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The Haar System as a Schauder Basis in Spaces of Hardy–Sobolev Type

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Abstract We show that, for suitable enumerations, the Haar system is a Schauder basis in the classical Sobolev spaces in $\mathbb{R}^d$ with integrability $1 < p < \infty$ and smoothness $1/p - 1 < s < 1/p$. This complements earlier work by the last two authors on the unconditionality of the Haar system and implies that it is a conditional Schauder basis for a nonempty open subset of the $(1/p, s)$-diagram. The results extend to (quasi-)Banach spaces of Hardy–Sobolev and Triebel–Lizorkin type in the range of parameters $\frac{d}{d+1} < p < \infty$ and $\max\{d(1/p - 1), 1/p - 1\} < s < \min\{1, 1/p\}$, which is optimal except perhaps at the end-points.

Keywords Schauder basis · Unconditional basis · Haar system · Sobolev space · Triebel–Lizorkin space

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1 Introduction

We recall the definition of the (inhomogeneous) Haar system in $\mathbb{R}^d$. Consider the 1-variable functions

$$h^{(0)} = \mathbb{1}_{[0,1)} \quad \text{and} \quad h^{(1)} = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}.$$

For every $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0,1\}^d$ one defines

$$h^{(\boldsymbol{\varepsilon})}(x_1, \ldots, x_d) = h^{(\varepsilon_1)}(x_1) \cdots h^{(\varepsilon_d)}(x_d).$$

Finally, one sets

$$h^{(\boldsymbol{\varepsilon})}_{k,\ell}(x) = h^{(\boldsymbol{\varepsilon})}(2^k x - \ell), \quad k \in \mathbb{N}_0, \ \ell \in \mathbb{Z}^d,$$

Denoting $\Upsilon = \{0,1\}^d \setminus \{\vec{0}\}$, the Haar system is then given by

$$\mathscr{H}_d = \Big\{h^{(\vec{0})}_{0,\ell}\Big\}_{\ell \in \mathbb{Z}^d} \cup \Big\{h^{(\boldsymbol{\varepsilon})}_{k,\ell} \mid k \in \mathbb{N}_0, \ \ell \in \mathbb{Z}^d, \ \boldsymbol{\varepsilon} \in \Upsilon\Big\}.$$

Observe that supp $h^{(\boldsymbol{\varepsilon})}_{k,\ell}$ is the dyadic cube $I_{k,\ell} := 2^{-k}(\ell + [0,1]^d)$.

In this paper we consider basis properties of $\mathscr{H}_d$ in Besov spaces $B^s_{p,q}$, and Triebel–Lizorkin spaces $F^s_{p,q}$ in $\mathbb{R}^d$. We refer to [9,10] for definitions and properties of these spaces, and to [1] for terminology and general facts about bases in Banach spaces.

In the 1970s, Triebel [7,8] proved that the Haar system $\mathscr{H}_d$ is a Schauder basis on $B^s_{p,q}(\mathbb{R}^d)$ if

$$\tfrac{d}{d+1} < p < \infty, \quad 0 < q < \infty, \quad \max\Big\{d(\tfrac{1}{p}-1), \tfrac{1}{p}-1\Big\} < s < \min\Big\{1, \tfrac{1}{p}\Big\}, \tag{1}$$

and that this range is maximal, except perhaps at the endpoints. Moreover, the basis is unconditional when (1) holds; see [11, Theorem 2.21]. Concerning $F^s_{p,q}$ spaces, however, in [11] it is only shown that $\mathscr{H}_d$ is an unconditional basis for $F^s_{p,q}(\mathbb{R}^d)$ when, besides (1), the additional assumption

$$\max\Big\{d(\tfrac{1}{q}-1), \tfrac{1}{q}-1\Big\} < s < \tfrac{1}{q} \tag{2}$$

is satisfied. Recently, two of the authors showed in [5,6] that the additional restriction (2) is in fact necessary, at least when $d = 1$. It was left open whether suitable enumerations of the Haar system can form a Schauder basis in $F^s_{p,q}$ in the larger range (1). We shall answer this question affirmatively.

Given an enumeration $\{u_1, u_2, \ldots\}$ of the system $\mathscr{H}_d$, we let P_N be the orthogonal projection onto the subspace spanned by $u_1, \ldots, u_N$, i.e.

$$P_N f = \sum_{n=1}^{N} \|u_n\|_2^{-2} \langle f, u_n \rangle u_n \,. \tag{3}$$

The sequence $\{u_n\}_{n=1}^{\infty}$ is a Schauder basis on $F^s_{p,q}$ if

$$\lim_{N\to\infty} \|P_N f - f\|_{F^s_{p,q}} = 0, \quad \text{for all } f \in F^s_{p,q}. \tag{4}$$

In view of the uniform boundedness principle, density theorems and the result for Besov spaces, (4) follows if we can show that the operators P_N have uniform $F^s_{p,q} \to F^s_{p,q}$ operator norms. Note, that the condition $s < 1/p$ is necessary since the Haar functions need to belong to $F^s_{p,q}$. By duality, if $1 < p < \infty$, the condition $s > 1/p - 1$ becomes also necessary, so the range in (1) is optimal in this case. If $p \le 1$, then an interpolation argument shows that (1) is also a maximal range, except perhaps at the end-points; see Sect. 4 below.

Definition An enumeration $\mathcal{U} = \{u_1, u_2, ...\}$ of the Haar system $\mathscr{H}_d$ is *admissible* if the following condition holds for each cube $I_\nu = \nu + [0,1]^d$, $\nu \in \mathbb{Z}^d$. If u_n and $u_{n'}$ are both supported in I_ν and $|\text{supp}(u_n)| > |\text{supp}(u_{n'})|$, then necessarily $n < n'$.
The table in Fig. 1 shows how to obtain an admissible (natural) enumeration of $\mathscr{H}_d$ via a diagonalization of the intervals I_ν versus the levels k. We first label the set $\mathbb{Z}^d = \{\nu_1, \nu_2, \ldots\}$. Then, we follow the order indicated by the table, where being at position (ν_i, k) means to pick all the Haar functions with support contained in I_{ν_i} and size 2^{-kd}, arbitrarily enumerated, before going to the subsequent table entry.

Our main result reads as follows.

Theorem 1.1 *Let $\mathcal{U} = \{u_n\}_{n=1}^{\infty}$ be an admissible enumeration of the Haar system $\mathscr{H}_d$. Assume that*

(i) $\frac{d}{d+1} < p < \infty$,
(ii) $0 < q < \infty$,
(iii) $\max\{d(\frac{1}{p} - 1), \frac{1}{p} - 1\} < s < \min\{1, \frac{1}{p}\}$.

Then $\mathcal{U}$ is a Schauder basis in $F^s_{p,q}(\mathbb{R}^d)$.

In the left part of Fig. 2, the trapezoid is the parameter domain for which the Haar system is a Schauder basis in the Hardy–Sobolev space $H^s_p(\mathbb{R})$ $(= F^s_{p,2}(\mathbb{R}))$ while the shaded part represents the parameter domain for which the Haar system is an unconditional basis in $H^s_p(\mathbb{R})$. The right figure shows the respective parameter domain for $H^s_p(\mathbb{R}^d)$.

The heart of the matter is a boundedness result for the dyadic averaging operators $\mathbb{E}_N$ given by

$$\mathbb{E}_N f(x) = \sum_{\mu\in\mathbb{Z}^d} \mathbb{1}_{I_{N,\mu}}(x)\, 2^N \int_{I_{N,\mu}} f(t)dt \tag{5}$$

Fig. 1 An admissible enumeration of $\mathscr{H}_d$

$k\backslash I_\nu$	I_{ν_0}	I_{ν_1}	I_{ν_2}	I_{ν_3}	I_{ν_4}	$\ldots$
0	1	2	4	7	11	
1	3	5	8	12		
2	6	9	13			
3	10	14				
4	15					

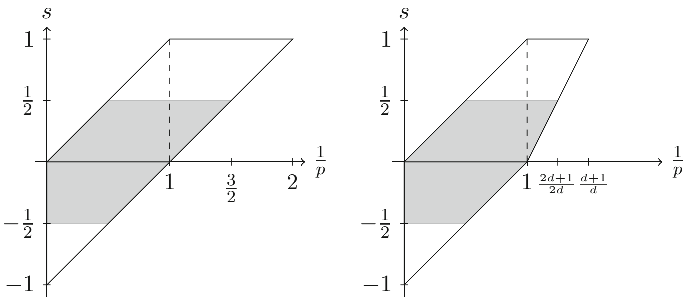

Fig. 2 Unconditionality of the Haar system in Hardy–Sobolev spaces in $\mathbb{R}$ and $\mathbb{R}^d$

with

$$I_{N,\mu} = 2^{-N}(\mu + [0,1)^d), \quad \mu \in \mathbb{Z}^d, \; N = 0, 1, 2, \ldots$$

Note that $\mathbb{E}_N f$ is just the conditional expectation of f with respect to the σ-algebra generated by the set $\mathscr{D}_N$ of all dyadic cubes of side length 2^{-N}. There is a well known relation between the Haar system and the dyadic averaging operators, namely for $N = 0, 1, 2, \ldots,$

$$\mathbb{E}_{N+1} f - \mathbb{E}_N f = \sum_{\varepsilon \in \Upsilon} \sum_{\mu \in \mathbb{Z}^d} 2^{Nd} \langle f, h_{N,\mu}^{(\varepsilon)} \rangle h_{N,\mu}^{(\varepsilon)}, \tag{6}$$

i.e. $\mathbb{E}_{N+1} - \mathbb{E}_N$ is the orthogonal projection onto the space generated by the Haar functions with Haar frequency 2^N.

Now let η_0 be a Schwartz function on $\mathbb{R}^d$, supported in $\{|\xi| < 3/8\}$ and so that $\eta_0(\xi) = 1$ for $|\xi| \le 1/4$. Let Π_N be defined by

$$\widehat{\Pi_N f}(\xi) = \eta_0(2^{-N}\xi)\widehat{f}(\xi). \tag{7}$$

There is a basic standard inequality (almost immediate from the definition of Triebel–Lizorkin spaces)

$$\sup_N \|\Pi_N f\|_{F^s_{p,q}} \le C(p,q,s)\|f\|_{F^s_{p,q}} \tag{8}$$

which is valid for all $s \in \mathbb{R}$ and for $0 < p < \infty$, $0 < q \le \infty$. Moreover, (8) and the fact that $\|\Pi_N g - g\|_{F^s_{p,q}} \to 0$ for Schwartz functions g gives

$$\lim_{N \to \infty} \|\Pi_N f - f\|_{F^s_{p,q}} = 0 \tag{9}$$

if $f \in F^s_{p,q}$ and $0 < p, q < \infty$. The main tool in proving Theorem 1.1 is a similar bound for the operators $\mathbb{E}_N$ which of course follows from the corresponding bound for $\mathbb{E}_N - \Pi_N$. It turns out that the operators $\mathbb{E}_N - \Pi_N$ enjoy better mapping properties in Besov spaces.

Similar bounds are also satisfied by projection operators into sets of Haar functions with fixed Haar frequency. Namely, for $N \in \mathbb{N}$ and functions $a \in \ell^\infty(\mathbb{Z}^d \times \Upsilon)$, we define

$$T_N[f,a] = \sum_{\boldsymbol{\varepsilon}\in\Upsilon} \sum_{\mu\in\mathbb{Z}^d} a_{\mu,\boldsymbol{\varepsilon}} 2^{Nd} \langle f, h^{(\boldsymbol{\varepsilon})}_{N,\mu}\rangle h^{(\boldsymbol{\varepsilon})}_{N,\mu}. \tag{10}$$

Observe that the choice $a_{\mu,\boldsymbol{\varepsilon}} \equiv 1$ recovers the operator $\mathbb{E}_{N+1} - \mathbb{E}_N$. Then, we shall prove the following.

Theorem 1.2 *Let* $d/(d+1) < p \le \infty$, $0 < r \le \infty$, *and*

$$\max\{d(1/p-1), 1/p-1\} < s < \min\{1, 1/p\}. \tag{11}$$

Then there is a constant $C := C(p,r,s) > 0$ *such that for all* $f \in B^s_{p,\infty}$

$$\sup_N \|\mathbb{E}_N f - \Pi_N f\|_{B^s_{p,r}} \le C\|f\|_{B^s_{p,\infty}}. \tag{12}$$

Moreover,

$$\sup_N \|T_N[f,a]\|_{B^s_{p,r}} \lesssim \|a\|_\infty \|f\|_{B^s_{p,\infty}}. \tag{13}$$

We have the embedding $F^s_{p,q} \subset F^s_{p,\infty} \subset B^s_{p,\infty}$ which we use on the function side. For $r \le p$ we have the embedding $B^s_{p,r} \subset F^s_{p,r}$ (by Minkowski's inequality in $L^{p/r}$) and if also $r < q$ we have $F^s_{p,r} \subset F^s_{p,q}$; these two are used for $\mathbb{E}_N f - \Pi_N f$, or $T_N[f,a]$. In particular we conclude from Theorem 1.2 that $\mathbb{E}_N - \Pi_N$ is bounded on $F^s_{p,q}$, uniformly in N. Hence

Corollary 1.3 *Let* p, s *be as in* (11) *and* $0 < q \le \infty$. *Then*

$$\sup_N \|\mathbb{E}_N f\|_{F^s_{p,q}} + \sup_N \sup_{\|a\|_{\ell^\infty}\le 1} \|T_N[f,a]\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}}. \tag{14}$$

The proofs in this paper use basic principles in the theory of function spaces, such as L^p inequalities for the Peetre maximal functions. A different approach to Corollary 1.3 via wavelet theory is presented in the subsequent paper [2]. The main arguments and the proof of Theorem 1.2 are contained in Sect. 2. In Sect. 3 we show how estimates in the proof of Theorem 1.2 are used to deduce Theorem 1.1. Finally, in Sect. 4 we discuss the optimality of the results.

2 Proof of Theorem 1.2

We start with some preliminaries on convolution kernels which are used in Littlewood-Paley type decompositions. Let β_0, β be Schwartz functions on $\mathbb{R}^d$, compactly supported in $(-1/2, 1/2)^d$ such that $|\widehat{\beta_0}(\xi)| > 0$ when $|\xi| \le 1$ and $|\widehat{\beta}(\xi)| > 0$ when $1/8 \le |\xi| \le 1$. Moreover assume β has vanishing moments up to a large order

$$M > \frac{d}{p} + |s|, \tag{15}$$

that is,

$$\int_{\mathbb{R}^d} \beta(x)\, x_1^{m_1} \cdots x_d^{m_d}\, dx = 0 \quad \text{when} \quad m_1 + \ldots + m_d < M\,. \tag{16}$$

For $k = 1, 2, \ldots$ let $\beta_k := 2^{kd}\beta(2^k\cdot)$ and $L_k f = \beta_k * f$. We shall use the inequality

$$\|g\|_{B^s_{p,r}} \lesssim \Big(\sum_{k=0}^{\infty} 2^{ksr}\|L_k g\|_p^r\Big)^{1/r} \tag{17}$$

and apply it to $g = \mathbb{E}_N f - \Pi_N f$. Inequality (17) is of course just one part of a characterization of $B^s_{p,r}$ spaces by sequences of compactly supported kernels (or 'local means'), with sufficient cancellation assumptions, see for example [10, Sect. 2.5.3].

Let $\eta_0 \in C_c^\infty(\mathbb{R}^d)$ be as in (7), that is, supported on $\{|\xi| < 3/8\}$ and such that $\eta_0(\xi) = 1$ when $|\xi| \le 1/4$. Define Λ_0, and Λ_k for $k \ge 1$ by

$$\widehat{\Lambda_0 f}(\xi) = \frac{\eta_0(\xi)}{\widehat{\beta}_0(\xi)}\widehat{f}(\xi)$$
$$\widehat{\Lambda_k f}(\xi) = \frac{\eta_0(2^{-k}\xi) - \eta_0(2^{-k+1}\xi)}{\widehat{\beta}(2^{-k}\xi)}\widehat{f}(\xi), \quad k \ge 1.$$

Then $\sum_{j=0}^\infty L_j\Lambda_j = Id$ with convergence in $\mathcal{S}'$, and

$$\sup_{j\ge 0} 2^{js}\|\Lambda_j f\|_p \lesssim \|f\|_{B^s_{p,\infty}}\,.$$

Moreover $\Pi_N = \sum_{j=0}^N L_j\Lambda_j$, and therefore

$$\mathbb{E}_N f - \Pi_N f = \sum_{j=0}^N (\mathbb{E}_N L_j\Lambda_j f - L_j\Lambda_j f) + \sum_{j=N+1}^\infty \mathbb{E}_N L_j\Lambda_j f. \tag{18}$$

If we use the convenient notation

$$\mathbb{E}_N^\perp := I - \mathbb{E}_N,$$

then the asserted estimate (12) will follow from

$$\Big(\sum_{k=0}^\infty 2^{ksr}\Big\|\sum_{j=N+1}^\infty L_k\mathbb{E}_N L_j\Lambda_j f\Big\|_p^r\Big)^{1/r} \lesssim \sup_j 2^{js}\|\Lambda_j f\|_p\,. \tag{19}$$

and

$$\Big(\sum_{k=0}^\infty 2^{ksr}\Big\|\sum_{j=0}^N L_k\mathbb{E}_N^\perp L_j\Lambda_j f\Big\|_p^r\Big)^{1/r} \lesssim \sup_j 2^{js}\|\Lambda_j f\|_p\,. \tag{20}$$

Below we shall use variants of the Peetre maximal functions, which are a standard tool in the study of Besov and Triebel–izorkin spaces. We define

$$\mathfrak{M}_j g(x) = \sup_{|h|_\infty \le 2^{-j+1}} |g(x+h)|, \tag{21a}$$

$$\mathfrak{M}_j^* g(x) = \sup_{|h|_\infty \le 2^{-j+5}} |g(x+h)|, \tag{21b}$$

$$\mathfrak{M}_{A,j}^{**} g(x) = \sup_{h \in \mathbb{R}^d} \frac{|g(x+h)|}{(1+2^j|h|)^A}, \tag{21c}$$

where $|h|_\infty = \max\{|h_1|, \dots, |h_d|\}$, $h = (h_1, \dots, h_d) \in \mathbb{R}^d$. These different versions are introduced for technical purposes in the proofs. They satisfy obvious pointwise inequalities,

$$\mathfrak{M}_j g(x) \le \mathfrak{M}_j^* g(x) \le C_A \mathfrak{M}_{A,j}^{**} g(x),$$

and

$$\begin{aligned} \mathfrak{M}_j g(x) &\le \inf_{|h|_\infty \le 2^{-j+4}} \mathfrak{M}_j^* g(x+h) \\ &\le \Big(2^{(j-4)d} \int_{|h|_\infty \le 2^{-j+4}} [\mathfrak{M}_j^* g(x+h)]^r dh\Big)^{1/r}, \quad 0 < r \le \infty. \end{aligned} \tag{22}$$

Below we shall use Peetre's inequality ([3], see also [9, Sect. 1.3.1])

$$\|\mathfrak{M}_{A,j}^{**} f\|_p \le C_{p,A} \|f\|_p, \quad 0 < p \le \infty, \quad A > d/p, \tag{23}$$

for $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying

$$\operatorname{supp}(\widehat{f}) \subset \{\xi : |\xi| \le 2^{j+1}\}. \tag{24}$$

Throughout we shall assume that $M \gg A$; we require specifically

$$d/p < A < M - |s|.$$

The main estimates needed in the proof of (19) and (20) are summarized in

Proposition 2.1 *Let* $0 < p \le \infty$ *and*

$$B(j,k,N) = \begin{cases} 2^{N-j}\, 2^{\frac{j-k}{p}}\, 2^{(j-N)(d-1)(\frac{1}{p}-1)_+} & \text{if } j,k \ge N+1, \\ 2^{\frac{N-k}{p}}\, 2^{j-N} & \text{if } j \le N,\ k \ge N+1, \\ 2^{k-N} 2^{j-N} 2^{(N-k)d(\frac{1}{p}-1)_+} & \text{if } 0 \le j,k \le N, \\ 2^{k-j+\frac{j-N}{p}+[N-k+(j-k)(d-1)](\frac{1}{p}-1)_+} & \text{if } j \ge N+1,\ k \le N. \end{cases} \tag{25}$$

Then the following inequalities hold for all $f \in \mathcal{S}'(\mathbb{R}^d)$ *whose Fourier transform is supported in* $\{|\xi| \le 2^{j+1}\}$.

(i) For $j \ge N+1$,

$$\|L_k \mathbb{E}_N[L_j f]\|_p \lesssim \begin{cases} B(j,k,N)\|f\|_p & \text{if } k \ge N+1, \\ [B(j,k,N) + 2^{-|j-k|(M-A)}]\|f\|_p & \text{if } 0 \le k \le N. \end{cases} \tag{26}$$

(ii) For $0 \le j \le N$,

$$\|L_k \mathbb{E}_N^{\perp}[L_j f]\|_p \lesssim \begin{cases} \big[B(j,k,N) + 2^{-|j-k|(M-A)}\big]\|f\|_p & \text{if } k \ge N+1, \\ B(j,k,N)\|f\|_p & \text{if } 0 \le k \le N. \end{cases} \tag{27}$$

(iii) The same bounds hold if the operators $\mathbb{E}_N$ *in (i) and* $\mathbb{E}_N^{\perp}$ *in (ii) are replaced by* $T_N[\cdot, a]$, *uniformly in* $\|a\|_\infty \le 1$.

We begin with two preliminary lemmata, the first a straightforward estimate for $L_k L_j$.

Lemma 2.2 *Let* $k, j \ge 0$ *and suppose that* f *is locally integrable. Let* M *be as in* (16) *with* $M > A > d/p$. *Then*

$$|L_k L_j f(x)| \lesssim 2^{-|k-j|(M-A)} \mathfrak{M}^{**}_{A,\max\{j,k\}} f(x). \tag{28}$$

If $f \in \mathcal{S}'(\mathbb{R}^d)$ *with* $\widehat{f}(\xi) = 0$ *for* $|\xi| \ge 2^{j+1}$ *then*

$$\|L_k L_j f\|_p \lesssim 2^{-|k-j|(M-A)} \|f\|_p .$$

Proof The second assertion is an immediate consequence of (28), by (23). We have $L_k L_j f = \gamma_{j,k} * f$ where $\gamma_{j,k} = \beta_k * \beta_j$. By symmetry we may assume $k \le j$. Using the cancellation assumption (16) on the β_j we get

$$|\gamma_{j,k}(x)| = \Big| \int 2^{kd} \Big[\beta(2^k(x-y) - \sum_{m=0}^{M-1} \frac{1}{m!} \langle -2^k y, \nabla\rangle^m \beta(2^k x)\Big] 2^{jd} \beta(2^j y) dy \Big|$$
$$= \Big| \int 2^{kd} \int_0^1 \frac{(1-s)^{M-1}}{(M-1)!} \langle -2^k y, \nabla\rangle^M \beta(2^k x - s 2^k y)\, ds\, 2^{jd} \beta(2^j y) dy \Big|$$
$$\lesssim 2^{(k-j)M}\, 2^{kd}\, \mathbb{1}_{[-2^{-k}, 2^{-k}]^d}(x),$$

and thus

$$2^{(j-k)M} |\gamma_{j,k} * f(x)| \lesssim 2^{kd} \int_{|h|_\infty \le 2^{-k}} |f(x-h)|\, dh$$
$$\lesssim 2^{kd} \int_{|h|_\infty \le 2^{-k}} \frac{2^{(j-k)A} |f(x-h)|}{(1 + 2^j|h|)^A} dh \lesssim 2^{(j-k)A}\, \mathfrak{M}^{**}_{A,j} f(x).$$

Hence (28) holds. □

Some Notation

(i) Below, when $j > N$ we use the notation

$$\mathcal{U}_{N,j} = \Big\{(y_1,\dots,y_d) \in \mathbb{R}^d \mid \min_{1\le i\le d} \operatorname{dist}(y_i, 2^{-N}\mathbb{Z}) \le 2^{-j-1}\Big\}.$$

That is, $\mathcal{U}_{N,j}$ is a 2^{-j-1}-neighborhood of the set $\cup_{I\in\mathscr{D}_N} \partial I$.

(ii) For a dyadic cube I of side length 2^{-N} and $l > N$ we shall denote by $\mathscr{D}_l[\partial I]$ the set of dyadic cubes $J \in \mathscr{D}_l$ such that $\bar{J} \cap \partial I \neq \emptyset$.

(iii) For a dyadic cube I of side length 2^{-N} denote by $\mathscr{D}_N(I)$ the neighboring cubes of I, that is, the cubes $I' \in \mathscr{D}_N$ with $\bar{I} \cap \bar{I}' \neq \emptyset$.

Lemma 2.3 *(i) Let $k > N \ge 1$ and g be locally integrable. Then*

$$L_k(\mathbb{E}_N g)(x) = 0, \quad \text{for all } x \in \mathcal{U}^{\complement}_{N,k} = \mathbb{R}^d \setminus \mathcal{U}_{N,k}\,. \tag{29}$$

(ii) Let $j > N \ge 1$, and f locally integrable. Then

$$\mathbb{E}_N[L_j f] = \mathbb{E}_N[L_j(\mathbb{1}_{\mathcal{U}_{N,j}} f)]. \tag{30}$$

Moreover,

$$\big|\mathbb{E}_N(L_j f)\big| \lesssim 2^{(N-j)d} \sum_{I\in\mathscr{D}_N} \sum_{J\in\mathscr{D}_{j+1}[\partial I]} \|f\|_{L^\infty(J)}\, \mathbb{1}_I. \tag{31}$$

Proof (i) We use the support and cancellation properties of β_k. Note that

$$L_k(\mathbb{E}_N g)(x) = \int \beta_k(x-y)\, \mathbb{E}_N g(y)\, dy,$$

and $\operatorname{supp}\beta_k(x-\cdot) \subset x + 2^{-k}[-1/2,1/2]^d$. So, if $I \in \mathscr{D}_N$ and $x \in I \cap \mathcal{U}^{\complement}_{N,k}$, then $\operatorname{supp}\beta_k(x-\cdot) \subset I$, and hence

$$L_k(\mathbb{E}_N g)(x) = (\mathbb{E}_N g)_{|_I}(x) \int_I \beta_k(x-y)\, dy \;=\; 0.$$

(ii) One argues similarly. First note that, changing the order of integration,

$$\mathbb{E}_N(L_j f) = \sum_{I\in\mathscr{D}_N} \int_{\mathbb{R}^d} f(y)\Big[\int_I \beta_j(x-y)\, dx\Big]\, dy\, \frac{\mathbb{1}_I}{|I|}. \tag{32}$$

Now if $J \in \mathscr{D}_N$ and $y \in J \cap \mathcal{U}^{\complement}_{N,k}$ then $\operatorname{supp}\beta_j(\cdot - y) \subset J$, and hence $\int_J \beta_j(x-y)\,dx = 0$. Thus $\mathbb{E}_N[L_j(\mathbb{1}_{\mathcal{U}^{\complement}_{N,j}} f)] = 0$. Finally, to prove (31) note that, if $I \in \mathscr{D}_N$

and $x \in I$, then from (32) it follows

$$\begin{aligned}\left|\mathbb{E}_N(L_j f)(x)\right| &= |I|^{-1}\left|\sum_{J\in\mathscr{D}_{j+1}[\partial I]}\int_J f(y)\Big[\int_I \beta_j(x-y)\,dx\Big]\,dy\right|\\ &\le 2^{Nd}\sum_{J\in\mathscr{D}_{j+1}[\partial I]}\|f\|_{L^\infty(J)}2^{-(j+1)d}\|\beta_j\|_1,\end{aligned}$$

which gives the asserted (31).

□

Proof of Proposition 2.1

Proof of (26) *in the Case* $j, k \ge N+1$

By Lemma 2.3. i, $L_k\mathbb{E}_N[L_j f](x) = 0$ if $x \in \mathcal{U}_{N,K}^{\complement}$, so we assume that $x \in \mathcal{U}_{N,k}\cap I$, for some $I \in \mathscr{D}_N$. Recall that $\mathscr{D}_N(I)$ consists of the neighboring cubes of I. Then (31) and the support property of β_k give

$$\begin{aligned}|L_k\mathbb{E}_N[L_j f](x)| &\le \int |\beta_k(x-y)|\left|\mathbb{E}_N(L_j f)(y)\right|dy\\ &\lesssim 2^{(N-j)d}\sum_{I'\in\mathscr{D}_N(I)}\sum_{J\in\mathscr{D}_{j+1}[\partial I']}\|f\|_{L^\infty(J)}\,\|\beta_k\|_1.\end{aligned}$$

Hence

$$\begin{aligned}\|L_k\mathbb{E}_N[L_j f]\|_p &= \Big[\sum_{I\in\mathscr{D}_N}\int_{I\cap\mathcal{U}_{N,k}}|L_k(\mathbb{E}_N L_j f)|^p\,dx\Big]^{\frac{1}{p}}\\ &\lesssim 2^{(N-j)d}\Big[\sum_{I\in\mathscr{D}_N}\Big(\sum_{J\in\mathscr{D}_{j+1}[\partial I]}\|f\|_{L^\infty(J)}\Big)^p\,|I\cap\mathcal{U}_{N,k}|\Big]^{\frac{1}{p}}. \quad (33)\end{aligned}$$

Now, $|I\cap\mathcal{U}_{N,k}| \approx 2^{-k}2^{-N(d-1)}$, and card $\mathscr{D}_{j+1}[\partial I] \approx 2^{(j-N)(d-1)}$. Also, if we write $J = 2^{-j-1}(\ell_J + [0,1]^d)$, then

$$\|f\|_{L^\infty(J)} \le \inf_{|h|_\infty\le 2^{-j-1}}\mathfrak{M}_j^* f(\ell_J+h) \le \Big[2^{jd}\int_{|h|_\infty\le 2^{-j-1}}\mathfrak{M}_j^* f(\ell_J+h)^p\,dh\Big]^{\frac{1}{p}}.$$

Therefore, using either Hölder's inequality (if $p>1$), or the embedding $\ell^p \hookrightarrow \ell^1$ (if $p\le 1$), we have

$$\Big[\sum_{I\in\mathscr{D}_N}\Big(\sum_{J\in\mathscr{D}_{j+1}[\partial I]}\|f\|_{L^\infty(J)}\Big)^p\Big]^{\frac{1}{p}}$$

$$\lesssim 2^{(j-N)(d-1)(1-\frac{1}{p})_+}\Big[\sum_{I\in\mathscr{D}_N}\sum_{J\in\mathscr{D}_{j+1}[\partial I]}\|f\|^p_{L^\infty(J)}\Big]^{\frac{1}{p}}$$

$$\lesssim 2^{(j-N)(d-1)(1-\frac{1}{p})_+}\Big[\sum_{J\in\mathscr{D}_{j+1}} 2^{jd}\int_{|h|_\infty\le 2^{-j-1}}\mathfrak{M}^*_j f(\ell_J+h)^p\,dh\Big]^{\frac{1}{p}}$$

$$\lesssim 2^{(j-N)(d-1)(1-\frac{1}{p})_+}\,2^{\frac{jd}{p}}\,\big\|\mathfrak{M}^*_j f\big\|_{L^p(\mathbb{R}^d)}\,. \tag{34}$$

Finally, inserting (34) into (33), and using (23), yields

$$\begin{aligned}\|L_k\mathbb{E}_N[L_j f]\|_p &\lesssim 2^{(N-j)d}2^{(j-N)(d-1)(1-\frac{1}{p})_+}\,2^{\frac{jd}{p}}\,\|f\|_p\,2^{-\frac{k}{p}}2^{-\frac{N(d-1)}{p}}\\ &= 2^{N-j}\,2^{\frac{j-k}{p}}\,2^{(j-N)(d-1)(\frac{1}{p}-1)_+}\,\|f\|_p\,,\end{aligned}$$

using in the last step the trivial identity $(1-\frac{1}{p})_+ = (\frac{1}{p}-1)_+-(\frac{1}{p}-1)$. This establishes (26) for $j,k\ge N+1$. □

Proof of (27) *in the Case* $j\le N$, $k\ge N+1$

For $w\in I$ with $I\in\mathscr{D}_N$ we have

$$\begin{aligned}&\big|\mathbb{E}^\perp_N(L_j f)(w)\big| = \big|\mathbb{E}_N[L_j f](w)-L_j f(w)\big|\\ &\quad= 2^{Nd}\Big|\int_I\int_{\mathbb{R}^d}2^{jd}\big[\beta(2^j(v-y))-\beta(2^j(w-y))\big]f(y)dy\,dv\Big|\\ &\quad= 2^{(N+j)d}\Big|\int_I\int_{\mathbb{R}^d}\int_0^1\nabla\beta(2^j[(1-t)w+tv-y])\cdot 2^j(v-w)\,dt\,f(y)\,dy\,dv\Big|\\ &\quad\le 2^{(N+j)d}2^{j-N}\int_I\int_0^1\int_{\mathbb{R}^d}|f(y)|\,|\nabla\beta(2^j[(1-t)w+tv-y])|\,dy\,dt\,dv\\ &\quad\lesssim 2^{j-N}\,\mathfrak{M}_j f(w),\end{aligned}$$

since for fixed w,v,t the expression involving $\nabla\beta$ is supported in the set $\{y\mid |y-w|_\infty\le 2^{-j-1}+2^{-N}\}$. Moreover, since $k>N$, when $w\in I_{N,\mu}$ and $|z|_\infty\le 2^{-k-1}$ we have

$$\big|\mathbb{E}^\perp_N[L_j f](w-z)\big|\lesssim 2^{j-N}\inf_{|h|_\infty\le 2^{-j}}\mathfrak{M}^*_j f(2^{-N}\mu+h)\,, \tag{35}$$

and therefore,

$$\begin{aligned}\Big|L_k\left(\mathbb{E}^\perp_N[L_j f]\right)(w)\Big| &\le \int|\mathbb{E}^\perp_N(w-z)|\,|\beta_k(z)|\,dz\\ &\lesssim 2^{j-N}\Big[\fint_{|h|_{\infty\le 2^{-j}}}\mathfrak{M}^*_j f(2^{-N}\mu+h)^p\,dh\Big]^{\frac{1}{p}}.\end{aligned} \tag{36}$$

Now Lemma 2.3. i gives

$$\begin{aligned}&\big\|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)\big\|_p\\&\lesssim \|L_k L_j f\|_{L^p(\mathcal{U}_{N,k}^{\complement})}+\Big[\sum_{\mu\in\mathbb{Z}^d}\|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)\|^p_{L^p(\mathcal{U}_{N,k}\cap I_{N,\mu})}\Big]^{\frac1p}.\end{aligned} \tag{37}$$

Using (36), the last term is controlled by

$$\begin{aligned}&2^{j-N}\Big[\sum_{\mu\in\mathbb{Z}^d}|I_{N,\mu}\cap\mathcal{U}_{N,k}|\fint_{|h|_\infty\le 2^{-j}}\mathfrak{M}_j^* f(2^{-N}\mu+h)^p\,dh\Big]^{\frac1p}\\&\quad\lesssim 2^{j-N}\,[2^{-k}2^{-N(d-1)}]^{\frac1p}\,2^{\frac{Nd}{p}}\,\|\mathfrak{M}_j^* f\|_p\;\lesssim\;2^{j-N}\,2^{\frac{N-k}{p}}\,\|f\|_p.\end{aligned}$$

Finally, the first term in (37) is controlled by Lemma 2.2, so overall one obtains

$$\big\|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)\big\|_p\lesssim\Big[2^{-(M-A)|k-j|}+2^{j-N}\,2^{\frac{N-k}{p}}\Big]\,\|f\|_p,$$

establishing (27) in the case $j\le N,\ k\ge N+1$. □

Proof of (27) *in the Case* $0\le j,k\le N$

We use

$$\int_I \mathbb{E}_N^{\perp}[L_j f](y)\,dy=0,\quad I\in\mathscr{D}_N,$$

to write

$$L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)(x)=\sum_{\mu}\int_{I_{N,\mu}}\big(\beta_k(x-y)-\beta_k(x-2^{-N}\mu)\big)\,\mathbb{E}_N^{\perp}[L_j f](y)\,dy\,.$$

For fixed x, we say that

$$\mu\in\Lambda(x)\text{ if }|x-2^{-N}\mu|_\infty\le 2^{-N}+2^{-k-1}. \tag{38}$$

Observe that only these μ's contribute to the above sum. Notice also that

$$|\beta_k(x-y)-\beta_k(x-2^{-N}\mu)|\lesssim 2^{kd}\,2^{k-N},\quad\text{if } y\in I_{N,\mu},$$

and since $j\le N$, the estimate in (35) gives

$$\big|\mathbb{E}_N^{\perp}[L_j f](y)\big|\lesssim 2^{j-N}\inf_{|h|_\infty\le 2^{-j}}\mathfrak{M}_j^* f(2^{-N}\mu+h)\,,\quad y\in I_{N,\mu}.$$

Combining all these bounds we obtain

$$
\begin{aligned}
|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)(x)| &\lesssim 2^{(k-N)(d+1)}2^{j-N}\sum_{\mu\in\Lambda(x)}\Big(\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^* f]^p\Big)^{\frac{1}{p}}\\
&\lesssim 2^{(k-N)(d+1)}2^{j-N}2^{(N-k)d(1-\frac{1}{p})_+}\Big(\sum_{\mu\in\Lambda(x)}\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^* f]^p\Big)^{\frac{1}{p}},
\end{aligned}
$$

using in the last step Hölder's inequality (or $\ell^p \hookrightarrow \ell^1$ if $p \le 1$) and the fact that $\operatorname{card}\Lambda(x) \approx 2^{(N-k)d}$. Observe also that the L^p-quasinorm of the last bracketed expression satisfies

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^d}\sum_{\mu\in\Lambda(x)}\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^* f]^p\Big)^{\frac{1}{p}} &\approx \Big(\sum_{\mu\in\mathbb{Z}^d}2^{-kd}\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^* f]^p\Big)^{\frac{1}{p}}\\
&\lesssim 2^{(N-k)d/p}\,\|\mathfrak{M}_j^* f\|_p.
\end{aligned}
$$

Thus, overall we obtain

$$
\begin{aligned}
\big\|L_k\mathbb{E}_N^{\perp}[L_j f]\big\|_p &\lesssim 2^{(k-N)(d+1)}2^{j-N}2^{(N-k)d(1-\frac{1}{p})_+}\,2^{(N-k)d/p}\,\|f\|_p\\
&= 2^{k-N}2^{j-N}2^{(N-k)d(\frac{1}{p}-1)_+}\,\|f\|_p,
\end{aligned}
$$

after simplifying the indices in the last step. This establishes (27) in the case $0 \le j, k \le N$. □

Proof of (26) *in the Case* $j \ge N+1$, $k \le N$

This condition and (30) in Lemma 2.3 imply that $\mathbb{E}_N[L_j f] = \mathbb{E}_N[L_j(f\mathbb{1}_{\mathcal{U}_{N,j}})]$. For simplicity, we denote $\widetilde{f} = f\mathbb{1}_{\mathcal{U}_{N,j}}$, and write

$$
L_k\mathbb{E}_N[L_j f] = L_k(\mathbb{E}_N[L_j\widetilde{f}] - L_j\widetilde{f}) + L_kL_j\widetilde{f}. \tag{39}
$$

Observe that, by Lemma 2.2,

$$
\|L_kL_j\widetilde{f}\|_p \lesssim 2^{-(M-A)|j-k|}\,\big\|\mathfrak{M}_{A,j}^{**}f(x)\big\|_p \lesssim 2^{-(M-A)|j-k|}\,\|f\|_p.
$$

So, we only need to estimate $\|L_k\mathbb{E}_N^{\perp}[L_j\widetilde{f}]\|_p$. Proceeding as in the proof of the case $j, k \le N$, we write (with $\Lambda(x)$ as in (38))

$$
\begin{aligned}
&|L_k\big(\mathbb{E}_N^{\perp}[L_j\widetilde{f}]\big)(x)|\\
&\quad\le \sum_{\mu\in\Lambda(x)}\int_{I_{N,\mu}}\big|\beta_k(x-y)-\beta_k(x-2^{-N}\mu)\big|\,\big|\mathbb{E}_N^{\perp}[L_j\widetilde{f}](y)\big|\,dy
\end{aligned}
$$

$$\lesssim 2^{kd}2^{k-N}\sum_{\mu\in\Lambda(x)}\int_{I_{N,\mu}}\Big(|\mathbb{E}_N[L_j\widetilde{f}]|+|L_j(\widetilde{f})|\Big)$$
$$=\mathcal{A}_1(x)+\mathcal{A}_2(x). \tag{40}$$

Now, using again (31), we have

$$|\mathcal{A}_1(x)|\lesssim 2^{(k-N)(d+1)}2^{(N-j)d}\sum_{\mu\in\Lambda(x)}\sum_{J\in\mathscr{D}_{j+1}[\partial I_{N,\mu}]}\|f\|_{L^\infty(J)}$$
$$\lesssim 2^{k-N}2^{(k-j)d}\,2^{(N-k)d(1-\frac{1}{p})_+}\Big[\sum_{\mu\in\Lambda(x)}\Big(\sum_{J\in\mathscr{D}_{j+1}[\partial I_{N,\mu}]}\|f\|_{L^\infty(J)}\Big)^p\Big]^{\frac{1}{p}}, \tag{41}$$

since card $\Lambda(x)\approx 2^{(N-k)d}$. Taking the L^p-quasinorm of the last bracketed expression gives

$$\Big[\int_{x\in\mathbb{R}^d}\sum_{\mu\in\Lambda(x)}\Big(\sum_{J\in\mathscr{D}_{j+1}[\partial I_{N,\mu}]}\|f\|_{L^\infty(J)}\Big)^p\,dx\Big]^{\frac{1}{p}}$$
$$\lesssim\Big[\sum_{I\in\mathscr{D}_N}2^{-kd}\Big(\sum_{J\in\mathscr{D}_{j+1}[\partial I]}\|f\|_{L^\infty(J)}\Big)^p\Big]^{\frac{1}{p}}$$
$$\lesssim 2^{\frac{(j-k)d}{p}}\,2^{(j-N)(d-1)(1-\frac{1}{p})_+}\,\big\|\mathfrak{M}_j^*f\big\|_{L^p(\mathbb{R}^d)}\quad\text{by (34)}. \tag{42}$$

Therefore, combining exponents in (41) and (42) one obtains

$$\|\mathcal{A}_1\|_p\lesssim 2^{k-N}2^{(k-j)d}\,2^{(N-k)d(1-\frac{1}{p})_+}\,2^{\frac{(j-k)d}{p}}\,2^{(j-N)(d-1)(1-\frac{1}{p})_+}\,\|f\|_p$$
$$=2^{k-j}\,2^{\frac{j-N}{p}}\,2^{(N-k)(\frac{1}{p}-1)_+}\,2^{(j-k)(d-1)(\frac{1}{p}-1)_+}\,\|f\|_p. \tag{43}$$

Finally, we estimate the term $\mathcal{A}_2(x)$ in (40). First notice that

$$|L_j(\widetilde{f})(y)|\le\int_{\mathcal{U}_{N,j}}|\beta_j(y-z)||f(z)|\,dz=0,\quad\text{if } y\in\mathcal{U}^{\complement}_{N,j-1},$$

since supp$\beta_j(y-\cdot)\subset y+2^{-j}[-\frac{1}{2},\frac{1}{2}]^d\subset\mathcal{U}^{\complement}_{N,j}$. Moreover, if $I\in\mathscr{D}_N$, then for every cube $J\in\mathscr{D}_j$ such that $J\subset I\cap\mathcal{U}_{N,j-1}$ we have

$$|L_j(\widetilde{f})(y)|\le\int|\beta_j(z)||f(y-z)|\,dz\lesssim\|f\|_{L^\infty(J^*)},\quad\text{if } y\in J$$

where $J^* = J + 2^{-j}[-\frac{1}{2}, \frac{1}{2}]^d$. Therefore,

$$\int_I |L_j(\widetilde{f})(y)| \lesssim \sum_{J \in \mathscr{D}_j[\partial I]} \|f\|_{L^\infty(J^*)} |J|,$$

and overall we obtain

$$|\mathcal{A}_2(x)| \lesssim 2^{(k-j)d} 2^{k-N} \sum_{\mu \in \Lambda(x)} \sum_{J \in \mathscr{D}_{j-1}[\partial I_{N,\mu}]} \|f\|_{L^\infty(J)}.$$

But this is essentially the same expression we obtained in (41) for the term $|\mathcal{A}_1(x)|$, so the same argument will give an estimate of $\|\mathcal{A}_2\|_p$ in terms of the quantity in (43). This concludes the proof of (26) in the case $j \geq N+1$, $k \leq N$.

Finally, concerning (iii) in Proposition 2.1, we remark that the previous proofs can easily be adapted replacing the operators $\mathbb{E}_N$ and $\mathbb{E}_N^\perp$ by $T_N[\cdot, a]$, keeping in mind that $T_N[g, a]$ is now constant in cubes $I \in \mathscr{D}_{N+1}$, and enjoys an additional cancellation, $\int_{I_{N,\mu}} T_N[g, a](x)dx = 0$, which simplifies some of the previous steps. □

Proof of Theorem 1.2, Conclusion

It remains to prove inequalities (19) and (20). By the embedding properties for the sequence spaces ℓ^r it suffices to verify both inequalities for very small r, say

$$r \leq \min\{p, 1\}. \tag{44}$$

In view of the embedding $\ell^r \hookrightarrow \ell^1$ and Minkowski's inequality (in $L^{p/r}$) it suffices then to prove

$$\sup_N \Big(\sum_{k=0}^\infty 2^{ksr} \sum_{j=N+1}^\infty \big\| L_k \mathbb{E}_N L_j \Lambda_j f \big\|_p^r \Big)^{1/r} \lesssim \sup_j 2^{js} \|\Lambda_j f\|_p \tag{45}$$

and

$$\sup_N \Big(\sum_{k=0}^\infty 2^{ksr} \sum_{j=0}^N \big\| L_k (\mathbb{E}_N^\perp L_j \Lambda_j f) \big\|_p^r \Big)^{1/r} \lesssim \sup_j 2^{js} \|\Lambda_j f\|_p . \tag{46}$$

If we apply Proposition 2.1 to each of the functions $\Lambda_j f$, we reduce matters to observe that

$$\sup_N \sum_{k=0}^\infty 2^{ksr} \sum_{j=0}^\infty \big[2^{-js} B(j, k, N)\big]^r < \infty, \tag{47}$$

with $B(j, k, N)$ as in (25), and that

$$\Big(\sum_{j=N+1}^\infty \sum_{k=0}^N + \sum_{k=N+1}^\infty \sum_{j=0}^N \Big) 2^{-|j-k|(M-A)} < \infty$$

which is trivial. The verification of (47) under the assumptions in (11) is also elementary, but we carry out some details to clarify how the conditions on p and s are used.

When $j, k > N$, we have $B(j,k,N) = 2^{N-j}\, 2^{\frac{j-k}{p}}\, 2^{(j-N)(d-1)(\frac{1}{p}-1)_+}$ and thus

$$\sum_{k>N} 2^{ksr} \sum_{j>N} \left[2^{-js} B(j,k,N)\right]^r$$
$$= \Big(\sum_{k>N} 2^{-kr(\frac{1}{p}-s)}\Big)\Big(\sum_{j>N} 2^{-rj[s+1-\frac{1}{p}-(d-1)(\frac{1}{p}-1)_+]}\Big)\, 2^{Nr[1-(d-1)(\frac{1}{p}-1)_+]}, \quad (48)$$

and the series converge provided $s < 1/p$ and

$$s > \tfrac{1}{p} - 1 + (d-1)(\tfrac{1}{p}-1)_+ \;=\; \max\Big\{d(\tfrac{1}{p}-1),\, \tfrac{1}{p}-1\Big\}. \quad (49)$$

Further, being geometric sums, the final outcome in (48) is bounded uniformly in N.

Next assume $j \le N < k$, then $B(j,k,N) = 2^{\frac{N-k}{p}} 2^{j-N}$ and hence

$$\sum_{k>N} 2^{ksr} \sum_{j\le N} \left[2^{-js} B(j,k,N)\right]^r = \Big(\sum_{k>N} 2^{-kr(\frac{1}{p}-s)}\Big)\Big(\sum_{j\le N} 2^{rj(1-s)}\Big)\, 2^{Nr(\frac{1}{p}-1)},$$

which are finite expressions provided $s < \min\{1, 1/p\}$.

Consider $j, k \le N$, with $B(j,k,N) = 2^{k-N} 2^{j-N} 2^{(N-k)d(\frac{1}{p}-1)_+}$. Then

$$\sum_{k\le N} 2^{ksr} \sum_{j\le N} \left[2^{-js} B(j,k,N)\right]^r =$$

$$= \Big(\sum_{k\le N} 2^{kr[s+1-d(\frac{1}{p}-1)_+]}\Big)\Big(\sum_{j\le N} 2^{rj(1-s)}\Big)\, 2^{-Nr[2-d(\frac{1}{p}-1)_+]},$$

which leads to uniform expressions in N under the assumptions $s < 1$ and

$$s > d(\tfrac{1}{p}-1)_+ - 1, \quad (50)$$

the latter being weaker than (49).

When $k \le N < j$ we have $B(j,k,N) = 2^{k-j+\frac{j-N}{p}+[N-k+(j-k)(d-1)](\frac{1}{p}-1)_+}$ and

$$\sum_{k\le N} 2^{ksr} \sum_{j>N} \left[2^{-js} B(j,k,N)\right]^r =$$

$$= \Big(\sum_{k\le N} 2^{kr[s+1-d(\frac{1}{p}-1)_+]}\Big)\Big(\sum_{j>N} 2^{-rj[s+1-\frac{1}{p}-(d-1)(\frac{1}{p}-1)_+]}\Big)\, 2^{-Nr[\frac{1}{p}-(\frac{1}{p}-1)_+]},$$

where in the first series we would use (50) and in the second series (49). We have verified (47) in all cases. This finishes the proof of Theorem 1.2. □

3 Schauder Bases

Let P_N be defined as in (3). For the proof of Theorem 1.1 we need to prove that $\|P_N f - f\|_{F^s_{p,q}} \to 0$ for every $f \in F^s_{p,q}$, with (p, s) as in (11) and $0 < q < \infty$. We first discuss some preliminaries about localization and pointwise multiplication by characteristic functions of cubes, then prove uniform bounds for the $F^s_{p,q} \to F^s_{p,q}$ operator norms of the P_N and then establish the asserted limiting property.

Preliminaries

For $\nu \in \mathbb{Z}^d$ let χ_ν be the characteristic function of $\nu + [0, 1)^d$.

Lemma 3.1 *Assume that*

$$\tfrac{d-1}{d} < p < \infty, \quad 0 < q \le \infty, \quad \textit{and} \quad \max\{d(\tfrac{1}{p} - 1), \tfrac{1}{p} - 1\} < s < \tfrac{1}{p}. \tag{51}$$

Then, the following holds for all g_ν and $f \in F^s_{p,q}$:

$$\Big\| \sum_{\nu \in \mathbb{Z}^d} \chi_\nu g_\nu \Big\|_{F^s_{p,q}} \lesssim \Big(\sum_{\nu \in \mathbb{Z}^d} \|g_\nu\|^p_{F^s_{p,q}} \Big)^{1/p}$$

and

$$\Big(\sum_{\nu \in \mathbb{Z}^d} \|f \chi_\nu\|^p_{F^s_{p,q}} \Big)^{1/p} \lesssim \|f\|_{F^s_{p,q}} .$$

Proof Let $\varsigma \in C_c^\infty(\mathbb{R}^d)$ so that $\mathrm{supp}(\varsigma) \subset (-1, 1)^d$ and $\sum_{\nu \in \mathbb{Z}^d} \varsigma(x - \nu) = 1$ for all $x \in \mathbb{R}^d$. Let $\varsigma_\nu = \varsigma(\cdot - \nu)$. We have, for all $s \in \mathbb{R}$,

$$\|g\|_{F^s_{p,q}} \asymp \Big(\sum_\nu \|\varsigma_\nu g\|^p_{F^s_{p,q}} \Big)^{1/p} ; \tag{52}$$

see [10, 2.4.7]. Hence

$$\begin{aligned} \Big\| \sum_{\nu \in \mathbb{Z}^d} \chi_\nu g_\nu \Big\|_{F^s_{p,q}} &= \Big\| \sum_{\nu'} \varsigma_{\nu'} \sum_\nu \chi_\nu g_\nu \Big\|_{F^s_{p,q}} \lesssim \Big(\sum_{\nu'} \Big\| \varsigma_{\nu'} \sum_{|\nu - \nu'|_\infty \le 1} \chi_\nu g_\nu \Big\|^p_{F^s_{p,q}} \Big)^{1/p} \\ &\lesssim \Big(\sum_{\nu'} \sum_{|\nu - \nu'|_\infty \le 1} \|g_\nu\|^p_{F^s_{p,q}} \Big)^{1/p} \lesssim \Big(\sum_\nu \|g_\nu\|^p_{F^s_{p,q}} \Big)^{1/p} . \end{aligned}$$

Here we have used that $\varsigma_{\nu'}\chi_\nu$ are pointwise multipliers of $F^s_{p,q}$, with uniform bounds in (ν,ν'), in the range given by (51); see [4, Thm. 4.6.3/1]. This proves the first inequality.

For the second inequality we first observe that, by (52),

$$\|f\chi_\nu\|_{F^s_{p,q}} \lesssim \Big(\sum_{\nu'}\|f\chi_\nu\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p}, \quad \nu\in\mathbb{Z}^d,$$

which yields

$$\begin{aligned}\Big(\sum_\nu\|f\chi_\nu\|^p_{F^s_{p,q}}\Big)^{1/p} &\lesssim \Big(\sum_\nu\sum_{\nu'}\|f\chi_\nu\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p}\\ &\lesssim \Big(\sum_{\nu'}\sum_{|\nu-\nu'|_\infty\le 1}\|f\chi_\nu\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p}\\ &\lesssim \Big(\sum_{\nu'}\|f\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p} \lesssim \|f\|_{F^s_{p,q}},\end{aligned}$$

where we have used the pointwise multiplier assertion [4, Thm. 4.6.3/1] and then again (52) in the last step. □

Uniform Boundedness of the P_N

Observe that by the localization property of the Haar functions we have $P_N f = \sum_{\nu\in\mathbb{Z}^d}\chi_\nu P_N f = \sum_\nu \chi_\nu P_N[f\chi_\nu]$. Thus by Lemma 3.1

$$\|P_N f\|_{F^s_{p,q}} \lesssim \Big(\sum_\nu\big\|P_N[f\chi_\nu]\big\|^p_{F^s_{p,q}}\Big)^{1/p}.$$

Since the enumeration of the Haar system is assumed to be admissible we have

$$P_N[f\chi_\nu] = \mathbb{E}_{N_\nu}[f\chi_\nu] + T_{N_\nu}[f\chi_\nu, a^{N,\nu}] \tag{53}$$

for some $N_\nu\in\mathbb{N}$, with $N_\nu\le N$ and appropriate sequences $a^{N,\nu}$ assuming only the values 1 and 0. We remark that for each ν, $N_\nu = N_\nu(N)$ with

$$\lim_{N\to\infty} N_\nu(N) = \infty\,. \tag{54}$$

By Theorem 1.2

$$\begin{aligned}&\Big(\sum_\nu\big\|P_N[f\chi_\nu]\big\|^p_{F^s_{p,q}}\Big)^{1/p}\\ &\lesssim \Big(\sum_\nu\big\|\mathbb{E}_{N_\nu}[f\chi_\nu]\big\|^p_{F^s_{p,q}}\Big)^{1/p} + \Big(\sum_\nu\big\|T_{N_\nu}[f\chi_\nu, a^{N,\nu}]\big\|^p_{F^s_{p,q}}\Big)^{1/p}\end{aligned}$$

$$\lesssim \Big(\sum_{\nu} \| f \chi_\nu \|^p_{F^s_{p,q}} \Big)^{1/p} \lesssim \| f \|_{F^s_{p,q}} ,$$

where for the last inequality we have used Lemma 3.1 again.

Proof (Proof of Theorem 1.1, Conclusion) Let $f \in F^s_{p,q}$, with (p, s) as in (11) and $0 < q < \infty$. Let $C = \max\{1, \sup_N \|P_N\|_{F^s_{p,q} \to F^s_{p,q}}\}$. Since Schwartz functions are dense in $F^s_{p,q}$ when $0 < p, q < \infty$ there is $\tilde{f} \in \mathcal{S}(\mathbb{R})$ such that $\|f - \tilde{f}\|_{F^s_{p,q}} < (3C)^{-1}\epsilon$ and hence $\|P_N f - P_N \tilde{f}\|_{F^s_{p,q}} < \epsilon/3$. Choose s_1 so that $s < s_1 < \max\{1/p, 1\}$ then $\tilde{f} \in B^{s_1}_{p,q} \hookrightarrow F^s_{p,q}$. Since the Haar system is an unconditional basis on $B^{s_1}_{p,q}$ ([11]) we have $\lim_{N\to\infty} \|P_N \tilde{f} - \tilde{f}\|_{B^{s_1}_{p,q}} = 0$ and therefore $\lim_{N\to\infty} \|P_N \tilde{f} - \tilde{f}\|_{F^s_{p,q}} = 0$. Combining these facts we get $\|P_N f - f\|_{F^s_{p,q}} < \epsilon$ for sufficiently large N which shows that $P_N f \to f$ in $F^s_{p,q}$. □

4 Optimality Away from the End-Points

Proposition 4.1 *Let $0 < q < \infty$. Then, the Haar system $\mathscr{H}_d$ is not a Schauder basis of $F^s_{p,q}(\mathbb{R}^d)$ in each of the following cases:*

(i) if $1 < p < \infty$ and $s \geq 1/p$ or $s < 1/p - 1$,
(ii) if $d/(d+1) \leq p \leq 1$ and $s > 1$ or $s < d(1/p - 1)$,
(iii) if $0 < p < d/(d+1)$ and $s \in \mathbb{R}$.

The same result for the spaces $B^s_{p,q}(\mathbb{R}^d)$ was proved by Triebel in [8]; see also [11, Proposition 2.24]. Proposition 4.1 can be obtained from this and Theorem 1.1 by suitable interpolation.

Indeed, assertion (i) was already discussed in the paragraph following (4), so we restrict to $p \leq 1$. Assume next that $\mathscr{H}_d$ is a basis for $F^s_{p,q}$ for some $d/(d+1) < p < 1$ and $s > 1$ or $s < d(1/p - 1)$. By Theorem 1.1, $\mathscr{H}_d$ is also a basis for $F^{s_0}_{p,q}$, for any $d(1/p - 1) < s_0 < 1$. By real interpolation, see e.g. [9, Theorem 2.4.2(ii)], for all $0 < \theta < 1$, the system $\mathscr{H}_d$ will then be a basis of

$$\big(F^{s_0}_{p,q}, F^s_{p,q}\big)_{\theta,q} = B^{s_\theta}_{p,q}, \quad \text{with } s_\theta = (1-\theta)s_0 + \theta s.$$

But when θ is close to 1 this would contradict Triebel's result. The remaining cases, $p = 1$ and $p \geq d/(d+1)$ can be proved similarly using complex interpolation of F-spaces; see [10, 1.6.7].

We remark that, in the paper [8], the failure of the Schauder basis property in the B-spaces is sometimes due to the fact that span $\mathscr{H}_d$ fails to be dense in $B^s_{p,q}$. This is the case, for instance, in the region

$$(d-1)/d < p < 1 \quad \text{and} \quad \max\big\{1, d(1/p - 1)\big\} < s < 1/p; \tag{55}$$

see [8, Corollary 2]. Here we show that also a quantitative bound holds, therefore ruling out the possibility that $\mathscr{H}_d$ could be a basic sequence.

Proposition 4.2 *Let* $0 < q \leq \infty$*, and* (p, s) *be as in* (55)*. Then,*

$$\|\mathbb{E}_N\|_{B^s_{p,q} \to B^s_{p,q}} \gtrsim 2^{(s-1)N}.$$

Proof Let $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta \equiv 1$ on $[-2, 2]^d$, and consider the Schwartz function $f(x) = x_1\,\eta(x)$. It suffices to show that

$$\big\|\mathbb{E}_N f\big\|_{B^s_{p,q}} \gtrsim 2^{(s-1)N}. \tag{56}$$

Under (55) we have $s > \sigma_p := d(1/p-1)_+$. Assume first that $s < 2$ (which is always the case if $d > 1$). Then we can use the equivalence of quasi-norms

$$\|g\|_{B^s_{p,q}(\mathbb{R}^d)} \approx \|g\|_p + \sum_{j=1}^{d} \Big(\int_0^1 \frac{\|\Delta^2_{he_j} g\|_p^q}{h^{sq}} \frac{dh}{h} \Big)^{1/q},$$

with the usual modification in the case $q = \infty$, see [10, 2.6.1]. In particular

$$\big\|\mathbb{E}_N f\big\|_{B^s_{p,q}} \gtrsim \left(\int_0^{2^{-N-1}} \frac{\big\|\Delta^2_{he_1}\big(\mathbb{E}_N f\big)\big\|^q_{L^p([0,1]^d)}}{h^{sq}} \frac{dh}{h} \right)^{1/q}. \tag{57}$$

Now, it is easily checked that, when $x \in [0, 1)^d$, one has

$$\mathbb{E}_N f = \sum_{0 \leq k < 2^N} \tfrac{k+1/2}{2^N}\, \mathbb{1}_{[\frac{k}{2^N}, \frac{k+1}{2^N}) \times [0,1)^{d-1}},$$

and likewise, if we additionally assume $0 < h < 2^{-N-1}$, then

$$\Delta_{he_1}\big(\mathbb{E}_N f\big) = 2^{-N-1} \sum_{k=1}^{2^N} \mathbb{1}_{[\frac{k}{2^N}-h, \frac{k}{2^N}) \times [0,1)^{d-1}}.$$

and

$$\Delta^2_{he_1}\big(\mathbb{E}_N f\big) = 2^{-N-1} \sum_{k=1}^{2^N} \Big[\mathbb{1}_{[\frac{k}{2^N}-2h, \frac{k}{2^N}-h) \times [0,1)^{d-1}} - \mathbb{1}_{[\frac{k}{2^N}-h, \frac{k}{2^N}) \times [0,1)^{d-1}} \Big].$$

Therefore,

$$\|\Delta^2_{he_1} \mathbb{E}_N f\|_{L^p([0,1]^d)} = 2^{(N+1)(1/p-1)}\, h^{1/p},$$

which, inserted into (57), gives (56). If $d = 1$ and $s \geq 2$, one applies a similar argument to the functions $\Delta^L_{he_1}(\mathbb{E}_N f)$ with $L = \lfloor s \rfloor + 1$ and $h < 2^{-N}/L$. □

By interpolation one obtains as well a quantitative bound for the relevant cases in Proposition 4.1(ii).

Corollary 4.3 *Let* $0 < q \leq \infty$*,* $d/(d+1) < p < 1$ *and* $1 < s < 1/p$*. Then, for all* $\varepsilon > 0$*,*

$$\|\mathbb{E}_N\|_{F^s_{p,q} \to F^s_{p,q}} \gtrsim c_\varepsilon\, 2^{(s-1-\varepsilon)N}. \tag{58}$$

Proof If $d(1/p-1) < s_0 < 1$ and $\theta \in (0,1)$, then the real interpolation inequalities give

$$\big\|\mathbb{E}_N\big\|^{1-\theta}_{F^{s_0}_{p,q} \to F^{s_0}_{p,q}} \, \big\|\mathbb{E}_N\big\|^{\theta}_{F^{s}_{p,q} \to F^{s}_{p,q}} \;\geq\; c_\theta\, \big\|\mathbb{E}_N\big\|_{B^{s_\theta}_{p,q} \to B^{s_\theta}_{p,q}},$$

with $s_\theta = (1-\theta)s_0 + \theta s$. By Proposition 4.2 the right hand side is larger than a constant times $2^{N(s_\theta - 1)}$, while by Corollary 1.3 we have $\big\|\mathbb{E}_N\big\|_{F^{s_0}_{p,q} \to F^{s_0}_{p,q}} \approx 1$. Choosing θ sufficiently close to 1 one derives (58). □

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