HILBERT-TYPE INEQUALITIES IN HOMOGENEOUS CONES

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ABSTRACT. We prove L^p-L^q bounds for the class of Hilbert-type operators associated with generalized powers $Q^{\boldsymbol{\alpha}}$ in a homogeneous cone Ω. Our results extend and slightly improve earlier work from [16], where the problem was considered for scalar powers $\boldsymbol{\alpha} = (\alpha, \dots, \alpha)$ and symmetric cones. We give a more transparent proof, provide new examples, and briefly discuss the open question regarding characterization of L^p boundedness for the case of vector indices $\boldsymbol{\alpha}$. Some applications are given to boundedness of Bergman projections in the tube domain over Ω.

1. Introduction

Let Ω be a homogeneous open convex cone in $\mathbb{R}^n$, and consider the associated generalized powers

$$Q^{\boldsymbol{\alpha}}(x) = \prod_{j=1}^{r} Q_j^{\alpha_j}(x), \quad x \in \Omega, \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r,$$

where r is the rank of the cone, and $Q_j(x)$, $j = 1, \dots, r$, are the basic power functions with respect to a fixed coordinate system; see §2 below for precise definitions. We shall also denote the invariant measure in Ω by $d\sigma(x) = Q^{-\boldsymbol{\tau}}(x)dx$, with $\boldsymbol{\tau}$ as in (2.3).

In this paper we shall be interested in the following Hilbert-type operators

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} f(x) = Q^{\boldsymbol{\alpha}}(x) \int_\Omega \frac{Q^{\boldsymbol{\beta}}(y)}{Q^{\boldsymbol{\gamma}}(x+y)} f(y)\, d\sigma(y), \quad x \in \Omega, \tag{1.1}$$

for general multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^r$. More precisely, we wish to determine the validity of the inequalities

$$\Big[\int_\Omega |S_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} f(x)|^q \, Q^{\boldsymbol{\mu}}(x)\, d\sigma(x) \Big]^{\frac{1}{q}} \le C \Big[\int_\Omega |f(y)|^p \, Q^{\boldsymbol{\nu}}(y)\, d\sigma(y) \Big]^{\frac{1}{p}} \tag{1.2}$$

for general exponents $1 \le p, q < \infty$ and all multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\mu} \in \mathbb{R}^r$. Note that when $n = r = 1$ and $\Omega = (0, \infty)$, these are versions of the classical Hilbert inequalities, as they are called in [13, Ch IX].

When Ω is a symmetric cone, this question has been addressed in [16] in the special case of *scalar* multi-indices, that is when $\boldsymbol{\alpha} = (\alpha, \dots, \alpha)$ and likewise for $\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\mu}$.

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In that case, a suitable variation of the classical Schur test provides a characterization of the exponents for which (1.2) holds, see [16, Theorem 2.1], at least under the constraints $1 \leq p \leq q < \infty$ and

$$\frac{\nu}{p'} + \frac{\mu}{q} > 0. \tag{1.3}$$

The situation for vector multi-indices, however, has additional difficulties, as more complicated test functions are expected in the Schur test, even when Ω is a symmetric cone. Moreover, if $r \geq 3$ then the known necessary and sufficient conditions do not match in general, even when $p = q$ and $\boldsymbol{\nu} = \boldsymbol{\mu}$; see the comments in [17, §8] (which implicitly go back to [8] and [9]). Although this last phenomenon seems a harder question, it actually suggests that a better understanding of the general (vector indexed) inequalities is needed.

In this note we present a first step in this direction, and obtain necessary conditions and sufficient conditions so that (1.2) holds in the case of vector multi-indices. More precisely, using the notation in §2, we shall prove the following.

Theorem 1.4. *Let $1 \leq p, q < \infty$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\mu} \in \mathbb{R}^r$. Let $\Omega \subset \mathbb{R}^n$ be a homogeneous convex cone, and let $\mathbf{g}$ and $\mathbf{g}'$ be the associated indices defined in* (2.3).

i) Suppose that $1 \leq p \leq q < \infty$ and that

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} + \frac{\boldsymbol{\mu}}{q} + \boldsymbol{\beta} - \frac{\boldsymbol{\nu}}{p} \tag{A1}$$

$$\boldsymbol{\alpha} + \frac{\boldsymbol{\mu}}{q} > \frac{\mathbf{g}}{q} + \frac{\mathbf{g}'}{p'} \quad \textit{and} \quad \boldsymbol{\beta} - \frac{\boldsymbol{\nu}}{p} > \frac{\mathbf{g}}{p'} + \frac{\mathbf{g}'}{q}\ ; \tag{A2}$$

then the inequality (1.2) *holds for all non-negative f.*

ii) Assume the validity of (1.2) *for all $f \geq 0$. Then necessarily $p \leq q$ and the conditions (A1) and (A2') must hold, where*

$$\boldsymbol{\alpha} + \frac{\boldsymbol{\mu}}{q} > \max\left\{\frac{\mathbf{g}}{q}, \frac{\mathbf{g}'}{p'}\right\} \quad \textit{and} \quad \boldsymbol{\beta} - \frac{\boldsymbol{\nu}}{p} > \max\left\{\frac{\mathbf{g}}{p'}, \frac{\mathbf{g}'}{q}\right\} \tag{A2'}$$

iii) The conditions (A2) and (A2') coincide in each of the following cases

- *if $p = 1$*
- *if $r \in \{1, 2\}$, or if $r = 3$ and Ω is the Vinberg cone*
- *if Ω is an irreducible symmetric cone and both, $\boldsymbol{\alpha} + \frac{\boldsymbol{\mu}}{q}$ and $\boldsymbol{\beta} - \frac{\boldsymbol{\nu}}{p}$, are scalars.*

Remark 1.5. It is easily checked that letting

$$\mathbf{a} = \boldsymbol{\alpha} + \frac{\boldsymbol{\mu}}{q} \quad \text{and} \quad \mathbf{b} = \boldsymbol{\beta} - \frac{\boldsymbol{\nu}}{p}\ ,$$

then, the validity of (1.2) for all $f \geq 0$ is equivalent to the boundedness of

$$S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} : L^p(\Omega, d\sigma) \longrightarrow L^q(\Omega, d\sigma).$$

So, below it will suffice to look at this case, which involves a simpler notation. A version of Theorem 1.4 for the case $q = \infty$ is also given in Corollary 4.12 below.

We make some remarks about Theorem 1.4 and its comparison with [16, Theorem 2.1]. Our proof is also based on a Schur test strategy, however we find a simpler and slightly more efficient approach than in [2, 16], which in particular removes the artificial constraint in (1.3). We provide a correction of an unclear statement in [16, p. 510] concerning the class of test functions that are needed in these proofs; see Remark 3.15 below. We also provide new examples that disregard the cases $p > q$, which were not considered in the scalar setting of [16].

Finally, we consider homogeneous cones Ω as a natural framework for this problem. The new required tools are based on the Vinberg theory of T-algebras (as in [17]), and a key explicit identity for beta-type integrals due to Gindikin, see Lemma 2.7 below.

To conclude the paper, we briefly discuss some applications to the boundedness of Bergman projections in the tube domain $T_\Omega = \mathbb{R}^n + i\Omega$ of $\mathbb{C}^n$. As in earlier papers [3, 6, 19, 5, 16, 7] this is a main motivation for the study of Hilbert-type inequalities. Letting $z = x + iy \in T_\Omega$, we consider the measure

$$dV_{\boldsymbol{\nu}}(z) = Q^{\boldsymbol{\nu}}(y)\, dx\, d\sigma(y),$$

and denote by $L^p_{\boldsymbol{\nu}}(T_\Omega)$, $1 \leq p \leq \infty$, the Lebesgue space $L^p(T_\Omega, dV_{\boldsymbol{\nu}})$. The (weighted) Bergman space $A^p_{\boldsymbol{\nu}}(T_\Omega)$ is the closed subspace of $L^p_{\boldsymbol{\nu}}(T_\Omega)$ consisting of holomorphic functions. In order that $A^2_{\boldsymbol{\nu}} \neq \{0\}$, we must take $\boldsymbol{\nu} > \mathbf{g}$; see [8, II.2, II.3].

The (weighted) Bergman projection $P_{\boldsymbol{\nu}}$ is the orthogonal projection of the Hilbert space $L^2_{\boldsymbol{\nu}}(T_\Omega)$ onto its subspace $A^2_{\boldsymbol{\nu}}(T_\Omega)$. It is defined by the integral

$$P_{\boldsymbol{\nu}} f(z) = \int_{T_\Omega} B_{\boldsymbol{\nu}}(z, w) f(w) dV_{\boldsymbol{\nu}}(w), \quad z \in T_\Omega,$$

where the associated Bergman kernel is explicitly given by

$$B_{\boldsymbol{\nu}}(z, w) = \mathfrak{c}_{\boldsymbol{\nu}}\, Q^{-\boldsymbol{\nu}-\boldsymbol{\tau}}\big((z - \bar{w})/i\big)\,, \quad z, w \in T_\Omega,$$

for a suitable constant $\mathfrak{c}_{\boldsymbol{\nu}} > 0$; see e.g. [17, p. 499]. An important problem in the field is to determine when $P_{\boldsymbol{\nu}}$ extends as a bounded operator from $L^p_{\boldsymbol{\nu}}$ into $A^p_{\boldsymbol{\nu}}$; see [3, 8, 6, 4, 17, 7].

Let us now introduce mixed norm spaces. For $1 \leq s, p \leq \infty$, let $L^{s,p}_{\boldsymbol{\nu}}(T_\Omega)$ be the set of all measurable functions f on T_Ω such that

$$||f||_{L^{s,p}_{\boldsymbol{\nu}}(T_\Omega)} := \left(\int_\Omega \left(\int_{\mathbb{R}^n} |f(x + iy)|^s dx \right)^{\frac{p}{s}} Q^{\boldsymbol{\nu}}(y) d\sigma(y) \right)^{\frac{1}{p}} < \infty$$

(with obvious modifications if s or p are ∞). Note that for $s=p$, we have $L^{p,p}_{\boldsymbol\nu}=L^p_{\boldsymbol\nu}$.

Consider now the positive operator $P^+_{\boldsymbol\nu}$ defined by

$$P^+_{\boldsymbol\nu} f(z)=\int_{T_\Omega}|B_{\boldsymbol\nu}(z,w)|\,f(w)\,dV_{\boldsymbol\nu}(w),\quad z\in T_\Omega.$$

Clearly the boundedness of $P^+_{\boldsymbol\nu}$ implies the boundedness of $P_{\boldsymbol\nu}$, but the converse is in general not true. More generally, consider the class of operators

$$T^+_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}f(z)=Q^{\boldsymbol\alpha}(\Im z)\int_{T_\Omega}\frac{f(w)\,dV_{\boldsymbol\beta}(w)}{|Q^{\boldsymbol{\gamma}+\boldsymbol{\tau}}((z-\bar w)/i)|},\quad z\in T_\Omega.$$

Observe that $P^+_{\boldsymbol\nu}=\mathfrak{c}_{\boldsymbol\nu}\,T^+_{\mathbf{0},\boldsymbol\nu,\boldsymbol\nu}$. These operators appear in various papers [6, 19, 5, 16, 7], and are linked to the Hilbert-type operators $S_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ by the following result. Below we denote $L^p_{\boldsymbol\nu}(\Omega)=L^p\big(\Omega,Q^{\boldsymbol\nu}(y)d\sigma(y)\big)$.

Theorem 1.6. *Let $1\le p,q<\infty$ and $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\nu},\boldsymbol{\mu}\in\mathbb{R}^r$ be such that*

$$\boldsymbol\gamma>\mathbf{g}'. \tag{1.7}$$

Then, the following are equivalent

(i) $S_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}:L^p_{\boldsymbol\nu}(\Omega)\to L^q_{\boldsymbol\mu}(\Omega)$ *is a bounded operator*
(ii) $T^+_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}:L^{s,p}_{\boldsymbol\nu}(T_\Omega)\to L^{s,q}_{\boldsymbol\mu}(T_\Omega)$ *is bounded for all* $1\le s\le\infty$
(iii) $T^+_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}:L^{s,p}_{\boldsymbol\nu}(T_\Omega)\to L^{s,q}_{\boldsymbol\mu}(T_\Omega)$ *is bounded for some* $1\le s\le\infty$.

As a corollary of Theorems 1.4 and 1.6, we can state the following special case, which seems new in this generality. The "diagonal" case, corresponding to $\lambda=1$, can be found in [17, Theorem 6.2 and (8.1)].

Corollary 1.8. *Let $\boldsymbol\nu>\max\{\mathbf{g},\mathbf{g}'\}$ and $1\le p,q,s<\infty$.*

(i) Then $P^+_{\boldsymbol\nu}:L^{s,p}_{\boldsymbol\nu}(T_\Omega)\longrightarrow L^{s,q}_{\boldsymbol\mu}(T_\Omega)$ is bounded whenever

$$q=\lambda p,\quad \boldsymbol\mu=\lambda\boldsymbol\nu,\quad \textit{for some } \lambda\ge1, \tag{1.9}$$

and

$$1+\frac{\mathbf{g}'/\lambda}{\boldsymbol\nu-\mathbf{g}}\;<\;p\;<\;1+\frac{\boldsymbol\nu-\mathbf{g}/\lambda}{\mathbf{g}'}. \tag{1.10}$$

(ii) If $P^+_{\boldsymbol\nu}:L^{s,p}_{\boldsymbol\nu}(T_\Omega)\longrightarrow L^{s,q}_{\boldsymbol\mu}(T_\Omega)$ is bounded then necessarily (1.9) *holds and*

$$1+\frac{\mathbf{g}'/\lambda}{\boldsymbol\nu}\;<\;p\;<\;1+\frac{\boldsymbol\nu}{\mathbf{g}'}. \tag{1.11}$$

2. Preliminaries

2.1. **Homogeneous cones.** A theorem of Vinberg [20, Theorem III.4] establishes that every convex homogeneous cone Ω can be described in a unique way (modulo isomorphisms) as the cone arising from a *T-algebra* structure. Next, we briefly describe how these are defined; we refer to [20, §III.1] or [17, §2] for further details and bibliography on the subject.

A *matrix algebra of rank* r is a real algebra $\mathcal{U}$ (not necessarily associative) bigraded by subspaces

$$\mathcal{U} = \bigoplus_{1\leq i,j\leq r} \mathcal{U}_{ij} \,,$$

such that the following product rules hold for all $i, j, k \in \{1, \ldots, r\}$

$$\mathcal{U}_{ij}\mathcal{U}_{jk} \subset \mathcal{U}_{ik}, \quad \text{and} \quad \mathcal{U}_{ij}\mathcal{U}_{\ell k} = \{0\} \;\text{ if } \ell \neq j.$$

An *involution* in $\mathcal{U}$ is a linear mapping $x \mapsto x^\star$ such that for all $x, y \in \mathcal{U}$ it holds

$$(x^\star)^\star = x, \quad (xy)^\star = y^\star x^\star \;, \quad \text{and additionally} \quad (\mathcal{U}_{ij})^\star = \mathcal{U}_{ji}, \; \forall\, i, j.$$

The elements $x \in \mathcal{U}$ can be represented by formal matrices $(x_{ij})_{1\leq i,j\leq r}$ with $x_{ij} \in \mathcal{U}_{ij}$. Then $x^\star$ corresponds to the formal transpose matrix, that is $(x^\star)_{ij} = (x_{ji})^\star$.

A *T-algebra* is a matrix algebra with an involution $\star$ satisfying the following axioms, (T1) through (T7).

(T1) The subalgebras $\mathcal{U}_{ii}$ are 1-dimensional, and there are (unique) idempotents $c_i = c_i^2$ such that

$$\mathcal{U}_{ii} = \mathbb{R}c_i, \quad i = 1, \ldots, r.$$

We denote by $\rho_{ii} : \mathcal{U}_{ii} \to \mathbb{R}$ the algebra isomorphism so that $\rho_{ii}(c_i) = 1$. More generally, we let $\rho_{ii}(x) = \rho_{ii}(x_{ii})$, if $x \in \mathcal{U}$.

(T2) For every $x_{ij} \in \mathcal{U}_{ij}$ it holds

$$x_{ij}c_j = c_i x_{ij} = x_{ij}.$$

In particular, the unit element in $\mathcal{U}$ is given by $\mathbf{e} := \sum_{i=1}^r c_i$.

Consider the "trace" operator defined by

$$\operatorname{tr}(x) = \sum_{j=1}^r \rho_{ii}(x), \quad x \in \mathcal{U}.$$

Then it must hold

(T3) $\operatorname{tr}(xy) = \operatorname{tr}(yx)$, $x, y \in \mathcal{U}$

(T4) $\operatorname{tr}\big(x(yz)\big) = \operatorname{tr}\big((xy)z\big)$, $x, y, z \in \mathcal{U}$

(T5) $\operatorname{tr}(xx^\star) > 0$, if $x \in \mathcal{U}$ and $x \neq 0$.

Consider the subalgebra of *upper triangular matrices*

$$\mathcal{T} = \bigoplus_{1\leq i\leq j\leq r} \mathcal{U}_{ij}\,.$$

Then it must hold

(T6) $t(uv) = (tu)v, \quad \forall\, t,u,v \in \mathcal{T}$

(T7) $t(uu^\star) = (tu)u^\star, \quad \forall\, t,u \in \mathcal{T}.$

In particular, by (T6), $\mathcal{T}$ is associative. The open subalgebra of elements with positive diagonal entries

$$H = \big\{t \in \mathcal{T} \;:\; \rho_{ii}(t) > 0,\ i = 1,\dots,r\big\}$$

contains no divisors of zero, and hence it is a Lie group; [20, p. 383]. Finally, consider the real vector space of *hermitian matrices* in $\mathcal{U}$

$$V = \{x \in \mathcal{U} \;:\; x^\star = x\},$$

endowed with the inner product $\langle x|y\rangle = \mathrm{tr}(xy)$. We define the cone Ω associated with the T-algebra structure by

$$\Omega = \big\{\, tt^\star \;:\; t \in H\big\} \subset V.$$

It can be shown that Ω is a homogeneous convex cone in V, with no straight lines, and that the group H acts simply and transitively in Ω, via the transformations

$$\pi(s)[tt^\star] = (st)(st)^\star\,, \quad s,t \in H;$$

see [20, Prop III.1]. In particular, to every $y \in \Omega$ it corresponds a unique $t \in H$ such that

$$y = \pi(t)[\mathbf{e}] = t\cdot\mathbf{e} = tt^\star. \tag{2.1}$$

All these concepts have a clear meaning when $\mathcal{U}$ consists of real $r\times r$ matrices, in which case $V = \mathrm{Sym}\ (r,\mathbb{R})$ and Ω is the cone of positive definite symmetric matrices; see more examples in §2.3 below. In general, all homogeneous cones (modulo isomorphisms) can be obtained by this procedure; see [20, Theorem III.4].

2.2. **Generalized powers in Ω.** We set some further notation from [11]; see also [17]. Let $n_{ij} = \dim\mathcal{U}_{ij} = \dim\mathcal{U}_{ji}$, $1 \leq i,j \leq r$, and consider the numbers

$$n_i = \sum_{j=1}^{i-1} n_{ij} \quad \text{and} \quad m_i = \sum_{j=i+1}^{r} n_{ij}, \quad i = 1,\dots,r. \tag{2.2}$$

Consider also the parameters

$$\tau_i = 1 + \tfrac{1}{2}(n_i + m_i), \quad i = 1,\dots,r,$$

and note that

$$n = \dim V = r + \sum_{i=1}^{r} m_i = r + \sum_{i=1}^{r} n_i = \sum_{i=1}^{r} \tau_i.$$

From these quantities we define the following distinguished multi-indices

$$\mathbf{g} = \tfrac{1}{2}(m_1, \ldots, m_r), \quad \mathbf{g}' = \tfrac{1}{2}(n_1, \ldots, n_r), \quad \boldsymbol{\tau} = (\tau_1, \ldots, \tau_r). \tag{2.3}$$

Observe in particular that $\mathbf{g} + \mathbf{g}' = \boldsymbol{\tau} - \mathbf{1}$, with the usual convention $\mathbf{1} = (1, \ldots, 1)$.

We turn to the definition of the generalized powers in Ω. If $y = tt^\star \in \Omega$, for some (unique) $t \in H$, we let

$$Q_j(y) = Q_j(tt^\star) \;=\; \rho_{jj}(t)^2, \quad j = 1, \ldots, r,$$

see [17, p 482] or [20, (III.27)]. This coincides with the quantity denoted by $\chi_j(y)$ in Gindikin's work; see e.g. [12, (2.21)]. It can be shown that these are rational functions of y (ie, quotients of polynomials), and that they can be extended to $\Omega + iV$. These functions verify the following homogeneity under the action of H

$$Q_j(t \cdot y) = Q_j(t \cdot \mathbf{e})\, Q_j(y), \quad t \in H, \quad y \in \Omega. \tag{2.4}$$

Finally, given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ (or even in $\mathbb{C}^r$) one defines

$$Q^{\boldsymbol{\alpha}}(y) := \prod_{j=1}^{r} Q_j^{\alpha_j}(y), \quad y \in \Omega.$$

It can be shown that $\pi(t)$, extended as a linear map in V, satisfies

$$\det \pi(t) = Q^{\boldsymbol{\tau}}(t \cdot \mathbf{e}), \quad \text{if } t \in H \text{ and } t \cdot \mathbf{e} = tt^\star \in \Omega,$$

see [17, (2.10)]. It follows that

$$d\sigma(y) = Q^{-\boldsymbol{\tau}}(y)\, dy$$

is a (left)-invariant measure in Ω under the action of the group H.

2.3. **Some examples.** The following examples are discussed in [11, pp.17-19]; see also [12, Chapter 2, §1.8].

2.3.1. **Cones of positive-definite symmetric matrices.** Let $\mathcal{U}$ be the algebra of real $r \times r$ matrices. Then $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$ is the cone of positive definite symmetric matrices. The representation of $y \in \Omega$ as $y = tt^\star$, $t \in H$, see (2.1), corresponds to the standard decomposition of a positive-definite symmetric matrix as a product of an upper triangular matrix and its transpose. The parameters in (2.3) take the form

$$\mathbf{g} = \frac{1}{2}\,(r-1, \ldots, 1, 0), \quad \mathbf{g}' = \frac{1}{2}\,(0, 1, \ldots, r-1), \quad \text{and} \quad \boldsymbol{\tau} \equiv 1 + \tfrac{r-1}{2},$$

while the basic power functions associated with the cone are given by

$$Q_j(y) = \frac{\Delta_{r-j+1}(y)}{\Delta_{r-j}(y)}, \quad j = 1, \dots, r,$$

where $\Delta_i(y)$ is the principal lower corner minor of the matrix y (with $\Delta_0 = 1$). In particular, when $r = 2$ we have

$$Q_1(y) = \frac{y_{11}y_{22} - y_{12}^2}{y_{22}}, \quad \text{and} \quad Q_2(y) = y_{22}, \quad \text{if} \quad y = \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}.$$

2.3.2. **Vinberg cone.** Let $\mathcal{U}$ consist of real 3×3 matrices with

$$\mathcal{U}_{23} = \mathcal{U}_{32} = \{0\}.$$

When $x, y \in \mathcal{U}$ its product $z = xy$ is defined by $z_{ij} = \sum_{k=1}^{3} x_{ik}y_{kj}$ if $i + j \neq 5$, and $z_{23} = z_{32} = 0$ (that is, the $\mathcal{U}$-projection of the usual matrix product). It is easily seen that this product is compatible with the T-algebra axioms. In this case we have $n_{12} = n_{13} = 1$ and $n_{23} = 0$, so that

$$\mathbf{g} = (1, 0, 0), \quad \mathbf{g}' = (0, \tfrac{1}{2}, \tfrac{1}{2}), \quad \text{and} \quad \boldsymbol{\tau} = (2, \tfrac{3}{2}, \tfrac{3}{2}).$$

The associated cone Ω is the set of all positive-definite matrices of the form

$$y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & 0 \\ y_{13} & 0 & y_{33} \end{pmatrix},$$

and the basic power functions are given by

$$Q_1(y) = y_{11} - \frac{y_{12}^2}{y_{22}} - \frac{y_{13}^2}{y_{33}}, \quad Q_2(y) = y_{22}, \quad \text{and} \quad Q_3(y) = y_{33}.$$

2.4. **A beta integral formula.** Below we use standard conventions for multi-indices, namely if $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r) \in \mathbb{R}^r$, then

$$\boldsymbol{\alpha} > \boldsymbol{\beta} \quad \text{means} \quad \alpha_i > \beta_i, \quad \forall\, i = 1, \dots, r, \tag{2.5}$$

and

$$\boldsymbol{\alpha}\boldsymbol{\beta} = (\alpha_1\beta_1, \dots, \alpha_r\beta_r) \in \mathbb{R}^r. \tag{2.6}$$

A key result in the later computations will be the following lemma due to Gindikin. We remark that the formulation in [11, Proposition 2.6] contains a mistake, so we present the correct statement given in [17, Lemma 4.19]; see also [9, Corollary 2.19].

Lemma 2.7. *Let* $\mathbf{s}, \mathbf{t} \in \mathbb{R}^r$ *be such that*

$$\mathbf{t} > \mathbf{g} \quad \textit{and} \quad \mathbf{s} > \mathbf{g}'.$$

Then, and only then, the following integrals are finite and take the value

$$\int_\Omega \frac{Q^{\mathbf{t}-\boldsymbol{\tau}}(y)}{Q^{\mathbf{s}+\mathbf{t}}(x+y)}\, dy = c\, Q^{-\mathbf{s}}(x), \quad x \in \Omega,$$

for some constant $c = \beta_\Omega(\mathbf{s}+\mathbf{g}-\mathbf{g}',\mathbf{t}) > 0$.

We quote two more integral estimates that can be found in [17]. In these statements the euclidean vector space $(V, \langle\cdot|\cdot\rangle)$ is identified with $\mathbb{R}^n$.

Lemma 2.8. [17, Lemma 4.20] *Let* $\boldsymbol{\gamma} \in \mathbb{R}^r$. *Then the integral*

$$J_{\boldsymbol{\gamma}}(y) = \int_V \left|Q^{-(\boldsymbol{\gamma}+\boldsymbol{\tau})}(y+ix)\right| dx, \quad y \in \Omega \tag{2.9}$$

converges if and only if $\boldsymbol{\gamma} > \mathbf{g}'$. *In this case, there exists* $c_{\boldsymbol{\gamma}} > 0$ *such that*

$$J_{\boldsymbol{\gamma}}(y) = c_{\boldsymbol{\gamma}}\, Q^{-\boldsymbol{\gamma}}(y).$$

Lemma 2.10. [17, Lemma 4.21] *Let* $\boldsymbol{\gamma} \in \mathbb{R}^r$. *Then there is a constant* $C_{\boldsymbol{\gamma}} > 0$ *such that for all* $y \in \Omega$, $|y| \le 1/4$,

$$\int_{\{x\in V:\,|x|<1\}} \left|Q^{-(\boldsymbol{\gamma}+\boldsymbol{\tau})}(y+ix)\right| dx \;\ge\; C_{\boldsymbol{\gamma}}\, Q^{-\boldsymbol{\gamma}}(y).$$

2.5. **A Schur type test.** Below we use a known test for $L^p - L^q$ boundedness of positive operators. When $p = q$ it is the usual Schur test, see e.g. [10, Lemma 3.1], or [1, Theorem 6.3] for a statement closer to the notation below. For $p \le q$, this test (in a slightly weaker form) can be found in the work of Okikiolu [18]. We reproduce the elementary proof for completeness. We consider abstract (σ-finite) Lebesgue spaces $L^p = L^p(Y, d\nu)$ and $L^q = L^q(X, d\mu)$, and as usual, $1/p + 1/p' = 1$.

Lemma 2.11. *Let* $1 \le p \le q \le \infty$. *Given a non-negative kernel* $K(x,y) \ge 0$, *consider the formal operator*

$$Tf(x) := \int_Y K(x,y)\, f(y)\, d\nu(y), \quad x \in X.$$

Assume that

$$0 \le K(x,y) \le G(x,y)H(x,y), \tag{2.12}$$

and that there exist functions $\phi_1(y) > 0$ *and* $\phi_2(x) > 0$ *and constants* $c_1, c_2 > 0$, *such that*

$$\left[\int_Y |G(x,y)\phi_1(y)|^{p'}\, d\nu(y)\right]^{\frac{1}{p'}} \le\; c_2\, \phi_2(x), \qquad \forall\, x \in X \tag{2.13}$$

$$\left[\int_X |H(x,y)\phi_2(x)|^{q}\, d\mu(x)\right]^{\frac{1}{q}} \le\; c_1\, \phi_1(y), \qquad \forall\, y \in Y. \tag{2.14}$$

Then T *maps* $L^p(Y, d\mu) \to L^q(X, d\mu)$ *boundedly, and*

$$\|Tf\|_{L^q(X,d\mu)} \;\le\; c_1\, c_2\, \|f\|_{L^p(Y,d\nu)}, \quad \forall\, f \in L^p(Y, d\nu).$$

Proof. If $f \geq 0$ and $x \in X$, by (2.12) and Hölder's inequality we have

$$\begin{aligned} Tf(x) &= \int_Y K(x,y) f(y)\, d\nu(y) \leq \int_Y \phi_1(y)\, G(x,y)\, H(x,y)\, \phi_1^{-1}(y) f(y)\, d\nu(y) \\ &\leq \big\|\phi_1\, G(x,\cdot)\big\|_{L^{p'}(Y)} \Big[\int_Y \Big|H(x,y)\phi_1^{-1}(y) f(y)\Big|^p d\nu(y)\Big]^{\frac{1}{p}}. \end{aligned}$$

By (2.13), the first factor is bounded by $c_2\phi_2(x)$, so taking L^q-norms we obtain

$$\|Tf\|_{L^q(X)} \leq c_2 \Big(\int_X \Big[\int_Y \Big|H(x,y)\phi_1^{-1}(y)\phi_2(x) f(y)\Big|^p d\nu(y)\Big]^{\frac{q}{p}} d\mu(x)\Big)^{\frac{1}{q}}.$$

Since $q \geq p$ we can use the Minkowski integral inequality to deduce

$$\begin{aligned} \|Tf\|_{L^q(X)} &\leq c_2 \Big[\int_Y \Big(\int_X \Big|H(x,y)\phi_1^{-1}(y)\phi_2(x) f(y)\Big|^q d\mu(x)\Big)^{\frac{p}{q}} d\nu(y)\Big]^{\frac{1}{p}} \\ &= c_2 \Big\|\phi_1^{-1}(y)\, f(y)\, \big\|H(x,y)\,\phi_2(x)\big\|_{L^q(d\mu(x))}\Big\|_{L^p(d\nu(y))} \\ &\leq c_1\, c_2\, \big\|f\big\|_{L^p(Y)}, \end{aligned}$$

using in the last step the assumption (2.14). Observe that the above argument for $1 < p \leq q < \infty$, remains also valid if $p = 1$ or $q = \infty$. □

3. Proof of Theorem 1.4: sufficient conditions

3.1. **Proof of part (i) for $p > 1$.** As observed in Remark 1.5, it suffices to show that if $1 < p \leq q < \infty$, then

$$S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} : L^p(\Omega, d\sigma) \longrightarrow L^q(\Omega, d\sigma),$$

under the conditions $\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}$ and

$$\mathbf{a} > \frac{\mathbf{g}}{q} + \frac{\mathbf{g}'}{p'} \qquad \text{and} \qquad \mathbf{b} > \frac{\mathbf{g}}{p'} + \frac{\mathbf{g}'}{q}\,. \tag{3.1}$$

We apply the Schur test to

$$Tf(x) = \int_\Omega K(x,y) f(y)\, d\sigma(y), \quad \text{with} \quad K(x,y) = \frac{Q^{\mathbf{a}}(x)\, Q^{\mathbf{b}}(y)}{Q^{\mathbf{a}+\mathbf{b}}(x+y)}.$$

To do so, we shall split the kernel as

$$K(x,y) = K(x,y)^{\mathbf{t}}\, K(x,y)^{\mathbf{1}-\mathbf{t}}, \tag{3.2}$$

for some $\mathbf{t} \in \mathbb{R}^r$ with $\mathbf{0} < \mathbf{t} < \mathbf{1}$, where we use the multi-index conventions in (2.5) and (2.6). It then suffices to find test functions

$$\phi_1(y) = Q^{-\boldsymbol{\sigma}}(y) \quad \text{and} \quad \phi_2(x) = Q^{-\boldsymbol{\delta}}(x), \tag{3.3}$$

for suitable $\boldsymbol{\sigma}, \boldsymbol{\delta} \in \mathbb{R}^r$, such that

$$\Big[\int_\Omega \Big|\frac{Q^{\mathbf{a}\mathbf{t}}(x)\, Q^{\mathbf{b}\mathbf{t}}(y)}{Q^{(\mathbf{a}+\mathbf{b})\mathbf{t}}(x+y)}\, Q^{-\boldsymbol{\sigma}}(y)\Big|^{p'} d\sigma(y)\Big]^{1/p'} \lesssim Q^{-\boldsymbol{\delta}}(x), \quad x \in \Omega, \tag{3.4}$$

and

$$\Big[\int_\Omega \Big|\frac{Q^{\mathbf{a}(\mathbf{1}-\mathbf{t})}(x)\,Q^{\mathbf{b}(\mathbf{1}-\mathbf{t})}(y)}{Q^{(\mathbf{a}+\mathbf{b})(\mathbf{1}-\mathbf{t})}(x+y)}\,Q^{-\boldsymbol{\delta}}(x)\Big|^q\,d\sigma(x)\Big]^{1/q} \lesssim Q^{-\boldsymbol{\sigma}}(y), \quad y\in\Omega, \tag{3.5}$$

To do so we use Lemma 2.7. The expression in the left hand side of (3.4) is finite iff

$$(\mathbf{bt}-\boldsymbol{\sigma})p' > \mathbf{g} \quad \text{and} \quad (\mathbf{at}+\boldsymbol{\sigma})p' > \mathbf{g}'\,,$$

in which case it takes the value $c_1\,Q^{-\boldsymbol{\sigma}}(x)$, for some $c_1>0$. Similarly, the left expression in (3.5) is finite iff

$$(\mathbf{a}(\mathbf{1}-\mathbf{t})-\boldsymbol{\delta})q > \mathbf{g} \quad \text{and} \quad (\mathbf{b}(\mathbf{1}-\mathbf{t})+\boldsymbol{\delta})q > \mathbf{g}'\,,$$

in which case it takes the value $c_2\,Q^{-\boldsymbol{\delta}}(y)$, for some $c_2>0$. Thus, (3.4) and (3.5) will hold if we take $\boldsymbol{\delta}=\boldsymbol{\sigma}$, and if we can select $\boldsymbol{\sigma}$ and $\mathbf{t}$ such that

$$\frac{\mathbf{g}'}{p'}-\mathbf{at}<\boldsymbol{\sigma}<\mathbf{bt}-\frac{\mathbf{g}}{p'} \qquad \text{and} \qquad \frac{\mathbf{g}'}{q}-\mathbf{b}(\mathbf{1}-\mathbf{t})<\boldsymbol{\sigma}<\mathbf{a}(\mathbf{1}-\mathbf{t})-\frac{\mathbf{g}}{q}. \tag{3.6}$$

For each of these two "intervals" to be non-empty we must impose the conditions

$$\frac{\mathbf{g}+\mathbf{g}'}{p'}<(\mathbf{a}+\mathbf{b})\mathbf{t} \qquad \text{and} \qquad \frac{\mathbf{g}+\mathbf{g}'}{q}<(\mathbf{a}+\mathbf{b})(\mathbf{1}-\mathbf{t}),$$

which solving for $\mathbf{t}$ lead to

$$\frac{\mathbf{g}+\mathbf{g}'}{(\mathbf{a}+\mathbf{b})\,p'}<\mathbf{t}<\mathbf{1}-\frac{\mathbf{g}+\mathbf{g}'}{(\mathbf{a}+\mathbf{b})\,q}, \tag{3.7}$$

interpreted as pointwise inequalities for each coordinate $i=1,\ldots,r$. By the assumptions in (3.1), it is always possible to find such a $\mathbf{t}$.

Once $\mathbf{t}$ is chosen, we check that a multi-index $\boldsymbol{\sigma}$ as in (3.6) exists. To do so we conveniently denote by $(\mathbf{u}_1,\mathbf{v}_1)$ and $(\mathbf{u}_2,\mathbf{v}_2)$ the two "intervals" of multi-indices in (3.6) (formally, rectangles in $\mathbb{R}^r$). We must show that

$$\big(\mathbf{u}_1,\mathbf{v}_1\big)\cap\big(\mathbf{u}_2,\mathbf{v}_2\big)\neq\emptyset. \tag{3.8}$$

Indeed, the conditions in (3.1) can be written as

$$\mathbf{u}_1<\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{u}_2<\mathbf{v}_1, \tag{3.9}$$

regardless of the value of $\mathbf{t}$, and this in turn is easily seen to be equivalent to (3.8). This completes the proof of part (i) in Theorem 1.4 when $p>1$.

3.2. **Proof of part (i) for $p=1$.** The only difference is that, rather than (3.4), we need to require

$$\sup_{y\in\Omega}\frac{Q^{\mathbf{at}}(x)\,Q^{\mathbf{bt}-\boldsymbol{\sigma}}(y)}{Q^{(\mathbf{a}+\mathbf{b})\mathbf{t}}(x+y)} \lesssim Q^{-\boldsymbol{\delta}}(x), \quad x\in\Omega. \tag{3.10}$$

Letting $x = t \cdot \mathbf{e}$ with $t \in H$, we may as well take the sup over $y = t \cdot u$, $u \in \Omega$; thus, the homogeneity in (2.4) shows that (3.10) can be written as

$$Q^{-\boldsymbol{\sigma}}(x) \sup_{u \in \Omega} \frac{Q^{\mathbf{bt}-\boldsymbol{\sigma}}(u)}{Q^{(\mathbf{a}+\mathbf{b})\mathbf{t}}(\mathbf{e}+u)} \lesssim Q^{-\boldsymbol{\delta}}(x), \quad x \in \Omega.$$

For this to hold we need $\boldsymbol{\delta} = \boldsymbol{\sigma}$ and

$$-\mathbf{at} \leq \boldsymbol{\sigma} \leq \mathbf{bt}.$$

This is less restrictive than the left expression in (3.6) with $p = 1$. So the same arguments given above can be used in this case, providing the existence of the required multi-index $\boldsymbol{\sigma}$ under the assumptions (3.1), which now take the simpler form

$$\mathbf{a} > \mathbf{g}/q \qquad \text{and} \qquad \mathbf{b} > \mathbf{g}'/q\,. \tag{3.11}$$

3.3. **The case $q = \infty$.** We have excluded this case to avoid end-point situations, however, the same argument as in §3.2 can be applied when $q = \infty$, at least if $1 < p \leq \infty$. The sufficient conditions in (3.1) remain valid, and now take the form

$$\mathbf{a} > \mathbf{g}'/p' \qquad \text{and} \qquad \mathbf{b} > \mathbf{g}/p'\,. \tag{3.12}$$

Alternatively, this could also be proved by duality.

Finally, we mention the special case corresponding to $p = 1$ and $q = \infty$, for which a direct inequality shows that $S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}$ (with $\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}$) maps L^1 into L^∞ in the larger range

$$\mathbf{a} \geq \mathbf{0} \quad \text{and} \quad \mathbf{b} \geq \mathbf{0}. \tag{3.13}$$

3.4. **Some remarks on the sufficient conditions.** We make some comments about the previous proof

Remark 3.14. We have used a multi-index $\mathbf{t} \in \mathbb{R}^r$ in (3.2), in order to allow for greater generality. However, the proof works as well with a scalar $\mathbf{t} = (t, \ldots, t)$. Indeed, in view of (3.7) (and using $\mathbf{g} + \mathbf{g}' = \boldsymbol{\tau} - \mathbf{1}$), it suffices to pick t such that

$$\frac{\vartheta}{p'} < t < 1 - \frac{\vartheta}{q}, \quad \text{where} \quad \vartheta := \max_{1 \leq i \leq r} \frac{\tau_i - 1}{a_i + b_i}.$$

Now, (3.1) implies $\mathbf{a} + \mathbf{b} > (\boldsymbol{\tau} - \mathbf{1})(\frac{1}{p'} + \frac{1}{q})$, and thus $(\frac{1}{p'} + \frac{1}{q})\vartheta < 1$, so such a t can always be chosen. In particular, when $p = q$ we could take $t = 1/p'$, which is the usual choice in Schur's test.

Remark 3.15. Even in the special case when $\Omega = \mathrm{Sym}_+(2, \mathbb{R})$, $p = q$ and $\mathbf{a}, \mathbf{b}$ are scalars, the optimal sufficient conditions in (3.1) cannot be obtained merely with test functions ϕ_1, ϕ_2 involving a scalar parameter $\boldsymbol{\sigma} = (\sigma, \ldots, \sigma)$. Indeed, in this special case we have

$$\mathbf{g} = (1/2, 0) \quad \text{and} \quad \mathbf{g}' = (0, 1/2)$$

so the condition in (3.6) becomes

$$\frac{1}{2p'} - at < \sigma < bt - \frac{1}{2p'} \quad \text{and} \quad \frac{1}{2p} - (1-t)b < \sigma < (1-t)a - \frac{1}{2p}.$$

However, arguing as in (3.8) and (3.9), one sees that these two conditions cannot simultaneously hold if $a \leq 1/2$ or $b \leq 1/2$. So one would not cover the whole range of sufficient conditions in (3.1), which in this case should be

$$a > \max\left\{\frac{1}{2p}, \frac{1}{2p'}\right\} \quad \text{and} \quad b > \max\left\{\frac{1}{2p}, \frac{1}{2p'}\right\}.$$

This suggests that the proof of [16, Lemma 4.1] may not be correct, and must be modified according to Remark 4.2 in that paper.

4. Proof of Theorem 1.4: necessary conditions

As observed in Remark 1.5, to prove part (ii) of Theorem 1.4 it suffices to find necessary conditions for the boundedness of

$$S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} : L^p(\Omega, d\sigma) \longrightarrow L^q(\Omega, d\sigma), \tag{4.1}$$

for fixed $1 \leq p, q < \infty$ and $\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma} \in \mathbb{R}^r$. Observe that $S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f(x)$ is always well-defined for non-negative f. So, the boundedness of the mapping (4.1) must be understood as

$$\left\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\right\|_q \leq C \left\|f\right\|_p, \quad \forall\, f \geq 0, \tag{4.2}$$

for some constant $C > 0$. The p-norms will always refer to the spaces $L^p(\Omega, d\sigma)$.

4.1. Necessary condition on $\boldsymbol{\gamma}$.

Lemma 4.3. *Let $1 \leq p, q \leq \infty$ and $\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma} \in \mathbb{R}^r$. Assume that (4.2) holds. Then $\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}$.*

Proof. We use the homogeneity under the H-action in Ω. Let $t \in H$ and define

$$f_t(y) := f(t \cdot y), \quad y \in \Omega.$$

Then, by the left-invariance of $d\sigma$ and (2.4) we have

$$\begin{aligned} S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}(f_t)(x) &= Q^{\mathbf{a}}(x) \int_\Omega f(y) \frac{Q^{\mathbf{b}}(t^{-1} \cdot y)}{Q^{\boldsymbol{\gamma}}(x + t^{-1} \cdot y)}\, d\sigma(y) \\ &= Q^{\mathbf{a}+\mathbf{b}-\boldsymbol{\gamma}}(t^{-1} \cdot \mathbf{e}) \left(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\right)(t \cdot x)\,. \end{aligned}$$

Thus, if (4.2) holds we will have

$$Q^{\mathbf{a}+\mathbf{b}-\boldsymbol{\gamma}}(t^{-1} \cdot \mathbf{e}) \left\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}(f)\right\|_q = \left\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}(f_t)\right\|_q \leq C \left\|f_t\right\|_p = C \left\|f\right\|_p.$$

Fixing a positive $f \in L^p$, and letting t vary in the set of diagonal matrices whose entries go to 0 or $+\infty$, we see that necessarily $\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}$. □

4.2. **Necessity of $p \le q$.** We begin with an elementary observation, which is similar to the one used in [14, Theorem 1.1].

Lemma 4.4. *Let $1 \le p < \infty$, and for $R > 0$ let $f_R(y) := f(y/R)$, $y \in \Omega$. Then*

$$\lim_{R\to\infty} \big\|f + f_R\big\|_p = 2^{1/p}\,\|f\|_p, \quad \forall\, f \in L^p(\Omega, d\sigma). \tag{4.5}$$

Proof. If we assume that $\operatorname{supp} f \Subset \Omega$, then f and f_R will have disjoint supports for large enough R, and thus

$$\|f + f_R\|_p^p = \int_\Omega \big(|f(y)|^p + |f_R(y)|^p\big)\, d\sigma(y) = 2\|f\|_p^p,$$

by the invariance of the measure. For general $f \in L^p$, given $\varepsilon > 0$, one finds h with $\operatorname{supp} h \Subset \Omega$ and $\|f - h\|_p < \varepsilon$. Then, by the triangle inequality

$$\begin{aligned}\Big|\|f + f_R\|_p - 2^{\frac1p}\|f\|_p\Big| &\le 2\|f-h\|_p + \Big|\|h + h_R\|_p - 2^{\frac1p}\|h\|_p\Big| + 2^{\frac1p}\Big|\|h\|_p - \|f\|_p\Big| \\ &\le (2 + 2^{\frac1p})\varepsilon + \Big|\|h + h_R\|_p - 2^{\frac1p}\|h\|_p\Big| \le (2+2^{\frac1p})\varepsilon,\end{aligned}$$

if R is sufficiently large. The result now follows by the definition of limit. □

Lemma 4.6. *Let $1 \le p, q < \infty$ and $\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma} \in \mathbb{R}^r$. Assume that (4.2) holds. Then $p \le q$.*

Proof. Assume that $p > q$ and that (4.2) holds. By Lemma 4.3 we must have $\boldsymbol{\gamma} = \mathbf{a}+\mathbf{b}$, so with the notation $f_R(y) := f(y/R)$, from the previous lemma, one easily sees that

$$\big(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f_R\big)(x) = \big(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\big)(x/R).$$

Thus,

$$\big\|\big(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\big) + \big(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\big)(\cdot/R)\big\|_q = \big\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}(f + f_R)\big\|_q \le \|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}\|\,\|f + f_R\|_p,$$

where $\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}\|$ denotes the infimum of all constants C such that (4.2) holds. Letting $R \nearrow \infty$ and using Lemma 4.4 we conclude that

$$2^{\frac1q}\,\big\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\big\|_q \le 2^{\frac1p}\,\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}\|\,\|f\|_p.$$

Since this is valid for all $f \ge 0$ we conclude that

$$\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}\| \le 2^{\frac1p - \frac1q}\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}\|,$$

which is not possible when $p > q$ (since $\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}\| > 0$). □

4.3. **Necessary conditions on a and b.** We first recall some notions from [17, §4.2]. For $y_0 \in \Omega$, we let $B_r(y_0) := \{y \in \Omega \ : \ d(y, y_0) < r\}$, where d denotes the associated riemannian distance in Ω (which is H-invariant). The following result is known for symmetric cones; see [6, Corollary 2.3]. In the appendix we sketch a different proof, which is valid as well for homogeneous cones.

LEMMA 4.7. *Let Ω be a homogeneous cone. Then, there are constants $c_1, c_2 > 0$ such that*

$$c_1 \leq \frac{Q_j(x+y)}{Q_j(x+y_0)} \leq c_2, \quad \textit{if} \;\; d(y, y_0) < 1, \quad x \in \Omega, \quad j = 1, \ldots, r.$$

LEMMA 4.8. *Let $1 \leq p \leq q < \infty$, and let $\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma} \in \mathbb{R}^r$ with $\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}$. Assume that* (4.2) *holds. Then*

$$\mathbf{a} > \max\left\{\frac{\mathbf{g}}{q}, \frac{\mathbf{g}'}{p'}\right\} \quad \textit{and} \quad \mathbf{b} > \max\left\{\frac{\mathbf{g}}{p'}, \frac{\mathbf{g}'}{q}\right\}. \tag{4.9}$$

Proof. Let $f = \chi_{B_1(\mathbf{e})}$. Then, by the previous lemma

$$(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f)(x) = \int_{B_1(\mathbf{e})} \frac{Q^{\mathbf{a}}(x)\, Q^{\mathbf{b}}(y)}{Q^{\mathbf{a}+\mathbf{b}}(x+y)}\, d\sigma(y) \;\approx\; \frac{Q^{\mathbf{a}}(x)}{Q^{\mathbf{a}+\mathbf{b}}(x+\mathbf{e})}, \quad x \in \Omega.$$

So, if (4.2) holds we deduce that

$$\left[\int_\Omega \left|\frac{Q^{\mathbf{a}}(x)}{Q^{\mathbf{a}+\mathbf{b}}(x+\mathbf{e})}\right|^q d\sigma(x)\right]^{\frac{1}{q}} \lesssim \left\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\right\|_q \leq C\, \|f\|_p < \infty,$$

which by Gindinkin's result, Lemma 2.7, implies that

$$\mathbf{a} > \mathbf{g}/q \quad \text{and} \quad \mathbf{b} > \mathbf{g}'/q.$$

The other condition follows by duality, since $S^*_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} = S_{\mathbf{b},\mathbf{a},\boldsymbol{\gamma}}$ and (4.2) implies that

$$\left\|S_{\mathbf{b},\mathbf{a},\boldsymbol{\gamma}} f\right\|_{p'} \leq C\, \|f\|_{q'}.$$

Hence, testing with $f = \chi_{B_1(\mathbf{e})}$ as above leads to

$$\mathbf{a} > \mathbf{g}'/p' \quad \text{and} \quad \mathbf{b} > \mathbf{g}/p',$$

at least if $p > 1$. When $p = 1$ the above inequalities are no longer strict, and become

$$\mathbf{a} \geq \mathbf{0} \quad \text{and} \quad \mathbf{b} \geq \mathbf{0}. \tag{4.10}$$

But in this case $\mathbf{a} > \mathbf{g}/q$ is the same as $\mathbf{a} > \max\left\{\mathbf{g}/q, \mathbf{0} = \mathbf{g}'/p'\right\}$, and likewise for $\mathbf{b}$. So, the result is proved in all cases. □

REMARK 4.11. Lemma 4.8 is also valid in the case $q = \infty$, as long as $1 < p \leq \infty$. The only special case happens when $p = 1$ and $q = \infty$, for which the necessary condition would just be (4.10). This matches the sufficient condition discussed in (3.13). Thus, overall we can state the following

COROLLARY 4.12 **(Case $q = \infty$).** *Let $1 < p \le \infty$, and $\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma} \in \mathbb{R}^r$. Then $S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} : L^p(d\sigma) \to L^\infty(d\sigma)$ if and only if $\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}$ and*

$$\mathbf{a} > \mathbf{g}'/p' \qquad \text{and} \qquad \mathbf{b} > \mathbf{g}/p' \,. \tag{4.13}$$

If $p = 1$ the same characterization holds with (4.13) replaced by

$$\mathbf{a} \ge \mathbf{0} \quad \text{and} \quad \mathbf{b} \ge \mathbf{0}.$$

5. Comparison of necessary and sufficient conditions

Here we prove the statements in part iii) of Theorem 1.4. Namely, for the operators $S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} : L^p \to L^q$ (with $\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}$) we compare the necessary conditions

$$\mathbf{a} > \max\left\{\frac{\mathbf{g}}{q}, \frac{\mathbf{g}'}{p'}\right\} \quad \text{and} \quad \mathbf{b} > \max\left\{\frac{\mathbf{g}}{p'}, \frac{\mathbf{g}'}{q}\right\} \tag{A2'}$$

and the sufficient conditions

$$\mathbf{a} > \frac{\mathbf{g}}{q} + \frac{\mathbf{g}'}{p'} \qquad \text{and} \qquad \mathbf{b} > \frac{\mathbf{g}}{p'} + \frac{\mathbf{g}'}{q}\,; \tag{A2}$$

see (3.1) and (4.9) above. These clearly coincide when $p = 1$ (or $q = \infty$), so we will assume $1 < p \le q < \infty$.

5.1. **(A2) and (A2') for a general cone Ω.** When $r = 1$, then $\Omega = (0, \infty)$ and the conditions trivially coincide, since $\mathbf{g} = \mathbf{g}' = 0$. This is the classical setting for the Hilbert inequalities, see also [2]. For $r \ge 2$ we use the following simple observation.

LEMMA 5.1. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^r$. Then*

$$\mathbf{u} + \mathbf{v} = \max\{\mathbf{u}, \mathbf{v}\}$$

if and only if $\operatorname{supp} \mathbf{u} \cap \operatorname{supp} \mathbf{v} = \emptyset$, that is, iff $\quad u_i v_i = 0, \ \forall\, i = 1, \ldots, r$.

The previous lemma has the following immediate consequence. Recall that $\mathbf{g}$ and $\mathbf{g}'$ are defined in (2.2) and (2.3) in terms of the indices $n_{ij} = \dim \mathcal{U}_{ij}$.

COROLLARY 5.2. *Let $1 < p \le q < \infty$. Let Ω be a homogeneous cone associated with a T-algebra $\mathcal{U} = \bigoplus_{1 \le i,j \le r} \mathcal{U}_{ij}$. Then the set of indices $\mathbf{a}, \mathbf{b} \in \mathbb{R}^r$ satisfying (A2) coincides with (A2') if and only if*

$$\Big(\sum_{j=1}^{i-1} n_{ij}\Big)\Big(\sum_{j=i+1}^{r} n_{ij}\Big) = 0, \quad \forall\, i \in \{1, \ldots, r\}. \tag{5.3}$$

We discuss some examples.

(1) **Case** $r = 2$. There is only one relevant index, namely, $n_{12} = d \in \mathbb{N} \cup \{0\}$. Therefore, we have

$$\mathbf{g} = (d/2, 0) \quad \text{and} \quad \mathbf{g}' = (0, d/2).$$

By Lemma 5.1 conditions (A2) and (A2') coincide, and take the form

$$\mathbf{a} > \Big(\frac{d}{2q}\,,\,\frac{d}{2p'}\Big) \quad \text{and} \quad \mathbf{b} > \Big(\frac{d}{2p'}\,,\,\frac{d}{2q}\Big),$$

Note that if $d = 0$, the cone is isomorphic to $\Omega = (0,\infty)\times(0,\infty)$ and we are again in a trivial situation. If $d \geq 1$ then Ω is isomorphic to a light-cone in $\mathbb{R}^{d+2}$, which is the only irreducible homogeneous cone of rank 2.

(2) **Vinberg cone.** Let Ω be the cone of rank 3 defined in §2.3.2; see also [17, Example 2.9]. This is the first relevant example of a homogeneous non-symmetric cone. In this case we have $n_{12} = n_{13} = 1$ and $n_{23} = 0$, so that

$$\mathbf{g} = (1,0,0) \quad \text{and} \quad \mathbf{g}' = (0, \tfrac{1}{2}, \tfrac{1}{2}).$$

Thus, also in this case (A2) and (A2') coincide, and take the form

$$\mathbf{a} > \Big(\frac{1}{q}\,,\,\frac{1}{2p'}\,,\,\frac{1}{2p'}\Big), \quad \mathbf{b} > \Big(\frac{1}{p'}\,,\,\frac{1}{2q}\,,\,\frac{1}{2q}\Big).$$

(3) **Symmetric cones.** Let Ω be an irreducible symmetric cone of rank r. Then, there is a constant d such that $n_{ij} = d$, $\forall\ i \neq j$. The values of $\mathbf{g}$ and $\mathbf{g}'$ are given by

$$\mathbf{g} = \Big((r-i)\frac{d}{2}\Big)_{i=1}^{r} \qquad \text{and} \qquad \mathbf{g}' = \Big((i-1)\frac{d}{2}\Big)_{i=1}^{r}, \tag{5.4}$$

so the conditions of Lemma 5.1 will never be met if $r \geq 3$ and $d \geq 1$. So in these cases, conditions (A2) and (A2') will not agree in general.

5.2. **(A2) and (A2') for scalar parameters.** We prove the last assertion in Theorem 1.4. Suppose that Ω is an irreducible symmetric cone and that

$$\mathbf{a} = (a,\dots,a) \quad \text{and} \quad \mathbf{b} = (b,\dots,b).$$

In view of (5.4), we can write the first condition in (A2) as

$$a \;>\; \max_{1\leq i\leq r}\Big(\frac{r-i}{q} + \frac{i-1}{p'}\Big)\frac{d}{2} \;=\; (r-1)\frac{d}{2}\,\max\Big\{\frac{1}{q}, \frac{1}{p'}\Big\}, \tag{5.5}$$

where the last equality follows from the fact that a linear expression in i must attain its maximum either at $i=1$ or $i=r$. Note that (5.5) clearly coincides with the condition in (A2'). The situation is similar for $\mathbf{b}$, leading also to the expression

$$b \;>\; (r-1)\frac{d}{2}\,\max\Big\{\frac{1}{q}, \frac{1}{p'}\Big\}. \tag{5.6}$$

Finally, using that $(r-1)d/2 = \frac{n}{r} - 1$, we obtain the following version for [16, Theorem 2.1].

Corollary 5.7. *Let $1 \leq p, q < \infty$. Let Ω be an irreducible symmetric cone in $\mathbb{R}^n$. Then $S_{a,b,\gamma}$ maps $L^p(\Omega, d\sigma)$ into $L^q(\Omega, d\sigma)$ if and only if $p \leq q$, $\gamma = a + b$ and*

$$\min\{a,b\} > \Big(\frac{n}{r} - 1\Big)\,\max\Big\{\frac{1}{q}, \frac{1}{p'}\Big\}.$$

6. Positive Bergman operators: proof of Theorem 1.6

As in Remark 1.5, letting

$$\mathbf{a} = \boldsymbol{\alpha} + \frac{\boldsymbol{\mu}}{q} \quad \text{and} \quad \mathbf{b} = \boldsymbol{\beta} - \frac{\boldsymbol{\nu}}{p} \,,$$

it suffices to prove Theorem 1.6 for the operators $S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}$ and $T^+_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}$, assuming that $\boldsymbol{\nu} = \boldsymbol{\mu} = \mathbf{0}$. So throughout the proof we assume this case, and for simplicity we denote $L^{s,p} = L^{s,p}(T_\Omega, dx\, d\sigma(y))$.

6.1. **Proof of "(i) $\Rightarrow$ (ii)" of Theorem 1.6.** We shall use the following notation: for $f \in L^{s,p}(T_\Omega)$ and $v \in \Omega$ we let

$$f_v(u) = f(u + iv), \; u \in \mathbb{R}^n.$$

For fixed $y \in \Omega$, using Minkowski's integral inequality and Lemma 2.8 (since $\boldsymbol{\gamma} > \mathbf{g}'$), one obtains

$$\begin{aligned} \left\| T^+_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f(x+iy) \right\|_{L^s(dx)} &\leq Q^{\mathbf{a}}(y) \int_\Omega \int_{\mathbb{R}^n} \frac{\|f_v(x-u)\|_{L^s(dx)}}{|Q^{\boldsymbol{\gamma}+\boldsymbol{\tau}}(y+v-iu)|} \, Q^{\mathbf{b}}(v)\, du\, dv \\ &= c_{\boldsymbol{\gamma}}\, Q^{\mathbf{a}}(y) \int_\Omega \frac{\|f_v\|_{L^s}\, Q^{\mathbf{b}}(v)\, dv}{Q^{\boldsymbol{\gamma}}(y+v)} \; = \; c_{\boldsymbol{\gamma}}\, S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}\big(\|f_v\|_{L^s}\big)(y). \end{aligned}$$

Thus, taking $L^q(\Omega, d\sigma)$-norms in both sides and using (i) we obtain

$$\left\| T^+_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f \right\|_{L^{s,q}(T_\Omega)} \lesssim \left\| S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}}(\|f_v\|_{L^s}) \right\|_{L^q(d\sigma)} \lesssim \left\| \|f_v\|_{L^s} \right\|_{L^p(d\sigma)} = \|f\|_{L^{s,p}(T_\Omega)}.$$

Since s here is arbitrary, we obtain the assertion in (ii) of Theorem 1.6.

6.2. **Proof of "(ii) $\Rightarrow$ (iii)" of Theorem 1.6.** This assertion is trivial.

6.3. **Proof of "(iii) $\Rightarrow$ (i)" of Theorem 1.6.** We first observe that (iii) implies, by a homogeneity argument as in Lemma 4.3, that necessarily

$$\boldsymbol{\gamma} = \mathbf{a} + \mathbf{b}. \tag{6.1}$$

Now, let $f \in L^p(\Omega, d\sigma)$ be such that $f \geq 0$ and

$$\operatorname{supp} f \subset \Omega \cap B_{1/8}(0),$$

and define $g(u+iv) = \chi_{B_2(0)}(u)\, f(v)$, for $u + iv \in T_\Omega$. If $|x| \leq 1$ and $y \in \Omega \cap B_{1/8}(0)$ then, by Lemma 2.10, there is a constant $C_{\boldsymbol{\gamma}} > 0$ such that

$$\begin{aligned} \big(T^+_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} g\big)(x+iy) &= Q^{\mathbf{a}}(y) \int_\Omega \int_{B_2(x)} \frac{du}{|Q^{\boldsymbol{\gamma}+\boldsymbol{\tau}}(y+v-iu)|}\, f(v)\, Q^{\mathbf{b}}(v)\, dv \\ &\geq C_{\boldsymbol{\gamma}}\, Q^{\mathbf{a}}(y) \int_\Omega \frac{f(v)\, Q^{\mathbf{b}}(v)\, dv}{Q^{\boldsymbol{\gamma}}(y+v)} \; = \; C_{\boldsymbol{\gamma}} \big(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\big)(y), \end{aligned}$$

since $B_2(x) \supset \{|u| < 1\}$. So taking first the L^s-norm over $|x| \leq 1$, and then the $L^q(d\sigma)$-norm over $y \in \Omega \cap B_{1/8}(0)$, and using the assumption (iii), we see that

$$\left\| S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f \right\|_{L^q(\Omega \cap B_{1/8}(0))} \lesssim \left\| T^+_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} g \right\|_{L^{s,q}(T_\Omega)} \lesssim \|g\|_{L^{s,p}(T_\Omega)} = c\, \|f\|_{L^p(d\sigma)}.$$

Let now f be an arbitrary function in $L^p(\Omega, d\sigma)$ with compact support, and pick any large R such that $\operatorname{supp} f \subset \Omega \cap B_{R/8}(0)$. Then, we can apply the previous reasoning to the rescaled function $f_R = f(R\cdot)$ to obtain

$$\left\| S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f_R \right\|_{L^q(\Omega \cap B_{1/8}(0))} \lesssim \|f_R\|_{L^p(d\sigma)} = \|f\|_{L^p(d\sigma)}.$$

Using (6.1) one easily sees that $(S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f_R)(y) = (S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f)(Ry)$, so changing variables the above inequality becomes

$$\left\| S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f \right\|_{L^q(\Omega \cap B_{R/8}(0))} \lesssim \|f\|_{L^p(d\sigma)}.$$

Letting $R \to \infty$ we conclude that $\|S_{\mathbf{a},\mathbf{b},\boldsymbol{\gamma}} f\|_{L^q(\Omega, d\sigma)} \lesssim \|f\|_{L^p(d\sigma)}$, which implies the assertion in (i). $\square$

6.4. **Proof of Corollary 1.8.** Since $P^+_{\boldsymbol{\nu}} = \mathfrak{c}_{\boldsymbol{\nu}}\, T^+_{\mathbf{0},\boldsymbol{\nu},\boldsymbol{\nu}}$ and we assume $\boldsymbol{\nu} > \mathbf{g}'$, it follows from Theorem 1.6 that the boundedness of

$$P^+_{\boldsymbol{\nu}} : L^{s,p}_{\boldsymbol{\nu}}(T_\Omega) \longrightarrow L^{s,q}_{\boldsymbol{\mu}}(T_\Omega)$$

is equivalent to the boundedness of

$$S_{\mathbf{0},\boldsymbol{\nu},\boldsymbol{\nu}} : L^p_{\boldsymbol{\nu}}(\Omega) \longrightarrow L^q_{\boldsymbol{\mu}}(\Omega).$$

Now one uses Theorem 1.4 and finds that in this case the conditions $p \leq q$ and (A1) are the same as (1.9), while (A2) and (A2') can be written, respectively, as (1.10) and (1.11). $\square$

7. Appendix: Proof of Lemma 4.7

We give a proof of Lemma 4.7, as we did not find one in the literature which is valid for homogeneous cones. The proof below only uses the H-invariance of the riemannian metric, and hence of the associated distance d in Ω. It is based on the following facts:

(i) There exists $\delta_0 \in (0,1)$ such that

$$B_1(\mathbf{e}) \subset \delta_0 \mathbf{e} + \Omega, \tag{7.1}$$

Indeed, recall that such a distance d is complete and hence every d-bounded set is relatively compact; see e.g. Theorems 4.1 and 4.5 in Chapter IV of [15]. The result then follows by compactness from $\overline{B_1}(\mathbf{e}) \subset \cup_{\delta > 0}(\delta \mathbf{e} + \Omega) = \Omega$.

(ii) The following equivalence holds for each $t \in H$,

$$t \cdot \mathbf{e} \in B_1(\mathbf{e}) \quad \text{if and only if} \quad t^{-1} \cdot \mathbf{e} \in B_1(\mathbf{e})$$

Indeed, this is just a consequence of $d(t \cdot \mathbf{e}, \mathbf{e}) = d(\mathbf{e}, t^{-1} \cdot \mathbf{e})$, by the H-invariance.

(iii) The following inequalities hold for every $x, y \in \Omega$

$$Q_j(x+y) \geq Q_j(x), \quad j = 1, \ldots, r.$$

This has been shown in [17, Lemma 4.13].

We now establish Lemma 4.7, that is, the existence of $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{Q_j(x+y)}{Q_j(x+y_0)} \leq c_2, \quad \text{if } d(y, y_0) < 1, \quad x \in \Omega, \quad j = 1, \ldots, r. \tag{7.2}$$

By H-invariance we may assume that $y_0 = \mathbf{e}$. Now, if $y \in B_1(\mathbf{e})$, facts (i) and (iii) above imply

$$\begin{aligned} Q_j(x+y) &= Q_j(x + y - \delta_0 \mathbf{e} + \delta_0 \mathbf{e}) \;\geq\; Q_j(x + \delta_0 \mathbf{e}) \\ &= Q_j\big((1-\delta_0)x + \delta_0 x + \delta_0 \mathbf{e}\big) \;\geq\; \delta_0 \, Q_j(x + \mathbf{e}). \end{aligned}$$

On the other hand, if $y = t \cdot \mathbf{e} \in B_1(\mathbf{e})$, then $t^{-1} \cdot \mathbf{e} \in B_1(\mathbf{e})$, so (2.4) and the previous case give

$$\begin{aligned} Q_j(x+y) &= Q_j(t \cdot \mathbf{e}) \, Q_j(t^{-1} \cdot x + \mathbf{e}) \\ &\leq \delta_0^{-1} Q_j(t \cdot \mathbf{e}) \, Q_j(t^{-1} \cdot x + t^{-1} \cdot \mathbf{e}) \;=\; \delta_0^{-1} Q_j(x + \mathbf{e}). \end{aligned}$$

Thus, we have shown (7.2) with $c_1 = 1/c_2 = \delta_0$.

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