BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality *

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Abstract

We identify the dual space of the Hardy-type space $H^1_{\mathcal{L}}$ related to the time independent Schrödinger operator $\mathcal{L} = -\Delta + V$, with V a potential satisfying a reverse Hölder inequality, as a BMO-type space $BMO_{\mathcal{L}}$. We prove the boundedness in this space of the versions of some classical operators associated to $\mathcal{L}$ (Hardy-Littlewood, semigroup and Poisson maximal functions, square function, fractional integral operator). We also get a characterization of $BMO_{\mathcal{L}}$ in terms of Carlesson measures.

1 Introduction

Let V be a fixed non-negative function on $\mathbb{R}^d$, $d \geq 3$, satisfying a *reverse Hölder inequality* $RH_s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$; that is, there exists $C = C(s, V) > 0$ such that

$$\left(\fint_B V(x)^s \, dx \right)^{\frac{1}{s}} \leq C \fint_B V(x) \, dx, \tag{1.1}$$

for every ball $B \subset \mathbb{R}^d$. Consider the time independent Schrödinger operator with the potential V:

$$\mathcal{L} = -\Delta + V,$$

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and its associated semigroup:

$$T_tf(x) = e^{-t\mathcal{L}}f(x) = \int_{\mathbb{R}^d} k_t(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0. \tag{1.2}$$

A Hardy-type space related to $\mathcal{L}$ is naturally defined by:

$$H^1_{\mathcal{L}} = \{f \in L^1(\mathbb{R}^d) \; : \; \mathcal{T}^*f(x) = \sup_{t>0} |T_tf(x)| \in L^1(\mathbb{R}^d)\},$$
$$\text{with} \quad \|f\|_{H^1_{\mathcal{L}}} := \|\mathcal{T}^*f\|_{L^1(\mathbb{R}^d)}. \tag{1.3}$$

For the above class of potentials, it was shown in [2] that $H^1_{\mathcal{L}}$ admits a special atomic characterization, where cancellation conditions are only required for atoms with small supports. In this paper we shall be interested in properties of the dual space of $H^1_{\mathcal{L}}$, which we shall identify with a subclass of BMO functions, namely:

$$BMO_{\mathcal{L}} = \left\{ f \in BMO \; : \; \frac{1}{|B|}\int_B |f| \leq C, \text{ for all } B = B_R(x) \; : \; R > \rho(x) \right\}. \tag{1.4}$$

The precise definition of the norm in this space is given in Definition 3.5. The critical radii above are determined by the function $\rho(x;V) = \rho(x)$ which takes the explicit form

$$\rho(x) = \sup\left\{ r > 0 \; : \; \frac{1}{r^{d-2}}\int_{B(x,r)} V(y)dy \leq 1 \right\}. \tag{1.5}$$

Throughout the paper shall assume that $V \not\equiv 0$, so that $0 < \rho(x) < \infty$ (see [7]). This $BMO_{\mathcal{L}}$ space turns out to be the suitable extreme point for $p = \infty$ concerning the boundedness of the classical operators associated to the operator $\mathcal{L}$. We shall use the following notations:

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|\,dy, \tag{1.6}$$

$$\mathcal{T}^*f(x) = \sup_{t>0} |e^{-t\mathcal{L}}f(x)|, \tag{1.7}$$

$$\mathcal{P}^*f(x) = \sup_{t>0}\Big|P_tf(x)\Big|, \text{ where } P_t = e^{-t\sqrt{\mathcal{L}}} = \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}\,du, \tag{1.8}$$

$$s_Qf(x) = \Big(\int_0^\infty |Q_tf(x)|^2 \tfrac{dt}{t}\Big)^{\frac{1}{2}}, \tag{1.9}$$

$$\mathcal{I}_\alpha f(x) = \mathcal{L}^{-\alpha/2}f(x) = \int_0^\infty e^{-t\mathcal{L}}f(x)\,t^{\alpha/2-1}\,dt \quad \text{for} \quad 0 < \alpha < d. \tag{1.10}$$

These notations correspond, respectively, to the Hardy-Littlewood maximal function, the semigroup and Poisson-semigroup maximal functions, the $\mathcal{L}$-square function and the $\mathcal{L}$-fractional integral operator. We observe that in the classical case (i.e. $V \equiv 0$) these operators fail to be bounded in BMO, in fact may be identically infinity for functions with certain growth (see §5 below). However in our case it turns out that they behave correctly in $BMO_{\mathcal{L}}$ as the following results shows.

Theorem 1.11 *Let $V \not\equiv 0$ be a non-negative potential in $RH_s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$. The operators M, $\mathcal{T}^*$, $\mathcal{P}^*$ and s_Q are well-defined and bounded in $BMO_{\mathcal{L}}$. For all $0 < \alpha < d$, the operator $\mathcal{I}_\alpha$ is bounded from $L^{d/\alpha}(\mathbb{R}^d)$ into $BMO_{\mathcal{L}}$.*

We also show a characterization of $BMO_{\mathcal{L}}$ in terms of Carleson measures. A positive measure μ on $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty)$ is said to be a Carleson measure if

$$\|\mu\|_{\mathcal{C}} := \sup_{x\in\mathbb{R}^d,\ r>0} \frac{\mu(B_r(x)\times(0,r))}{|B_r(x)|} < \infty. \tag{1.12}$$

Our result characterizes the elements of $BMO_{\mathcal{L}}$ by a Carleson measure condition related to appropriate square function. To be more precise let

$$(Q_t f)(x) = t^2 \left(\left.\frac{dT_s}{ds}\right|_{s=t^2} f \right)(x), \quad (x,t) \in \mathbb{R}^{d+1}_+, \tag{1.13}$$

then the following theorems holds

Theorem 1.14 *Let $V \not\equiv 0$ be a non-negative potential in $RH_s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$ and $\rho(x) = \rho(x)$ be the weight defined in (1.5).*

1. *If $f \in BMO_{\mathcal{L}}$, then $d\mu_f(x,t) := |Q_t f(x)|^2\, dxdt/t$ is a Carleson measure.*
2. *Conversely, if $f \in L^1((1+|x|)^{-(d+1)}dx)$ and $d\mu_f(x,t)$ is a Carleson measure, then $f \in BMO_{\mathcal{L}}$.*

Moreover, in either case, there exists $C > 0$ such that

$$\frac{1}{C}\,\|f\|^2_{BMO_{\mathcal{L}}} \le \|d\mu_f\|_{\mathcal{C}} \le C\,\|f\|^2_{BMO_{\mathcal{L}}}.$$

The outline of the paper is as follows. In Section 2 we gather the required estimates on the kernel $Q_t(x,y)$, which complement those presented for $k_t(x,y)$ in [2, 3, 4]. In Section 3 we recall the atomic decomposition for $H^1_{\mathcal{L}}$ and establish the identification $(H^1_{\mathcal{L}})^* \equiv BMO_{\mathcal{L}}$. In Section 5 we study the boundedness of classical operators in $BMO_{\mathcal{L}}$. Some of the techniques needed in the proofs of Section 5 appear naturally when proving Theorem 1.14, it is because of this reason that we present in Section 4 the proof of Theorem 1.14.

2 Estimates on the kernels

We begin by recalling some basic properties of the function $\rho(x)$ under the assumption (1.1) on V (see [7, Lemma 1.4]).

Proposition 2.1 *There exist $c > 0$ and $k_0 \geq 1$ so that, for all $x, y \in \mathbb{R}^d$*

$$c^{-1}\,\rho(x)\Big(1+\tfrac{|x-y|}{\rho(x)}\Big)^{-k_0} \;\leq\; \rho(y) \;\leq\; c\,\rho(x)\,\Big(1+\tfrac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}. \tag{2.2}$$

In particular, $\rho(x) \sim \rho(y)$ when $y \in B_r(x)$ and $r \leq C\rho(x)$.

From the Feynman-Kac formula, it is well-known that the semigroup kernels $k_t(x, y)$, associated with $T_t = e^{-t\mathcal{L}}$, satisfy the estimates

$$0 \leq k_t(x,y) \leq h_t(x-y) := (4\pi t)^{-\frac{d}{2}}\,\exp(-\tfrac{|x-y|^2}{4t}). \tag{2.3}$$

These estimates can be improved in time when $V \not\equiv 0$ satisfies the reverse Hölder condition RH_s for some $s > d/2$. The function $\rho(x)$ arises naturally in this context.

Proposition 2.4 (see [3], [6]) *For every N, there is a constant C_N such that*

$$0 \leq k_t(x,y) \leq C_N\,t^{-\frac{d}{2}}\,\exp(-\tfrac{|x-y|^2}{5t})\,\Big(1+\tfrac{\sqrt{t}}{\rho(x)}+\tfrac{\sqrt{t}}{\rho(y)}\Big)^{-N}. \tag{2.5}$$

Below, we shall also make use of the Lipschitz regularity of the kernels, which is again a consequence of (1.1) (see [4, Proposition 4.11]).

Proposition 2.6 *If $V \in RH_s(\mathbb{R}^d)$, $s > d/2$, then there exists $\delta = \delta(s) > 0$ and $c > 0$ such that for every $N > 0$ there is a constant C_N so that, for all $|h| \leq \sqrt{t}$*

$$|k_t(x+h,y)-k_t(x,y)| \leq C_N\,\Big(\tfrac{|h|}{\sqrt{t}}\Big)^{\delta}\,t^{-\frac{d}{2}}\,\exp(-\tfrac{c|x-y|^2}{t})\Big(1+\tfrac{\sqrt{t}}{\rho(x)}+\tfrac{\sqrt{t}}{\rho(y)}\Big)^{-N}. \tag{2.7}$$

We will also need estimates for the integral kernels of the operators Q_t in (1.13):

$$Q_t(x,y) = t^2\,\frac{\partial k_s(x,y)}{\partial s}\bigg|_{s=t^2} \tag{2.8}$$

Proposition 2.9 *There exist constants $c, \delta > 0$ such that for every N there is a constant C_N so that*

(a) $\quad |Q_t(x,y)| \leq C_N\,t^{-d}\,\exp(-\tfrac{c|x-y|^2}{t^2})\,\Big(1+\tfrac{t}{\rho(x)}+\tfrac{t}{\rho(y)}\Big)^{-N};$

(b) $\quad |Q_t(x+h,y)-Q_t(x,y)| \leq C_N\,\Big(\tfrac{|h|}{t}\Big)^{\delta}\,t^{-d}\,\exp(-\tfrac{c|x-y|^2}{t^2})\Big(1+\tfrac{t}{\rho(x)}+\tfrac{t}{\rho(y)}\Big)^{-N},$
for all $\;|h| \leq t$;

(c) $\quad \Big|\displaystyle\int_{\mathbb{R}^d} Q_t(x,y)dy\Big| \leq C_N\,\dfrac{(t/\rho(x))^{\delta}}{(1+t/\rho(x))^N}.$

Proof: Corollary 6.4 of [4] asserts that the integral kernels $k_\zeta(x,y)$ of the extension of $\{T_t\}_{t>0}$ to the holomorphic semigroup $\{T_\zeta\}_{\zeta\in\Delta_{\pi/4}}$ satisfy

$$|k_\zeta(x,y)| \le C_N(\Re\zeta)^{-d/2}\Big(1+\tfrac{\Re\zeta}{\rho(x)^2}+\tfrac{\Re\zeta}{\rho(y)^2}\Big)^{-N}\exp\big(-c|x-y|^2/\Re\zeta\big). \tag{2.10}$$

The Cauchy integral formula combined with (2.10) gives

$$\begin{aligned}\Big|\frac{d}{dt}k_t(x,y)\Big| &= \Big|\frac{1}{2\pi}\int_{|\zeta-t|=t/10}\frac{k_\zeta(x,y)}{(\zeta-t)^2}\,d\zeta\Big| \\ &\le \frac{C_N}{t}t^{-d/2}\Big(1+\tfrac{t}{\rho(x)^2}+\tfrac{t}{\rho(y)^2}\Big)^{-N}\exp\big(-c'|x-y|^2/t\big)\end{aligned}\tag{2.11}$$

which, by (2.8), implies (a).

We now turn to prove (b). By the semigroup property, Proposition 2.6, and the part (a) already proved, we obtain

$$\begin{aligned}\big|Q_t(x+h,y)-Q_t(x,y)\big| &= \Big|2\int\Big(k_{t^2/2}(x+h,z)-k_{t^2/2}(x,z)\Big)Q_{t/\sqrt{2}}(z,y)\,dz\Big| \\ &\le C_N\Big(\tfrac{|h|}{t}\Big)^{\delta}\int\Big(1+\tfrac{t}{\rho(x)}\Big)^{-N}t^{-d}e^{-c|x-z|^2/t^2}t^{-d}e^{-c|z-y|^2/t^2}\Big(1+\tfrac{t}{\rho(y)}\Big)^{-N}dz \\ &\lesssim C_N\Big(\tfrac{|h|}{t}\Big)^{\delta}t^{-d}\exp\big(-\tfrac{c'|x-y|^2}{t^2}\big)\Big(1+\tfrac{t}{\rho(x)}+\tfrac{t}{\rho(y)}\Big)^{-N}\end{aligned}$$

which establishes (b).

It follows from [7] Lemmas 1.2 and 1.8 that there is a constant C_0 such that for a nonnegative Schwartz class function φ there exists a constant C such that

$$\int\varphi_t(x-y)V(y)\,dy \le \begin{cases} Ct^{-1}\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\delta} & \text{for } t\le\rho(x)^2 \\ C\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{C_0+2-d} & \text{for } t>\rho(x)^2,\end{cases}\tag{2.12}$$

where $\varphi_t(x)=t^{-d/2}\varphi(x/\sqrt{t})$. Therefore

$$\Big|\int\frac{d}{dt}k_t(x,y)\,dy\Big| = \Big|T_t\mathcal{L}1(x)\Big| = \int k_t(x,y)V(y)\,dy \tag{2.13}$$

and (c) follows by first using (2.5) with N sufficiently large and then (2.12). □

Finally, we recall some results about covering $\mathbb{R}^d$ by critical balls. These depend entirely on the estimates in Proposition 2.1, and can be found in [2]. Throughout the paper, given a ball B we denote by B^* the ball with same center and twice radius.

Proposition 2.14 (see [2, Lemma 2.3]). *There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^d$, so that the family of "critical balls" $\mathcal{Q} = \{Q_k\}_{k=1}^{\infty}$, defined by $Q_k := \{|x - x_k| < \rho(x_k)\}$, satisfy*

(a) $\bigcup_k Q_k = \mathbb{R}^d$.

(b) There exists $N = N(\rho)$ so that, for every $k \geq 1$, $\mathrm{card}\,\{j \;:\; Q_j^{**} \cap Q_k^{**} \neq \emptyset\} \leq N$.

Combining the previous with (2.2) one easily gets

Corollary 2.15 *There is a constant $c = c(\rho)$ so that, for every ball $B_R(x)$ with $R > \rho(x)$, we have*

$$|B_R(x)| \leq \sum_{Q_k \cap B_R(x) \neq \emptyset} |Q_k| \leq c\,|B_R(x)|.$$

Corollary 2.16 *There exists a family of C^{∞} functions φ_k such that* $\mathrm{supp}\,\varphi_k \subset Q_k^*$, $0 \leq \varphi_k \leq 1$, $|\nabla \varphi_k| \leq C/\rho(x_k)$, $\sum_k \varphi_k = 1$.

3 The dual space of $H^1_{\mathcal{L}}$

We shall assume that $V \not\equiv 0$ is a potential in the reverse Hölder class as stated in the introduction. For such potentials it was shown in [2] the following characterization of $H^1_{\mathcal{L}}$.

A function $a\colon \mathbb{R}^d \to \mathbb{C}$ is an *$H^1_{\mathcal{L}}$-atom associated with a ball $B_r(x_0)$* when

$$\mathrm{supp}\,a \subset B_r(x_0), \qquad \|a\|_{\infty} \leq 1/|B_r(x_0)|, \tag{3.1}$$

and in addition,

$$\int a(x)\,dx = 0, \qquad \text{whenever } 0 < r < \rho(x_0). \tag{3.2}$$

Theorem 3.3 *An integrable function f in $\mathbb{R}^d$ belongs to $H^1_{\mathcal{L}}$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^1_{\mathcal{L}}$-atoms and $\sum_j |\lambda_j| < \infty$. Moreover, there exists a constant $c > 0$ such that*

$$c^{-1}\,\|f\|_{H^1_{\mathcal{L}}} \leq \inf\left\{\textstyle\sum_j |\lambda_j| \;:\; f = \sum_j \lambda_j a_j\right\} \leq c\,\|f\|_{H^1_{\mathcal{L}}}.$$

Remark 3.4 We note that in the above atomic decomposition we may restrict to atoms supported by balls $B_r(x)$ with $r \leq \rho(x)$. Indeed, if we are given an $H^1_{\mathcal{L}}$-atom a associated with a ball $B_r(y_0)$ with $r > \rho(y_0)$ then it easily follows from Proposition 2.14 that the atom a can be written as $a = \sum_j \lambda_j a_j$, where $\sum_j |\lambda_j| \lesssim 1$, and a_j are $H^1_{\mathcal{L}}$-atoms supported by critical balls.

Definition 3.5 *We shall say that a locally integrable function f belongs to $BMO_{\mathcal{L}}$ whenever there is a constant $C \geq 0$ so that*

$$\frac{1}{|B_s|}\int_{B_s} |f - f_{B_s}| \leq C \quad \text{and} \quad \frac{1}{|B_r|}\int_{B_r} |f| \leq C, \tag{3.6}$$

for all balls $B_s = B_s(x)$, $B_r = B_r(x)$ such that $s \leq \rho(x) \leq r$. We let $\|f\|_{BMO_{\mathcal{L}}}$ denote the smallest C in (3.6) above. Here and subsequently, $f_B = |B|^{-1}\int_B f(x)\,dx$.

We observe that $\|f\|_{BMO_{\mathcal{L}}}$ is actually a norm (and not only a seminorm) making $BMO_{\mathcal{L}}$ a Banach space. Moreover, $\|f\|_{BMO} \leq 2\|f\|_{BMO_{\mathcal{L}}}$. We also observe that, by Proposition 2.14 and its corollary, it is enough to consider the condition on the right hand side of (3.6) only for balls from the family $\mathcal{Q}$.

From the previous atomic decomposition, and the definition of $f \in BMO_{\mathcal{L}}$, it is clear that

$$\Phi_f(a) := \int_{\mathbb{R}^d} f(x)a(x)\,dx, \quad \text{acting on } H^1_{\mathcal{L}}\text{-atoms } a,$$

defines a linear functional on $H^1_{\mathcal{L}}$ with $\|\Phi_f\| \leq c\|f\|_{BMO_{\mathcal{L}}}$. Our first result shows that all linear functionals on $H^1_{\mathcal{L}}$ do actually arise in this way.

Theorem 3.7 *The correspondence*

$$BMO_{\mathcal{L}} \ni f \longmapsto \Phi_f \in (H^1_{\mathcal{L}})^*$$

is a linear isomorphism of Banach spaces.

We shall need a lemma about the size of $H^1_{\mathcal{L}}$-functions.

Lemma 3.8 *The space $L^2_c(\mathbb{R}^d)$, of square-integrable functions with compact support is contained in $H^1_{\mathcal{L}}$. Moreover, there is a constant $C = C(\mathcal{L}) > 0$ so that, for large balls $B = B_R(x_0)$ with $R \geq \rho(x_0)$, we have*

$$\|g\|_{H^1_{\mathcal{L}}} \leq C\,|B|^{\frac{1}{2}}\,\|g\|_{L^2(B)}, \quad \forall\, g \in L^2(B). \tag{3.9}$$

Proof: Assume that $g \in L^2(B_R(x_0))$, $R \geq \rho(x_0)$. Then, using Corollary 2.16, we can write $g = \sum_k \varphi_k g = \sum_k g_k$ and $\|g\|^2_{L^2} \sim \sum_k \|g_k\|^2_{L^2}$. By Schwarz's inequality and Corollary 2.15 it suffices to prove (3.9) for $g = g_k$. Obviously,

$$\|\mathcal{T}^* g_k\|_{L^1(Q_k^{**})} \leq C\,|Q_k|^{1/2}\|\mathcal{T}^* g_k\|_{L^2(Q_k^{**})} \leq C\,|Q_k|^{1/2}\|g_k\|_{L^2}. \tag{3.10}$$

If $x \notin Q_k^{**}$, then applying Propositions 2.4 and 2.1, we get

$$\begin{aligned} |T_t g_k(x)| &\leq C_N \int \Big(1 + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N} t^{-d/2} e^{-c|x-y|^2/t} |g_k(y)|\,dy \\ &\leq C_N \frac{\rho(x_k)^N}{|x - x_k|^{d+N}} |Q_k|^{1/2}\|g_k\|_{L^2}. \end{aligned} \tag{3.11}$$

Now (3.9) for $g = g_k$ follows from (3.11) and (3.10).

□

PROOF of Theorem 3.7: Let $\Phi \in (H^1_{\mathcal{L}})^*$, and denote $B_N = B_N(0)$, $N > \rho(0)$. By Lemma 3.8, there exists a unique $f_N \in L^2(B_N)$ with $\|f_N\|_{L^2(B_N)} \le C\,|B_N|^{\frac12}\,\|\Phi\|$, and so that

$$\Phi(g) = \int_{B_N} f_N\, g, \quad \forall\, g \in L^2(B_N).$$

Iterating in N, and noticing that $f_{N+1}|_{B_N} = f_N$, one defines a unique locally square-integrable function f in $\mathbb{R}^d$ so that

$$\Phi(g) = \int_{\mathbb{R}^d} f(x)\, g(x)\, dx, \quad \text{for all } g \in L^2_c(\mathbb{R}^d).$$

Thus, $\Phi = \Phi_f$, and only remains to show that $\|f\|_{BMO_{\mathcal{L}}} \le C\|\Phi\|$. If we test first on atoms supported by large balls $B = B_R(x)$, $R > \rho(x)$, and use again Lemma 3.8, we see that $\|f\|_{L^2(B)} \le C\,|B|^{\frac12}\,\|\Phi\|$. Thus, from Hölder's inequality we conclude

$$\tfrac{1}{|B|}\int_B |f| \le \Big(\tfrac{1}{|B|}\int_B |f|^2\Big)^{\frac12} \le C\,\|\Phi\|. \tag{3.12}$$

On the other hand, since classical H^1-atoms are particular cases of $H^1_{\mathcal{L}}$-atoms, it follows that $\Phi|_{H^1} \in (H^1)^*$, and thus $\Phi|_{H^1} = \Phi_h$ for a unique (modulo constants) $h \in BMO(\mathbb{R}^d)$. Now, testing on H^1-atoms over a fixed ball B we see that $f = h + c_B$, for some constant c_B, and therefore $f \in BMO(\mathbb{R}^d)$ with $\|f\|_{BMO} \le 2\,\|h\|_{BMO} \le C'\,\|\Phi\|$. This establishes the theorem.

□

Observe that we have shown, for large balls, the local square integrability estimate (3.12). The exponent 2 can actually be replaced by any $1 \le p < \infty$, by just using in the proof above the corresponding p'-version of Lemma 3.8. This, together with the John-Nirenberg inequality gives the following corollary.

COROLLARY 3.13 *For every $p \in [1, \infty)$ there exists $c = c(p, \rho) > 0$ such that, for all $f \in BMO_{\mathcal{L}}$ we have*

$$\Big(\tfrac{1}{|B|}\int_B |f - f_B|^p\Big)^{\frac1p} \le c\,\|f\|_{BMO_{\mathcal{L}}}, \quad \textit{for all balls } B,$$

$$\Big(\tfrac{1}{|B|}\int_B |f|^p\Big)^{\frac1p} \le c\,\|f\|_{BMO_{\mathcal{L}}}, \quad \textit{for } B = B_r(x) \;:\; r \ge \rho(x).$$

We conclude with the following lemma which will be used often below. Its proof is elementary and left to the reader.

LEMMA 3.14 *There exists $c > 0$ so that, for all $f \in BMO_{\mathcal{L}}$ and $B = B_r(x)$ with $r < \rho(x)$, then*

$$|f_{B^*}| \le c\,(1 + \log \tfrac{\rho(x)}{r})\,\|f\|_{BMO_{\mathcal{L}}}.$$

4 Proof of Theorem 1.14

Lemma 4.1 *For all $f \in L^2(\mathbb{R}^d)$ we have $\|s_Q f\|_2 = \frac{1}{\sqrt{8}}\|f\|_2$. Moreover,*

$$f(x) = 8 \lim_{\varepsilon\to 0,\ N\to\infty} \int_\varepsilon^N Q_t^2 f(x)\,\frac{dt}{t}, \qquad \text{in } L^2(\mathbb{R}^d). \tag{4.2}$$

Proof: The proof is standard and follows by using spectral techniques, as in [8, Chapter 3]. For completeness we provide some details. Since $T_t = e^{-t\mathcal{L}} = \int_0^\infty e^{-t\lambda}\,dE(\lambda)$, we have

$$t\,\frac{dT_t}{dt} = -t\mathcal{L}\,T_t = -\int_0^\infty t\lambda\,e^{-t\lambda}\,dE(\lambda).$$

Thus, for all $f \in L^2(\mathbb{R}^d)$, using the self-adjointness of Q_t, we get

$$\begin{aligned}\|s_Q f\|_2^2 &= \int_{\mathbb{R}^d}\int_0^\infty |Q_t f(x)|^2\,\frac{dt}{t}\,dx\\ &= \int_0^\infty \langle\, t^4\big(\tfrac{dT_s}{ds}|_{s=t^2}\big)^2 f, f\rangle\,\frac{dt}{t}\\ &= \int_0^\infty \Big[\int_0^\infty t^4\lambda^2\,e^{-2t^2\lambda}\,\frac{dt}{t}\Big]\,dE_{f,f}(\lambda) \;=\; \tfrac{1}{8}\,\|f\|_2^2.\end{aligned}$$

For the second part, it suffices to show that, for every pair of sequences $n_k \nearrow \infty$, $\varepsilon_k \searrow 0$

$$\lim_{k\to\infty}\int_{n_k}^{n_{k+m}} Q_t^2 f\,\frac{dt}{t} = \lim_{k\to\infty}\int_{\varepsilon_{k+m}}^{\varepsilon_k} Q_t^2 f\,\frac{dt}{t} = 0, \quad \forall\, m \geq 1. \tag{4.3}$$

Indeed, when this is case we can find $h \in L^2(\mathbb{R}^d)$ so that $\lim_{k\to\infty}\int_{\varepsilon_k}^{n_k} Q_t^2 f\,\frac{dt}{t} = h$, and therefore, by using also a polarized version of the first part

$$\begin{aligned}\langle h, g\rangle &= \lim_{k\to\infty}\int_{\varepsilon_k}^{n_k}\langle Q_t f, Q_t g\rangle\,\frac{dt}{t}\\ &= \int_0^\infty \langle Q_t f, Q_t g\rangle\,\frac{dt}{t} \;=\; \tfrac{1}{8}\,\langle f, g\rangle, \quad \forall\, g \in L^2(\mathbb{R}^d),\end{aligned} \tag{4.4}$$

which implies $h = \frac{1}{8}f$. To prove (4.3) we use again functional calculus, so that

$$\Big\|\int_{n_k}^{n_{k+m}} Q_t^2 f\,\frac{dt}{t}\Big\|^2 \leq \int_0^\infty \Big|\int_{n_k}^{n_{k+m}} t^4\lambda^2\,e^{-2t^2\lambda}\,\tfrac{dt}{t}\Big|^2\,dE_{f,f}(\lambda).$$

Computing the integral inside one is led to estimate

$$\int_0^\infty (1+2\lambda n_k^2)e^{-2\lambda n_k^2}dE_{f,f}(\lambda), \quad \text{as } n_k \to \infty,$$

which by dominated convergence tends to 0. Observe that the last step makes use of the fact that 0 is not an eigenvalue of $\mathcal{L}$ because, $V(x) > 0$ for almost every x, and $\langle \mathcal{L}f, f\rangle \geq \langle Vf, f\rangle > 0$ unless $f \equiv 0$). One proceeds similarly when $\varepsilon_k \to 0$. □

4.1 Proof of part (1) of Theorem 1.14

Our first observation is that

$$Q_t f(x) = \int_{\mathbb{R}^d} Q_t(x,y) f(y)\, dy$$

is a well-defined absolutely convergent integral for all $(x,t) \in \mathbb{R}^{d+1}_+$, as it follows from the kernel decay in Proposition 2.9 and the integrability of $(1+|y|)^{-d-1}|f(y)|$ (see [9, page 141]). Let us fix a ball $B = B_r(x_0)$. We wish to show that

$$\frac{1}{|B|} \int_0^r \int_B |Q_t f(x)|^2 \frac{dxdt}{t} \leq C\, \|f\|^2_{BMO_{\mathcal{L}}}. \tag{4.5}$$

To do this, we split the function f into local, global, and constant parts as follows

$$\begin{aligned} f &= (f - f_{B^*})\, \chi_{B^*} + (f - f_{B^*})\, \chi_{(B^*)^c} + f_{B^*} \\ &= f_1 + f_2 + f_{B^*}, \end{aligned}$$

As we shall see, the novelty of the $BMO_{\mathcal{L}}$ condition appears in the control of the constant term. For the other two terms the proof follows from standard arguments (cf. e.g. [9]) and estimates for the kernel $Q_t(x,y)$. Indeed, in the local case, a simple use of Lemma 4.1 gives us

$$\begin{aligned} \frac{1}{|B|} \int_0^r \int_B |Q_t f_1(x)|^2 \frac{dxdt}{t} &\leq \frac{C}{|B|} \int_B |s_Q f_1(x)|^2\, dx \\ &\leq \frac{C}{|B|}\, \|f_1\|_2^2 = \frac{1}{|B|} \int_{B^*} |f - f_{B^*}|^2, \end{aligned}$$

which is smaller than a multiple of $\|f\|^2_{BMO}$ by Corollary 3.13.

To estimate the global term, we only need a very mild decay of $|Q_t(x,y)|$: if $x \in B = B_r(x_0)$ and $t < r$ then

$$\begin{aligned} |Q_t f_2(x)| &\lesssim \int_{\mathbb{R}^d} |f_2(y)|\, \frac{t^{-d}}{(1 + \frac{|x-y|}{t})^{d+1}}\, dy \\ &\lesssim \int_{(B^*)^c} |f(y) - f_{B^*}|\, \frac{t}{|x_0 - y|^{d+1}}\, dy \\ &\lesssim \sum_{k=1}^{\infty} \frac{t}{(2^k r)^{d+1}} \left[\int_{|y-x_0|\sim 2^k r} |f(y) - f_{B_{2^k r}}|\, dy \,+\, (2^k r)^d\, |f_{B_{2^k r}} - f_{B^*}| \right] \\ &\lesssim \frac{t}{r} \sum_{k=1}^{\infty} 2^{-k}\, [\, \|f\|_{BMO} \,+\, k\, \|f\|_{BMO}] \lesssim \frac{t}{r}\, \|f\|_{BMO}. \end{aligned}$$

Thus, integrating over $B \times (0,r)$ we obtain

$$\frac{1}{|B|} \int_0^r \int_B |Q_t f_2(x)|^2 \frac{dxdt}{t} \lesssim \int_0^r \frac{t^2}{r^2} \frac{dt}{t}\, \|f\|^2_{BMO} = \tfrac{1}{2}\, \|f\|^2_{BMO} \lesssim \|f\|^2_{BMO_{\mathcal{L}}}.$$

It remains to estimate the constant term, for which we shall make use of part (c) of Proposition 2.9. Assuming first that $r < \rho(x_0)$, and using $\rho(x) \sim \rho(x_0)$ for $x \in B$ (cf. Proposition 2.1), we have

$$\begin{aligned}
\tfrac{1}{|B|} \int_0^r \int_B |Q_t(f_{B^*}\mathbf{1})(x)|^2 \tfrac{dxdt}{t} &= \tfrac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \Big| \int Q_t(x,y)dy \Big|^2 \tfrac{dxdt}{t} \\
&\lesssim \tfrac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \big(t/\rho(x)\big)^{2\delta} \frac{dxdt}{t} \\
&\lesssim |f_{B^*}|^2 \big(r/\rho(x_0)\big)^{2\delta} \\
&\lesssim \|f\|^2_{BMO_{\mathcal{L}}} \big(1+\log \tfrac{\rho(x_0)}{r}\big)^2 \big(r/\rho(x_0)\big)^{2\delta} \lesssim \|f\|^2_{BMO_{\mathcal{L}}},
\end{aligned} \tag{4.6}$$

where the last line follows from Lemma 3.14.

Suppose finally that $r \geq \rho(x_0)$, and select from Corollary 2.15 a finite family of critical balls $\{Q_\ell\}$ so that $B \subset \cup Q_\ell$ and $\sum |Q_\ell| \lesssim |B|$. Then, using again part (c) of Proposition 2.9 and $|f_{B^*}| \leq \|f\|_{BMO_{\mathcal{L}}}$, we can bound the left hand side of (4.6) by

$$\begin{aligned}
&\tfrac{\|f\|^2_{BMO_{\mathcal{L}}}}{|B|} \sum_\ell \left(\int_0^{\rho(x_\ell)} \int_{Q_\ell} (t/\rho(x_\ell))^{2\delta} \tfrac{dxdt}{t} + \int_{\rho(x_\ell)}^\infty \int_{Q_\ell} \frac{dx}{(1+t/\rho(x_\ell))^{2N-2\delta}} \tfrac{dt}{t} \right) \\
\lesssim\ & C\|f\|^2_{BMO_{\mathcal{L}}} |B|^{-1} \sum_\ell |Q_\ell| \lesssim \|f\|^2_{BMO_{\mathcal{L}}},
\end{aligned}$$

which, by Corollary 2.15, establishes the first part of Theorem 1.14.

Remark 4.7 It is worthwhile to notice that, from the previous proof, it actually follows that

$$\sup_{t>0} \|Q_t f\|_\infty = \sup_{t>0, x\in\mathbb{R}^d} |Q_t f(x)| \lesssim \|f\|_{BMO_{\mathcal{L}}}.$$

That is, the solution to the evolution equation

$$\begin{cases} u_t(t,x) = -\mathcal{L}u(t,x), & (t,x) \in \mathbb{R}^{d+1}_+ \\ u(0,\cdot) = f \end{cases} \tag{4.8}$$

with initial data $f \in BMO_{\mathcal{L}}$, satisfies the regularity estimate

$$\|\mathcal{L}u(t,\cdot)\|_\infty \lesssim t^{-1} \|f\|_{BMO_{\mathcal{L}}}.$$

4.2 Proof of part (2) of Theorem 1.14

Now, let us fix $f \in L^1((1+|x|)^{-(d+1)}dx)$ so that $\mu_f(x,t) := |Q_t f(x)|^2 dxdt/t$ is a Carleson measure. We wish to show that such f must belong to $BMO_{\mathcal{L}}$. By Theorem 3.7 it suffices to show that the linear functional

$$H^1_{\mathcal{L}} \ni g \longmapsto \Phi_f[g] := \int_{\mathbb{R}^d} f(x)g(x)\,dx,$$

defined at least over finite linear combinations of $H^1_{\mathcal{L}}$-atoms, satisfies the estimate

$$\big|\,\Phi_f[g]\,\big| \;\leq\; c\,\|\mu_f\|_{\mathcal{C}}^{\frac{1}{2}}\,\|g\|_{H^1_{\mathcal{L}}}. \tag{4.9}$$

To do this, we shall proceed in three steps. First, we shall write Φ_f in terms of the extended functions

$$F(x,t) := Q_t f(x) \quad \text{and} \quad G(x,t) := Q_t g(x), \quad (x,t) \in \mathbb{R}^{d+1}_+.$$

More precisely, we shall show the following identity.

Lemma 4.10 *Let $f \in L^1((1+|x|)^{-(d+1)}dx)$ and g be an $H^1_{\mathcal{L}}$-atom. Then*

$$\tfrac{1}{8}\int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx \;=\; \int_{\mathbb{R}^{d+1}_+} F(x,t)\,\overline{G(x,t)}\,\frac{dxdt}{t}. \tag{4.11}$$

In our second step we shall bound the right hand side of (4.11) using a general result about tent spaces.

Lemma 4.12 : (see [9, p. 162]). *Let $F(x,t)$, $G(x,t)$ be measurable functions on $\mathbb{R}^{d+1}_+$ satisfying*

$$\mathcal{I}(F)(x) := \sup_{x\in B}\left(\frac{1}{|B|}\int_0^{r(B)}\int_B |F(y,t)|^2\,\frac{dydt}{t}\right)^{\frac{1}{2}} \in L^\infty(\mathbb{R}^d),$$

$$\mathcal{G}(G)(x) := \left(\int\int_{\Gamma(x)} |G(y,t)|^2\,\frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}} \in L^1(\mathbb{R}^d),$$

where $r(B)$ denotes the radius of B and $\Gamma(x) = \{(y,t)\in\mathbb{R}^{d+1}_+ \;:\; |y-x|<t\}$. Then, there is a universal $c>0$ so that

$$\begin{aligned}\int\int_{\mathbb{R}^{d+1}_+} |F(y,t)G(y,t)|\,\frac{dydt}{t} \;&\leq\; c\int_{\mathbb{R}^d}\mathcal{I}(F)(x)\,\mathcal{G}(G)(x)\,dx \\ &\leq\; c\,\|\mathcal{I}(F)\|_{L^\infty}\,\|\mathcal{G}(G)\|_{L^1}.\end{aligned}$$

Observe that $\|\mu_f\|_{\mathcal{C}} = \|\mathcal{I}(F)\|^2_{L^\infty}$. Thus, in order to establish (4.9) we just need to show that $\|\mathcal{G}(G)\|_{L^1} \leq C\,\|g\|_{H^1_{\mathcal{L}}}$. Note that $\mathcal{G}(G)(x) = S_Q g(x)$, where S_Q is the following area integral operator,

$$S_Q g(x) := \left(\int_0^\infty\int_{|x-y|<t} |Q_t g(y)|^2\,\frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}}, \quad x\in\mathbb{R}^d.$$

Then, the following result about the boundedness of the area integral operator gives us the desired inequality.

Lemma 4.13 *There exists $c > 0$ so that $\|S_Q g\|_{L^1} \le c\,\|g\|_{H^1_{\mathcal{L}}}$ for every g being a finite linear combination of $H^1_{\mathcal{L}}$-atoms.*

Thus, we have reduced the proof of the theorem to show Lemmas 4.10 and 4.13. Let us start with the proof of the second one.

Proof of Lemma 4.13: By Theorem 1.5 in [2], it is enough to consider sums of atoms associated to balls $B_r(x_0)$ with $r \lesssim \rho(x_0)$. Let us fix an $H^1_{\mathcal{L}}$-atom, $g(x)$, associated with a ball $B = B_r(x_0)$. Now, observe that

$$\begin{aligned}
\|S_Q g\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left[\int\int_{\mathbb{R}^{d+1}_+} |Q_t g(y)|^2 \,\chi_{\Gamma(x)}(y,t)\, \frac{dydt}{t^{d+1}} \right] dx \\
&= \int\int_{\mathbb{R}^{d+1}_+} |Q_t g(y)|^2 \left[\int_{|x-y|<t} dx \right] \frac{dydt}{t^{d+1}} \\
&= c_d \int\int_{\mathbb{R}^{d+1}_+} |Q_t g(y)|^2 \,\frac{dydt}{t} \\
&= c_d \,\|s_Q g\|^2_{L^2(\mathbb{R}^d)} = \frac{c_d}{8}\,\|g\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

after using Lemma 4.1 in the last step. Therefore,

$$\int_{B^{***}} S_Q g(x)\,dx \le |B^{***}|^{\frac12} \left(\int_{B^{***}} S_Q g(x)^2\,dx \right)^{\frac12} \lesssim |B|^{\frac12}\,\|g\|_{L^2} \lesssim 1.$$

To complete the proof of Lemma 4.13, we must find a uniform bound for

$$I = \int_{(B^{***})^c} S_Q g(x)\,dx.$$

We consider first the case when $r < \rho(x_0)$. Then, by the moment condition on g,

$$\begin{aligned}
S_Q g(x) &= \left[\int_0^\infty \int_{|x-y|<t} \left(\int_{\mathbb{R}^d} (Q_t(y,x') - Q_t(y,x_0)) g(x')\,dx' \right)^2 \frac{dydt}{t^{d+1}} \right]^{\frac12} \\
&\le \left[\int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} \left(\int_B |Q_t(y,x') - Q_t(y,x_0)|\,\frac{dx'}{|B|} \right)^2 \frac{dydt}{t^{d+1}} \right]^{\frac12} \\
&\quad + \left[\int_{\frac{|x-x_0|}{2}}^\infty \int_{|x-y|<t} \Big(\cdots \Big)^2 \frac{dydt}{t^{d+1}} \right]^{\frac12} = I_1(x) + I_2(x). \qquad (4.14)
\end{aligned}$$

We now use the smoothness of $Q_t(x,y) = Q_t(y,x)$ established in Proposition 2.9. In the first range of integration we have $|y - x'| \sim |y - x_0| \sim |x - x_0|$ and $|x' - x_0| <$

$|y-x_0|/4$, so that

$$\begin{aligned}
I_1(x) &\lesssim \left[\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}\Big(\int_B \Big(\tfrac{|x'-x_0|}{t}\Big)^{\delta} t^{-d}\,(1+\tfrac{|y-x_0|}{t})^{-(d+1)}\,\tfrac{dx'}{|B|}\Big)^2\,\frac{dydt}{t^{d+1}}\right]^{\frac12}\\
&\le \left[\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}(\tfrac{r}{t})^{2\delta}\,t^{-2d}\,(1+\tfrac{|y-x_0|}{t})^{-2(d+1)}\,\frac{dydt}{t^{d+1}}\right]^{\frac12}\\
&\lesssim \left[\int_0^{\frac{|x-x_0|}{2}}(\tfrac{r}{t})^{2\delta}\,t^{-2d}\,\Big(\tfrac{t}{|x-x_0|}\Big)^{2(d+1)}\,\frac{dt}{t}\right]^{\frac12}\\
&\simeq \frac{r^\delta}{|x-x_0|^{d+1}}\left[\int_0^{\frac{|x-x_0|}{2}} t^{2-2\delta}\,\frac{dt}{t}\right]^{\frac12} \simeq \frac{r^\delta}{|x-x_0|^{d+\delta}}. \qquad (4.15)
\end{aligned}$$

Thus, integrating over $(B^{***})^c$,

$$\int_{(B^{***})^c} I_1(x)\,dx \lesssim \int_{|x-x_0|>8r}\frac{r^\delta}{|x-x_0|^{d+\delta}}\,dx \lesssim 1.$$

For $I_2(x)$ we have $|x'-x_0|\le r<|x-x_0|/2\le t$, so that Proposition 2.9 gives

$$|Q_t(y,x')-Q_t(y,x_0)| \lesssim \Big(\tfrac{|x'-x_0|}{t}\Big)^{\delta} t^{-d}.$$

Thus, for $x\in(B^{***})^c$ a similar argument to that presented above leads to

$$\begin{aligned}
I_2(x) &\lesssim \left[\int_{\frac{|x-x_0|}{2}}^{\infty}\int_{|x-y|<t}\Big(\int_B \tfrac{|x'-x_0|^\delta}{t^\delta}\,t^{-d}\,\tfrac{dx'}{|B|}\Big)^2\,\frac{dydt}{t^{d+1}}\right]^{\frac12}\\
&\le \left[\int_{\frac{|x-x_0|}{2}}^{\infty}\int_{|x-y|<t}\tfrac{r^{2\delta}}{t^{2\delta}}\,t^{-2d}\,\frac{dydt}{t^{d+1}}\right]^{\frac12}\\
&\simeq r^\delta\left[\int_{\frac{|x-x_0|}{2}}^{\infty}\frac{dt}{t^{2d+2\delta+1}}\right]^{\frac12} \simeq \frac{r^\delta}{|x-x_0|^{d+\delta}}.
\end{aligned}$$

Integrating over $(B^{***})^c$ one gets the required bound.

We now turn to the estimate of $\int_{(B^{***})^c} S_Q g(x)\,dx$ when r is comparable to $\rho(x_0)$. As before, we shall estimate pointwise $S_Q g(x)$, for each $x\in(B^{***})^c$. Proceeding as in (4.14), we break up the integral in $t>0$ defining $S_Q g(x)$ into three parts

$$S_Q g(x)^2 = \int_0^{\frac r2}\cdots + \int_{\frac r2}^{\frac{|x-x_0|}{4}}\cdots + \int_{\frac{|x-x_0|}{4}}^{\infty}\cdots = I_1'(x)+I_2'(x)+I_3'(x).$$

In the first integrand we have $|x'-y|\sim|x-x_0|$, so applying the estimate (a) of Proposition 2.9, we get

$$I_1'(x)\lesssim\frac{r^2}{|x-x_0|^{2(d+1)}},\quad x\in(B^{***})^c.$$

For the second term we use the extra decay in time, together with $|x'-y|\sim|x-x_0|$ and $\rho(x')\sim\rho(x_0)\sim r$ (see Propositions 2.1 and 2.9):

$$\begin{aligned}
I_2'(x) &\lesssim \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}}\int_{|x-y|<t}\Big(\int_B \tfrac{t^{-d}\,(1+|x-x_0|/t)^{-(d+M+1)}}{(1+t/\rho(x_0))^M}\,\tfrac{dx'}{|B|}\Big)^2\,\frac{dydt}{t^{d+1}}\\
&\lesssim \int_{\frac{r}{2}}^{|x-x_0|} t^{-2d}\Big(\tfrac{t}{|x-x_0|}\Big)^{2(d+M+1)}\Big(\tfrac{\rho(x_0)}{t}\Big)^{2M}\,\frac{dt}{t}\\
&\simeq \frac{r^{2(M+1)}}{|x-x_0|^{2(d+M+1)}}\int_1^{\frac{2|x-x_0|}{r}} t^2\,\frac{dt}{t}\simeq\frac{r^{2M}}{|x-x_0|^{2(d+M)}}.
\end{aligned}$$

Finally, for the last term the extra decay just gives

$$\begin{aligned}
I_3'(x) &\lesssim \int_{\frac{|x-x_0|}{4}}^{\infty}\int_{|x-y|<t}\Big(\int_B t^{-d}\,(1+t/\rho(x_0))^{-M}\,\tfrac{dx'}{|B|}\Big)^2\,\frac{dydt}{t^{d+1}}\\
&\lesssim \int_{\frac{|x-x_0|}{4}}^{\infty} t^{-2d}\Big(\tfrac{\rho(x_0)}{t}\Big)^{2M}\,\frac{dt}{t}\\
&\simeq \frac{\rho(x_0)^{2M}}{|x-x_0|^{2(d+M)}}\simeq\frac{r^{2M}}{|x-x_0|^{2(d+M)}}.
\end{aligned}$$

So, in the three cases we obtain bounds that lead to $\int_{(B^{***})^c}S_Qg(x)\,dx\lesssim 1$. This completes the proof of Lemma 4.13.

□

To conclude with the proof of Theorem 1.14, it only remains to justify the identity in Lemma 4.10. We observe that such identity is clearly valid when $f,g\in L^2(\mathbb{R}^d)$ (see (4.4)), while we must justify the convergence of the integrals in the case when $f\in L^1((1+|x|)^{-(d+1)}dx)$ and g is an $H^1_{\mathcal{L}}$-atom. As in the classical case (when $V\equiv 0$) this requires further estimates of the kernels (cf. [9]). A sketch of the modifications needed in this situation is the following.

Proof of Lemma 4.10: First one observes that, by Lemmas 4.12 and 4.13, and the dominated convergence theorem, the following integral is absolutely convergent and satisfies

$$I=\int_{\mathbb{R}^{d+1}_+}F(x,t)\,\overline{G(x,t)}\,\frac{dxdt}{t}=\lim_{\substack{\varepsilon\to0\\ N\to\infty}}\int_\varepsilon^N\int_{\mathbb{R}^d}Q_tf(x)\,\overline{Q_tg(x)}\,\frac{dxdt}{t}.$$

Next, a formal use of Fubini's theorem allows us to write, for each $t > 0$,

$$\int_{\mathbb{R}^d} Q_t f(x)\,\overline{Q_t g(x)}\,dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q_t(x,y) f(y)\,\overline{Q_t g(x)}\,dydx = \int_{\mathbb{R}^d} f(y)\,\overline{Q_t^2 g(y)}\,dy,$$

and, consequently, again formally

$$\begin{aligned} I &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \left[\int_{\mathbb{R}^d} f(y)\,\overline{Q_t^2 g(y)}\,dy\right]\frac{dt}{t} \\ &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{R}^d} f(y)\left[\int_\varepsilon^N \overline{Q_t^2 g(y)}\,\frac{dt}{t}\right] dy. \end{aligned} \tag{4.16}$$

The absolute integrability justifying these steps is a simple exercise, which combines the hypothesis $f \in L^1((1+|x|)^{-(d+1)}dx)$, the kernel decay $|Q_t(x,y)| \lesssim t^{-d}(1+|x-y|/t)^{-N}$, and the following general estimate on $H^1_{\mathcal{L}}$-atoms:

Lemma 4.17 *Let $q_t(x,y)$ be a function satisfying*

$$|q_t(x,y)| \le c_N\left(1+\tfrac{t}{\rho(x)}+\tfrac{t}{\rho(y)}\right)^{-N} t^{-d}(1+|x-y|/t)^{-N}. \tag{4.18}$$

Then, for every $H^1_{\mathcal{L}}$-atom g supported by $B_r(y_0)$, there is $C_{y_0,r} > 0$ such that

$$\mathcal{M}_q g(x) = \sup_{t>0}\left|\int_{\mathbb{R}^d} q_t(x,y)g(y)\,dy\right| \le C_{y_0,r}\,(1+|x|)^{-(d+1)}, \quad x\in\mathbb{R}^d. \tag{4.19}$$

Proof: There is no loss of generality in assuming that $r < 2\rho(y_0)$. Obviously,

$$\left|\int q_t(x,y)g(y)\,dy\right| \le C\|g\|_{L^\infty}. \tag{4.20}$$

If $x \notin B_{2r}(y_0)$ then for $y \in B_r(y_0)$ we have $|x-y| \sim |x-y_0|$, $\rho(y_0)\sim\rho(y)$. Hence, applying (4.18), we get

$$\begin{aligned} \left|\int q_t(x,y)g(y)\,dy\right| &\le C_N\|g\|_{L^1}\Big(1+\frac{t}{\rho(y_0)}\Big)^{-N} t^{-d}\Big(1+\frac{|x-y_0|}{t}\Big)^{-N} \\ &\le C_N|x-y_0|^{-d-N}\rho(y_0)^N \end{aligned} \tag{4.21}$$

Now (4.19) easily follows from (4.20) and (4.21).

□

Finally, in order to complete proof of the lemma we also need to justify the estimate

$$\sup_{\varepsilon,N>0}\left|\int_\varepsilon^N Q_t^2 g(y)\,\frac{dt}{t}\right| \le C_{y_0,r}\,(1+|y|)^{-(d+1)}, \quad y\in\mathbb{R}^d. \tag{4.22}$$

Indeed, such bound allows passing the limit inside the integral in (4.16), and conclude from (4.2) that

$$I=\frac{1}{8}\int_{\mathbb{R}^d} f(y)\,\overline{g(y)}\,dy.$$

For the justification of (4.22) one defines a new kernel $H_\varepsilon(x,y)$ as the one associated to the operator $\int_\varepsilon^\infty Q_t^2 g(y)\,\frac{dt}{t}$ and observes that

$$\Big|\int_\varepsilon^N Q_t^2 g(y)\,\tfrac{dt}{t}\Big| \le \sup_{\varepsilon>0}\Big|\int_{\mathbb{R}^d} H_\varepsilon(x,y)g(y)\,dy\Big| + \sup_{N>0}\Big|\int_{\mathbb{R}^d} H_N(x,y)g(y)\,dy\Big|.$$

So, it suffices to verify that $H_\varepsilon(x,y)$ satisfies the assumption of Lemma 4.17. But this is an immediate consequence of the same properties for the kernels $T_{t^2}(x,y)$ and $Q_t(x,y)$, due to the identity

$$H_\varepsilon(x,y)=\tfrac{1}{8}\big(T_{2\varepsilon^2}(x,y)-Q_{\sqrt{2}\,\varepsilon}(x,y)\big).$$

The verification of this last identity is left to the reader, which may use spectral techniques as in the proof of (4.2). This establishes Lemma 4.10, and completes the proof of Theorem 1.14.

□

5 Operators acting on $BMO_\mathcal{L}$

5.1 The Hardy-Littlewood maximal operator

Bennet, DeVore and Sharpley [1] proved that for a function $f\in BMO$, the *(uncentered) Hardy-Littlewood maximal function* $Mf(x)$ (see (1.6)) is either identically infinity or belongs to BMO, with norm $\|Mf\|_{BMO}\le C\|f\|_{BMO}$. The first situation occurs when f grows at infinity, e.g. with $f(x)=\log_+|x|$.

Below we show that $Mf(x)$ is always well-defined when $f\in BMO_\mathcal{L}$, and moreover, that M preserves this space. This extends some known properties about the action of M over the local *bmo* space of Goldberg.

Theorem 5.1 *The operator M maps $BMO_\mathcal{L}$ into $BMO_\mathcal{L}$. Moreover, there exists $C>0$ such that $\|Mf\|_{BMO_\mathcal{L}}\le C\|f\|_{BMO_\mathcal{L}}$ for all $f\in BMO_\mathcal{L}$.*

Proof: We first show that, for every $f\in BMO_\mathcal{L}$ the maximal function is finite *a.e.* This is a consequence of the following lemma.

Lemma 5.2 *Given $f\in BMO_\mathcal{L}$, $x_0\in\mathbb{R}^d$ and any $C_0\ge 1$, then $Mf(x)<\infty$ at almost every $x\in B_0=B(x_0,C_0\rho(x_0))$.*

Proof: Let us split $f = f_1 + f_2$, with $f_1(x) = f(x)\chi_{B_0^*}$. Since every function in $BMO_{\mathcal{L}}$ is locally integrable, we have $Mf_1(x) < \infty$, for $a.e.\, x \in \mathbb{R}^d$. To bound the second term, we use that supp $f_2 \subset (B_0^*)^c$. Then, we may compute $Mf_2(x)$ at $x \in B_0$ by just considering the integrals over balls $B \ni x$ for which $B \cap (B_0^*)^c \neq \emptyset$, and therefore with diameter $2r \geq C_0\rho(x_0)$. Thus, we have $B \subset B_{4r}(x_0) = \tilde{B}$, and consequently, by the definition of $BMO_{\mathcal{L}}$ (observe that $r(\tilde{B}) = 4r > \rho(x_0)$)

$$\frac{1}{|B|}\int_B |f_2(y)|\,dy \leq \frac{4^d}{|B_{4r}(x_0)|}\int_{B_{4r}(x_0)} |f(y)|\,dy \leq c\,\|f\|_{BMO_{\mathcal{L}}}. \tag{5.3}$$

□

We turn to the boundedness of M in $BMO_{\mathcal{L}}$. By invoking the results in [1] and the definition of $BMO_{\mathcal{L}}$, it suffices to prove that for every $B = B_r(x_0)$ with $r \geq \rho(x_0)$

$$\frac{1}{|B|}\int_B |Mf(y)|\,dy \leq C\|f\|_{BMO_{\mathcal{L}}}. \tag{5.4}$$

To show this, we split $f = f_1 + f_2$, with $f_1(x) = f(x)\chi_{B^*}$. From (5.3) it follows that $Mf_2(x) \leq c\|f\|_{BMO_{\mathcal{L}}}$ for all $x \in B$, so (5.4) follows with f replaced by f_2. For the other term, we may use the boundedness of M in L^2 to obtain:

$$\begin{aligned} \frac{1}{|B|}\int_B |Mf_1(y)|\,dy &\leq \Big(\frac{1}{|B|}\int_B |Mf_1(y)|^2\,dy\Big)^{1/2} \\ &\leq C\Big(\frac{1}{|B|}\int_{B^*} |f(y)|^2\,dy\Big)^{1/2} \lesssim \|f\|_{BMO_{\mathcal{L}}}, \end{aligned}$$

where in the last inequality we have used Corollary 3.13.

□

5.2 Semigroup maximal functions

In our setting, we are also interested in maximal functions arising from the semigroup T_t. These are more naturally related with the definition of our spaces (recall the definition of $H^1_{\mathcal{L}}$ in §1), and give us information about the solution to the evolution equation in (4.8).

More precisely, we shall consider $\mathcal{T}^*$ and $\mathcal{P}^*$, that is, the "heat" and the "Poisson" maximal functions related to $\mathcal{L}$ (see (1.7) and (1.8)). We observe that, in general, these maximal operators *are not bounded in the classical* $BMO(\mathbb{R}^d)$. E.g., when $V(x) = |x|^2$ (Hermite operator), then $\mathcal{T}^*\mathbf{1}(x)$ is not a constant function, and so $\mathcal{T}^*$ cannot be bounded in BMO (see [11]). Our next result shows that the natural space for boundedness of $\mathcal{T}^*$ and $\mathcal{P}^*$ is $BMO_{\mathcal{L}}$.

Theorem 5.5 *Let $f \in BMO_{\mathcal{L}}$. Then $\mathcal{T}^*f$ and $\mathcal{P}^*f$ belong to $BMO_{\mathcal{L}}$, and moreover, there exists $C > 0$ such that*

$$\|\mathcal{T}^*f\|_{BMO_{\mathcal{L}}} + \|\mathcal{P}^*f\|_{BMO_{\mathcal{L}}} \leq C\,\|f\|_{BMO_{\mathcal{L}}}.$$

For a fixed ball $B \subset \mathbb{R}^d$ we say that $f \in BMO(B)$ if

$$\begin{aligned} \|f\|_{BMO(B)} &= \sup_{B_r(x)\subset B} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f_{B_r(x)}|\,dy \\ &\sim \sup_{B_r(x)\subset B} \inf_{c\in\mathbb{C}} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-c|\,dy. \end{aligned} \tag{5.6}$$

is finite.

Proof of Theorem 5.5: We will use systematically the following elementary lemma.

Lemma 5.7 *Let $h \in BMO(Q_k^*)$ and g_1 and g_2 be functions in L^∞. If f is any measurable function satisfying*

$$h - g_1 \le f \le h + g_2, \quad a.e.$$

then $f \in BMO(Q_k^*)$ and $\|f\|_{BMO(Q_k^*)} \le \|h\|_{BMO(Q_k^*)} + \max\{\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty}\}$.

We shall consider first the operator $\mathcal{T}^*$. By the definition of $BMO_{\mathcal{L}}$ and Proposition 2.1 it suffices to prove the following: there exists a constant $C > 0$ such that for every fixed $Q_k \in \mathcal{Q}$ (see Proposition 2.14), we have

(i) $\displaystyle \frac{1}{|Q_k|}\int_{Q_k} |\mathcal{T}^* f(x)|\,dx \le C\,\|f\|_{BMO_{\mathcal{L}}}$;

(ii) $\|\mathcal{T}^* f\|_{BMO(Q_k^*)} \le C\,\|f\|_{BMO_{\mathcal{L}}}$.

Observe that (i) implies the almost everywhere finiteness of the operator. This part is immediate from Lemma 5.2 and (5.4), since $\mathcal{T}^* f(x) \le \sup_{t>0} |f| * h_t(x) \lesssim M|f|(x)$, and therefore

$$\frac{1}{|Q_k|}\int_{Q_k} |\mathcal{T}^* f(x)|\,dx \lesssim \frac{1}{|Q_k|}\int_{Q_k} M|f|(x)\,dx \le C\,\|f\|_{BMO_{\mathcal{L}}}.$$

We just need to show (ii), which we shall split into three different estimates:

$$\Big\| \sup_{t\ge\rho(x_k)^2} |T_t f(x)| \Big\|_{L^\infty(Q_k^*)} \le C\,\|f\|_{BMO_{\mathcal{L}}}, \tag{5.8}$$

$$\Big\| \sup_{t\le\rho(x_k)^2} |(T_t - \tilde{T}_t) f(x)| \Big\|_{L^\infty(Q_k^*)} \le C\,\|f\|_{BMO_{\mathcal{L}}}, \tag{5.9}$$

$$\Big\| \sup_{t\le\rho(x_k)^2} |\tilde{T}_t f(x)| \Big\|_{BMO(Q_k^*)} \le C\,\|f\|_{BMO_{\mathcal{L}}}, \tag{5.10}$$

where $\tilde{T}_t f = f * h_t$. It is not difficult to verify that (ii) follows from these three estimates by using Lemma 5.7.

The proof of each of them requires a different use of the decay and the smoothness of the kernels. For instance, (5.8) follows easily from (2.5). More precisely,

$$\begin{aligned} |T_t f(x)| &\lesssim \int_{\mathbb{R}^d} |f(y)|\, t^{-d/2}(1+|x-y|/\sqrt{t})^{-M}\, dy \\ &\lesssim \frac{1}{t^{d/2}} \int_{|x-y|\leq\sqrt{t}} |f(y)|\, dy + \sum_{j=1}^{\infty} \frac{1}{2^{jN}} \frac{1}{t^{d/2}} \int_{|x-y|\sim 2^j\sqrt{t}} |f(y)|\, dy. \end{aligned}$$

Now, for $j \geq 0$, we have $2^j\sqrt{t} \geq \rho(x_k) \sim \rho(x)$ for every $x \in Q_k^*$, thus

$$\frac{1}{t^{d/2}} \int_{|x-y|\sim 2^j\sqrt{t}} |f(y)|\, dy \lesssim \frac{2^{jd}}{|B_{2^j\sqrt{t}}(x)|} \int_{B_{2^j\sqrt{t}}(x)} |f(y)|\, dy \leq 2^{jd}\, \|f\|_{BMO_{\mathcal{L}}}.$$

Therefore,

$$\sup_{t\geq\rho(x_k)^2} |T_t f(x)| \lesssim \sum_{j=0}^{\infty} \frac{1}{2^{j(N-d)}} \|f\|_{BMO_{\mathcal{L}}} = C\|f\|_{BMO_{\mathcal{L}}}. \tag{5.11}$$

To prove (5.9) we shall need the following estimates on the difference $k_t - h_t$, which can be found in [3].

Lemma 5.12 (See Proposition 2.16 in [3].) *There exists a nonnegative Schwartz class function w in $\mathbb{R}^d$ so that*

$$|h_t(x-y) - k_t(x,y)| \leq \begin{cases} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} w_t(x-y), & \text{for } \sqrt{t} \leq \rho(x) \\ \left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta} w_t(x-y), & \text{for } \sqrt{t} \leq \rho(y) \\ w_t(x-y), & \text{elsewhere,} \end{cases} \tag{5.13}$$

where $w_t(x-y) = t^{-d/2} w((x-y)/\sqrt{t})$.

Going back to (5.9), since $\rho(x) \sim \rho(x_k)$ for all $x \in Q_k^*$, and in this case $\sqrt{t} \leq \rho(x_k)$, we can use Lemma 5.12 and proceed as in the previous estimate to obtain:

$$\begin{aligned} |(T_t - \tilde{T}_t) f(x)| &\leq \int_{\mathbb{R}^d} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} w_t(x-y) |f(y)|\, dy \\ &\lesssim \left(\frac{\sqrt{t}}{\rho(x_k)}\right)^{\delta} \sum_{j=0}^{\infty} 2^{-j(N-d)} \frac{1}{|B_{2^j\sqrt{t}}(x)|} \int_{B_{2^j\sqrt{t}}(x)} |f(y)|\, dy \\ &= \left(\frac{\sqrt{t}}{\rho(x_k)}\right)^{\delta} \Bigg(\sum_{1\leq 2^j \leq \frac{\rho(x_k)}{\sqrt{t}}} 2^{-j(N-d)} \frac{1}{|B_{2^j\sqrt{t}}(x)|} \int_{B_{2^j\sqrt{t}}(x)} |f(y)|\, dy \\ &\qquad + \sum_{2^j > \frac{\rho(x_k)}{\sqrt{t}}} 2^{-j(N-d)} \|f\|_{BMO_{\mathcal{L}}} \Bigg). \end{aligned}$$

Now, by Lemma 3.14, for j such that $1 \le 2^j \le \frac{\rho(x_k)}{\sqrt{t}}$, we have

$$\frac{1}{|B_{2^j\sqrt{t}}(x)|}\int_{B_{2^j\sqrt{t}}(x)} |f(y)|\,dy \lesssim 1+\log\frac{\rho(x)}{2^j\sqrt{t}} \lesssim 1+\log\frac{\rho(x_k)}{\sqrt{t}}.$$

Therefore,

$$|(T_t-\tilde{T}_t)f(x)| \lesssim \left(\frac{\sqrt{t}}{\rho(x_k)}\right)^{\delta}\left(1+\log\frac{\rho(x_k)}{\sqrt{t}}\right)\|f\|_{BMO_{\mathcal{L}}}\sum_{j=0}^{\infty}2^{-j(N-d)} \lesssim \|f\|_{BMO_{\mathcal{L}}}.$$

Finally, let us sketch the proof of (5.10). This seems to be a classical result, following from a vector-valued singular integral theory. Consider $B=B_r(x_0)\subset Q_k^*$ and split f as

$$f=(f-f_B)\chi_{B^*}+\left[(f-f_B)\chi_{(B^*)^c}+f_B\right]=f_1+f_2. \tag{5.14}$$

Denote $\|\tilde{T}_tf(x)\|_{\ell^\infty(t)}=\sup_{t\le\rho(x_k)^2}|\tilde{T}_tf(x)|$ and choose $c_B=\|\tilde{T}_tf_2(x_0)\|_{\ell^\infty(t)}$, which is a finite real number by (5.4). Then,

$$\begin{aligned}\frac{1}{|B|}\int_B\Big|\sup_{t\le\rho(x_k)^2}|\tilde{T}_tf(x)|-c_B\Big|\,dx &\le \frac{1}{|B|}\int_B\|\tilde{T}_tf(x)-\tilde{T}_tf_2(x_0)\|_{\ell^\infty(t)}\,dx\\ &\le \frac{1}{|B|}\int_B\|\tilde{T}_tf_1(x)\|_{\ell^\infty(t)}\,dx+\frac{1}{|B|}\int_B\|\tilde{T}_tf_2(x)-\tilde{T}_tf_2(x_0)\|_{\ell^\infty(t)}\,dx=I+II.\end{aligned}$$

For the first integral, observe that by the L^2 boundedness of $\sup_{t\le\rho(x_k)^2}|\tilde{T}_tf(x)|$, we have

$$I\le\left(\frac{1}{|B|}\int_{B^*}|f(x)-f_B|^2\,dx\right)^{1/2}\le C\|f\|_{BMO}\le C\|f\|_{BMO_{\mathcal{L}}}.$$

For the second integral, the standard arguments of singular integrals and the smoothness of the kernel will give:

$$\begin{aligned}&\frac{1}{|B|}\int_B\|\tilde{T}_tf_2(x)-\tilde{T}_tf_2(x_0)\|_{\ell^\infty(t)}\,dx\\ &\le \frac{1}{|B|}\int_B\left\|\int_{(B^*)^c}(h_t(x,y)-h_t(x_0,y))(f(y)-f_B)\,dy\right\|_{\ell^\infty(t)}dx\le C\|f\|_{BMO}.\end{aligned}$$

It should be observed that in this last step the constant C is independent of Q_k.

We now turn to the Poisson maximal function $\mathcal{P}^*f$, for which we indicate the main differences with respect to the previous proof. As before, it is enough to prove (i)–(ii) with $\mathcal{T}^*$ replaced by $\mathcal{P}^*$. By subordination we have $\mathcal{P}^*f(x)\lesssim\mathcal{T}^*f(x)$. Hnence there is no problem in verifying (i). It thus suffices to prove that $\|\mathcal{P}^* f\|_{BMO(Q_k^*)} \leq C\|f\|_{BMO_\mathcal{L}}$ for any fixed $Q_k \in \mathcal{Q}$. Set $\rho = \rho(x_k)$. Observe that

$$\mathcal{P}_1^* f(x) - \mathcal{P}_2^* f(x) \leq \mathcal{P}^* f(x) \leq \mathcal{P}_1^* f(x) + \mathcal{P}_2^* f(x),$$

where

$$\begin{aligned} \mathcal{P}_1^* f(x) &= \sup_{t\geq 0} \left| \int_{t^2/4\rho^2}^{\infty} \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x)\, du \right|, \\ \mathcal{P}_2^* f(x) &= \sup_{t\geq 0} \left| \int_0^{t^2/4\rho^2} \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x)\, du \right|. \end{aligned}$$

Moreover, $|\mathcal{P}_2^* f(x)| \leq \sup_{s\geq \rho^2} |T_s f(x)|$, which by (5.8), belongs to $L^\infty(Q_k^*)$. An application of Lemma 5.7 reduces matters to show that $\mathcal{P}_1^* f$ is in $BMO(Q_k^*)$. We shall repeat this argument two more times. Indeed

$$\mathcal{P}_{11}^* f(x) - \mathcal{P}_{12}^* f(x) \leq \mathcal{P}_1^* f(x) \leq \mathcal{P}_{11}^* f(x) + \mathcal{P}_{12}^* f(x),$$

where

$$\begin{aligned} \mathcal{P}_{11}^* f(x) &= \sup_{t\geq 0} \left| \int_{t^2/4\rho^2}^{\infty} \frac{e^{-u}}{\sqrt{u}}\, \tilde{T}_{t^2/4u} f(x)\, du \right|, \\ \mathcal{P}_{12}^* f(x) &= \sup_{t\geq 0} \left| \int_{t^2/4\rho^2}^{\infty} \frac{e^{-u}}{\sqrt{u}}\, (T_{t^2/4u} - \tilde{T}_{t^2/4u}) f(x)\, du \right|. \end{aligned}$$

Now $\mathcal{P}_{12}^* f(x) \leq C \sup_{s\leq \rho(x_k)^2} |(T_s - \tilde{T}_s) f(x)|$, which belongs to L^∞ by (5.9). Next for the remaining term $\mathcal{P}_{11}^* f(x)$ we have

$$\mathcal{P}_{111}^* f(x) - \mathcal{P}_{112}^* f(x) \leq \mathcal{P}_{11}^* f(x) \leq \mathcal{P}_{111}^* f(x) + \mathcal{P}_{112}^* f(x),$$

where

$$\begin{aligned} \mathcal{P}_{111}^* f(x) &= \sup_{t\geq 0} \left| \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}}\, \tilde{T}_{t^2/4u} f(x)\, dx \right|, \\ \mathcal{P}_{112}^* f(x) &= \sup_{t\geq 0} \left| \int_0^{t^2/4\rho^2} \frac{e^{-u}}{\sqrt{u}} \tilde{T}_{t^2/4u} f(x)\, dx \right|. \end{aligned}$$

Again, $|\mathcal{P}_{112}^* f(x)| \leq \sup_{s\geq \rho^2} |\tilde{T}_s f(x)|$ which is bounded by the same reasoning as in (5.8). Thus, we have reduced matters to show that $\mathcal{P}_{111}^* f$, which is the classical Poisson maximal function, belongs to $BMO(Q_k^*)$. This will again follow from classical arguments of vector-valued singular integrals. Skipping these details, the proof of the theorem is now complete.

□

Remark 5.18 It is interesting to observe that an analog of the previous theorem holds as well for the *non-tangential maximal operator*

$$\mathcal{T}^{**}f(x) = \sup_{|x-y|<t} |T_t f(y)|, \quad x \in \mathbb{R}^d.$$

In fact, the reader can easily check that the same proof above goes along in this more general situation.

5.3 Fractional integrals

In this section we shall be interested in the behavior of the fractional integral operator $\mathcal{I}_\alpha = \mathcal{L}^{-\alpha/2}$ (see (1.10)).

Recall that in the classical setting $I_\alpha = (-\Delta)^{-\alpha/2}$ maps $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ for $0 < \alpha < d$ and $0 < p < q < \infty$ with $1/p = 1/q - \alpha/d$. Moreover, there is a dichotomy in the limiting case $q = \infty$: for every $f \in L^{d/\alpha}(\mathbb{R}^d)$, either $I_\alpha f \equiv \infty$ or $I_\alpha f \in BMO(\mathbb{R}^d)$ with $\|I_\alpha f\|_{BMO} \leq C\,\|f\|_{L^{d/\alpha}(\mathbb{R}^d)}$ (see page 221 in [10]). Simple examples like $f(x) = \frac{1}{|x|^\alpha \log|x|}$ show that $I_\alpha f$ may be identically ∞.

Our next result shows that such pathological behaviors cannot happen to the operator $\mathcal{I}_\alpha$, when the potential $V \not\equiv 0$. Moreover, the natural target space is now $BMO_\mathcal{L}$.

Theorem 5.19 *For all $0 < \alpha < d$, the operator $\mathcal{I}_\alpha$ is bounded from $L^{d/\alpha}(\mathbb{R}^d)$ into $BMO_\mathcal{L}$, that is, there is a constant $C > 0$ so that*

$$\|\mathcal{I}_\alpha f\|_{BMO_\mathcal{L}} \leq C\,\|f\|_{L^{d/\alpha}}, \quad \textit{for every } f \in L^{d/\alpha}(\mathbb{R}^d).$$

Proof: The proof follows the same scheme as in the previous section. We shall try to establish the analogs of (i)-(ii) with $\mathcal{T}^*$ replaced by $\mathcal{I}_\alpha$. To see (i), let us split

$$\mathcal{I}_\alpha f(x) = \int_0^{\rho(x)^2} T_t f(x) t^{\alpha/2-1}\,dt + \int_{\rho(x)^2}^\infty T_t f(x) t^{\alpha/2-1}\,dt = I_1 f(x) + I_2 f(x). \quad (5.20)$$

For the first integral, we use the trivial estimate

$$|I_1 f(x)| \leq \int_0^{\rho(x)^2} \mathcal{T}^* f(x) t^{\alpha/2-1}\,dt \lesssim \rho(x)^\alpha M|f|(x). \quad (5.21)$$

For the second integral, the extra decay in time of $k_t(x,y)$ gives

$$\begin{aligned} |I_2 f(x)| &\leq C_N \int_{\rho(x)^2}^\infty \int_{\mathbb{R}^d} \left(\frac{\rho(x)}{\sqrt{t}}\right)^N h_{ct}(x-y)|f(y)|\,dy\,t^{\alpha/2-1}\,dt \\ &\lesssim M|f|(x)\,\rho(x)^N \int_{\rho(x)^2}^\infty t^{\alpha/2-N/2-1}\,dt \lesssim \rho(x)^\alpha M|f|(x). \quad (5.22) \end{aligned}$$

Thus, combining these two estimates, and using $\rho(x) \sim \rho(x_k)$ if $x \in Q_k$, we obtain

$$\frac{1}{|Q_k|}\int_{Q_k} |\mathcal{I}_\alpha f(x)|\,dx \lesssim \frac{1}{|Q_k|^{1-\alpha/d}}\int_{Q_k} |Mf(x)|\,dx \le \|Mf\|_{L^{d/\alpha}} \lesssim \|f\|_{L^{d/\alpha}},$$

where in the last two steps we used Hölder's inequality and the boundedness of M in $L^{d/\alpha}$.

We pass now to prove the analog of assertion (ii), that is, $\|\mathcal{I}_\alpha f\|_{BMO(Q_k^*)} \le C\,\|f\|_{BMO_\mathcal{L}}$. As before, the strategy is based on an iterative use of Lemma 5.7. Splitting $\mathcal{I}_\alpha f = I_1 f(x) + I_2 f(x)$ as in (5.20), we shall show that $I_1 f \in BMO(Q_k^*)$ and $I_2 f \in L^\infty(Q_k^*)$, with norms controlled by $\|f\|_{L^{d/\alpha}}$. For the second term we must slightly refine the argument in (5.22): if $x \in Q_k^*$ then

$$|I_2 f(x)| \lesssim \rho(x_k)^N \int_{\rho(x_k)^2}^\infty \int_{\mathbb{R}^d} \frac{1}{t^{(d-\alpha)/2}} e^{-\frac{|x-y|^2}{4ct}} |f(y)|\,dy\, t^{-N/2-1}\,dt.$$

Now, observe that

$$\begin{aligned}\int_{\mathbb{R}^d} t^{-(d-\alpha)/2}\, e^{-\frac{|x-y|^2}{4ct}} |f(y)|\,dy \;&\lesssim\; \sum_{j=0}^\infty \frac{2^{j(d-\alpha)}\, e^{-c' 2^{2j}}}{(2^j\sqrt{t})^{d-\alpha}} \int_{\frac{|x-y|}{\sqrt{t}}\le 2^j} |f(y)|\,dy \\ &\lesssim\; \sup_{r>0} \frac{1}{|B_r(x)|^{1-\alpha/d}} \int_{B_r(x)} |f(y)|\,dy =: M_\alpha f(x),\end{aligned}$$

where M_α denotes the *fractional maximal operator*, which trivially maps $L^{d/\alpha}(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$. Thus,

$$|I_2 f(x)| \;\lesssim\; \rho(x_k)^N\, M_\alpha f(x) \int_{\rho(x_k)^2}^\infty t^{-N/2-1}\,dt \lesssim \|f\|_{L^{d/\alpha}(\mathbb{R}^d)}. \tag{5.23}$$

To deal with $I_1 f$ we make some further splittings:

$$I_1 f = \int_0^{\rho(x_k)^2} \tilde{T}_t f\, t^{\alpha/2-1}\,dt + \int_0^{\rho(x_k)^2} (T_t - \tilde{T}_t) f\, t^{\alpha/2-1}\,dt = I_{11} f \;+\; I_{12} f.$$

A new use of Lemma 5.12, Proposition 2.1, and reasoning with a similar estimate to that presented above gives

$$\begin{aligned}|I_{12} f(x)| \;&\lesssim\; \int_0^{\rho(x_k)^2} \int_{\mathbb{R}^d} \left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta w_t(x-y)|f(y)|\,dy\, t^{\alpha/2-1}\,dt \\ &\lesssim\; \int_0^{\rho(x_k)^2} \frac{t^{\delta/2-1}}{\rho(x)^\delta} M_\alpha f(x)\,dt \lesssim \|f\|_{L^{d/\alpha}(\mathbb{R}^d)}\end{aligned}$$

for all $x \in Q_k^*$.

It remains to control the term $I_{11}f$, for which we must show that, given any ball $B = B_r(x_0) \subset Q_k^*$ there is a constant c_B so that

$$\tfrac{1}{|B|} \int_B |\, I_{11}f(x) - c_B|\, dx \le C\, \|f\|_{L^{d/\alpha}(\mathbb{R}^d)}. \tag{5.24}$$

This is elementary to verify with $c_B = 0$ when the radius of the ball B is comparable to $\rho(x_k)$. Indeed, in this case Hölder's and Minkowski's inequalities give

$$\begin{aligned} \tfrac{1}{|B|} \int_B |\, I_{11}f(x)|\, dx &\le \left(\tfrac{1}{|B|} \int_B |\, I_{11}f(x)|^{d/\alpha}\, dx \right)^{\alpha/d} \\ &\le \tfrac{1}{|B|^{\alpha/d}} \int_0^{\rho(x_k)^2} \|\tilde{T}_t f\|_{L^{d/\alpha}} t^{\alpha/2-1}\, dt \lesssim \|f\|_{L^{d/\alpha}}. \end{aligned} \tag{5.25}$$

Suppose instead we are given a ball with $r \ll \rho(x_k)$. In this case we must further split the integral defining $I_{11}f(x)$ into two new pieces:

$$I_{11}f(x) = \int_0^{r^2} \tilde{T}_t f(x) t^{\alpha/2-1}\, dt + \int_{r^2}^{\rho(x_k)^2} \tilde{T}_t f(x) t^{\alpha/2-1}\, dt = I_{111}f(x) + I_{112}f(x).$$

For the first piece we can repeat the previous calculation to obtain $\frac{1}{|B|}\int_B |I_{111}f(x)|\, dx \lesssim \|f\|_{L^{d/\alpha}}$. For the second piece we write $f = f_1 + f_2$ with $f_1(x) = f(x)\chi_{B^*}(x)$, and choose $c_B = I_{112}f_2(x_0)$ (which is a finite number for *a.e.* x_0). Then,

$$\tfrac{1}{|B|} \int_B |I_{112}f(x) - c_B|\, dx \le \tfrac{1}{|B|} \int_B |I_{112}f_1(x)|\, dx + \tfrac{1}{|B|} \int_B |I_{112}f_2(x) - I_{112}f_2(x_0)|\, dx.$$

When $x \in B$, the first integrand is uniformly bounded by

$$\begin{aligned} |I_{112}f_1(x)| &\lesssim \int_{r^2}^{\rho(x_k)^2} \int_{B^*} t^{-d/2} e^{-\frac{|x-y|^2}{4t}}\, |f(y)|\, dy\, t^{\alpha/2-1}\, dt \\ &\lesssim \int_{r^2}^{\rho(x_k)^2} \int_{|x-y|<3r} |f(y)|\, dy\, t^{-d/2+\alpha/2-1}\, dt \\ &\lesssim M_\alpha f(x)\, r^{d-\alpha} \int_{r^2}^{\infty} t^{-d/2+\alpha/2-1}\, dt \le \|f\|_{L^{d/\alpha}}. \end{aligned}$$

For the second integrand we use smoothness

$$\begin{aligned} \Big| I_{112}f(x) - I_{112}f(x_0) \Big| &\lesssim \int_{r^2}^{\rho(x_k)^2} \int_{(B^*)^c} \frac{1}{t^{d/2}} \left| e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x_0-y|^2}{4t}} \right| |f(y)|\, dy\, t^{\alpha/2-1}\, dt \\ &\lesssim \int_{r^2}^{\rho(x_k)^2} \sum_{j=1}^{\infty} \frac{1}{t^{d/2}} \int_{|x_0-y|\sim 2^j r} \left| e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x_0-y|^2}{4t}} \right| |f(y)|\, dy\, t^{\alpha/2-1}\, dt. \end{aligned}$$

By the mean value theorem (and the fact $|x-x_0|<r<\sqrt{t}$) we have:

$$\begin{aligned}\int_{|x_0-y|\sim 2^jr}\Big|e^{-\frac{|x-y|^2}{4t}}-e^{-\frac{|x_0-y|^2}{4t}}\Big||f(y)|\,dy &\lesssim \tfrac{|x-x_0|}{\sqrt{t}}\,e^{-c(2^jr)^2/t}\int_{B_{2^jr}(x_0)}|f(y)|\,dy\\ &\lesssim \tfrac{|x-x_0|}{\sqrt{t}}\,\frac{(2^jr)^{d-\alpha}\,M_\alpha f(x_0)}{(1+2^jr/\sqrt{t})^N},\end{aligned}$$

for a sufficiently large integer N. Hence,

$$\big|I_{112}f(x)-I_{112}f(x_0)\big|\lesssim\int_{r^2}^{\rho(x_k)^2}t^{-d/2+\alpha/2-1}\Big[\sum_{j=1}^{\infty}\tfrac{(2^jr)^{d-\alpha}}{(1+2^jr/\sqrt{t})^N}\Big]\tfrac{|x-x_0|}{\sqrt{t}}\,dt\,M_\alpha f(x_0)\,.$$

Breaking the inner sum into two parts it is easy to see that it is bounded by a constant times $(\sqrt{t})^{d-\alpha}$. Therefore

$$\big|I_{112}f(x)-I_{112}f(x_0)\big|\lesssim M_\alpha f(x_0)|x-x_0|\int_{r^2}^{\infty}t^{-1/2-1}\,dt\lesssim M_\alpha f(x_0)\lesssim\|f\|_{L^{d/\alpha}}.$$

From here it is immediate that $\frac{1}{|B|}\int_B\big|I_{112}f(x)-I_{112}f(x_0)\big|\,dx\lesssim\|f\|_{L^{d/\alpha}}$, establishing the assertion (ii).

□

5.4 Square functions

Consider the following square function associated with $\mathcal{L}$:

$$s(f)(x)=\Big(\int_0^\infty|t\tfrac{dT_t}{dt}f(x)|^2\,\tfrac{dt}{t}\Big)^{1/2}. \tag{5.26}$$

This is just a multiple of the square function $s_Qf(x)$ defined in (1.9).

Theorem 5.27 *There exists a constant $C>0$ such that*

$$\|s(f)\|_{BMO_\mathcal{L}}\le C\|f\|_{BMO_\mathcal{L}}. \tag{5.28}$$

Proof: Fix $f\in BMO_\mathcal{L}$ and $Q_k\in\mathcal{Q}$ with center x_k. It is not difficult to prove using (2.11) that

$$\frac{1}{|Q_k|}\int_{Q_k}|s(f)(x)|\,dx\lesssim\|f\|_{BMO_\mathcal{L}}. \tag{5.29}$$

In fact, if we split

$$|s(f)(x)|^2=|s_1f(x)|^2+|s_2f(x)|^2:=\int_0^{\rho(x_k)^2}|t\tfrac{dT_t}{dt}f(x)|^2\,\tfrac{dt}{t}+\int_{\rho(x_k)^2}^{\infty}|t\tfrac{dT_t}{dt}f(x)|^2\,\tfrac{dt}{t},$$

then (5.29) for $s_1(f)$ was shown in (4.5). The second term $s_2(f)(x)$ has a uniform bound when $x \in Q_k$. In fact, by using (a) in Proposition 2.9 and the same reasoning as in (5.11), we get

$$\begin{aligned} |s_2 f(x)|^2 &\lesssim \int_{\rho(x_k)^2}^{\infty} \Big[\int_{\mathbb{R}^d} \big(\tfrac{\rho(x)}{\sqrt{t}}\big)^N t^{-d/2} e^{-\frac{c|x-y|^2}{t}} |f(y)|\, dy \Big]^2 \tfrac{dt}{t} \\ &\lesssim \Big[\sup_{s \geq \rho(x_k)^2} |f| * h_s(x) \Big]^2 \lesssim \|f\|^2_{BMO_{\mathcal{L}}}. \end{aligned}$$

To complete the proof, it suffices to show that

$$\|s(f)(x)\|_{BMO(Q_k^*)} \leq C\|f\|_{BMO_{\mathcal{L}}} \tag{5.30}$$

with some constant C independent Q_k. Decompose $f = f_1 + f_2$ as in (5.14) and set $c_B = \Big(\int_0^{\rho(x_k)^2} |t \frac{d\widetilde{T}_t}{dt} f_2(x_0)|^2 \frac{dt}{t} \Big)^{1/2}$, which is a finite number. Then

$$\begin{aligned} |s(f)(x) - c_B| &\leq \Big(\int_{\rho(x_k)^2}^{\infty} |t \tfrac{dT_t}{dt} f(x)|^2 \tfrac{dt}{t} \Big)^{1/2} + \Big(\int_0^{\rho(x_k)^2} |t \tfrac{d\widetilde{T}_t}{dt} f(x) - t \tfrac{d\widetilde{T}_t}{dt} f_2(x_0)|^2 \tfrac{dt}{t} \Big)^{1/2} \\ &+ \Big(\int_0^{\rho(x_k)^2} |t \tfrac{dT_t}{dt} f(x) - t \tfrac{d\widetilde{T}_t}{dt} f(x)|^2 \tfrac{dt}{t} \Big)^{1/2} = S_1(x) + S_2(x) + S_3(x). \end{aligned}$$

One can easily check using standard arguments combined with (2.11) that

$$\frac{1}{|B|} \int_B \big(S_1(x) + S_2(x) \big)\, dx \lesssim \|f\|_{BMO_{\mathcal{L}}}. \tag{5.31}$$

The proof of (5.30) will be finished if we show that

$$S_3(x) \lesssim \|f\|_{BMO_{\mathcal{L}}} \qquad \text{for } x \in B. \tag{5.32}$$

The perturbation formula implies

$$\frac{d\widetilde{T}_t}{dt} - \frac{dT_t}{dt} = \widetilde{T}_{t/2} V T_{t/2} + \int_0^{t/2} (\frac{d}{dt} \widetilde{T}_{t-s}) V T_s\, ds + \int_{t/2}^{t} \widetilde{T}_{t-s} V \, \frac{dT_s}{ds}\, ds. \tag{5.33}$$

Thus (5.32) will be established if we verify that

$$\begin{aligned} &\int_0^{\rho(x_k)^2} |t\tilde{T}_{t/2} V T_{t/2} f(x)|^2 \tfrac{dt}{t} + \int_0^{\rho(x_k)^2} \Big| \int_0^{t/2} t (\frac{d}{dt} \tilde{T}_{t-s}) V T_s f(x)\, ds \Big|^2 \tfrac{dt}{t} \\ &+ \int_0^{\rho(x_k)^2} \Big| \int_{t/2}^{t} t\tilde{T}_{t-s} V \, \tfrac{dT_s}{ds} f(x)\, ds \Big|^2 \tfrac{dt}{t} = S_3'(x) + S_3''(x) + S_3'''(x) \lesssim \|f\|^2_{BMO_{\mathcal{L}}}. \end{aligned} \tag{5.34}$$

The following lemma is a direct consequence of the definition of $BMO_{\mathcal{L}}$ and Lemma 3.14

Lemma 5.35 *Let $q_t(x,y)$ be a function satisfying (4.18). Then, for every N there is a constant $C_N > 0$ such that*

$$\Big|\int q_t(x,y)f(y)\,dy\Big| \le C_N \|f\|_{BMO_{\mathcal{L}}}\Big(1+\log_+ \tfrac{\rho(x)}{t}\Big)\Big(1+\frac{t}{\rho(x)}\Big)^{-N}. \qquad (5.36)$$

Hence, from Propositions 2.4 and 2.1 we conclude

$$\begin{aligned} S_3'(x) &\lesssim \int_0^{\rho(x_k)^2} \Big|\int t h_{t/2}(x,z)V(z)\Big(1+\log_+ \tfrac{\rho(z)}{\sqrt{t}}\Big)\,dz\Big|^2 \frac{dt}{t}\,\|f\|^2_{BMO_{\mathcal{L}}} \\ &\lesssim \int_0^{\rho(x_k)^2} \Big|\int t h_{t/2}(x,z)V(z) \qquad\qquad (5.37) \\ &\quad \times \Big(1+\log_+ \big[\tfrac{\rho(x)}{\sqrt{t}}\big(1+\tfrac{|x-z|}{\sqrt{t}}\,\tfrac{\sqrt{t}}{\rho(x)}\big)^{\frac{k_0}{1+k_0}}\big]\Big)\,dz\Big|^2 \frac{dt}{t}\,\|f\|^2_{BMO_{\mathcal{L}}}. \end{aligned}$$

By using (2.12) and the fact that $\rho(x) \sim \rho(x_k)$ for $x \in B$, we have

$$S_3'(x) \lesssim \int_0^{\rho(x_k)^2} \Big|\Big(\tfrac{\sqrt{t}}{\rho(x)}\Big)^{\delta}\Big(1+\log_+ \tfrac{\rho(x)}{\sqrt{t}}\Big)\Big|^2 \frac{dt}{t}\,\|f\|^2_{BMO_\rho} \lesssim \|f\|^2_{BMO_\rho}.$$

The estimates for $S_3''(x)$ and $S_3'''(x)$ stated in (5.34) could be proved by applying the same arguments and the bounds for $k_s(x,y)$ and $\frac{d}{ds}k_s(x,y)$ established in Section 2. □

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