THE HAAR SYSTEM IN TRIEBEL-LIZORKIN SPACES: ENDPOINT RESULTS

GUSTAVO GARRIGÓS ANDREAS SEEGER TINO ULLRICH

Dedicated to Guido Weiss, with affection, on his 92nd birthday

Abstract. We characterize the Schauder and the unconditional basis properties for the Haar system in the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$, at the endpoint cases $s = 1$, $s = d/p - d$ and $p = \infty$. Together with the earlier results in [10, 4], this completes the picture for such properties in the Triebel-Lizorkin scale, and complements a similar study for the Besov spaces given in [5].

1. Introduction and statements of main results

In this paper we essentially complete the study of the basis properties for the (inhomogeneous) Haar system in the scale of Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$. In particular, we describe the behavior at the endpoint cases which was left open in our earlier work [4]. Similar endpoint questions for the family of Besov spaces have been presented in the companion paper [5]. We note that markedly different outcomes occur for each family, in both the non-endpoint situations ([12, 15, 10, 11, 4]) and the endpoint ([5], [6]) situations.

We now set the basic notation required to state the results. Given the one variable functions $h^{(0)} = \mathbb{1}_{[0,1)}$ and $h^{(1)} = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$, for each $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0,1\}^d$, $k \in \mathbb{N}_0$ and $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{Z}^d$, we define

$$h^\epsilon_{k,\nu}(x) := \prod_{i=1}^d h^{(\epsilon_i)}(2^k x_i - \nu_i), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Then, the *Haar system* is the collection of functions

$$\mathscr{H}_d = \Big\{ h^{\vec{0}}_{0,\nu} \Big\}_{\nu \in \mathbb{Z}^d} \cup \Big\{ h^\epsilon_{k,\nu} \;:\; k \in \mathbb{N}_0,\ \nu \in \mathbb{Z}^d,\ \epsilon \in \Upsilon \Big\},$$

where we denote $\Upsilon = \{0,1\}^d \setminus \{\vec{0}\}$.

Consider $F^s_{p,q}(\mathbb{R}^d)$ with the usual definition in [13, §2.3.1] or [3, §12]. To investigate the Schauder basis properties of $\mathscr{H}_d$, we initially assume that

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$0 < p, q < \infty$ (so that $\mathcal{S}$ is dense in $F^s_{p,q}$, and the latter is separable), and that

$$h^{\epsilon}_{k,\nu} \in F^s_{p,q} \quad \text{and} \quad h^{\epsilon}_{k,\nu} \in (F^s_{p,q})^*, \quad \forall\, \epsilon, k, \nu. \tag{1}$$

Given an enumeration $\mathcal{U} = \big\{u_n = h^{\epsilon(n)}_{k(n),\nu(n)}\big\}_{n=1}^{\infty}$ of $\mathscr{H}_d$, we consider the corresponding partial sum operators

$$S_R f = S^{\mathcal{U}}_R f = \sum_{n=1}^{R} u_n^*(f) u_n\,, \quad R \in \mathbb{N}, \tag{2}$$

where the linear functionals u_n^* are defined by

$$u_n^*(f) = 2^{k(n)d} \langle f, h^{\epsilon(n)}_{k(n),\nu(n)} \rangle\,, \quad f \in \mathcal{S}. \tag{3}$$

The condition in (1) ensures that these operators are well-defined and individually bounded in $F^s_{p,q}(\mathbb{R}^d)$. Also, $u_n^*(u_m) = \delta_{n,m}$, $n, m \geq 1$.

The basis properties of $\mathcal{U}$ are related to the validity of the bound

$$\sup_{R \in \mathbb{N}} \|S^{\mathcal{U}}_R\|_{F^s_{p,q} \to F^s_{p,q}} < \infty. \tag{4}$$

Indeed, if $\operatorname{span} \mathscr{H}_d$ is dense in $F^s_{p,q}$, then (4) is equivalent to $\mathcal{U}$ being a *Schauder basis* of $F^s_{p,q}$, that is

$$\lim_{R \to \infty} \|S^{\mathcal{U}}_R f - f\|_{F^s_{p,q}} = 0 \tag{5}$$

for every $f \in F^s_{p,q}$. Moreover, the basis is *unconditional* if and only if the bound in (4) is uniform in all enumerations $\mathcal{U}$. Finally, if $\operatorname{span} \mathscr{H}_d$ is not assumed to be dense, then (4) still implies that $\mathcal{U}$ is a *basic sequence* of $F^s_{p,q}$, meaning that (5) holds for all f in the $F^s_{p,q}$-closure of $\operatorname{span} \mathscr{H}_d$.

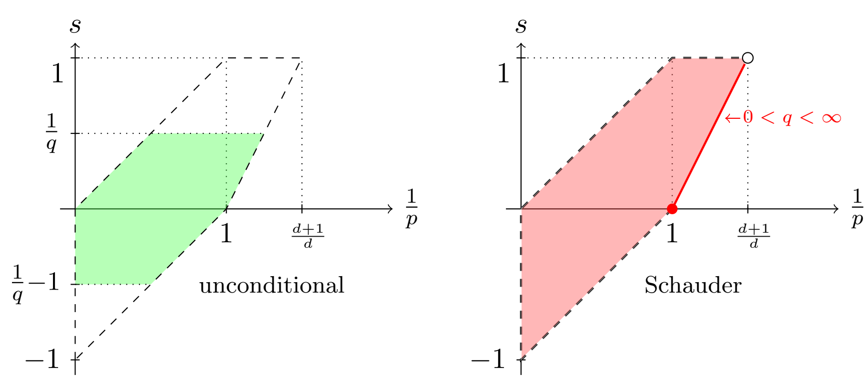

FIGURE 1. Parameter domain $\mathfrak{P}$ for $\mathscr{H}_d$ in $F^s_{p,q}(\mathbb{R}^d)$. The left region corresponds to unconditionality, and right region to the Schauder basis property.

The pentagon $\mathfrak{P}$ depicted in Figure 1 shows the natural index region for these problems; outside its closure either (1) or the density of span $\mathscr{H}_d$ fail. The open pentagon corresponds to the range $\frac{d}{d+1} < p < \infty$, $0 < q < \infty$, and

$$\max\left\{d(\tfrac{1}{p}-1), \tfrac{1}{p}-1\right\} < s < \min\left\{1, \tfrac{1}{p}\right\}. \tag{6}$$

Triebel showed in [15, Theorem 2.21] that $\mathscr{H}_d$ is an unconditional basis of $F^s_{p,q}(\mathbb{R}^d)$ in the green shaded region, where the additional restriction

$$\max\left\{d(\tfrac{1}{q}-1), \tfrac{1}{q}-1\right\} < s < \tfrac{1}{q} \tag{7}$$

is imposed. The necessity of condition (7) for unconditionality was established in [10, 11] (for $d = 1$). On the other hand, we recently showed in [4] that natural enumerations of $\mathscr{H}_d$ form a Schauder basis of $F^s_{p,q}(\mathbb{R}^d)$ in the full open pentagon $\mathfrak{P}$. Except for a few trivial cases, the behavior at the points $(1/p, s)$ lying in the boundary of $\mathfrak{P}$ was left unexplored.

In this paper we attempt to fill this gap giving an answer with a positive or negative outcome depending on the secondary index q. Moreover, when possible, the negative answer is replaced by a suitable basic sequence property.

We first state the complete range for unconditionality, which contains new negative cases and a multivariate extension of the examples in [10].

Theorem 1.1. *Let $0 < p, q < \infty$ and $s \in \mathbb{R}$. Then, $\mathscr{H}_d$ is an unconditional basis of $F^s_{p,q}(\mathbb{R}^d)$ if and only if the conditions* (6) *and* (7) *are both satisfied.*

In the next results we drop unconditionality, and consider the Schauder basis property for the following natural orderings of the Haar system $\mathscr{H}_d$; see [4, 5].

Definition 1.2. *(i)* An enumeration $\mathcal{U}$ is said to be *admissible* if for some constant $b \in \mathbb{N}$ the following holds: for each cube $I_\nu = \nu + [0, 1]^d$, $\nu \in \mathbb{Z}^d$, if u_n and $u_{n'}$ are both supported in I_ν and $|\operatorname{supp}(u_n)| \geq 2^{bd}|\operatorname{supp}(u_{n'})|$, then necessarily $n < n'$.

(ii) An enumeration $\mathcal{U}$ is *strongly admissible* if for some constant $b \in \mathbb{N}$ the following holds: for each cube I_ν, $\nu \in \mathbb{Z}^d$, if I^{**}_ν denotes the five-fold dilated cube with respect to its center, and if u_n and $u_{n'}$ are supported in I^{**}_ν with $|\operatorname{supp}(u_n)| \geq 2^{bd}|\operatorname{supp}(u_{n'})|$ then necessarily $n < n'$.

Our next theorem characterizes the Schauder basis property in $F^s_{p,q}$ for the class of strongly admissible enumerations of $\mathscr{H}_d$. A new positive result is obtained in the line $s = d/p - d$, when $\frac{d}{d+1} < p \leq 1$; see Figure 1. The special case $F^0_{1,2} = h^1$ is classical, and was established in [1, 17]. The negative results for $s = 1$ are also new.

Theorem 1.3. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then, the following statements are equivalent, i.e. (a) $\Longleftrightarrow$ (b):*

(a) Every strongly admissible enumeration $\mathcal{U}$ of $\mathscr{H}_d$ is a Schauder basis of $F^s_{p,q}(\mathbb{R}^d)$.

(b) One of the following three conditions is satisfied:

- *(i)* $1 < p < \infty, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \quad 0 < q < \infty,$
- *(ii)* $\frac{d}{d+1} < p \le 1, \quad \frac{d}{p} - d < s < 1, \quad 0 < q < \infty,$
- *(iii)* $\frac{d}{d+1} < p \le 1, \quad s = \frac{d}{p} - d, \quad 0 < q < \infty.$

As in [4, 5], a crucial tool in our analysis will be played by the *dyadic averaging operators* $\mathbb{E}_N$. That is, if $\mathscr{D}_N$ is the set of all dyadic cubes of length 2^{-N},

$$I_{N,\nu} = 2^{-N}(\nu + [0,1)^d), \quad \nu \in \mathbb{Z}^d,$$

then we define

$$\mathbb{E}_N f(x) = \sum_{\nu\in\mathbb{Z}^d} \mathbb{1}_{I_{N,\nu}}(x)\, 2^{Nd} \int_{I_{N,\nu}} f(y)dy\,, \tag{8}$$

at least for $f \in \mathcal{S}$. We shall also need the following companion operators involving Haar functions of a fixed frequency level. Namely, for $N \in \mathbb{N}$ and any $\mathfrak{a} = (a_{\nu,\varepsilon})_{\nu,\varepsilon} \in \ell^\infty(\mathbb{Z}^d \times \Upsilon)$ we set

$$T_N[f,\mathfrak{a}] = \sum_{\epsilon\in\Upsilon}\sum_{\nu\in\mathbb{Z}^d} a_{\nu,\epsilon} 2^{Nd} \langle f, h^\epsilon_{N,\nu}\rangle h^\epsilon_{N,\nu}. \tag{9}$$

For these operators one looks for estimates that are uniform in $\|\mathfrak{a}\|_\infty \le 1$.

The relation between the partial sums $S^{\mathcal{U}}_R$ and the operators $\mathbb{E}_N$ and $T_N[\cdot,\mathfrak{a}]$ is explained in §10 below; see also [4, 5]. In particular, their uniform boundedness in $F^s_{p,q}$ implies that (4) holds for all strongly admissible enumerations $\mathcal{U}$. The optimal region for the uniform boundedness for $\mathbb{E}_N$ and $T_N[\cdot,\mathfrak{a}]$ in $F^s_{p,q}$ is given in the next theorem, and depicted in Figure 2 below.

Theorem 1.4. *Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$.*

(a) The operators $\mathbb{E}_N$ admit an extension from $\mathcal{S}$ into $F^s_{p,q}(\mathbb{R}^d)$ such that

$$\sup_{N\ge0} \|\mathbb{E}_N\|_{F^s_{p,q}\to F^s_{p,q}} < \infty$$

if and only if one of the following five conditions is satisfied:

- *(i)* $1 < p \le \infty, \quad -1 + \frac{1}{p} < s < \frac{1}{p}, \quad 0 < q \le \infty,$
- *(ii)* $\frac{d}{d+1} \le p < 1, \quad s = 1, \quad 0 < q \le 2,$
- *(iii)* $\frac{d}{d+1} < p \le 1, \quad d(\frac{1}{p} - 1) < s < 1, \quad 0 < q \le \infty,$
- *(iv)* $\frac{d}{d+1} < p \le 1, \quad s = d(\frac{1}{p} - 1), \quad 0 < q \le \infty,$
- *(v)* $p = \infty, \quad s = 0, \quad 0 < q \le \infty.$

(b) If one of the conditions (i)-(v) is satisfied then the operators $T_N[\cdot,\mathfrak{a}]$ are uniformly bounded on $F^s_{p,q}(\mathbb{R}^d)$ when $N \ge 0$ and $\|\mathfrak{a}\|_{\ell^\infty(\mathbb{Z}^d\times\Upsilon)} \le 1$.

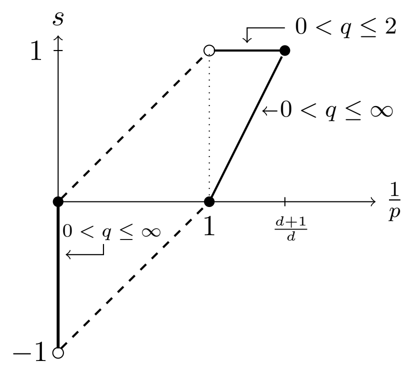

FIGURE 2. Region for uniform boundedness of $\mathbb{E}_N$ (hence for the *basic sequence* property) in the spaces $F^s_{p,q}(\mathbb{R}^d)$.

Regarding positive results, the cases (i) and (iii) were established in [4]. The novel cases appear at the end-point lines in (ii) and (iv), and the special point (v); see Figure 2.

The proof of (ii) will follow from a slightly stronger result which we state next. Let $\eta_0 \in C^\infty_c(\mathbb{R}^d)$ be supported in $\{|\xi| < 3/4\}$ with $\eta_0(\xi) = 1$ for $|\xi| \le 1/4$, and let $\varPi_N$ be defined by

$$\widehat{\varPi_N f}(\xi) = \eta_0(2^{-N}\xi)\widehat{f}(\xi). \tag{10}$$

Then we shall actually prove the following.

Theorem 1.5. *Let $d/(d+1) \le p < 1$ and $0 < r \le \infty$. Then*

$$\sup_N \|\mathbb{E}_N f - \varPi_N f\|_{B^1_{p,r}} \lesssim \|f\|_{F^1_{p,2}}. \tag{11}$$

Moreover, for $\|\mathfrak{a}\|_{\ell^\infty} \le 1$,

$$\sup_N \|T_N[f, \mathfrak{a}]\|_{B^1_{p,r}} \lesssim \|f\|_{F^1_{p,2}}. \tag{12}$$

Using the embeddings $F^s_{p,q} \subset F^s_{p,2}$ for $q \le 2$, and $B^s_{p,r} \subset F^s_{p,r} \subset F^s_{p,q}$ for $r \le \min\{p, q\}$, one deduces the uniform bounds in (ii) of Theorem 1.4.

Likewise, for the end-point cases in (iv) and (v) we shall establish the following stronger results.

Theorem 1.6. *Let $d/(d+1) < p \le 1$, $0 < r \le \infty$, and $s = \frac{d}{p} - d$. Then*

$$\sup_N \|\mathbb{E}_N f - \varPi_N f\|_{B^s_{p,r}} \lesssim \|f\|_{F^s_{p,\infty}}, \tag{13}$$

and likewise for the operators $T_N[\cdot, \mathfrak{a}]$, uniformly in $\|\mathfrak{a}\|_{\ell^\infty} \le 1$.

Theorem 1.7. *For every $r > 0$, it holds*

$$\sup_N \|\mathbb{E}_N f - \varPi_N f\|_{F^0_{\infty,r}} \lesssim \|f\|_{B^0_{\infty,\infty}}, \tag{14}$$

and likewise for the operators $T_N[\cdot, \mathfrak{a}]$, uniformly in $\|\mathfrak{a}\|_{\ell^\infty} \le 1$.

Finally, concerning the negative results in Theorem 1.4, the only non-trivial case appears when $s = 1$, for which we shall establish the following.

Theorem 1.8. *Let* $\frac{d}{d+1} \le p < 1$ *and* $2 < q \le \infty$. *Then,*

$$\|\mathbb{E}_N\|_{F^1_{p,q} \to F^1_{p,q}} \approx N^{\frac{1}{2} - \frac{1}{q}}.$$

This paper. In §2 we set the basic notation. In §3 and §4 we prove, respectively, Theorems 1.5 and 1.6, except for the special case $p = d/(d+1)$ which is treated in §5. Theorem 1.7 is shown in §6, and Theorem 1.8 in §7. In §8 we gather all these results and complete the proof of Theorem 1.4, explaining as well the meaning of the extensions of the operators $\mathbb{E}_N$ to the full spaces $F^s_{p,q}$. In §9 we study the failure of density for $\operatorname{span} \mathscr{H}_d$ in the case $s = 1$. In §10 and 11 we pass to the operators $S^{\mathcal{U}}_R$, showing their relation with $\mathbb{E}_N$ for admissible enumerations, and establishing Theorem 1.3. Finally, §12 is devoted to unconditionality, and the proof of Theorem 1.1.

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2. Preliminaries

2.1. *Besov and Triebel-Lizorkin quasi-norms.* Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$ be given. Throughout the paper we fix a number $A > d/p$ and an integer

$$M > A + |s| + 2. \tag{15}$$

Consider two functions $\beta_0, \beta \in C^\infty_c(\mathbb{R}^d)$, supported in $(-1/2, 1/2)^d$, with the properties $|\widehat{\beta}_0(\xi)| > 0$ if $|\xi| \le 1$, $|\widehat{\beta}(\xi)| > 0$ if $1/8 \le |\xi| \le 1$ and β has vanishing moments up to order M, that is

$$\int_{\mathbb{R}^d} \beta(x)\, x_1^{m_1} \cdots x_d^{m_d}\, dx = 0, \quad \forall\, m_i \in \mathbb{N}_0 \ \text{ with } \ m_1 + \ldots + m_d \le M. \tag{16}$$

The optimal value of M is irrelevant for the purposes of this paper, and (15) suffices for our results. We let $\beta_k := 2^{kd}\beta(2^k \cdot)$ for each $k \ge 1$, and denote

$$L_k f = \beta_k * f$$

whenever $f \in \mathcal{S}'(\mathbb{R}^d)$. These convolution operators, sometimes called *local means*, can be used to define equivalent quasi-norms in the $B^s_{p,q}$ and $F^s_{p,q}$ spaces. Namely,

$$\|g\|_{B^s_{p,q}} \approx \Big\|\big\{2^{ks}L_k g\big\}_{k=0}^{\infty}\Big\|_{\ell^q(L^p)} \tag{17}$$

and if $0 < p < \infty$,

$$\|g\|_{F^s_{p,q}} \approx \Big\|\big\{2^{ks}L_k g\big\}_{k=0}^{\infty}\Big\|_{L^p(\ell^q)} \tag{18}$$

see e.g. [14, 2.5.3 and 2.4.6]. For the latter spaces, when $p = \infty$ (and $q < \infty$) one defines instead

$$\|g\|_{F^s_{\infty,q}} \approx \sup_{n\geq 0}\sup_{I\in\mathscr{D}_n}\Big(\frac{1}{|I|}\int_I \sum_{k\geq n} 2^{ksq}|L_k g(x)|^q\,dx\Big)^{1/q}, \tag{19}$$

see [3, (12.8)], [2]. Finally, one lets $F^s_{\infty,\infty} = B^s_{\infty,\infty}$.

Next, let $\eta_0 \in C_c^\infty(\mathbb{R}^d)$ be supported on $\{\xi : |\xi| < 3/8\}$ and such that $\eta_0(\xi) = 1$ if $|\xi| \leq 1/4$. We define the operators

$$\widehat{\Lambda_0 f}(\xi) = \frac{\eta_0(\xi)}{\widehat{\beta}_0(\xi)}\widehat{f}(\xi)\,, \tag{20a}$$

$$\widehat{\Lambda_k f}(\xi) = \frac{\eta_0(2^{-k}\xi) - \eta_0(2^{-k+1}\xi)}{\widehat{\beta}(2^{-k}\xi)}\widehat{f}(\xi), \quad k \geq 1, \tag{20b}$$

so that

$$f = \sum_{j=0}^{\infty} L_j\Lambda_j f \tag{21}$$

with convergence in $\mathcal{S}'$. Of course, one obtains (the usual) equivalent norms if in (17), (18) and (19) the operators L_k are replaced by Λ_k. In particular, if we let $\mathit{\Pi}_N = \sum_{j=0}^N L_j\Lambda_j$, then

$$\sup_N \|\mathit{\Pi}_N f\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}}. \tag{22}$$

Below we shall be interested in uniformly bounded extensions of the dyadic averaging operators $\mathbb{E}_N$ defined in (8). We shall denote

$$\mathbb{E}_N^\perp = I - \mathbb{E}_N \quad \text{and} \quad \mathit{\Pi}_N^\perp = I - \mathit{\Pi}_N,$$

and write

$$\mathbb{E}_N - \mathit{\Pi}_N = \mathbb{E}_N\,\mathit{\Pi}_N^\perp \,-\, \mathbb{E}_N^\perp\,\mathit{\Pi}_N. \tag{23}$$

Then, using (17), we have

$$\begin{aligned} \big\|\mathbb{E}_N f - \mathit{\Pi}_N f\big\|_{B^s_{p,r}} \;\lesssim\; & \Big\|\big\{2^{ks}L_k\mathbb{E}_N\mathit{\Pi}_N^\perp f\big\}_{k=0}^{\infty}\Big\|_{\ell^r(L^p)} + \\ & + \Big\|\big\{2^{ks}L_k\mathbb{E}_N^\perp\mathit{\Pi}_N f\big\}_{k=0}^{\infty}\Big\|_{\ell^r(L^p)}. \end{aligned} \tag{24}$$

Following [4, 5], we shall prove Theorems 1.5, 1.6 and 1.7 using suitable estimates for the functions $L_k \mathbb{E}_N L_j g$ and $L_k \mathbb{E}_N^{\perp} L_j g$, for each $j, k \geq 0$, some of which will be new in this paper.

3. Proof of Theorem 1.5: The case $p > \frac{d}{d+1}$

Let $s = 1$ and $d/(d+1) < p < 1$. For these indices, Theorem 1.5 will be a consequence of the following two results. The first result is contained in [5] (Propositions 3.1 and 3.4), and was also implicit in [4] (proof of inequality (19)).

Proposition 3.1. *For $\frac{d}{d+1} < p < 1$ and $r > 0$, it holds*

$$\sup_N \Big(\sum_{k=0}^{\infty} 2^{kr} \big\| L_k \, \mathbb{E}_N \, \Pi_N^{\perp} f \big\|_p^r \Big)^{1/r} \lesssim \|f\|_{B^1_{p,\infty}}. \tag{25}$$

The same holds if $\mathbb{E}_N$ is replaced by $T_N[\cdot, \mathfrak{a}]$ with $\|\mathfrak{a}\|_{\ell^\infty} \leq 1$.

The second result is new, and it will require a few additional arguments compared to [4, 5]. The conditions on p are also less demanding. Here $h^p = F^0_{p,2}$ is the local Hardy space; see e.g. [13, 2.5.8].

Proposition 3.2. *For $\frac{d}{d+2} < p < 1$ and $r > 0$, it holds*

$$\sup_N \Big(\sum_{k=0}^{\infty} 2^{kr} \big\| L_k \, \mathbb{E}_N^{\perp} \, \Pi_N f \big\|_p^r \Big)^{1/r} \lesssim \|\nabla f\|_{h^p}. \tag{26}$$

The same holds if $\mathbb{E}_N^{\perp}$ is replaced by $T_N[\cdot, \mathfrak{a}]$ with $\|\mathfrak{a}\|_{\ell^\infty} \leq 1$.

We shall prove Proposition 3.2 in the next subsections, but we indicate now how (25) and (26) imply (11). Just use the Littlewood-Paley type inequality

$$\|\nabla f\|_{h^p} \lesssim \|f\|_{F^1_{p,2}};$$

(see *e.g.* [13, 2.3.8/3]) and the embedding $F^1_{p,2} \hookrightarrow B^1_{p,\infty}$.

3.1. *A pointwise estimate.* As in [4] we shall use the Peetre maximal functions

$$\mathfrak{M}^{**}_{A,j} g(x) = \sup_{h \in \mathbb{R}^d} \frac{|g(x+h)|}{(1 + 2^j |h|)^A}, \tag{27}$$

typically applied to scalar or Hilbert space valued $g \in \mathcal{S}'(\mathbb{R}^d)$ satisfying

$$\operatorname{supp} \widehat{g} \subset \{\xi : |\xi| \leq 2^{j+1}\}. \tag{28}$$

In [7] it was shown that for g satisfying (28),

$$\|\mathfrak{M}^{**}_{A,j} g\|_p \leq C_{p,A} \|g\|_p, \qquad 0 < p \leq \infty, \qquad A > d/p. \tag{29}$$

In what follows it will be convenient to use the notation

$$|x|_\infty = \max_{1 \leq i \leq d} |x_i|, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

The following lemma is a variation of [4, (35)]. The novelty here is that the operator $\mathbb{E}_N^\perp$ is acting on $\mathit{\Pi}_N f = \sum_{j\le N} L_j \Lambda_j f$, rather than in each $L_j\Lambda_j f$ separately.

Lemma 3.3. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then*

$$\big|\mathbb{E}_N^\perp[\mathit{\Pi}_N f](y)\big| \lesssim \inf_{|y'-y|_\infty \le 2^{1-N}} \mathfrak{M}_{A,N}^{**}(2^{-N}\mathit{\Pi}_N \nabla f)(y'), \quad y \in \mathbb{R}^d. \tag{30}$$

In particular, if $|y - \frac{\mu}{2^N}|_\infty \le 2^{1-N}$, then, for every $p > 0$,

$$|\mathbb{E}_N^\perp \mathit{\Pi}_N f(y)| \lesssim \left[\fint_{|h|_\infty \le 2^{2-N}} \big|\mathfrak{M}_{A,N}^{**}(2^{-N}\mathit{\Pi}_N\nabla f)(\tfrac{\mu}{2^N}+h)\big|^p\, dh\right]^{1/p}. \tag{31}$$

These bounds also hold if we replace $\mathbb{E}_N^\perp$ with $T_N(\cdot,\mathfrak{a})$ with $\|\mathfrak{a}\|_\infty \le 1$.

Proof. Recall that $\widehat{\mathit{\Pi}_N f}(\xi) = \eta_0(2^{-N}\xi)\widehat{f}(\xi)$. Let $\Phi \in \mathcal{S}$, with $\widehat{\Phi}(\xi) = 1$ when $|\xi| \le 1$, and let $\Phi_N(z) = 2^{Nd}\Phi(2^N z)$. Then

$$\mathit{\Pi}_N f = \Phi_N * \mathit{\Pi}_N f. \tag{32}$$

If $I \in \mathscr{D}_N$ is such that $y \in I$, we have

$$\begin{aligned}
\big|\mathbb{E}_N^\perp(\mathit{\Pi}_N f)(y)\big| &= \big|\mathbb{E}_N[\Phi_N * (\mathit{\Pi}_N f)](y) - \Phi_N * \mathit{\Pi}_N f(y)\big| \\
&= \Big|\fint_I \int_{\mathbb{R}^d} \Phi_N(z)\big[\mathit{\Pi}_N f(v-z) - \mathit{\Pi}_N f(y-z)\big] dz\, dv\Big| \\
&= \Big|\fint_I \int_{\mathbb{R}^d} \Phi_N(z) \int_0^1 \langle v-y, \nabla \mathit{\Pi}_N f(y+s(v-y)-z)\rangle ds dz\, dv\Big| \\
&\lesssim \int_{\mathbb{R}^d} |\Phi_N(z)|(1+2^N|z|)^A\, dz \ \sup_{\tilde z\in\mathbb{R}^d} \frac{|2^{-N}\nabla \mathit{\Pi}_N f(y'+\tilde z)|}{(1+2^N|\tilde z|)^A} \\
&\le C_A\, \mathfrak{M}_{A,N}^{**}[2^{-N}\nabla \mathit{\Pi}_N f](y'),
\end{aligned}$$

for any y' such that $|y-y'|_\infty \le 2^{1-N}$. This shows (30). The last assertion in (31) follows easily from here.

Finally, if we replace $\mathbb{E}_N^\perp$ with $T_N[\cdot,\mathfrak{a}]$, the cancellation of $\int_I h_I = 0$ implies that, for $w \in I$,

$$|T_N[\mathit{\Pi}_N f, a](w)| \le \Big|\tfrac{1}{|I|}\int_I h_I(v)\big[\Phi_N * \mathit{\Pi}_N f(v) - \Phi_N * \mathit{\Pi}_N f(w)\big]\, dv\Big|.$$

The rest of the proof is then carried out as above. □

3.2. *Norm estimates.* As in [4], we use the notation

$$\mathcal{U}_{N,k} = \Big\{ y \in \mathbb{R}^d \ : \ \min_{1\le i\le d} \operatorname{dist}(y_i, 2^{-N}\mathbb{Z}) \le 2^{-k-1}\Big\}, \quad k > N. \tag{33}$$

Roughly speaking, this is the set of points at distance $O(2^{-k})$ from $\bigcup_{I\in\mathscr{D}_N} \partial I$.

Note (or recall from [4, Lemma 2.3.i]) that if $k > N$ then

$$L_k(\mathbb{E}_N g)(x) = 0, \quad \forall\, x \in \mathcal{U}_{N,k}^{\complement} = \mathbb{R}^d \setminus \mathcal{U}_{N,k}. \tag{34}$$

The next two results will be obtained using Lemma 3.3.

Lemma 3.4. *Let $0 < p \le 1$. Then for every $k > N$ and $\|\mathfrak{a}\|_\infty \le 1$,*

$$2^k \|L_k \mathbb{E}_N^\perp \Pi_N f\|_p + 2^k \|L_k T_N[\Pi_N f, \mathfrak{a}]\|_p \lesssim 2^{-(k-N)(\frac{1}{p}-1)} \|\nabla \Pi_N f\|_p.$$

Proof. The observation in (34) implies that

$$\begin{aligned} \big\|L_k \mathbb{E}_N^\perp \Pi_N f\big\|_p &\lesssim \big\|L_k \mathbb{E}_N^\perp \Pi_N f\big\|_{L^p(\mathcal{U}_{N,k}^\complement)} + \big\|L_k \mathbb{E}_N^\perp \Pi_N f\big\|_{L^p(\mathcal{U}_{N,k})} \\ &\lesssim \|L_k \Pi_N f\|_{L^p(\mathcal{U}_{N,k}^\complement)} + \Big[\sum_{\mu \in \mathbb{Z}^d} \|L_k \mathbb{E}_N^\perp \Pi_N f\|^p_{L^p(\mathcal{U}_{N,k} \cap I_{N,\mu})}\Big]^{\frac{1}{p}}. \end{aligned} \tag{35}$$

Using (31) and the fact that $\operatorname{supp} \beta_k(x - \cdot) \subset \mu 2^{-N} + O(2^{-N})$ for $x \in I_{N,\mu}$, the last term is controlled by

$$\begin{aligned} &\Big[\sum_{\mu \in \mathbb{Z}^d} |I_{N,\mu} \cap \mathcal{U}_{N,k}| \fint_{|h|_\infty \le 2 2^{-N}} \big|\mathfrak{M}^{**}_{N,A}(2^{-N} \nabla \Pi_N f)(2^{-N}\mu + h)\big|^p \, dh\Big]^{\frac{1}{p}} \\ &\lesssim \big[2^{-k} 2^{-N(d-1)}\big]^{\frac{1}{p}} \, 2^{\frac{Nd}{p}} \, \big\|\mathfrak{M}^{**}_{N,A}(2^{-N} \nabla \Pi_N f)\big\|_p \lesssim 2^{-N} \, 2^{\frac{N-k}{p}} \, \|\nabla \Pi_N f\|_p. \end{aligned}$$

To estimate the first term on the right hand side of (35), we shall use the bound

$$|L_k \Pi_N f(x)| \lesssim 2^{-k} \, 2^{-(k-N)M} \, \mathfrak{M}^{**}_{A,N}\big(\nabla \Pi_N f\big)(x), \quad \text{if } k > N. \tag{36}$$

This can be proved as in [4, Lemma 2.2] using the vanishing moments of β_k; see Appendix 13 below. Then, from (36) and Peetre's inequality (29) it follows that

$$\begin{aligned} 2^k \|L_k \Pi_N f\|_{L^p(\mathcal{U}_{N,k}^\complement)} &\lesssim 2^{-(k-N)M} \, \big\|\mathfrak{M}^{**}_{A,N}\big(\nabla \Pi_N f\big)\big\|_{L^p(\mathbb{R}^d)} \\ &\lesssim 2^{-(k-N)M} \|\nabla \Pi_N f\|_p. \end{aligned}$$

Analogous arguments apply for $T_N[\Pi_N f, \mathfrak{a}]$ in place of $\mathbb{E}_N^\perp \Pi_N f$. ☐

Lemma 3.5. *Let $0 < p \le 1$. Then, for every $k \le N$, $\|\mathfrak{a}\|_\infty \le 1$,*

$$2^k \|L_k \mathbb{E}_N^\perp \Pi_N f\|_p + 2^k \|L_k T_N[\Pi_N f, \mathfrak{a}]\|_p \lesssim 2^{(N-k)(\frac{d}{p}-d-2)} \|\nabla \Pi_N f\|_p. \tag{37}$$

Proof. Since $\int_I \mathbb{E}_N^\perp[\Pi_N f](y)\, dy = 0$ for $I \in \mathscr{D}_N$, we may write

$$\begin{aligned} &L_k\big(\mathbb{E}_N^\perp[\Pi_N f]\big)(x) \\ &\qquad = \sum_{\mu \in \mathcal{Z}_{k,N}(x)} \int_{I_{N,\mu}} \big(\beta_k(x-y) - \beta_k(x - 2^{-N}\mu)\big) \, \mathbb{E}_N^\perp[\Pi_N f](y)\, dy \end{aligned}$$

where

$$\mathcal{Z}_{k,N}(x) = \{\mu \in \mathbb{Z}^d : |x - 2^{-N}\mu|_\infty \le 2^{-N} + 2^{-k-1}\}. \tag{38}$$

Note that $\operatorname{card} \mathcal{Z}_{k,N}(x) \approx 2^{(N-k)d}$. Now use

$$|\beta_k(x-y) - \beta_k(x - 2^{-N}\mu)| \lesssim 2^{kd} \, 2^{k-N}, \quad \text{if } y \in I_{N,\mu},$$

in combination with Lemma 3.3 to obtain

$$
\begin{aligned}
&|L_k\big(\mathbb{E}_N^{\perp}[\varPi_N f]\big)(x)| \lesssim \\
&\quad\lesssim 2^{(k-N)(d+1)} \sum_{\mu\in\mathcal{Z}_{k,N}(x)} \Big(\fint_{|h|_\infty\le 2^{2-N}} \big|\mathfrak{M}^{**}_{A,N}(2^{-N}\nabla\varPi_N f)\big|^p(\tfrac{\mu}{2^N}+h)\,dh\Big)^{\frac1p} \\
&\quad\lesssim 2^{(k-N)(d+1)}\Big(\sum_{\mu\in\mathcal{Z}_{k,N}(x)} \fint_{|h|_\infty\le 2^{2-N}} \big|\mathfrak{M}^{**}_{A,N}(2^{-N}\nabla\varPi_N f)\big|^p(\tfrac{\mu}{2^N}+h)\,dh\Big)^{\frac1p},
\end{aligned}
$$

the last step using the embedding $\ell^1 \hookrightarrow \ell^{1/p}$, since $p\le 1$. From this

$$
\begin{aligned}
&\|L_k\big(\mathbb{E}_N^{\perp}[\varPi_N f]\big)\|_p \\
&\lesssim 2^{(k-N)(d+1)}\Big[\int_{\mathbb{R}^d}\sum_{\mu\in\mathcal{Z}_{k,N}(x)}\fint_{|h|_\infty\le 2^{2-N}} \big|\mathfrak{M}^{**}_{A,N}(2^{-N}\nabla\varPi_N f)\big|^p(\tfrac{\mu}{2^N}+h)\,dh\,dx\Big]^{\frac1p} \\
&\lesssim 2^{(k-N)(d+1)}\Big[\sum_{\mu\in\mathbb{Z}^d} 2^{-kd}\fint_{|h|_\infty\le 2^{2-N}} \big|\mathfrak{M}^{**}_{A,N}(2^{-N}\nabla\varPi_N f)\big|^p(\tfrac{\mu}{2^N}+h)\,dh\Big]^{\frac1p} \\
&\lesssim 2^{(k-N)(d+1)}2^{(N-k)d/p}\,\big\|\mathfrak{M}^{**}_{A,N}(2^{-N}\nabla\varPi_N f)\big\|_p,
\end{aligned}
$$

and the assertion follows by the Peetre inequality for $\mathfrak{M}^{**}_{A,N}$. Analogous arguments apply for $T_N[\varPi_N f,\mathfrak{a}]$ in place of $\mathbb{E}_N^{\perp}\varPi_N f$. □

Proof of Proposition 3.2. Using the Lemmas 3.4 and 3.5, and noticing that we may sum in k since $\frac{d}{d+2}<p<1$, one easily obtains

$$
\Big(\sum_{k=0}^{\infty} 2^{kr}\big\|L_k\,\mathbb{E}_N^{\perp}\,\varPi_N f\big\|_p^r\Big)^{1/r} \lesssim \|\varPi_N\nabla f\|_p.
$$

The last quantity can be estimated further, applying to $g=\nabla f$ the inequality

$$
\|\varPi_N g\|_p \le \Big\|\sup_{N\ge 0}|\varPi_N g|\Big\|_p \lesssim \|g\|_{h^p} \approx \|g\|_{F^0_{p,2}}, \tag{39}
$$

which follows for example using the standard maximal function characterization of the h^p norm. This proves (26). The proof for the operators $T_N[\cdot,a]$ is exactly analogous. □

4. Proof of Theorem 1.6: the case $s=d/p-d$

Let $s=d/p-d$ and $d/(d+1)<p\le 1$ (we will take up the endpoint case $p=d/(d+1)$, when $s=1$ in §5). For these indices, Theorem 1.6 will be a consequence of the following two results. The first result was already established in [5] (Propositions 3.2 and 3.3), using the same type of analysis as in [4]. The inequality is slightly stronger than needed due to $F^s_{p,\infty}\hookrightarrow B^s_{p,\infty}$.

Proposition 4.1. *Let $\frac{d}{d+1} < p \le 1$ and $r > 0$. Then*

$$\Big(\sum_{k=0}^{\infty} 2^{k(d/p-d)r} \Big\| L_k \mathbb{E}_N^{\perp} \mathit{\Pi}_N f \Big\|_p^r \Big)^{1/r} \lesssim \|f\|_{B^{\frac{d}{p}-d}_{p,\infty}}. \tag{40}$$

The same holds if $\mathbb{E}_N^{\perp}$ is replaced by $T_N[\cdot, \mathfrak{a}]$ with $\|\mathfrak{a}\|_{\ell^\infty} \le 1$.

The second proposition is new, and its proof will require several additional refinements compared to the arguments given in [4].

Proposition 4.2. *Let $\frac{d-1}{d} < p \le 1$ and $r > 0$. Then,*

$$\Big(\sum_{k=N+1}^{\infty} 2^{k(d/p-d)r} \Big\| \sum_{j>N} L_k \mathbb{E}_N L_j \Lambda_j f \Big\|_p^r \Big)^{1/r} \lesssim \|f\|_{F^{\frac{d}{p}-d}_{p,\infty}} \tag{41}$$

$$\Big(\sum_{k=0}^{N} 2^{k(d/p-d)r} \Big\| \sum_{j>N} L_k \mathbb{E}_N L_j \Lambda_j f \Big\|_p^r \Big)^{1/r} \lesssim \|f\|_{F^{\frac{d}{p}-d}_{p,\infty}}. \tag{42}$$

The same holds if $\mathbb{E}_N$ is replaced by $T_N[\cdot, a]$ with $\|a\|_{\ell^\infty} \le 1$.

4.1. *Notation and observations on dyadic cubes.* Recall that every dyadic cube I is contained in a unique parent cube of double side length. Also each dyadic cube has 2^d children cubes of half side length. It will be useful to single out one of the children cubes according to the following definition.

Definition 4.3. Let I be a dyadic cube. We denote by $\omega(I)$ the unique child of I with the property that its closure contains the center of the parent cube of I.

We need some further notation (taken from [4]). For each dyadic cube $I \in \mathscr{D}_N$, we denote by $\mathscr{D}_N(I)$ the set of all its neighboring 2^{-N}-cubes, that is, $I' \in \mathscr{D}_N$ with $\bar{I} \cap \bar{I}' \neq \emptyset$. Likewise, if $\ell > N$ we denote by $\mathscr{D}_\ell[\partial I]$ the set of all $J \in \mathscr{D}_\ell$ such that $\bar{J} \cap \partial I \neq \emptyset$.

Lemma 4.4. *(i) Let $J \in \mathscr{D}_\ell[\partial I]$. Then*

$$2^{-\ell-1} \le \mathrm{dist}(x, \partial I)_\infty \le 2^{-\ell} \text{ for all } x \in \omega(J).$$

(ii) Let $I \in \mathscr{D}_N$, let $\ell_1, \ell_2 > N$ and consider two distinct cubes $J_1 \in \mathscr{D}_{\ell_1}[\partial I]$, $J_2 \in \mathscr{D}_{\ell_2}[\partial I]$. Then $\omega(J_1)$, $\omega(J_2)$ have disjoint interiors.

Proof. The upper bound in (i) is true for all $x \in J$, by definition of $\mathscr{D}_\ell[\partial I]$ and the lower bound follows from the definition of $\omega(J)$ since the parent cube of J is contained in I or one of its neighbors of equal side length. To see (ii) first observe that J_1, J_2 are disjoint if $\ell_1 = \ell_2$ (and hence $\omega(J_1)$ and $\omega(J_2)$ are disjoint). If $\ell_1 \neq \ell_2$ then (ii) follows from (i). □

4.2. *Proof of Proposition 4.2.* We make a preliminary observation about maximal functions. If g is continuous, for each $j \geq 0$ we let

$$\mathfrak{M}_j^* g(x) := \sup_{|h|_\infty \leq 2^{-j}} |g(x+h)|.$$

Then, if a cube $J \in \mathscr{D}_{j+1}$ has center c_J, we have

$$\sup_{x \in J} |g(x)| \leq \inf_{|h|_\infty \leq 2^{-j-1}} \mathfrak{M}_j^* g(c_J + h) \leq \Big[\fint_{\omega(J)} |\mathfrak{M}_j^* g|^p \Big]^{\frac{1}{p}}. \tag{43}$$

Proof of (41). Let $j, k > N$. By (34),

$$L_k \mathbb{E}_N [L_j \Lambda_j f](x) = 0 \quad \text{ if } x \in \mathcal{U}_{N,k}^{\complement}.$$

Moreover, by [4, Lemma 2.3 (ii)] we have

$$|\mathbb{E}_N (L_j \Lambda_j f)(y)| \lesssim 2^{(N-j)d} \sum_{I \in \mathscr{D}_N} \mathbb{1}_I(y) \sum_{J \in \mathscr{D}_{j+1}[\partial I]} \|\Lambda_j f\|_{L^\infty(J)}. \tag{44}$$

Let $x \in \mathcal{U}_{N,k} \cap I$, for some $I \in \mathscr{D}_N$. Then $\operatorname{supp} \beta_k(x - \cdot) \subset \bigcup_{I' \in \mathscr{D}_N(I)} I'$, and therefore (44) implies

$$\begin{aligned} |L_k \mathbb{E}_N [L_j \Lambda_j f](x)| &\leq \int |\beta_k(x-y)| \, \big|\mathbb{E}_N (L_j \Lambda_j f)(y)\big| \, dy \\ &\lesssim 2^{(N-j)d} \sum_{I' \in \mathscr{D}_N(I)} \sum_{J \in \mathscr{D}_{j+1}[\partial I']} \|\Lambda_j f\|_{L^\infty(J)} \, \|\beta_k\|_1. \end{aligned}$$

Using the inequality in (43), this in turn implies (since $p \leq 1$)

$$\begin{aligned} A_k(x)^p &:= \Big[\sum_{j>N} |L_k \mathbb{E}_N L_j \Lambda_j f(x)| \Big]^p \\ &\lesssim \sum_{j>N} 2^{(N-j)dp} \sum_{I' \in \mathscr{D}_N(I)} \sum_{J \in \mathscr{D}_{j+1}[\partial I']} \fint_{\omega(J)} |\mathfrak{M}_j^*(\Lambda_j f)|^p \\ &\lesssim 2^{Ndp} \sum_{I' \in \mathscr{D}_N(I)} \sum_{j>N} \sum_{J \in \mathscr{D}_{j+1}[\partial I']} \int_{\omega(J)} \sup_{\ell > N} |2^{\ell(\frac{d}{p} - d)} \mathfrak{M}_\ell^*(\Lambda_\ell f)|^p. \end{aligned} \tag{45}$$

By Lemma 4.4 the sets $\omega(J)$ for $J \in \mathscr{D}[\partial I']$ are disjoint. Also since $\#\mathscr{D}_N[I] = 2^d$ we obtain

$$A_k(x)^p \lesssim 2^{Ndp} \int_{I^{**}} \sup_{\ell > N} \big|2^{\ell(\frac{d}{p} - d)} \mathfrak{M}_\ell^*(\Lambda_\ell f)\big|^p,$$

with I^{**} the five-fold dilation of I with respect to c_I. Thus, if we write

$$G = \sup_{\ell > N} |2^{\ell(\frac{d}{p} - d)} \mathfrak{M}_\ell^*(\Lambda_\ell f)|, \tag{46}$$

we obtain

$$\begin{aligned}\|A_k\|_p^p &\lesssim 2^{Ndp} \sum_{I\in\mathscr{D}_N} |I\cap\mathcal{U}_{N,k}| \int_{I^{**}} |G|^p \\ &\lesssim 2^{N(d-\frac{d-1}{p})p}\, 2^{-k}\, \|G\|^p_{L^p(\mathbb{R}^d)},\end{aligned}$$

and therefore

$$2^{k(\frac{d}{p}-d)}\|A_k\|_p \lesssim 2^{-(k-N)(d-\frac{d-1}{p})}\, \|G\|_p. \tag{47}$$

When $p>(d-1)/d$ we can sum in $k>N$, and hence the left hand side of (42) is controlled by $\|G\|_p$. Now, Peetre's inequalities imply that

$$\mathfrak{M}^*_\ell[\Lambda_\ell f](x) \lesssim \mathfrak{M}^{**}_{A,\ell}[\Lambda_\ell f](x) \lesssim M_\sigma[\Lambda_\ell f](x),$$

for $\sigma=d/A$, where

$$M_\sigma g(x) = \sup_{R>0}\Big[\fint_{B_R(x)} |g|^\sigma\Big]^{1/\sigma}.$$

Thus,

$$\begin{aligned}\|G\|_p &\lesssim \Big\|\sup_{\ell>N} 2^{\ell(\frac{d}{p}-d)} M_\sigma[\Lambda_\ell f]\Big\|_p \le \Big\|M_\sigma\Big[\sup_{\ell>N} 2^{\ell(\frac{d}{p}-d)}|\Lambda_\ell f|\Big]\Big\|_p \\ &\lesssim \Big\|\sup_{\ell>N} 2^{\ell(\frac{d}{p}-d)}|\Lambda_\ell f|\Big\|_p \lesssim \|f\|_{F^{\frac{d}{p}-d}_{p,\infty}} \end{aligned}\tag{48}$$

using the boundedness of the Hardy-Littlewood maximal operator, since $\sigma=d/A<p$. This finishes the proof of (41), for the stated version involving $\mathbb{E}_N$. The analogous version for $T_N[\cdot,\mathfrak{a}]$ follows similarly, by replacing (44) with the corresponding version for the T_N, as in [4]. □

Proof of (42). Let $j>N$ and $k\le N$. Again, we shall follow the proof of [4, (26)], applying the same changes as in the previous subsection. Namely, let

$$\widetilde{f}_j = (\Lambda_j f)\mathbb{1}_{\mathcal{U}_{N,j}}$$

and note that $\mathbb{E}_N[L_j\Lambda_j f]=\mathbb{E}_N[L_j\widetilde{f}_j]$; see [4, Lemma 2.3]. Then we write

$$L_k\mathbb{E}_N[L_j\Lambda_j f] = L_k(\mathbb{E}_N[L_j\widetilde{f}_j] - L_j\widetilde{f}_j) + L_kL_j\widetilde{f}_j. \tag{49}$$

As in [4], the last term is harmless since

$$\|L_kL_j\widetilde{f}_j\|_p \lesssim 2^{-(M-A)|j-k|}\,\big\|\mathfrak{M}^{**}_{A,j}(\Lambda_j f)(x)\big\|_p \lesssim 2^{-(M-A)|j-k|}\,\|\Lambda_j f\|_p,$$

by [4, Lemma 2.2]. Thus, assuming $r\le p$ (as we may), and using the r-triangle inequality,

$$\begin{aligned}&\Big(\sum_{k=0}^N 2^{k(\frac{d}{p}-d)r}\|\sum_{j>N} L_kL_j\widetilde{f}_j\|_p^r\Big)^{1/r} \lesssim \\ &\Big(\sum_{k=0}^N\sum_{j>N} 2^{(k-j)(\frac{d}{p}-d+(M-A))r} 2^{j(\frac{d}{p}-d)r}\|\Lambda_j f\|_p^r\Big)^{1/r} \lesssim \|f\|_{B^{\frac{d}{p}-d}_{p,\infty}}.\end{aligned}$$

Hence it remains to prove

$$\Big(\sum_{k=0}^{N} 2^{k(d/p-d)r}\Big\|\sum_{j>N} L_k\mathbb{E}_N^{\perp}L_j\widetilde{f}_j\Big\|_p^r\Big)^{1/r}\lesssim\|f\|_{F^{\frac{d}{p}-d}_{p,\infty}}. \tag{50}$$

Following [4], and letting $\mathscr{Z}_{k,N}(x)$ be as in (38), we write

$$\begin{aligned}\mathcal{A}_{j,k}(x) &:= |L_k\big(\mathbb{E}_N^{\perp}[L_j\widetilde{f}_j]\big)(x)|\\ &\leq \sum_{\mu\in\mathscr{Z}_{k,N}(x)}\Big|\int_{I_{N,\mu}}\big(\beta_k(x-y)-\beta_k(x-2^{-N}\mu)\big)\,\mathbb{E}_N^{\perp}[L_j\widetilde{f}_j](y)\,dy\Big|\\ &\lesssim 2^{kd}2^{k-N}\sum_{\mu\in\mathscr{Z}_{k,N}(x)}\int_{I_{N,\mu}}\Big(|\mathbb{E}_N[L_j\widetilde{f}_j](y)|+|L_j\widetilde{f}_j(y)|\Big)\,dy.\end{aligned}$$

In [4, p. 1332], the terms corresponding to the two summands in the integral are estimated differently, but produce essentially the same outcome, namely

$$\mathcal{A}_{j,k}(x)\lesssim 2^{k-N}2^{(k-j)d}\Big(\sum_{\mu\in\mathscr{Z}_{k,N}(x)}\sum_{J\in\mathscr{D}_{j+1}[\partial I_{N,\mu}]}\|\Lambda_j f\|^p_{L^\infty(J)}\Big)^{\frac{1}{p}}, \tag{51}$$

see [4, (41)]. At this point we argue as in the previous subsection. That is, we use (43) to have

$$\|\Lambda_j f\|_{L^\infty(J)}\leq\Big[\fint_{\omega(J)}\mathfrak{M}_j^*[\Lambda_j f](y)^p\,dy\Big]^{\frac{1}{p}}, \tag{52}$$

and conclude that

$$\begin{aligned}\mathcal{A}_k(x)^p &:= \Big[\sum_{j>N}|L_k\mathbb{E}_N^{\perp}L_j\widetilde{f}_j(x)|\Big]^p\\ &\lesssim\sum_{j>N}2^{(k-N)p}2^{(k-j)dp}\sum_{\mu\in\mathscr{Z}_{k,N}(x)}\sum_{J\in\mathscr{D}_{j+1}[\partial I_{N,\mu}]}\fint_{\omega(J)}|\mathfrak{M}_j^*(\Lambda_j f)|^p\\ &\lesssim 2^{(k-N)p}2^{kdp}\sum_{\mu\in\mathscr{Z}_{k,N}(x)}\int_{I^{**}_{N,\mu}}|G|^p\end{aligned} \tag{53}$$

with G as in (46), and using the disjointness of the sets $\omega(J)$ as before. Thus, integrating the above expression

$$\begin{aligned}\|\mathcal{A}_k\|_p^p &\lesssim 2^{(k-N)p}2^{kdp}\sum_{\mu\in\mathbb{Z}^d}2^{-kd}\int_{I^{**}_{N,\mu}}|G|^p\\ &\lesssim 2^{(k-N)p}2^{k(d-\frac{d}{p})p}\int_{\mathbb{R}^d}|G|^p\end{aligned}$$

and therefore

$$2^{k(\frac{d}{p}-d)}\|\mathcal{A}_k\|_p\lesssim 2^{k-N}\,\|G\|_p. \tag{54}$$

Therefore, one can sum in $k\leq N$, and obtain the desired expression in (50) using the estimate for $\|G\|_p$ in (48). This finishes the proof of (42). The corresponding version for T_N is proved similarly (notice that in (49) the analysis of the last summand becomes unnecessary, due to the additional cancellation of T_N). □

5. Proof of Theorem 1.5: The case $p = \frac{d}{d+1}$

The end-point case $p = d/(d+1)$ and $s = 1$, was excluded from the previous proofs because of the restrictions imposed in Propositions 3.1 and 4.1. However, one can use instead Propositions 4.2 and 3.2, which are valid at this endpoint. Namely, they imply the inequalities

$$\sup_N \Big(\sum_{k=0}^{\infty} 2^{kr}\Big\| \sum_{j>N} L_k \mathbb{E}_N L_j \Lambda_j f\Big\|_p^r\Big)^{1/r} \lesssim \|f\|_{F^1_{p,\infty}}, \quad p = \frac{d}{d+1}, \tag{55}$$

and

$$\sup_N \Big(\sum_{k=0}^{\infty} 2^{kr}\Big\| L_k \mathbb{E}_N^{\perp} \Pi_N f\Big\|_p^r\Big)^{1/r} \lesssim \|f\|_{F^1_{p,2}} \quad p = \frac{d}{d+1}. \tag{56}$$

Then, the result stated in Theorem 1.5 follows using additionally the embedding $F^1_{p,2} \hookrightarrow F^1_{p,\infty}$ in (55). The same argument applies to $T_N[\cdot, \mathfrak{a}]$ with $\|\mathfrak{a}\|_\infty \leq 1$ if we use the corresponding versions of Propositions 4.2 and 3.2. □

6. Proof of Theorem 1.7: the case $s = 0$ and $p = \infty$

In view of (23), it suffices to prove the following.

Proposition 6.1. *Let $r > 0$. Then*

$$\big\|\mathbb{E}_N^{\perp} \Pi_N f\big\|_{F^0_{\infty,r}} + \big\|\mathbb{E}_N \Pi_N^{\perp} f\big\|_{F^0_{\infty,r}} \lesssim \|f\|_{B^0_{\infty,\infty}}. \tag{57}$$

One part of the estimates will be derived from the following inequalities, proved in [5, (36a), (37a)]:

$$\Big(\sum_{k\leq N} \big\|L_k \mathbb{E}_N^{\perp} \Pi_N f\big\|_\infty^r\Big)^{1/r} \lesssim \|f\|_{B^0_{\infty,\infty}} \tag{58}$$

$$\Big(\sum_{k\leq N} \big\|L_k \mathbb{E}_N \Pi_N^{\perp} f\big\|_\infty^r\Big)^{1/r} \lesssim \|f\|_{B^0_{\infty,\infty}}. \tag{59}$$

We remark that these same inequalities with $\sum_{k\leq N}$ replaced by $\sum_{k>N}$ are only true if $r = \infty$. This necessitates the use of $F^0_{\infty,r}$-norms on the left hand side of (57)

To establish the proposition, let $f \in B^0_{\infty,\infty}$ be such that $\|f\|_{B^0_{\infty,\infty}} = 1$. We shall prove separately each of the two inequalities.

6.1. *Proof of* $\big\|\mathbb{E}_N^{\perp}\mathit{\Pi}_N f\big\|_{F^0_{\infty,r}} \lesssim 1$. By (19) we can write

$$\big\|\mathbb{E}_N^{\perp}\mathit{\Pi}_N f\big\|^r_{F^0_{\infty,r}} = \sup_{\ell\geq 0}\sup_{I\in\mathscr{D}_\ell} A_I^{(\ell)}$$
$$\text{where } A_I^{(\ell)} := \fint_I \sum_{k\geq \ell} |L_k\mathbb{E}_N^{\perp}\mathit{\Pi}_N f|^r.$$

If $0\leq \ell\leq N$, then, for each $I\in\mathscr{D}_\ell$,

$$A_I^{(\ell)} \leq \sum_{k=\ell}^{N} \big\|L_k\mathbb{E}_N^{\perp}\mathit{\Pi}_N f\big\|^r_\infty + \fint_I \sum_{k>N} \big|L_k\mathbb{E}_N^{\perp}\mathit{\Pi}_N f\big|^r =: A_{I,1}^{(\ell)} + A_{I,2}^{(\ell)}. \tag{60}$$

By (58) we have $A_{I,1}^{(\ell)} \lesssim 1$. For the second term, one can split I into $2^{(N-\ell)d}$ disjoint cubes $J\in\mathscr{D}_N$, so that

$$A_{I,2}^{(\ell)} \leq \sup_{\substack{J\in\mathscr{D}_N\\ J\subset I}} A_J^{(N)}.$$

Thus, it suffices to show that

$$\sup_{\ell\geq N} A_I^{(\ell)} \lesssim 1. \tag{61}$$

Let $\ell\geq N$ and $I\in\mathscr{D}_\ell$. Then (34) gives $L_k(\mathbb{E}_N g)\equiv 0$ in $\mathcal{U}_{N,k}^{\complement}$, if $k\geq \ell$, so

$$A_I^{(\ell)} = \frac{1}{|I|}\sum_{k\geq\ell}\int_{I\cap\mathcal{U}_{N,k}} \big|L_k\mathbb{E}_N^{\perp}\mathit{\Pi}_N f\big|^r + \frac{1}{|I|}\sum_{k\geq\ell}\int_{I\cap\mathcal{U}_{N,k}^{\complement}} \big|L_k\mathit{\Pi}_N f\big|^r. \tag{62}$$

We shall show that

$$\big|L_k\mathbb{E}_N^{\perp}\mathit{\Pi}_N f(x)\big| \lesssim \|f\|_{B^0_{\infty,\infty}} = 1, \quad \text{for } x\in I. \tag{63}$$

This inequality combined with $|I\cap\mathcal{U}_{N,k}|\approx 2^{-(d-1)\ell}2^{-k}$ will imply that the first summand in (62) is bounded by a multiple of

$$\frac{1}{|I|}\sum_{k\geq\ell}|I\cap\mathcal{U}_{N,k}| \lesssim \sum_{k\geq\ell}2^{\ell-k}\lesssim 1.$$

We now show (63). Let $Q\in\mathscr{D}_N$ be such that $I\subset Q$. By (31)

$$\big|\mathbb{E}_N^{\perp}\mathit{\Pi}_N f(y)\big| \lesssim \fint_{Q^{**}} \mathfrak{M}^{**}_{A,N}\big(2^{-N}\mathit{\Pi}_N\nabla f\big), \quad \text{for } y\in I^*. \tag{64}$$

Now,

$$\begin{aligned}\Big\|\mathfrak{M}^{**}_{A,N}\big(2^{-N}\mathit{\Pi}_N\nabla f\big)\Big\|_\infty &\leq \big\|2^{-N}\mathit{\Pi}_N\nabla f\big\|_\infty \lesssim \sum_{j\leq N}2^{j-N}\|\Lambda_j f\|_\infty\\ &\lesssim \sup_{m\geq 0}\|\Lambda_m f\|_\infty \lesssim \|f\|_{B^0_{\infty,\infty}} = 1.\end{aligned} \tag{65}$$

So, if $x\in I$, then $\operatorname{supp}\beta_k(x-\cdot)\subset x+O(2^{-k})\subset I^*$, and using (64) and (65) one deduces (63). Finally, the second summand in (62) is simpler to estimate, using instead the pointwise bound in (36). This completes the proof of (61).

6.2. *Proof of* $\big\|\mathbb{E}_N \Pi_N^{\perp} f\big\|_{F^0_{\infty,r}} \lesssim 1$. We now must bound

$$\big\|\mathbb{E}_N \Pi_N^{\perp} f\big\|^r_{F^0_{\infty,r}} = \sup_{\ell\ge 0}\sup_{I\in\mathscr{D}_\ell} B_I^{(\ell)}$$
$$\text{where } B_I^{(\ell)} = \fint_I \sum_{k\ge\ell} |L_k \mathbb{E}_N \Pi_N^{\perp} f|^r.$$

The cases $0\le\ell\le N$ are handled with the same argument as in (60), this time using the inequality (59). If $\ell\ge N$ and $I\in\mathscr{D}_\ell$ we shall use

$$B_I^{(\ell)} \le \frac{1}{|I|}\sum_{k\ge\ell} |I\cap\mathcal{U}_{N,k}|\, \big\|L_k\mathbb{E}_N\Pi_N^{\perp} f\big\|^r_\infty,$$

so that it will suffice to show

$$\big\|L_k\mathbb{E}_N\Pi_N^{\perp} f\big\|_\infty \lesssim \|f\|_{B^0_{\infty,\infty}} = 1. \tag{66}$$

If $Q\in\mathscr{D}_N$, then (44) and the argument in (45) (with $\omega(J)$ as in §4.1) implies

$$\begin{aligned} \big|\mathbb{E}_N L_j\Lambda_j f(y)\big| &\lesssim 2^{(N-j)d}\sum_{J\in\mathscr{D}_{j+1}(\partial Q)} \|\Lambda_j f\|_{L^\infty(J)} \\ &\lesssim 2^{(N-j)d}\sum_{J\in\mathscr{D}_{j+1}(\partial Q)} \fint_{\omega(J)} \mathfrak{M}_j^*(\Lambda_j f), \quad y\in Q, \end{aligned} \tag{67}$$

using in the last step (43). So, summing up in $j>N$ and using the disjointness properties of the sets $\omega(J)$ we obtain

$$\begin{aligned} \sum_{j>N}\big|\mathbb{E}_N L_j\Lambda_j f(y)\big| &\lesssim 2^{Nd}\int_{Q^{**}} \sup_{m\ge0}\mathfrak{M}_m^*(\Lambda_m f) \\ &\lesssim \sup_{m\ge 0}\|\Lambda_m f\|_\infty \lesssim \|f\|_{B^0_{\infty,\infty}} = 1. \end{aligned}$$

Finally, taking the convolution with β_k one easily deduces (66). This completes the proof of (57). The corresponding version for the $T_N[\cdot,\mathfrak{a}]$ is proved similarly. Thus the proof of Proposition 6.1 is complete, and so is the proof of Theorem 1.7.

Remark 6.2. The above proof also shows that, if $f\in B^0_{\infty,\infty}$, and N is fixed, then the series $\sum_{j=0}^\infty \mathbb{E}_N(L_j\Lambda_j f)$ converges in the norm of $F^0_{\infty,r}$, for all $r>0$. This is a consequence of the crude bound

$$\|\textstyle\sum_{j=J_1}^{J_2}\mathbb{E}_N L_j\Lambda_j f\|_{F^0_{\infty,r}} \lesssim_N 2^{-J_1}\,\|f\|_{F^0_{\infty,\infty}},$$

which can be obtained from (67); see also [5, Remark 4.5].

7. Proof of Theorem 1.8: Necessary conditions for $s = 1$

Here we show the assertion in Theorem 1.8, which corresponds to the optimality of the range of q stated in (ii) of Theorem 1.4. More precisely, we establish the following.

Theorem 7.1. *Let $d/(d+1) \le p < 1$ and $2 < q \le \infty$. Then*

$$\|\mathbb{E}_N\|_{F^1_{p,q} \to F^1_{p,q}} \approx N^{\frac{1}{2}-\frac{1}{q}}. \tag{68}$$

Moreover, for every $N \ge 1$ there exists $g_N \in C_c^\infty((0,1)^d)$ such that

$$\|g_N\|_{F^1_{p,q}} \le 1 \quad \text{and} \quad \|\mathbb{E}_N(g_N)\|_{F^1_{p,\infty}} \gtrsim N^{\frac{1}{2}-\frac{1}{q}}. \tag{69}$$

7.1. *Proof of Theorem 7.1: upper bounds.* Let $s = 1$. Using Proposition 3.1 when $d/(d+1) < p < 1$, or Proposition 4.2 when $p = d/(d+1)$, one has the inequality

$$\Big\| \big\{ 2^{kq} \sum_{j>N} L_k \mathbb{E}_N L_j \Lambda_j f \big\}_{k=0}^\infty \Big\|_{L^p(\ell^q)} \lesssim \|f\|_{F^1_{p,\infty}}.$$

On the other hand, the proof of Proposition 3.2 gives

$$\Big\| \big\{ 2^{kq} L_k \mathbb{E}_N^\perp \Pi_N f \big\}_{k=0}^\infty \Big\|_{L^p(\ell^q)} \lesssim \|\Pi_N f\|_{F^1_{p,2}},$$

and by Hölder's inequality one has

$$\|\Pi_N f\|_{F^1_{p,2}} \lesssim \Big\| \big(\sum_{j=0}^N |2^j \Lambda_j f|^2 \big)^{\frac{1}{2}} \Big\|_p \lesssim N^{\frac{1}{2}-\frac{1}{q}} \|f\|_{F^1_{p,q}}.$$

Combining the above inequalities one obtains $\|\mathbb{E}_N\|_{F^1_{p,q} \to F^1_{p,q}} \lesssim N^{\frac{1}{2}-\frac{1}{q}}$. □

Remark 7.2. If $1 < s < 1/p$, the upper bound becomes exponential:

$$\big\|\mathbb{E}_N\big\|_{F^s_{p,q} \to F^s_{p,q}} \lesssim 2^{(s-1)N},$$

for $(d-1)/d < p < 1$. This is a consequence of the simpler estimates for $\mathbb{E}_N - \Pi_N : B^s_{p,\infty} \to B^s_{p,r}$ shown in [5, Propositions 3.1 through 3.4]. From [4] we have also corresponding matching lower bounds, see the discussion in §8 below.

7.2. *Proof of Theorem 7.1: lower bounds.* To make the notation simpler, the counterexample is first presented in the 1-dimensional case, and later extended to $\mathbb{R}^d$ with a tensor product argument.

7.2.1. *The case $d = 1$.* Consider, for $s > 0$ and $\Lambda \subset \mathbb{N}$, a Weierstrass-type function

$$f(x) = \Big(\sum_{j \in \Lambda} \frac{a_j}{2^{sj}} e^{2\pi i 2^j x} \Big) \psi(x), \quad x \in \mathbb{R}, \tag{70}$$

with $\psi \in C_c^\infty(0,1)$, and say $\psi = 1$ on $[1/4, 3/4]$. These functions satisfy

$$\|f\|_{F^s_{p,q}(\mathbb{R})} \approx \Big\|\big(\sum_{k=0}^\infty |2^{ks}\beta_k * f|^q\big)^{\frac1q}\Big\|_p \lesssim \big(\sum_{j\in\Lambda} |a_j|^q\big)^{1/q}. \tag{71}$$

This can for instance be proved from Hardy's inequalities and the following lemma

Lemma 7.3. *Let β_k be as in §2, and $\psi_j(x) = e^{2\pi i 2^j x}\psi(x)$. Then*

$$|\beta_k * \psi_j(x)| \lesssim 2^{-|j-k|M}, \quad x \in \mathbb{R}, \quad j, k = 0, 1, 2, \dots$$

Proof. If $k > j$, using that β has M-vanishing moments,

$$\begin{aligned} |\beta_k * \psi_j(x)| &= \Big| \int_{\mathbb{R}} \beta(y)\big[\psi_j(x - 2^{-k}y) - \sum_{m=0}^{M-1} \psi_j^{(m)}(x)(-2^{-k}y)^m\big]\, dy\Big| \\ &\lesssim 2^{(j-k)M}, \end{aligned} \tag{72}$$

since $\|\psi_j^{(M)}\|_\infty \lesssim 2^{jM}$. If $k \le j$, then Fourier inversion gives, for any $M_1 > 1$,

$$\begin{aligned} |\beta_k * \psi_j(x)| &= \Big| \int_{\mathbb{R}} \hat\beta(\xi/2^k)\,\hat\psi(\xi - 2^j)\, e^{2\pi i x\xi}\, d\xi\Big| \\ &\lesssim \int_{\mathbb{R}} \frac{d\xi}{(1 + \frac{|\xi|}{2^k})^{M_1}\,(1 + |\xi - 2^j|)^{M_1}} \\ &\lesssim \int_{|\xi - 2^j| > 2^{j-1}} \frac{2^{-jM_1}\, d\xi}{(1 + \frac{|\xi|}{2^k})^{M_1}} + \int_{|\xi - 2^j| \le 2^{j-1}} \frac{2^{(k-j)M_1}\, d\xi}{(1 + |\xi - 2^j|)^{M_1}} \\ &\lesssim 2^k 2^{-jM_1} + 2^{(k-j)M_1} \end{aligned}$$

and we get the $O(2^{-|j-k|M})$ bound if we choose $M_1 \ge M$. □

We now let $s = 1$ and

$$\mathfrak{Z}_N = \{j \in \mathbb{N} : N/4 \le j \le N/2\},$$

and consider a randomized version of (70), namely

$$f_N(x,t) = \sum_{j\in\mathfrak{Z}_N} \frac{r_j(t)}{2^j}\, e^{2\pi i 2^j x}\, \psi(x), \tag{73}$$

where $r_j : [0,1] \to \{-1, 1\}$ is the sequence of Rademacher functions. Then, by (71),

$$\sup_{t\in[0,1]} \|f_N(\cdot,t)\|_{F^1_{p,q}} \lesssim N^{1/q}.$$

Below we shall show that

$$\Big(\int_0^1 \big\|\mathbb{E}_N[f_N(\cdot,t)]\big\|^p_{F^1_{p,\infty}}\, dt\Big)^{1/p} \ge cN^{1/2}. \tag{74}$$

The above inequality will be a consequence of the estimate

$$\Big(\int_0^1 \big\|2^N\,\beta_N*(\mathbb{E}_N[f_N(\cdot,t)])\big\|_p^p\,dt\Big)^{1/p}\geq cN^{1/2}, \tag{75}$$

where $\beta_N=2^{Nd}\beta(2^N\cdot)$ and β is a suitable test function satisfying the conditions in §2.1. Thus for some $t_0\in[0,1]$ the function

$$g_N\;=\;N^{-1/q}\,f_N(\cdot,t_0) \tag{76}$$

will satisfy

$$\|\mathbb{E}_N g_N\|_{F^1_{p,q}(\mathbb{R})}\gtrsim\big\|2^N\,\beta_N*(\mathbb{E}_N g_N)\big\|_{L^p(\mathbb{R})}\gtrsim N^{\frac12-\frac1q}, \tag{77}$$

and in particular

$$\big\|\mathbb{E}_N\big\|_{F^1_{p,q}\to F^1_{p,q}}\;\gtrsim\;N^{\frac12-\frac1q}. \tag{78}$$

By Fubini's theorem and Khintchine's inequality the expression in (75) is equivalent to

$$2^N\,\Big\|\big(\sum_{j\in\mathfrak{Z}_N}\big|2^{-j}\,\beta_N*(\mathbb{E}_N\psi_j)\big|^2\big)^{\frac12}\Big\|_{L^p(\mathbb{R})}\geq cN^{1/2}. \tag{79}$$

If the operator $\mathbb{E}_N$ is omitted in the left hand side, then this quantity becomes uniformly bounded by Lemma 7.3, so (79) is also equivalent to

$$2^N\,\Big\|\big(\sum_{j\in\mathfrak{Z}_N}\big|2^{-j}\,\beta_N*(\mathbb{E}_N^\perp\psi_j)\big|^2\big)^{\frac12}\Big\|_{L^p(\mathbb{R})}\geq cN^{1/2}. \tag{80}$$

Below we fix β such that $\operatorname{supp}\beta=(-1/8,1/8)$, and denote its primitive by $B(x)=\int_{-\infty}^x\beta(u)du$, which also belongs to $C_c^\infty(-1/8,1/8)$. The following lemma is similar to [5, Lemma 6.4], but we include its proof below for completeness.

Lemma 7.4. *Let $\mu\in\mathbb{Z}$ and let $\widetilde{I}_{N,\mu}=[\frac{\mu}{2^N},\frac{\mu+1/8}{2^N}]$. Then*

$$\beta_N*(\mathbb{E}_N^\perp\psi_j)(x)=-2^{-N}\psi_j'(\tfrac{\mu}{2^N})\,B(2^Nx-\mu)+O\big(2^{2(j-N)}\big),\quad x\in\widetilde{I}_{N,\mu}. \tag{81}$$

Moreover, if $\frac{\mu}{2^N}\in[\frac14,\frac34]$, then $\psi_j'(\frac{\mu}{2^N})=2\pi i2^je^{2\pi i2^jx}$.

Assuming the lemma, the pth power of the left hand side of (80) can be bounded from below by

$$\sum_{\substack{\mu\in\mathbb{Z}\\ \frac14\le\frac{\mu}{2^N}\le\frac34}} \int_{\widetilde I_{N,\mu}} \Big(\sum_{j\in\Lambda}\big|2^{N-j}\,\beta_N*(\mathbb{E}_N^\perp\psi_j)\big|^2\Big)^{\frac p2}\,dx$$

$$\ge (2\pi)^p \sum_{\substack{\mu\in\mathbb{Z}\\ \frac14\le\frac{\mu}{2^N}\le\frac34}} \int_{\widetilde I_{N,\mu}} \Big(\sum_{j\in\Lambda}\big|B(2^Nx-\mu)\big|^2\Big)^{\frac p2} - c\big(\sum_{j\in\Lambda}|2^{j-N}|^2\big)^{\frac p2}\,dx$$

$$\gtrsim N^{\frac p2} \sum_{\substack{\mu\in\mathbb{Z}\\ \frac14\le\frac{\mu}{2^N}\le\frac34}} \int_{\widetilde I_{N,\mu}} \big|B(2^Nx-\mu)\big|^p\,dx \;-\; c'\,2^{-Np/2}$$

$$\gtrsim N^{\frac p2}\int_0^{1/8}\big|B(u)\big|^p\,du \;-\; c'\,2^{-Np/2} \;\gtrsim\; N^{\frac p2},$$

using in the last step that β (hence B) is not identically null in $(0,1/8)$. This finishes the proof modulo Lemma 7.4.

Proof of Lemma 7.4. For simplicity we write $I^+=I_{N,\mu}$ and $I^-=I_{N,\mu-1}$. If $x\in\widetilde I_{N,\mu}$, then $\operatorname{supp}\beta_N(x-\cdot)\subset I^+\cup I^-$, and thus

$$\beta_N*(\mathbb{E}_N^\perp\psi_j)(x)=\sum_{\pm}\int_{I^\pm}\beta_N(x-y)\Big[\psi_j(y)-⨍_{I^\pm}\psi_j\Big]\,dy. \tag{82}$$

Now, if $y\in I^\pm$, using the linear Taylor's expansion of ψ_j around y and the bound $\|\psi_j''\|_\infty\lesssim 2^{2j}$, the inner bracketed expression becomes

$$\begin{aligned}
⨍_{I^\pm}(\psi_j(y)-\psi_j(w))dw &= ⨍_{I^\pm}\psi_j'(y)(y-w)\,dw+O(2^{2(j-N)})\\
&= \psi_j'(\tfrac{\mu}{2^N})⨍_{I^\pm}(y-w)\,dw+O(2^{2(j-N)})\\
&= \psi_j'(\tfrac{\mu}{2^N})\,(y-c_{I^\pm})\;+O(2^{2(j-N)})\\
&= \psi_j'(\tfrac{\mu}{2^N})\,(y-\tfrac{\mu}{2^N})\;-\psi_j'(\tfrac{\mu}{2^N})\,\tfrac{\pm1}{2^{N+1}}+O(2^{2(j-N)}).
\end{aligned}$$

Putting these quantities into (82), and using the support and the moment condition of β, we are left with

$$\beta_N*(\mathbb{E}_N^\perp\psi_j)(x)=-\sum_{\pm}\psi_j'(\tfrac{\mu}{2^N})\,\tfrac{\pm1}{2^{N+1}}\int_{I_\pm}\beta_N(x-y)\,dy\;+O(2^{2(j-N)}). \tag{83}$$

Now, an elementary computation using the primitive, $B(u)$, of $\beta(u)$ shows that the two integrals substructed above can be written as

$$\int_{\frac{\mu}{2^N}}^{\frac{\mu+1}{2^N}}-\int_{\frac{\mu-1}{2^N}}^{\frac{\mu}{2^N}}\beta_N(x-y)dy \quad=\int_{\mu}^{\mu+1}-\int_{\mu-1}^{\mu}\beta(2^Nx-u)\,du=2B(2^Nx-\mu),$$

since $B(2^N x - \mu \pm 1) = 0$ by the support condition. Thus, placing this expression into (83) implies the asserted identity (81). □

7.2.2. *The d-dimensional case.* Consider $G_N(x_1, x') = g_N(x_1)\chi(x')$, where g_N is the 1-dimensional function in (76), and $\chi \in C_c^\infty(0,1)^{d-1}$ with $\chi \equiv 1$ in $[\frac{1}{8}, \frac{7}{8}]^{d-1}$. We shall show that

$$\|G_N\|_{F^1_{p,q}(\mathbb{R}^d)} \lesssim 1 \tag{84}$$

and

$$\big\|\mathbb{E}_N G_N\big\|_{F^1_{p,q}(\mathbb{R}^d)} \gtrsim N^{\frac{1}{2}-\frac{1}{q}}. \tag{85}$$

To do so, in the definition of the $F^s_{p,q}$-quasinorms we shall use suitable test functions of tensor product type; see also [5, §5.1] for a similar argument. Namely, for a fixed $M \in \mathbb{N}$ we consider a non-negative even function $\phi_0 \in C_c^\infty(-\frac{1}{8}, \frac{1}{8})$ such that $\phi_0^{(2M)}(t) > 0$ for all t in some interval $[-2\varepsilon, 2\varepsilon]$. Since $\widehat{\phi}_0(0) = \int \phi_0 > 0$, dilating if necessary we may also assume that $\widehat{\phi}_0 \neq 0$ on $[-1,1]$. Let $\varphi_0 \in C_c^\infty((-\frac{1}{8}, \frac{1}{8})^{d-1})$ be such that $\widehat{\varphi}_0 \neq 0$ on $[-1,1]^{d-1}$ and $\widehat{\varphi}_0(0) = 1$. For $M \geq 1$, let

$$\phi(t) := (\tfrac{d}{dt})^{2M}\phi_0(t), \qquad \varphi(x_2, \ldots, x_d) := \big(\tfrac{\partial^2}{\partial x_2^2} + \cdots + \tfrac{\partial^2}{\partial x_d^2}\big)^M \varphi_0(x').$$

Then, we define

$$\Psi(x) := \Delta^M[\phi_0 \otimes \varphi_0](x) = \phi(x_1)\varphi_0(x') + \phi_0(x_1)\varphi(x'). \tag{86}$$

Clearly,

$$\int_{\mathbb{R}^d} \Psi(y) y_1^{m_1} \cdots y_d^{m_d}\, dy = 0, \quad \text{when } m_1 + \ldots + m_d < 2M.$$

Finally, let $\Psi_0 = \phi_0 \otimes \varphi_0$, and $\Psi_k(x) = 2^{kd}\Psi(2^k x)$, $k \geq 1$. Then, if we choose M sufficiently large we have

$$\|f\|_{F^s_{p,q}} \approx \Big\|\big\{2^{ks}\Psi_k * f\big\}_{k\geq 0}\Big\|_{L^p(\ell^q)}, \quad \forall\, f \in F^s_{p,q}(\mathbb{R}^d); \tag{87}$$

see e.g. [14, 2.4.6]. Observe that, for $k \geq 1$ we can write

$$\Psi_k = \phi_k \otimes \varphi_{0,k} \, + \, \phi_{0,k} \otimes \varphi_k, \tag{88}$$

where we denote

$$\phi_k(x_1) = 2^k \phi(2^k x_1), \quad \varphi_k(x') = 2^{(d-1)k}\varphi(2^k x'),$$

and likewise for $\phi_{0,k}$ and $\varphi_{0,k}$.

With this notation the inequality in (84) is easily proved as follows. From (88) and $\|g_N\|_\infty, \|\chi\|_\infty \lesssim 1$ one obtains

$$|\Psi_k * G_N|(x_1, x') \lesssim |\phi_k * g_N|(x_1) + |\varphi_k * \chi|(x'), \quad k \geq 1,$$

and a similar (simpler) expression when $k = 0$. Therefore (87) and the compact support of the involved functions imply

$$\|G_N\|_{F^1_{p,q}(\mathbb{R}^d)} \lesssim \|g_N\|_{F^1_{p,q}(\mathbb{R})} \, + \, \|\chi\|_{F^1_{p,q}(\mathbb{R}^{d-1})} \lesssim 1.$$

In order to prove (85), we let $\mathbb{E}_N^{(1)}$ and $\mathbb{E}_N^{(d-1)}$ be the dyadic averaging operators on $\mathbb{R}$ and $\mathbb{R}^{d-1}$, respectively. For $N \geq 1$, we observe that

$$\Psi_N * (\mathbb{E}_N G_N)(x_1, x') = \phi_N * (\mathbb{E}_N^{(1)} g_N)(x_1), \text{ for } x' \in (\tfrac{1}{4}, \tfrac{3}{4})^{d-1}. \tag{89}$$

Indeed, for such x' one has

$$\varphi_{0,N} * (\mathbb{E}_N^{(d-1)} \chi)(x') = \int \varphi_0(y') dy' = 1,$$
$$\varphi_N * (\mathbb{E}_N^{(d-1)} \chi)(x') = \int \varphi(y') dy' = 0,$$

due to the support properties of $\varphi_{0,N}(x' - \cdot)$ and $\varphi_N(x' - \cdot)$. Therefore, (87) and (89) imply that

$$\|\mathbb{E}_N G_N\|_{F^1_{p,q}(\mathbb{R}^d)} \gtrsim \big\|2^N \phi_N * \mathbb{E}_N^{(1)} g_N\big\|_{L^p(\mathbb{R})} \gtrsim N^{\frac{1}{2} - \frac{1}{q}},$$

the last inequality due to (77). This proves (85), and concludes the proof of

$$\big\|\mathbb{E}_N\big\|_{F^1_{p,q}(\mathbb{R}^d) \to F^1_{p,q}(\mathbb{R}^d)} \gtrsim N^{\frac{1}{2} - \frac{1}{q}}, \quad 2 < q \leq \infty.$$

□

8. Boundedness of the dyadic averaging operators and the proof of Theorem 1.4

We now gather the results from the previous sections to complete the proof of Theorem 1.4. In §8.1 we first explain how the extension of $\mathbb{E}_N$, from $\mathcal{S}$ into $F^s_{p,q}$, should be defined (which is not obvious for all cases). We discuss sufficient conditions for uniform boundedness in §8.2 using theorems in previous chapters, and necessary conditions in §8.3. The proofs of some of the more tedious details about definability are given in §8.4. The proof of necessary conditions for the individual boundedness of the $\mathbb{E}_N$ is given in §8.5. In §8.6 we include a discussion when the characteristic function of a bounded interval can be defined as a linear functional on $F^s_{p,q}$.

8.1. *Extension of the operators $\mathbb{E}_N$ to the space $F^s_{p,q}$.* Let (s, p, q) be as in (i)-(v). For a distribution $f \in F^s_{p,q}$ we define

$$\mathbb{E}_N f := \sum_{j=0}^{\infty} \mathbb{E}_N (L_j \Lambda_j f). \tag{90}$$

We claim that this series always converges in the $F^s_{p,q}$-norm (actually, in all the $B^s_{p,r}$-norms, for $r > 0$). When $\max\{d/p - d, 1/p - 1\} < s < 1/p$ this fact was already justified in [5, Remarks 3.5 and 4.5]. When $s = d/p - d$, one can reach the same conclusion with a slight modification in the proof of Proposition 4.2. We present the details in Lemma 8.4 below. When $s = 0$ and $p = \infty$, the convergence holds in all $F^0_{\infty,r}$-norms, for $r > 0$, by Remark 6.2.

We also remark that when $f \in F^s_{p,q}$ is locally integrable with polynomial growth then the above extension coincides with the usual operator, that is

$$\sum_{j\geq 0} \mathbb{E}_N(L_j\Lambda_j f) = \sum_{I\in\mathscr{D}_N} \Big(\fint_I f\Big)\mathbb{1}_I;$$

see Lemma 8.5 below.

8.2. *Sufficient conditions in Theorem 1.4.* The uniform boundedness of $\mathbb{E}_N$ in $F^s_{p,q}$ in the cases (i) and (iii) was established in [4]. In the cases (ii), (iv) and (v) it is a consequence of the Theorems 1.5, 1.6 and 1.7, and elementary embeddings; see the discussion following (10) in section 1.

8.3. *Necessary conditions in Theorem 1.4.* We first identify the range of exponents for the continuity of an individual operator $\mathbb{E}_N$.

Definition 8.1. Let $\mathfrak{A}$ be the set of all (s,p,q) for which one of the following three conditions (i), (ii) or (iii) hold:

$$(91)\qquad \begin{cases} \text{(i)} & \max\{\frac{d}{p}-d,\, \frac{1}{p}-1\} < s < 1/p, \quad 0<p,q\leq\infty \\ \text{(ii)} & s=\frac{d}{p}-d, \quad 0<p\leq 1, \quad 0<q\leq\infty \\ \text{(iii)} & s=0, \quad p=\infty, \quad 0<q\leq\infty. \end{cases}$$

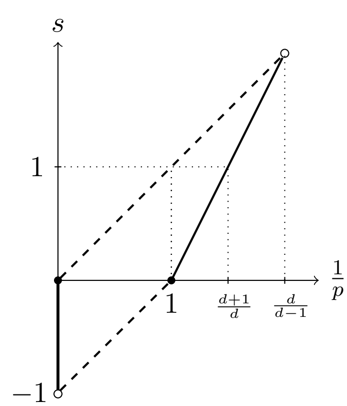

FIGURE 3. Range of exponents for the continuity of the individual operators $\mathbb{E}_N$ in $F^s_{p,q}(\mathbb{R}^d)$.

Proposition 8.2. *Let $N \in \mathbb{N}_0$ be fixed. Suppose that*

$$(92)\qquad \big\|\mathbb{E}_N\psi\big\|_{F^s_{p,q}} \leq c_N\|\psi\|_{F^s_{p,q}}, \quad \forall\, \psi\in\mathcal{S}(\mathbb{R}^d).$$

Then necessarily $(s,p,q)\in\mathfrak{A}$.

The proof of the proposition is based on the fact that $\mathbb{1}_{[0,1)^d}$ must belong to both $F^s_{p,q}$ and its dual space. We present the details in §8.5 below.

We now turn to the existence of uniformly bounded extensions of the operators $\mathbb{E}_N$ in the above region. This is equivalent to the uniform boundedness of the numbers

$$\mathrm{Op}_{\mathcal{S}}(\mathbb{E}_N, F^s_{p,q}) := \sup\Big\{\|\mathbb{E}_N f\|_{F^s_{p,q}} \;:\; f\in\mathcal{S},\ \|f\|_{F^s_{p,q}}\leq 1\Big\},$$

when $N = 0, 1, 2, \dots$ An example given in [4, Proposition 4.2] shows that,

$$\mathrm{Op}_{\mathcal{S}}(\mathbb{E}_N, F^s_{p,q}) \gtrsim \mathrm{Op}_{\mathcal{S}}(\mathbb{E}_N, B^s_{p,\infty}) \gtrsim 2^{(s-1)N},$$

when $1 < s < 1/p$ and $(d-1)/d < p < 1$. So the condition $s \leq 1$ is necessary. When $s = 1$, Theorem 7.1 shows that $0 < q \leq 2$ is also necessary. This comprises all the cases considered in Theorem 1.4, and completes the proof of all the assertions. Moreover, it also gives the following.

Corollary 8.3. *Let $N \in \mathbb{N}_0$ be fixed, and $(s, p, q) \in \mathfrak{A}$. Then the (extended) operator $\mathbb{E}_N$, as in (90), satisfies*

$$\mathrm{Op}_{\mathcal{S}}(\mathbb{E}_N, F^s_{p,q}) \approx \big\|\mathbb{E}_N\big\|_{F^s_{p,q}\to F^s_{p,q}} \approx \begin{cases} 2^{(s-1)N} & \text{if } 1 < s < 1/p \\ N^{1/2-1/q} & \text{if } s = 1,\ q \geq 2 \\ 1 & \text{otherwise.}\end{cases}$$

8.4. *Convergence of $\sum_j \mathbb{E}_N(L_j\Lambda_j f)$ when $f \in F^{d/p-d}_{p,\infty}$.*

Lemma 8.4. *Let $\frac{d-1}{d} < p \leq 1$, $s = d/p - d$ and $r > 0$. If $N \geq 0$ and $g \in F^s_{p,\infty}(\mathbb{R}^d)$, then*

$$\lim_{J_1\to\infty} \sup_{J_2\geq J_1} \Big\|\mathbb{E}_N\Big(\sum_{j=J_1}^{J_2} L_j\Lambda_j g\Big)\Big\|_{B^s_{p,r}} = 0.$$

In particular, the series

$$\mathbb{E}_N g := \sum_{j=0}^{\infty} \mathbb{E}_N(L_j\Lambda_j g)$$

converges in $F^s_{p,q}$ for all $0 < q \leq \infty$.

Proof. Pick a non-negative function $\zeta \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\zeta \geq 1 \quad \text{on } [-5,5]^d, \quad \text{and} \quad \mathrm{supp}\,\widehat{\zeta} \subset \{|\xi| \leq 1/8\}.$$

For each cube $Q = \{x \;:\; |x - x_0|_\infty \leq \delta\}$, let

$$\zeta_Q(x) := \zeta\Big(\frac{x - x_0}{\delta}\Big),$$

so that $\zeta_Q \geq 1$ in Q^{**} (the 5-fold dilate of Q), and $\widehat{\zeta}_Q$ has support in $\{|\xi| \leq (8\delta)^{-1}\}$. Finally, for $j \geq N + 1$, we define (with $\mathcal{U}_{N,j}$ as in (33))

$$\zeta_{j,N}(x) := \sum_{\substack{Q\in\mathscr{D}_{j+1}\\ Q\subset\mathcal{U}_{N,j}}} \zeta_Q(x).$$

This function satisfies the properties

$$|\zeta_{j,N}(x)| \le \frac{c_M}{\big(1+2^j \min_{1\le i\le d} \operatorname{dist}(x_i, 2^{-N}\mathbb{Z})\big)^M}, \tag{93}$$

for each $M>0$, and

$$\operatorname{supp} \widehat{\zeta}_{j,N} \subset \big\{|\xi| \le 2^{j-3}\big\}. \tag{94}$$

Let $f=\sum_{j=J_1}^{J_2} L_j\Lambda_j g$, and assume that $J_1 \ge L+3$ for some fixed $L>N$. Then, $f \in F^s_{p,\infty}$ and $\Lambda_j f \equiv 0$ unless $j \ge L$. We now follow the proof of Proposition 4.2, with the following modification. If $J \in \mathscr{D}_{j+1}$ is such that $J \subset \mathcal{U}_{N,j}$, then for all $y \in J$,

$$\mathfrak{M}^*_j(\Lambda_j f)(y) = \sup_{|h|_\infty \le 2^{-j}} |\Lambda_j f(y+h)| \le \mathfrak{M}^*_j\big(\zeta_{j,N}\Lambda_j f\big)(y).$$

One can use this estimate in (45) (or in (52)), so the same arguments which lead to (47) (or to (54)) can be applied with the function $\Lambda_\ell f$ replaced by $\zeta_{\ell,N}\Lambda_\ell f$. That is there exists $\gamma>0$ such that for A_k, $\mathcal{A}_k$ as in (45), (53), resp.,

$$2^{k(\frac{d}{p}-d)}(\|A_k\|_p + \|\mathcal{A}_k\|_p) \lesssim 2^{-|k-N|\gamma}\, \|G_L\|_p,$$

with

$$G_L(x) = \sup_{\ell \ge L}\; 2^{\ell(\frac{d}{p}-d)}\mathfrak{M}^*_\ell\big(\zeta_{\ell,N}\Lambda_\ell f\big).$$

Since the spectrum of $\zeta_{\ell,N}\Lambda_\ell f$ is contained in $\{|\xi|\le 2^\ell\}$, one can use Peetre's inequality and deduce as in (48) that

$$\|G_L\|_p \lesssim \Big\|\sup_{\ell\ge L} 2^{\ell(\frac{d}{p}-d)}\zeta_{\ell,N}\, |\Lambda_\ell f|\Big\|_p \lesssim \Big\|\zeta^*_{L,N}\, \sup_{\ell \ge 0} 2^{\ell(\frac{d}{p}-d)}\, |L_\ell \Lambda_\ell g|\Big\|_p,$$

where

$$\zeta^*_{L,N}(x) = \sup_{\ell\ge L}\zeta_{\ell,N}(x) \lesssim \Big(1+2^L \min_{1\le i\le d}(x_i, 2^{-N}\mathbb{Z})\Big)^{-M}.$$

Observe that

$$\lim_{L\to\infty} \zeta^*_{L,N}(x) = 0, \quad \forall\, x \in \bigcup_{n=1}^\infty \mathcal{U}^{\complement}_{N,n},$$

and therefore at almost every $x\in\mathbb{R}^d$. So, the assumption $g \in F^s_{p,\infty}$ and the Dominated Convergence Theorem imply that

$$\lim_{L\to\infty}\|G_L\|_p = 0.$$

□

Lemma 8.5. *Let (s,p,q) be as in (i)-(v) in Theorem 1.4. Let $f \in F^s_{p,q}$ be locally integrable with polynomial growth, that is, $f(x)/(1+|x|)^M \in L^1(\mathbb{R}^d)$ for some $M \ge 0$. Then*

$$\sum_{j\ge 0}\mathbb{E}_N(L_j\Lambda_j f) = \sum_{I\in\mathscr{D}_N}\Big(\fint_I f\Big)\mathbb{1}_I, \tag{95}$$

in the sense of tempered distributions.

Proof. In this lemma we restrict the notation

$$\mathbb{E}_N g(x) := \sum_{I\in\mathscr{D}_N} \Big(\fint_I g\Big)\mathbb{1}_I(x), \quad x\in\mathbb{R}^d,$$

only to locally integrable functions g with polynomial growth. In particular, $\mathbb{E}_N g$ is another such function, hence a tempered distribution. We write $\widetilde{\mathbb{E}}_N f$ for the distribution on the left hand side of (95). If $\psi\in\mathcal{S}(\mathbb{R}^d)$, then

$$(96) \qquad \big(\widetilde{\mathbb{E}}_N f,\psi\big) = \sum_{j=0}^{\infty}\big(\mathbb{E}_N(L_j\Lambda_j f),\psi\big) = \sum_{j=0}^{\infty}\int_{\mathbb{R}^d}(L_j\Lambda_j f)\,\mathbb{E}_N\psi.$$

The family of operators $\{\mathit{\Pi}_n = \sum_{j=0}^{n} L_j\Lambda_j\}_{n\geq 0}$ is a smooth approximation of the identity, and therefore

$$\mathit{\Pi}_n f\to f \quad \text{in } L^1(\mathbb{R}^d,(1+|x|)^{-M}dx),$$

by the condition on f. Therefore, using that $|\mathbb{E}_N\psi(x)|\lesssim (1+|x|)^{-M}$, we can pass the sum inside the integral in the last expression of (96), and continuing with Fubini's theorem obtain

$$\big(\widetilde{\mathbb{E}}_N f,\psi\big) = \int_{\mathbb{R}^d}(\sum_{j=0}^{\infty}L_j\Lambda_j f)\,\mathbb{E}_N\psi = \int_{\mathbb{R}^d} f\,\mathbb{E}_N\psi = \int_{\mathbb{R}^d}\mathbb{E}_N f\,\psi.$$

Hence $\widetilde{\mathbb{E}}_N f$ coincides with $\mathbb{E}_N f$ as distributions. □

Remark 8.6. One can extend the domain of $\mathbb{E}_N$ further, dropping the polynomial growth assumption in Lemma 8.5 if in the resolution of the identity (21) we replace the operator Λ_N by suitable compactly supported convolution kernels. Indeed there are, for $\varepsilon>0$, $M<\infty$, C^∞ functions ϕ, $\widetilde{\phi}$, ψ, $\widetilde{\psi}$ supported in $\{|x|\leq\varepsilon\}$ such that $\int\phi=1$, $\int\widetilde{\phi}=1$ and $1-\widehat{\phi}$, $1-\widehat{\widetilde{\phi}}$, $\widehat{\psi}$, $\widehat{\widetilde{\psi}}$ all vanish of order M at 0, and such that for distributions f

$$(97) \qquad L_0\widetilde{L}_0 f + \sum_{k=1}^{\infty} L_k\widetilde{L}_k f = f$$

in the sense of distributions; here L_0, $\widetilde{L}_0$ are the convolution operators with convolution kernels ϕ, $\widetilde{\phi}$, resp., and for $k\geq 1$, L_k and $\widetilde{L}_k$ are the convolution operators with convolution kernels $2^{(k-1)d}\psi(2^{k-1}\cdot)$, $2^{(k-1)d}\widetilde{\psi}(2^{k-1}\cdot)$, resp. The resolution in the form (97) is perhaps not widely known; a proof can be found in [9, Lemma 2.1], together with some extensions. For us the use of the nonlocal operators Λ_N has the advantage that we may apply the Peetre maximal inequalities in a straightforward way.

8.5. *Proof of Proposition 8.2***.** Since $\mathbb{E}_N[\psi](x) = \mathbb{E}_0[\psi(2^{-N}\cdot)](2^N x)$, we may assume that $N = 0$. Then (92) takes the form

$$\big\|\mathbb{E}_0\psi\big\|_{F^s_{p,q}} \le c_N \|\psi\|_{F^s_{p,q}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{98}$$

Let $\psi \in C_c^\infty((0,1)^d)$ such that $\int \psi = 1$. Then $\mathbb{E}_0(\psi) = \mathbb{1}_{[0,1)^d}$, and (98) implies

$$\mathbb{1}_{[0,1)^d} \in F^s_{p,q}(\mathbb{R}^d). \tag{99}$$

The validity of this property is well-known. If $0 < p < \infty$, then (99) holds iff $s < 1/p$. If $p = \infty$, then (99) holds iff $s \le 0$. See e.g. [16, Proposition 2.50]. This gives the required upper bounds on s.

We turn to the lower bounds for the exponent s. Consider the classes of test functions

$$\begin{aligned}\mathcal{F}_1 &= \operatorname{span}\Big\{\eta(x_1)\cdots\eta(x_d) \ : \ \eta \in C_c^\infty(-1,1) \ \text{ odd}\Big\}\\ \mathcal{F}_2 &= \Big\{\eta(x_1)\chi(x') \ : \ \eta \in C_c^\infty(-1,1) \ \text{ odd}, \ \chi \in C_c^\infty(0,1)^{d-1} \text{ with } \textstyle\int\chi = 1\Big\}.\end{aligned}$$

We first show that, if $f \in \mathcal{F}_i$, $i = 1,2$, then

$$\mathbb{E}_0(f) \ = \ \Big(\int_{[0,1]^d} f\Big) \cdot h_i, \tag{100}$$

for some fixed functions $h_1, h_2 \in \operatorname{span}\{\mathbb{1}_I \ : \ I \in \mathscr{D}_0\}$.

Let $f \in \mathcal{F}_1$. It suffices to show (100) for $f(x) = \eta(x_1)\cdots\eta(x_d)$, with $\eta \in C_c^\infty(-1,1)$ odd. Given $\varepsilon = (\varepsilon_1,\dots,\varepsilon_d) \in \{0,1\}^d$ we denote

$$Q_\varepsilon = [0,1)^d - \varepsilon \quad \text{and} \quad \operatorname{sign}(\varepsilon) = \prod_{i=1}^d (-1)^{\varepsilon_i}.$$

We claim that (100) holds with

$$h_1 = \sum_{\varepsilon \in \{0,1\}^d} \operatorname{sign}(\varepsilon)\, \mathbb{1}_{Q_\varepsilon}. \tag{101}$$

Indeed, for such f we have

$$\mathbb{E}_0(f) = \sum_{\varepsilon\in\{0,1\}^d} \fint_{Q_\varepsilon} f \cdot \mathbb{1}_{Q_\varepsilon},$$

and since η is odd

$$\fint_{Q_\varepsilon} f = \prod_{i=1}^d \int_{[0,1)-\varepsilon_i} \eta(x_i)\,dx_i = \prod_{i=1}^d (-1)^{\varepsilon_i} \int_0^1 \eta(x_i)\,dx_i = \operatorname{sign}(\varepsilon) \int_{[0,1)^d} f.$$

Thus, (100) follows.

Similarly, let $f \in \mathcal{F}_2$, and denote $Q_0 = [0,1)^d$ and $Q_1 = Q_0 - (1,0,\dots,0)$. Then

$$\mathbb{E}_0(f) = \fint_{Q_0} f \cdot \mathbb{1}_{Q_0} + \fint_{Q_1} f \cdot \mathbb{1}_{Q_1} = \Big(\int_{Q_0} f\Big)\big(\mathbb{1}_{Q_0} - \mathbb{1}_{Q_1}\big),$$

and hence (100) holds with $h_2 = \mathbb{1}_{Q_0} - \mathbb{1}_{Q_1}$. This completes the proof of (100), and reduces the proof of Proposition 8.2 to the following result.

Lemma 8.7. *Let $I = (0,1)^d$. Suppose that*

$$\Big| \int_I f \Big| \le C \|f\|_{F^s_{p,q}}, \quad \forall f \in \mathcal{F}_1 \cup \mathcal{F}_2. \tag{102}$$

Then one of the following two conditions must hold

(a) $s > \max\{\frac{d}{p} - d, \frac{1}{p} - 1\}, \quad 0 < p, q \le \infty$

(b) $s = \frac{d}{p} - d, \quad 0 < p \le 1, \quad 0 < q \le \infty.$

Proof. Let $\eta \in C^\infty_c(-1/2, 1/2)$, odd and such that $\int_0^1 \eta = 1$. Define

$$g_j(x_1, \dots, x_d) := 2^{jd} \eta(2^j x_1) \cdots \eta(2^j x_d), \quad j \ge 1.$$

Observe that $g_j \in \mathcal{F}_1$ and $\int_I g_j(x)\, dx = 1$. On the other hand, a standard computation shows that

$$\|g_j\|_{F^s_{p,q}(\mathbb{R}^d)} \lesssim 2^{-j(\frac{d}{p} - d - s)}. \tag{103}$$

So, (102) cannot hold unless $s \ge d/p - d$. Suppose now that $s = d/p - d$, and consider $g = \sum_{j=1}^N g_j \in \mathcal{F}_1$. In this case we have $\int_I g(x)\, dx = N$ and $\|g\|_{F^s_{p,q}(\mathbb{R}^d)} \lesssim N^{1/p}$. Thus, (102) cannot hold unless $0 < p \le 1$. Observe that when $d = 1$ this completes the proof of the lemma.

Suppose now that $d \ge 2$. Define now the functions

$$G_j(x_1, x') := g_j(x_1)\, \chi(x'), \quad j \ge 1,$$

where $\chi \in C^\infty_c(0,1)^{d-1}$ has $\int \chi = 1$. Then, $G_j \in \mathcal{F}_2$ and $\int_I G_j = 1$. On the other hand

$$\|G_j\|_{F^s_{p,q}(\mathbb{R}^d)} \lesssim \|g_j\|_{F^s_{p,q}(\mathbb{R})} \lesssim 2^{-j(\frac{1}{p} - 1 - s)}.$$

So (102) can only hold if $s \ge 1/p - 1$. In the case $s = 1/p - 1$, consider the function $G = \sum_{j=1}^N G_j \in \mathcal{F}_2$. Then we have

$$\int_I G = N \quad \text{and} \quad \|G\|_{F^s_{p,q}(\mathbb{R}^d)} \lesssim \|g\|_{F^s_{p,q}(\mathbb{R})} \lesssim N^{1/p}.$$

Thus, (102) can only hold if $0 < p \le 1$. This does not give a new region if $d \ge 2$. □

8.6. *Extension of $\mathbb{1}_I$ as a bounded functional in* $F^s_{p,q}$**.** The next result is a converse of Lemma 8.7, which in addition gives a continuous extension of the functional $\mathbb{1}_I$ to the whole space $F^s_{p,q}$.

Lemma 8.8. *Let (s,p,q) be numbers satisfying (a) or (b) in Lemma 8.7, and let $I \in \mathscr{D}_N$. Then, for each $f \in F^s_{p,q}$ the series*

$$\mathbb{1}^*_I(f) := \sum_{j=0}^\infty \int_I L_j \Lambda_j f \tag{104}$$

is absolutely convergent, and there exists a constant $C_N = C_N(s,p,q) > 0$ such that

$$\big|\mathbb{1}_I^*(f)\big| \le C_N \, \|f\|_{F^s_{p,q}}, \;\; \textit{for all } f\in F^s_{p,q}.$$

Moreover,

(1) *(104) does not depend on the specific decomposition $I = \sum_{j=0}^\infty L_j\Lambda_j$,*
(2) *If $f\in F^s_{p,q}(\mathbb{R}^d)$ is locally integrable with polynomial growth then*

$$\mathbb{1}_I^*(f) = \int_I f(x)\,dx.$$

(3) *For every $\zeta\in C_c^\infty$ such that $\zeta\equiv 1$ in a neighborhood of $\bar I$, it holds*

$$\mathbb{1}_I^*(\zeta f) \;=\; \mathbb{1}_I^*(f), \; \textit{for all } f\in F^s_{p,q}. \tag{105}$$

Proof. Since $F^s_{p,q}\hookrightarrow F^s_{p,\infty}$, it suffices to prove the result for $q=\infty$. First notice that

$$\Big|\sum_{j=0}^N \int_I L_j\Lambda_j f\Big| \le \int_I \big|\Pi_N f\big| \le |I|\,\big\|\Pi_N f\|_{L^\infty(I)}.$$

Now, using the Peetre maximal functions,

$$\big\|\Pi_N f\|_{L^\infty(I)} \lesssim \Big(\fint_{I^{**}} \big|\mathfrak{M}^*_N(\Pi_N f)\big|^p\Big)^{1/p} \lesssim 2^{\frac{Nd}{p}}\,\big\|\Pi_N f\big\|_p,$$

and letting $r=\min\{p,1\}$, the last factor is bounded by

$$\big\|\Pi_N f\big\|_p \le \Big(\sum_{j=0}^N \|L_j\Lambda_j f\|_p^r\Big)^{\frac1r} \lesssim \Big(\sum_{j=0}^N 2^{-jsr}\Big)^{\frac1r}\,\|f\|_{B^s_{p,\infty}}.$$

So, we are left with proving that

$$\sum_{j>N}\Big|\int_I L_j\Lambda_j f\Big| \lesssim C_N\,\|f\|_{F^s_{p,\infty}}.$$

Since $j>N$, we can use [4, Lemma 2.3.ii] and inequality (52) to obtain

$$\begin{aligned}\Big|\int_I L_j\Lambda_j f\Big| \;&\lesssim\; 2^{-jd}\sum_{J\in\mathscr{D}_{j+1}(\partial I)} \|\Lambda_j f\|_{L^\infty(J)}\\ &\lesssim\; 2^{-jd}\sum_{J\in\mathscr{D}_{j+1}(\partial I)} \Big(\fint_{w(J)}\big|\mathfrak{M}^*_j(\Lambda_j f)\big|^p\Big)^{1/p}.\end{aligned}$$

In the case (b), *i.e.* $s=d/p-d$ and $0<p\le 1$, we argue as in (45) and obtain

$$\sum_{j>N}\Big|\int_I L_j\Lambda_j f\Big|^p \lesssim \int_{I^{**}}\sup_{\ell>N}\big|2^{\ell(\frac dp-d)}\mathfrak{M}^*_\ell(\Lambda_\ell f)\big|^p \lesssim \|f\|^p_{F^{d/p-d}_{p,\infty}}.$$

In the case (a), *i.e.* $s > \max\{\frac{d}{p} - d, \frac{1}{p} - 1\}$, one can prove in a similar fashion the stronger estimate

$$\sum_{j>N} \Big| \int_I L_j \Lambda_j f \Big|^r \lesssim C_N(s,p) \, \|f\|^r_{B^s_{p,\infty}}$$

with $r = \min\{p, 1\}$.

It is immediate to verify that (104) does not depend on the specific resolution of the identity $I = \sum_{j=0}^{\infty} L_j \Lambda_j$. The assertion (2) in the statement is a consequence of the convergence of the approximate identity $\varPi_n f \to f$ in $L^1_{\rm loc}(\mathbb{R}^d)$ when f is locally integrable with polynomial growth.

We finally verify the third assertion. Let $\varPi_n = \sum_{j=0}^{n} L_j \Lambda_j$ and $\chi = 1 - \zeta$. Then,

$$\big| \mathbb{1}^*_I(f) - \mathbb{1}^*_I(\zeta f) \big| = \lim_{n\to\infty} \Big| \int_I \varPi_n(\chi f) \Big|.$$

Using distribution theory we can write

$$\int_I \varPi_n(\chi f)(x)\, dx = \int_I \big\langle \chi f, \varPi_n(x - \cdot) \big\rangle \, dx = \Big\langle f, \chi \textstyle\int_I \varPi_n(x - \cdot) dx \Big\rangle.$$

The result follows after checking that for

$$\Phi_n(y) := \chi(y) \int_I \varPi_n(x - y) dx$$

we have $\lim_{n\to\infty} \Phi_n = 0$ in the topology of the Schwartz class. □

Let $h \in \mathscr{H}_d$. The previous result can be applied to define h^* as a continuous linear functional in $F^s_{p,q}$. Namely,

$$h^*(f) \;=\; \tfrac{1}{\|h\|_1} \sum_{j=0}^{\infty} \int h(x)\, L_j \Lambda_j f(x)\, dx, \qquad \text{for } f \in F^s_{p,q}.$$

Then, Lemmas 8.7 and 8.8 imply the following.

Corollary 8.9. *The Haar functions, regarded as linear functionals, can be continuously extended from $\mathcal{S}$ into $F^s_{p,q}$ if and only if the indices (s, p, q) satisfy (a) or (b) in Lemma 8.7.*

9. Failure of density for $s = 1$

Proposition 9.1. *There exists a Schwartz function f supported in $(\frac{1}{16}, \frac{15}{16})^d$ such that, for all $0 < p \le 1$ and $0 < q \le \infty$,*

$$\liminf_{N\to\infty} \|\mathbb{E}_N f - f\|_{F^1_{p,q}} > 0. \tag{106}$$

Moreover, if $d/(d+1) \le p < 1$ and $0 < q \le 2$ then the span of $\mathscr{H}_d$ is not a dense set in $F^1_{p,q}(\mathbb{R}^d)$.

Proof. The proof of (106) uses the same function f as in [5, Proposition 8.3]. Namely, pick $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\eta \subset (\frac{1}{16}, \frac{15}{16})^d$ and $\eta(x) = 1$ on $(1/8, 7/8)^d$. Then consider $f(x) = x_1\, \eta(x)$. In [5, Proposition 8.3] it was shown that this function satisfies

$$\liminf_{N\to\infty} \|\mathbb{E}_N f - f\|_{B^1_{p,\infty}} > 0. \tag{107}$$

Therefore, (106) follows from here and the embeddings

$$F^1_{p,q} \hookrightarrow F^1_{p,\infty} \hookrightarrow B^1_{p,\infty}.$$

We next show that (106) implies the failure of the density of $\operatorname{span}\mathscr{H}_d$ in $F^1_{p,q}$, for all $0 < q \le 2$ and $d/(d+1) \le p < 1$. Indeed, assume for contradiction that such density holds, and given $f \in \mathcal{S}$ as in (106) and $\varepsilon > 0$, find $g \in \operatorname{span}\mathscr{H}_d$ such that $\|f - g\|_{F^1_{p,q}} < \varepsilon$. Let N_0 be large enough so that $\mathbb{E}_N(g) = g$ for all $N \ge N_0$. Then, the (quasi-)triangle inequality and the uniform boundedness of $\mathbb{E}_N$ in Theorem 1.4 gives

$$\|f - \mathbb{E}_N f\|_{F^1_{p,q}} \lesssim \|f - g\|_{F^1_{p,q}} + \|\mathbb{E}_N g - \mathbb{E}_N f\|_{F^1_{p,q}} \lesssim \varepsilon, \quad N \ge N_0,$$

which contradicts (106). □

Remark 9.2. It would be interesting to settle the question whether $\operatorname{span}\mathscr{H}_d$ is dense in the spaces $F^1_{p,q}$, when $d/(d+1) \le p < 1$ and $2 < q < \infty$. As the operators $\mathbb{E}_N$ are not uniformly bounded in this range our current argument is not sufficient to give an answer (*cf.* also [5, §8.1] for a similar discussion about the Besov space analogue of this question).

10. Localization and partial sums of admissible enumerations

Let $\mathcal{U} = \{u_n\}_{n=1}^\infty$ be a strongly admissible enumeration of $\mathscr{H}_d$, as in Definition 1.2 above. Explicit examples of such enumerations are not hard to construct; see e.g. [5, §11].

Here we quote a localization lemma for such enumerations, which relates the partial operators $S_R^{\mathcal{U}}$ and the dyadic averages $\mathbb{E}_N$ and $T_N[\cdot, \mathfrak{a}]$. We let $\varsigma \in C_c^\infty$ be supported in a 10^{-2} neighborhood of $[0,1)^d$ and so that

$$\sum_{\nu\in\mathbb{Z}^d} \varsigma(\cdot - \nu) \equiv 1, \tag{108}$$

and denote $\varsigma_\nu = \varsigma(\cdot - \nu)$, $\nu \in \mathbb{Z}^d$. The following identity has been proved in [5, Lemma 9.1].

Lemma 10.1. *Let $\mathcal{U}$ be a strongly admissible enumeration of $\mathscr{H}_d$. Then, for every $R \in \mathbb{N}$ and $\nu \in \mathbb{Z}^d$ there is an integer $N_\nu = N_\nu(R) \ge -1$ and $\{0,1\}$-sequences $\mathfrak{a}^{\kappa,\nu}$, $0 \le \kappa \le b$, such that for all $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have*

$$S_R^{\mathcal{U}}[g\varsigma_\nu] = \mathbb{E}_{N_\nu}[g\varsigma_\nu] + \sum_{\kappa=0}^{b} T_{N_\nu+\kappa}[g\varsigma_\nu, \mathfrak{a}^{\kappa,\nu}]. \tag{109}$$

We next recall a localization property of the $F^s_{p,q}$-quasinorms; see [14, 2.4.7] (and [16, 2.4.2] for $p=\infty$).

Lemma 10.2. *Let $0<p,q\le\infty$ and $s\in\mathbb{R}$. Then it holds*

$$\Big\|\sum_{\nu\in\mathbb{Z}^d}\varsigma_\nu g\Big\|_{F^s_{p,q}}\approx\Big(\sum_{\nu\in\mathbb{Z}^d}\big\|\varsigma_\nu\, g\big\|^p_{F^s_{p,q}}\Big)^{1/p}. \tag{110}$$

We are now ready to prove the uniform boundedness of the operators $S^{\mathcal{U}}_R$. We assume that $(s,p,q)\in\mathfrak{A}$, as defined in Definition 8.1, so that these operators can be continuously extended to the whole space $F^s_{p,q}$. More precisely, if $p,q<\infty$, condition (1) holds and S_R is well-defined as in section 1 (that is, extended from $\mathcal{S}$ to $F^s_{p,q}$ by density). In order to include as well the cases $p=\infty$ or $q=\infty$, one first considers extensions of the dual functionals u^*_n to the full space $F^s_{p,q}$ as follows

$$u^*_n(f):=\sum_{j=1}^\infty 2^{k(n)d}\int h^{\epsilon(n)}_{k(n),\nu(n)}\,L_j\Lambda_j f,\ \text{ for } f\in F^s_{p,q}; \tag{111}$$

see the details in §8.6. In this way, the identity in (109) remains valid for all $g\in F^s_{p,q}$.

Proposition 10.3. *Let $(s,p,q)\in\mathfrak{A}$. Suppose that*

$$\sup_{N\ge0}\|\mathbb{E}_N\|_{F^s_{p,q}\to F^s_{p,q}}+\sup_{N\ge0}\sup_{\|\mathfrak{a}\|_{\ell^\infty}\le1}\|T_N[\cdot,\mathfrak{a}]\|_{F^s_{p,q}\to F^s_{p,q}}<\infty. \tag{112}$$

Then, for every strongly admissible enumeration $\mathcal{U}$ it holds

$$\sup_{R\ge1}\|S^{\mathcal{U}}_R\|_{F^s_{p,q}\to F^s_{p,q}}<\infty.$$

Proof. Consider $S_R=S^{\mathcal{U}}_R$ as a continuous operator in $F^s_{p,q}$ (as described in §8.6). Then, the support properties of the extension, see (105), imply that

$$\varsigma_{\nu'}S_R(f\varsigma_\nu)=0,\quad \text{whenever}\quad |\nu-\nu'|_\infty\ge3.$$

Then, using (108) and (110),

$$\begin{aligned}\big\|S_Rf\big\|_{F^s_{p,q}}&\approx\Big(\sum_{\nu'}\big\|\varsigma_{\nu'}S_R\big(\sum_\nu\varsigma_\nu f\big)\big\|^p_{F^s_{p,q}}\Big)^{\frac1p}\\&\lesssim\Big(\sum_{\nu'}\sum_{\nu\,:\,|\nu-\nu'|_\infty\le2}\Big\|\varsigma_{\nu'}S_R(f\varsigma_\nu)\Big\|^p_{F^s_{p,q}}\Big)^{1/p}\\&\lesssim\Big(\sum_\nu\big\|S_R(f\varsigma_\nu)\big\|^p_{F^s_{p,q}}\Big)^{1/p},\end{aligned}$$

using in the last step that $\varsigma_{\nu'}$ is a uniform multiplier in $F^s_{p,q}$; see [14, 4.2.2]. Then Lemma 10.1 and (112) give

$$\begin{aligned}\|S_R f\|_{F^s_{p,q}} &\lesssim \Big(\sum_\nu \|\mathbb{E}_{N_\nu}(f\varsigma_\nu)\|^p_{F^s_{p,q}} + \|\sum_{\kappa=0}^{b} T_{N_\nu+\kappa}[f\varsigma_\nu, \mathfrak{a}^{\kappa,\nu}]\|^p_{F^s_{p,q}}\Big)^{1/p} \\ &\lesssim_b \Big(\sum_\nu \|f\varsigma_\nu\|^p_{F^s_{p,q}}\Big)^{1/p} \approx \|f\|_{F^s_{p,q}}. \qquad \square\end{aligned}$$

Remark 10.4. The equivalence in (110) is also true with ς replaced by $\mathbb{1}_{[0,1]^d}$ when

$$\max\Big\{\frac{d}{p}-1, \frac{1}{p}-1\Big\} < s < \frac{1}{p},$$

as in that case characteristic functions of cubes are multipliers in $F^s_{p,q}$. In particular, for those indices the assertion in Proposition 10.3 holds as well with the weaker notion of *admissible* enumeration; see [4, §3]. This is in particular the case when $s = 1$ and $d/(d+1) < p < 1$.

Finally, we conclude with the following observation, which we shall use to transfer negative results between the operators $\mathbb{E}_N$ and S_R. The explicit construction is given in [5, §11].

Lemma 10.5. *There exists a strongly admissible enumeration $\mathcal{U}$ with the following property: for every $m \geq 0$ there exists an integer $R(m) \geq 1$ such that*

$$S^{\mathcal{U}}_{R(m)} f \;=\; \mathbb{E}_m f, \quad f \in C^\infty_c((-5,5)^d). \tag{113}$$

11. The Schauder basis property: proof of Theorem 1.3

11.1. *Necessary conditions.* Suppose that every strongly admissible enumeration $\mathcal{U}$ of $\mathscr{H}_d$ is a Schauder basis of $F^s_{p,q}$. This implies that span $\mathscr{H}_d$ must be dense (hence $p, q < \infty$), and

$$C_{\mathcal{U}} := \sup_{R\geq 1} \|S^{\mathcal{U}}_R\|_{F^s_{p,q}\to F^s_{p,q}} < \infty. \tag{114}$$

Moreover, if we select $\mathcal{U}$ as in Lemma 10.5, then we must have

$$\sup_{m\geq 0} \mathrm{Op}_{\mathcal{S}}\big(\mathbb{E}_m, F^s_{p,q}\big) \;\leq\; C_{\mathcal{U}} < \infty.$$

In view of Proposition 9.1 and Corollary 8.3 this is only possible if (i), (ii) or (iii) in Theorem 1.3 hold.

11.2. *Sufficient conditions.* Under the assumptions in (i), (ii), and (iii) of Theorem 1.3, the operators $\mathbb{E}_N$ and $T_N[\cdot, \mathfrak{a}]$ are uniformly bounded in $F^s_{p,q}$, by Theorem (1.4). So we can use Proposition 10.3 and conclude that (114) must hold. The density of span $\mathscr{H}_d$ is also true in this range, so we conclude that $\mathcal{U}$ is a Schauder basis of $F^s_{p,q}$.

11.3. *Consequences for the basic sequence property.* Theorem 1.4 additionally implies convergence of basic sequences in the cases when span $\mathscr{H}_d$ is not dense. Namely, when $p = \infty$ or $q = \infty$, let $f^s_{p,q}$ denote the closure of the $\mathcal{S}$ in $F^s_{p,q}$. When $s < 1/p$ the subset span $\mathscr{H}_d$ is dense in $f^s_{p,q}$, so we deduce the following.

Corollary 11.1. *Let (s, p, q) be as in (i), (iii) or (iv) in Theorem 1.4. Then, every admissible enumeration $\mathcal{U}$ is a Schauder basis of $f^s_{p,q}$. That is,*

$$f = \sum_{n=1}^{\infty} u_n^*(f) u_n, \quad \textit{for all } \; f \in f^s_{p,q},$$

with convergence in the norm of $F^s_{p,q}$.

Remark 11.2. Observe that we have excluded the cases (ii) and (v) in Theorem 1.4. In these cases we can only say that $\mathcal{U}$ is a Schauder basis of the subspace

$$\overline{\operatorname{span} \mathscr{H}_d}^{F^s_{p,q}}.$$

A precise description of this subspace in those cases, however, is not clear. In the range (ii), ie $s = 1$ (and $q \le 2$) this subspace cannot contain the Schwartz class $\mathcal{S}$, as shown by Proposition 9.1. On the other hand, in the case (v), *i.e.* $s = 0$ and $p = \infty$, this subspace *strictly* contains $f^0_{\infty,q}$. Indeed, first of all one has

$$f^0_{\infty,\infty} \cap \operatorname{span} \mathscr{H}_d \; = \; \{0\};$$

see [5, Proposition 5.1]. Next, for all $q \le \infty$, the inclusion

$$C^\infty_c(\mathbb{R}^d) \subset \overline{\operatorname{span} \mathscr{H}_d}^{F^0_{\infty,q}}$$

follows, when $d = 1$, from the elementary embedding $B^{1/p}_{p,\infty}(\mathbb{R}) \hookrightarrow F^0_{\infty,q}(\mathbb{R})$ and the corresponding result for $B^{1/p}_{p,\infty}(\mathbb{R})$ in [5, Proposition 8.6]. When $d \ge 2$, one can approximate each $f \in C^\infty_c(\mathbb{R}^d)$ by a linear combination of functions $g^1(x_1) \cdots g^d(x_d)$ with $g^i \in C^\infty_c(\mathbb{R})$, and then use the previous result.

12. The unconditional basis property: proof of Theorem 1.1

The fact that $\mathscr{H}_d$ is an unconditional basis of $F^s_{p,q}$ when (6) and (7) hold was shown by Triebel in [15, Theorem 2.21]. We now indicate references for the *negative* end-point results, corresponding to the dotted or dashed lines around the green region in Figure 4.

The trivial cases correspond to the lines $p = \infty$, $s = 1/p$, and to the line $s = 1/p - 1$ with $p > 1$. In all of them not even the Schauder basis property may hold. Namely, if $p = \infty$ then $F^s_{\infty,q}$ is not separable, and hence span $\mathscr{H}_d$ is not dense (see however Remark 12.1 below for the validity of unconditionality in the subspace $f^s_{\infty,q}$). The other two cases are excluded because $(s, p, q) \notin \mathfrak{A}$, and hence (1) fails.

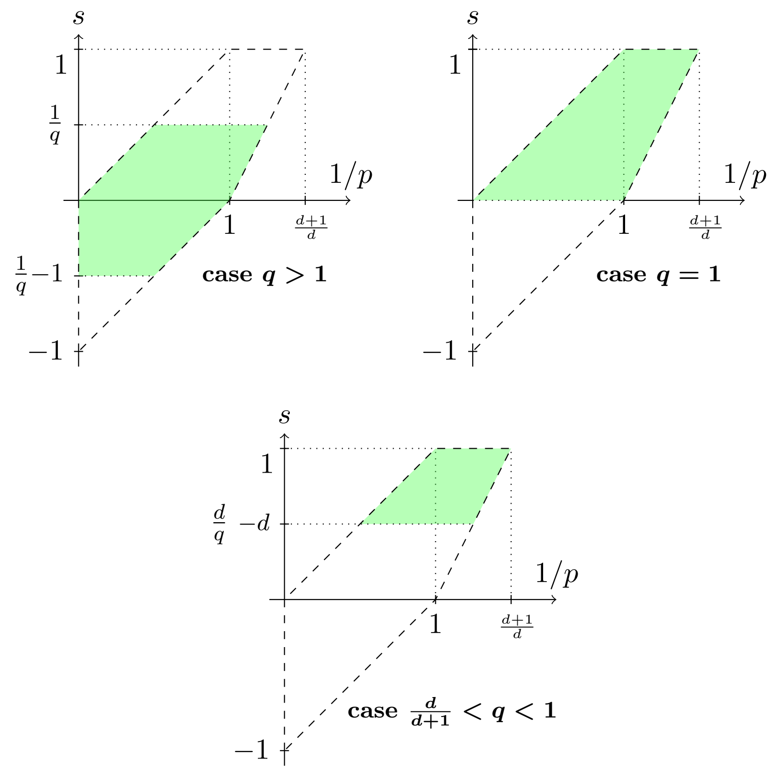

FIGURE 4. Parameter domain for unconditionality in the cases $q > 1$, $q = 1$ and $d/(d+1) < q < 1$, respectively.

Concerning the horizontal line $s = 1$, this is a borderline of the unconditionality region when $d/(d+1) \le p, q \le 1$. This case is excluded by Proposition 9.1, since $\operatorname{span} \mathscr{H}_d$ is not dense in $F^1_{p,q}$, so the Schauder basis property cannot hold here.

At the line $s = d - d/p$, for $d/(d+1) < p \le 1$, we have a positive Schauder basis result for strongly admissible $\mathcal{U}$, by Theorem 1.3. So we must prove that such a basis cannot be unconditional in $F^s_{p,q}$. This was already shown in [5, Theorem 13.1], based on an explicit example which works well in both the Besov and the Triebel-Lizorkin setting.

Finally, we consider the horizontal lines of the green region which lie inside the open pentagon $\mathfrak{P}$. In [10], the failure of unconditionality in these lines was shown in the case $q > 1$ and $d = 1$, indeed for all exponents $p \ge d/(d+1)$ (by [10, Remark 7.1]). Here we show how to modify the arguments in that paper to cover as well the cases $q \le 1$, and extend the construction to all $d \ge 1$.

We recall some notation from [10]. To each finite set $E \subset \mathscr{H}_d$ we associate the projection operator

$$P_E(f) = \sum_{h \in E} \langle f, h^* \rangle h,$$

where $h^* = 2^{kd} h$ is the dual functional of a Haar function $h \in \mathscr{H}_d$ of frequency 2^k. We also write $\mathrm{HF}(E)$ for the set of all Haar frequencies 2^k of elements $h \in E$.

We first remark that the results in [10, §6] remain valid when $q \le 1$. Namely, for each $N \ge 2$, an explicit construction is given of a function $f = f_N \in F^{1/q-1}_{p,q}(\mathbb{R})$ and a set $E = E_N \subset \mathscr{H}_1$ with $\#\mathrm{HF}(E) \le N 4^N$ such that[1]

$$\|f_N\|_{F^{1/q-1}_{p,q}(\mathbb{R})} \lesssim N^{1/q} \quad \text{and} \quad \|P_{E_N}(f_N)\|_{F^{1/q-1}_{p,q}(\mathbb{R})} \ge N^{1+\frac{1}{q}}, \tag{115}$$

if $0 < q \le p < \infty$. In particular, for $d = 1$,

$$\|P_{E_N}\|_{F^{1/q-1}_{p,q} \to F^{1/q-1}_{p,q}} \gtrsim N,$$

and hence $\mathscr{H}_1$ is not unconditional at the lower segment of the green region in Figure 4.

When $d \ge 2$, the above example can be adapted in two different ways. If $1 < q < \infty$, one considers the tensorized functions

$$F_N(x_1, x') := f_N(x_1) \otimes \chi(x'),$$

where f_N is as in (115) and $\chi \in C^\infty_c((-1,2)^{d-1})$ with $\chi \equiv 1$ in $[0,1]^{d-1}$, and defines the sets

$$\mathcal{E}_N := \Big\{ h \otimes \mathbb{1}_{[0,1]^{d-1}} \ : \ h \in E_N \Big\}.$$

Then, a standard computation (as in §7.2.2 above) gives

$$\|F_N\|_{F^{1/q-1}_{p,q}(\mathbb{R}^d)} \lesssim N^{1/q} \quad \text{and} \quad \|P_{\mathcal{E}_N}(F_N)\|_{F^{1/q-1}_{p,q}(\mathbb{R}^d)} \gtrsim N^{1+1/q}.$$

When $q \le 1$, one considers instead the natural generalization to $\mathbb{R}^d$ of the construction in [10, §6], namely using the test function $\prod_{i=1}^d \eta(x_i)$, in place of the one dimensional function η in [10, (46)]. More precisely, if $1 \le \kappa \le 4^{Nd}$, $b_\kappa = \kappa N$ and $1 \le \sigma \le N$, one defines the functions

$$\mathcal{Y}_{\kappa,\sigma}(x_1, \dots, x_d) = \sum_{\substack{(\nu_1,\dots,\nu_d) \in \mathbb{Z}^d \\ 0 \le \nu_i < 2^{b_\kappa - N - 2}}} 2^{-\sigma d} \prod_{i=1}^d \eta\big(2^{b_\kappa + N - \sigma}(x_i - 2^{N+2-b_\kappa} \nu_i)\big)$$

and, for $t \in [0,1]$,

$$f_t = \sum_{\kappa=1}^{4^{Nd}} r_\kappa(t)\, 2^{-(b_\kappa+N)(\frac{d}{q}-d)} \sum_{\sigma=1}^N \mathcal{Y}_{\kappa,\sigma},$$

[1] In the notation of [10, §6], one should consider sets A of *consecutive* Haar frequencies, so that the associated “density” number in [10, (43)] takes the value $Z = N$.

where $r_k(t)$ is a Rademacher function. Then, arguing as in [10, Lemma 6.2] and [10, Proposition 6.3], if $0 < q \le p$ one verifies that

$$\|f_t\|_{F^{d/q-d}_{p,q}(\mathbb{R}^d)} \lesssim N^{1/q},$$

and that for some t_0 and some $E \subset \mathscr{H}_d$ with $\mathrm{HF}(E) \subset \{2^k\}_{1\le k\le N4^{dN}}$,

$$\big\|P_E(f_{t_0})\big\|_{F^{d/q-d}_{p,q}(\mathbb{R}^d)} \gtrsim N^{1+\frac{1}{q}}.$$

This completes the proof of Theorem 1.1. □

Remark 12.1. If $p = \infty$ one can ask whether the Schauder basis property in the subspace $f^s_{\infty,q}$, for $-1 < s < 0$, can be upgraded to unconditional basis. This is certainly true when $1/q - 1 < s < 0$, by the uniform boundedness of the projection operators P_E (which follows by duality from the corresponding result for $F^{-s}_{1,q'}$), and by the density of $\operatorname{span}\mathscr{H}_d$ in $f^s_{\infty,q}$.

We now show that at the endpoint $s = 1/q - 1$ unconditionality must fail. If not, the operators P_E would be uniformly bounded in $f^{-1/q'}_{\infty,q}$, for all finite $E \subset \mathscr{H}_d$. Fix $p_0 \in (q,\infty)$, and for each $N \ge 1$, pick a set $A = A_N \subset \{2^n\}_{n\ge 0}$ of cardinality 2^N and such that $\log_2 A$ is N-separated. Then, by [10, Theorem 1.4], for every $E \subset \mathscr{H}_d$ with $\#\mathrm{HF}\,(E) \subset A$ it holds

$$\|P_E\|_{F^{-1/q'}_{p_0,q}} \lesssim N^{1/q'}. \tag{116}$$

By interpolation one then has, for $\theta \in (0,1)$,

$$\|P_E\|_{F^{-1/q'}_{p_0/\theta,q}} \lesssim \|P_E\|^{\theta}_{F^{-1/q'}_{p_0,q}} \lesssim N^{\frac{\theta}{q'}}.$$

But this contradicts the lower bound $N^{1/q'}$ (for the supremum of all such sets E) asserted in [10, Theorem 1.4.ii]. Similar arguments also disprove the unconditionality for s below the critical $1/q - 1$.

Finally consider the space $f^0_{\infty,q}$ for $1 \le q < \infty$, on which unconditionality fails (since otherwise it would hold on its dual $F^0_{1,q'}$, see [8, §2.1.5], on which unconditionality fails by [5, Prop. 13.3]).

Remark 12.2. We now consider the spaces $f^s_{p,\infty}$, when $1 < p < \infty$. The Schauder basis property holds for $1/p - 1 < s < 1/p$ while the unconditional basis property holds only for $1/p - 1 < s < 0$, already by the estimates in [15]. The unconditional basis property does not hold on $f^0_{p,\infty}$ since by duality ([8, §2.1.5]) it would imply it on $F^0_{p',1}$ where it fails by [10]. Finally when $p = q = \infty$ then $F^s_{\infty,\infty} = B^s_{\infty,\infty}$ hence $f^s_{\infty,\infty} = b^s_{\infty,\infty}$, and the unconditional basis property holds for $-1 < s < 0$ (for the dual statement see [5]).

Remark 12.3. It would be interesting to investigate the question of unconditionality of the Haar system as a basic sequence in $B^1_{p,q}$ and $F^1_{p,q}$ when $d/(d+1) < p < 1$.

13. Appendix

We give a detailed proof of the pointwise estimate asserted in (36) above.

Lemma 13.1. *Let $f \in \mathcal{S}'$ and $k > N$, then*

$$|L_k \Pi_N f(x)| \lesssim 2^{-k}\, 2^{-(k-N)M}\, \mathfrak{M}^{**}_{A,N}\big(\nabla \Pi_N f\big)(x). \tag{117}$$

Proof. Let $g = \Pi_N f$. Since $\int \beta = 0$ we can write

$$L_k g(x) = \int \beta_k(y)\big(g(x-y)-g(x)\big)\,dy = \int \beta_k(y) \int_0^1 \langle -y|(\nabla g)(x-sy)\rangle\, ds\; dy.$$

Using the $\mathbb{R}^d$-valued function $\tilde\beta(x) = -y\beta(y)$, and the fact that $g = g * \Phi_N$ (with Φ_N as in (32)), the above expression takes the form

$$\begin{aligned} L_k g(x) &= 2^{-k} \int\int_0^1 \langle \tilde\beta_k(y)|(\nabla g)(x-sy)\rangle\, ds\; dy \\ &= 2^{-k} \int \langle \gamma_{k,N}(z)|\nabla g(x-z)\rangle\, dz, \end{aligned} \tag{118}$$

where

$$\gamma_{k,N}(z) = \int_0^1 \int \tilde\beta_k(y)\, \Phi_N(z-sy)\, dy\, ds.$$

We now claim that, for each $L \in \mathbb{N}$,

$$|\gamma_{k,N}(z)| \le 2^{-(k-N)M} \frac{c_L\, 2^{Nd}}{(1+2^N|z|)^L}. \tag{119}$$

To prove so, notice that $\tilde\beta$ has vanishig moments up to order $M-1$, and hence, for each $s \in [0,1]$,

$$\begin{aligned} \int \tilde\beta_k(y)\, \Phi_N(z-sy)\, dy &= \int \tilde\beta_k(y) \Big[\Phi_N(z-sy) - \sum_{m=0}^{M-1} \frac{\langle -sy|\nabla\rangle^m}{m!}(\Phi_N)(z)\Big]\, dy \\ &= \int \tilde\beta_k(y) \Big[\int_0^1 \tfrac{(1-t)^{M-1}}{(M-1)!} \langle -sy|\nabla\rangle^M (\Phi_N)(z-tsy)\, dt\Big]\, dy. \end{aligned}$$

Since $\Phi_N(z) = 2^{Nd}\Phi(2^N z)$ and Φ is a Schwartz function, the modulus of the expression in brackets is bounded by

$$C_L\, |2^N y|^M\, 2^{Nd} \sup_{t\in[0,1]} (1+2^N|z-tsy|)^{-L} \lesssim C_L \frac{2^{(N-k)M}\, 2^{Nd}}{(1+2^N|z|)^L},$$

using in the last step that $y \in \operatorname{supp} \tilde\beta_k \subset B_{c2^{-k}}(0)$, with $k > N$. Inserting these bounds in the integral defining $\gamma_{k,N}(z)$ one easily obtains (119).

Finally, from (118) and (119) (with $L = A+d+1$), one concludes that

$$\begin{aligned} |L_k g(x)| &\lesssim 2^{-k}\, 2^{-(k-N)M} \int \frac{2^{Nd}\, |\nabla g(x-z)|}{(1+2^N|z|)^{A+d+1}}\, dz \\ &\lesssim 2^{-k}\, 2^{-(k-N)M}\, \mathfrak{M}^{**}_{A,N}\big(\nabla g\big)(x). \end{aligned}$$

□

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Gustavo Garrigós, Department of Mathematics, University of Murcia, 30100 Espinardo, Murcia, Spain

Email address: `gustavo.garrigos@um.es`

Andreas Seeger, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI,53706, USA

Email address: `seeger@math.wisc.edu`

Tino Ullrich, Fakultät für Mathematik, Technische Universität Chemnitz, 09107 Chemnitz, Germany

Email address: `tino.ullrich@mathematik.tu-chemnitz.de`