\nu}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}(x) \Big|^p dx \Big)^{1/p} \\ &\lesssim 2^{js} 2^{j+\ell-N} 2^{[N-(j+\ell)-j]/p}, \end{aligned} \tag{20}$$

where we took into account that the μ-sum consists of $O(2^{N-(j+\ell)})$ summands.

Step 3. Estimation of (11). Plugging (13) and (14) into the right- hand side of (12) yields

$$\Big[\sum_{j=N}^{\infty} \Big(\sum_{j+\ell \ge N} \Big\| 2^{js} \sum_\eta 2^j \langle \mathbb{E}_N f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \Big\|_p^\theta \Big)^{r/\theta} \Big]^{1/r} \tag{21}$$

$$\lesssim A_N \sup_{j,\ell} \Big(\sum_{\nu\in\mathbb{Z}} |2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \Big)^{1/p} \lesssim A_N \|f\|_{B^s_{p,\infty}}$$

with

$$A_N^r = \sum_{j\geq N} 2^{(N-j)r} \Big(\sum_{\ell\geq N-j} 2^{\theta\ell(1/p-1-s)} \Big)^{r/\theta} \lesssim 1$$

by the assumption $1/p > s > 1/p - 1$.

Plugging (13) and (15) into the right-hand side of (12) leads to a similar estimate as above, only the sums over j and ℓ change to

$$\widetilde{A}_N^r = \sum_{j\leq N} \Big(\sum_{\ell\geq N-j} 2^{\theta\ell(1/p-1-s)} \Big)^{r/\theta}$$

which is uniformly bounded in N if $s > 1/p - 1$.

Step 4. Estimation of (10). Combining (16), (18) and (19) we find

$$\Big[\sum_{j=0}^{N} \Big(\sum_{j+\ell\leq N} \Big\| 2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell} - f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \Big\|_p^\theta \Big)^{r/\theta} \Big]^{1/r}$$

$$\lesssim \Big[\Big(\sum_{j\leq N} 2^{(j-N)r} \Big(\sum_{\ell=-\infty}^{N-j} 2^{\theta\ell(1/p-s)} \Big)^{r/\theta} \Big]^{1/r} \|f\|_{B^s_{p,\infty}} .$$

The sums are finite and uniformly bounded if $1/p - 1 < s < 1/p$. Finally, we combine (12), (13) and (20) to obtain

$$\Big(\sum_{j=N}^{\infty} \Big(\sum_{j+\ell\leq N} \Big\| 2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell} - f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \Big\|_p^\theta \Big)^{r/\theta} \Big]^{1/r} \tag{22}$$

$$\lesssim \Big[\Big(\sum_{j\geq N} 2^{(j-N)\theta} 2^{(N-j)\theta/p} \Big(\sum_{\ell=-\infty}^{N-j} 2^{r\ell(1-s)} \Big)^{r/\theta} \Big]^{1/r} \|f\|_{B^s_{p,\infty}} ,$$

which is uniformly bounded if $s < 1$. This concludes the proof in the case $p \leq 1$. □

Proof in the case $1 \leq p \leq \infty$. We follow the proof in the case $p \leq 1$ until (12) and (16), respectively. Note, that we may use $\theta = 1$ now. Then we have to proceed differently.

Case 1.1. Assume $N < j, j+\ell$. Taking (12) into account we replace (13) by

$$\Big\| 2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta} \Big\|_p^p \tag{23}$$

$$\leq \int \Big[\sum_{\nu \in \mathbb{Z}} |2^{js} \lambda_{j+\ell,\nu}(f)| \cdot \Big| \sum_{\eta} 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta} \rangle \mathbb{1}_{j,\eta}(x) \Big| \Big]^p \, dx$$

$$\lesssim \sum_{\nu \in \mathbb{Z}} |2^{js} \lambda_{j+\ell,\nu}(f)|^p 2^{-j} \, 2^{(N-j-\ell)p}.$$

Indeed, since $\mathbb{E}_N \psi_{j+\ell,\nu} = 0$ if $\operatorname{supp} \psi_{j+\ell,\nu} \subset I_{N,\mu}$ the sum on the right-hand side of (23) is lacunary and the functions $\sum_\eta 2^j \langle \mathbb{E}_N \psi_{j+\ell,\nu}, \psi_{j,\eta} \rangle \mathbb{1}_{j,\eta}$ have essentially disjoint support (for different ν). Hence, we get

$$\Big\| 2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell}, \psi_{j,\eta} \rangle \mathbb{1}_{j,\eta} \Big\|_p \tag{24}$$

$$\lesssim 2^{-\ell s} 2^{N-j-\ell} 2^{\ell/p} \Big(\sum_{\nu \in \mathbb{Z}} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \Big)^{1/p}$$

$$\lesssim 2^{-\ell s} 2^{N-j-\ell} 2^{\ell/p} \|f\|_{B^s_{p,\infty}} .$$

For $1/p - 1 < s < 1/p$ the sum over the respective range of j and ℓ is uniformly bounded.

Case 1.2. We now deal with $j + \ell > N \geq j$. Due to the orthogonality of the wavelet system and $\ell > 0$ we obtain

$$\Big\| 2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell}, \psi_{j,\eta} \rangle \mathbb{1}_{j,\eta} \Big\|_p = \Big\| 2^{js} \sum_{\eta} 2^j \langle \mathbb{E}_N f_{j+\ell} - f_{j+\ell}, \psi_{j,\eta} \rangle \mathbb{1}_{j,\eta} \Big\|_p .$$

We continue exploiting the cancellation property

$$\mathbb{E}_N (f - \mathbb{E}_N f) = 0 \tag{25}$$

to estimate the right-hand side of (2). We obtain the following identities

$$\Big| 2^{js} \sum_{\eta} \mathbb{1}_{j,\eta}(x) 2^j \int \psi_{j,\eta}(y) (\mathbb{E}_N f_{j+\ell}(y) - f_{j+\ell}(y)) \, dy \Big| \tag{26}$$

$$= \Big| 2^{js} \sum_{\eta} \mathbb{1}_{j,\eta}(x) \sum_{\mu : |2^{-N}\mu - x| \lesssim 2^{-j}} 2^j \int_{I_{N,\mu}} \psi_{j,\eta}(y) (\mathbb{E}_N f_{j+\ell}(y) - f_{j+\ell}(y)) \, dy \Big|$$

$$= \Big| 2^{js} \sum_{\eta} \mathbb{1}_{j,\eta}(x) \sum_{\mu : |2^{-N}\mu - x| \lesssim 2^{-j}} 2^j$$

$$\times \int_{I_{N,\mu}} (\psi_{j,\eta}(y) - \psi_{j,\eta}(2^{-N}\mu)) (\mathbb{E}_N f_{j+\ell}(y) - f_{j+\ell}(y)) \, dy \Big| .$$

Let $\eta \in \mathbb{Z}$ such that $\mathbb{1}_{j,\eta}(x) = 1$. We continue estimating (26) by

$$\begin{aligned} &2^{js} \sum_{\mu:|2^{-N}\mu - x| \lesssim 2^{-j}} 2^j \int_{I_{N,\mu}} |(\psi_{j,\eta}(y) - \psi_{j,\eta}(2^{-N}\mu)) \cdot \mathbb{E}_N f_{j+\ell}(y)|\, dy \\ &+ \Big| 2^{js} \sum_{\mu:|2^{-N}\mu - x| \lesssim 2^{-j}} 2^j \int_{I_{N,\mu}} (\psi_{j,\eta}(y) - \psi_{j,\eta}(2^{-N}\mu)) \cdot f^{\mu,1}_{j+\ell}(y)\, dy \Big| \\ &+ \Big| 2^{js} \sum_{\mu:|2^{-N}\mu - x| \lesssim 2^{-j}} 2^j \int_{I_{N,\mu}} (\psi_{j,\eta}(y) - \psi_{j,\eta}(2^{-N}\mu)) \cdot f^{\mu,2}_{j+\ell}(y)\, dy \Big| \\ &=: F_0(x) + F_1(x) + F_2(x)\,, \end{aligned}$$

where

$$f^{\mu}_{j+\ell} := \sum_{\nu:\operatorname{supp}\psi_{j+\ell,\nu} \cap I_{N,\mu} \neq \emptyset} \lambda_{j+\ell,\nu}(f)\psi_{j+\ell,\nu}\,,$$

$$f^{\mu,1}_{j+\ell} := \sum_{\nu:\operatorname{supp}\psi_{j+\ell,\nu} \subset I_{N,\mu}} \lambda_{j+\ell,\nu}(f)\psi_{j+\ell,\nu}\,, \quad f^{\mu,2}_{j+\ell} := f^{\mu}_{j+\ell} - f^{\mu,1}_{j+\ell}\,.$$

Note, that the function F_1 vanishes since $\ell > 0$ (use orthogonality) and $j + \ell > 0$ (use vanishing moments).

$F_0(x)$ can be estimated by

$$2^{js} \sum_{\mu:|2^{-N}\mu - x| \lesssim 2^{-j}} 2^{2j-2N} \sup_{y \in I_{N,\mu}} \sum_{\nu:\operatorname{supp}\psi_{j+\ell,\nu} \cap I_{N,\mu} \neq \emptyset} |\lambda_{j+\ell,\nu}(f)\mathbb{E}_N(\psi_{j+\ell,\nu})(y)|\,.$$

Here $\mathbb{E}_N\psi_{j+\ell,\nu}$ is mostly vanishing, namely when $\operatorname{supp}\psi_{j+\ell,\nu} \subset I_{N,\mu}$. If it does not vanish then the boundary of $I_{N,\mu}$ intersects $\operatorname{supp}\psi_{j+\ell,\nu}$ and $|\mathbb{E}_N\psi_{j+\ell,\nu}| \lesssim 2^{N-(j+\ell)}$. This happens only for a bounded number of ν's (independently of j, ℓ). Thus for a fixed y only a bounded number of coefficients contribute. Hence, we have

(27)
$$F_0(x) \lesssim 2^{js} 2^{2j-2N} 2^{N-(j+\ell)} \sum_{\mu:|2^{-N}\mu - x| \lesssim 2^{-j}} \sup_{\nu:\operatorname{supp}\psi_{j+\ell,\nu} \cap \partial I_{N,\mu} \neq \emptyset} |\lambda_{j+\ell,\nu}(f)|\,.$$

Taking the L_p-norm and using Hölder's inequality with $1/p + 1/p' = 1$ yields

(28)
$$\|F_0\|_p \lesssim 2^{-\ell s} 2^{2j-2N} 2^{N-(j+\ell)} 2^{(N-j)/p'} 2^{\ell/p} \Big(\sum_{\nu} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \Big)^{1/p},$$

where again $\|f\|_{B^s_{p,\infty}}$ dominates the sum on the right-hand side, see (5). Finally, we deal with $F_2(x)$. Since to $f^{\mu,2}_{j+\ell}$ only a uniformly bounded number of coefficients $\lambda_{j+\ell,\nu}$ contribute to the sum and the integrals are taken over an interval of length $O(2^{-(j+\ell)})$ we obtain, similar as above, by Hölder's inequality

$$(29)\quad \|F_2\|_p \lesssim 2^{-\ell s}2^{-\ell+j-N}2^{(N-j)/p'}2^{\ell/p}\Big(\sum_{\nu}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}.$$

Putting the estimates from (2) to (29) together we observe that the sum over the respective range of j and ℓ (see (11)) is uniformly bounded with respect to N if $s > 1/p - 1$.

Case 2.1. Here we deal with $j+\ell, j \le N$. Starting from (16) (with $\theta = 1$) we continue similarly as after (25) and obtain the pointwise identity (26). Note, that we already start with $\mathbb{E}_N f_{j+\ell} - f_{j+\ell}$, so we do have to use the orthogonality argument (2), which does indeed not apply here since $\ell = 0$ is admitted.

Since $j+\ell \le N$ there is only a bounded number of coefficients $\lambda_{j+\ell,\nu}(f)$ contributing to $f_{j+\ell}$ on $I_{N,\mu}$. Using the mean value theorem in both factors of the integral in (26) we obtain

$$\begin{aligned}&\Big|2^{js}\sum_{\eta}2^j\langle \mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big| \\ &\lesssim 2^{js}2^{2j-2N}2^{j+\ell-N}\sum_{\mu:|2^{-N}\mu - x|\lesssim 2^{-j}}\ \sup_{|\nu 2^{-(j+\ell)}-2^{-N}\mu|\lesssim 1}|\lambda_{j+\ell,\nu}(f)|,\end{aligned}$$

which yields

$$\begin{aligned}&\Big|2^{js}\sum_{\eta}2^j\langle \mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p \\ &\lesssim 2^{-\ell s}2^{j+\ell-N}2^{2j-2N}2^{(N-j)/p'}2^{\ell/p}\Big(\sum_{\nu\in\mathbb{Z}}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}.\end{aligned}$$

The sum over the respective j and ℓ is uniformly bounded in N whenever $-1 < s < 1 + 1/p$.

Case 2.2. Finally $j + \ell \le N < j$. Using again the orthogonality relation of the wavelets we may estimate as follows (similar to (20))

$$(30)\qquad \Big\|2^{js}\sum_{\eta}2^j\langle \mathbb{E}_N f_{j+\ell} - f_{j+\ell}, \psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}\Big\|_p$$

$$\lesssim 2^{js}\Big(\sum_{\mu\in\mathbb{Z}}\int_{|x-2^{-N}\mu|\lesssim 2^{-j}}\Big|\sum_\eta 2^j\langle \mathbb{E}_N f_{j+\ell}-f_{j+\ell},\psi_{j,\eta}\rangle \mathbb{1}_{j,\eta}(x)\Big|^p\,dx\Big)^{1/p},$$

which is bounded by (see (20))

$$2^{-\ell s}2^{j+\ell-N}2^{(N-j)/p}\Big(\sum_{\nu\in\mathbb{Z}}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}. \tag{31}$$

Altogether we encounter the condition $1/p-1<s<1/p$ for any $0<r\le\infty$ for the uniform boundedness of $\mathbb{E}_N\colon B^s_{p,\infty}\to B^s_{p,r}$ in case $1\le p\le\infty$. □

3. On the Schauder basis property for the Haar system

Let $\{h_{N,\mu}:\mu\in\mathbb{Z}\}$ be the set of Haar functions with Haar frequency 2^{-N} and define for $N\in\mathbb{N}_0$ and sequences $a\in\ell^\infty(\mathbb{Z})$,

$$T_N[f,a]=\sum_{\mu\in\mathbb{Z}}a_\mu 2^N\langle f,h_{N,\mu}\rangle h_{N,\mu}. \tag{32}$$

In particular for the choice of $a=(1,1,1,\dots)$ one recovers the operator $\mathbb{E}_{N+1}-\mathbb{E}_N$. It was shown in [2] that Theorem 1.1 together with

$$\sup_{N\in\mathbb{N}}\sup_{\|a\|_\infty\le 1}\|T_N[f,a]\|_{B^s_{p,r}}\le C\|f\|_{B^s_{p,\infty}}, \tag{33}$$

$$1/2<p\le\infty,\ 0<r\le\infty,\ \text{and } 1/p-1<s<\min\{1/p,1\},$$

implies Schauder basis properties for suitable enumerations of the Haar system. For the sake of completeness we give a sketch of this inequality which relies on the arguments in the previous section.

Proof of (33). We may assume $\|a\|_\infty=1$. The modification of the proof of Proposition 2.1 is the fact that, due to the cancellation properties of the Haar functions participating in (32) we can work directly with $\|T_N[f,a]\|_{B^s_{p,r}}$ (instead of $\|\mathbb{E}_Nf-P_Nf\|_{B^s_{p,r}}$.

Case 1.1. Suppose $j+\ell, j>N$. The estimates in (23), (24) apply almost literally to $T_N[f,a]$ and yield estimates which are uniform for $\|a\|_\infty=1$. Note, that we did not yet need any cancellation of the Haar functions.

Case 1.2. Suppose $j+\ell>N\ge j$. We do not have to use (2) and work directly with $\|2^{js}T_N[f_{j+\ell},a]\|_p$. An analogous identity to (26) holds true with $\mathbb{E}_N(f_{j+\ell})-f_{j+\ell}$ replaced by $T_N[f_{j+\ell},a]$ due to the cancellation of the Haar functions $h_{N,\mu}$. In what follows we only have to care for a counterpart of F_0 since F_1 and F_2 do not show up. We end up with a counterpart of (28) for $\|2^{js}T_N[f_{j+\ell},a]\|_p$.

Case 2.1. Suppose $N \geq j + \ell, j$. Again, due to the cancellation of the Haar function, we obtain a version of (26) as in Case 1.2. The mean value theorem applied to the first factor in the integral gives the factor 2^{2j-2N}, whereas the cancellation of $h_{N,\mu}$ gives $|T_N(\psi_{j+\ell,\nu})(x)| \lesssim 2^{j+\ell-N}$. We continue as in the proof of Proposition 2.1.

Case 2.2. The remaining case $j+\ell \leq N < j$ goes analogously to Case 2.2. in the proof of Proposition 2.1. Note, that also here the splitting in (30) and the subsequent consideration for the second summand on the right-hand side is not necessary. □

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References

[1] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61, Society for Industrial and Applied Mathematics (SIAM) (Philadelphia, PA, 1992).

[2] G. Garrigós, A. Seeger and T. Ullrich, The Haar system as a Schauder basis in spaces of Hardy–Sobolev type, preprint, arXiv:1609.08225.

[3] A. Seeger and T. Ullrich, Haar projection numbers and failure of unconditional convergence in Sobolev spaces, *Math. Z.*, **285** (2017), 91–119.

[4] A. Seeger and T. Ullrich, Lower bounds for Haar projections: Deterministic Examples, *Constr. Approx.*, published online (2016), DOI: 10.1007/s00365-016-9353-3.

[5] H. Triebel, *Theory of Function Spaces*, III, Monographs in Mathematics, vol. 100, Birkhäuser Verlag (Basel, 2006).

[6] H. Triebel, *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*, EMS Tracts in Mathematics, vol. 11, European Mathematical Society (EMS) (Zürich, 2010).

[7] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, London Mathematical Society Student Texts, vol. 37, Cambridge University Press (Cambridge, 1997).