Analytic Besov spaces and Hardy-type inequalities in tube domains over symmetric cones.

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Abstract

We give various equivalent formulations to the (partially) open problem about L^p-boundedness of Bergman projections in tubes over cones. Namely, we show that such boundedness is equivalent to the duality identity between Bergman spaces, $A^{p'} = (A^p)^*$, and also to a Hardy type inequality related to the wave operator. We introduce analytic Besov spaces in tubes over cones, for which such Hardy inequalities play an important role. For $p \geq 2$ we identify as a Besov space the range of the Bergman projection acting on L^p, and also the dual of $A^{p'}$. For the Bloch space $\mathbb{B}^\infty$ we give in addition new necessary conditions on the number of derivatives required in its definition.

1 Introduction

Let T_Ω be a symmetric domain of tube type in $\mathbb{C}^n$, that is $T_\Omega = \mathbb{R}^n + i\Omega$ where Ω is an *irreducible symmetric cone* in $\mathbb{R}^n$. These domains can be seen as multidimensional analogues of the upper half plane in $\mathbb{C}$. A typical example arises when Ω is the forward light-cone of $\mathbb{R}^n$, $n \geq 3$,

$$\Lambda_n = \big\{ y \in \mathbb{R}^n \ : \ y_1^2 - y_2^2 - \ldots - y_n^2 > 0, \ \ y_1 > 0 \big\}.$$

Other examples correspond to the cones $\mathrm{Sym}_+(r, \mathbb{R})$ of positive definite symmetric $r \times r$-matrices. We refer to the text [15] for a general description of symmetric cones. Following the notation in [15] we write r for the rank of Ω and $\Delta(x)$ for the associated determinant function. In the above examples, light-cones have rank 2 and determinant equal to the Lorentz form $\Delta(y) = y_1^2 - y_2^2 - \ldots - y_n^2$, while the cones $\mathrm{Sym}_+(r, \mathbb{R})$ have rank r and the determinant is the usual determinant of $r \times r$ matrices. We shall denote by $\mathcal{H}(T_\Omega)$ the space of holomorphic functions on T_Ω.

A major open question in these domains concerns the L^p boundedness of *Bergman projections*, which can only hold for values of p sufficiently close to 2 [6, 11, 10]. More

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precisely, consider the (weighted) spaces

$$L^p_\nu(T_\Omega) \;=\; L^p(T_\Omega, \Delta(y)^{\nu-n/r}dx\,dy)$$

and let $A^p_\nu(T_\Omega)$ be the subspace of holomorphic functions. Denote by P_ν the orthogonal projection mapping $L^2_\nu(T_\Omega)$ into $A^2_\nu(T_\Omega)$. The usual (unweighted) Bergman spaces correspond to $\nu = \frac{n}{r}$, while the weighted cases can be considered when $\nu > \frac{n}{r} - 1$ (since otherwise $A^p_\nu = \{0\}$).

Conjecture 1 *Let $\nu > \frac{n}{r} - 1$. Then the Bergman projection P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if*

$$p'_\nu \;<\; p \;<\; p_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1} - \frac{(1-\nu)_+}{\frac{n}{r} - 1}.$$

This problem has only been settled in the case of light-cones for sufficiently large ν's [10]. In general, the known results can be described as follows (see [6, 11, 8, 10]). The fact that boundedness can only hold when $\tilde{p}'_\nu < p < \tilde{p}_\nu$, where

$$\tilde{p}_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1},$$

is trivially given by the $L^{p'}_\nu$-integrability of the Bergman kernel (which only happens when $p < \tilde{p}_\nu$) and duality. The necessity of the condition involving $(1-\nu)_+$ was established in [10], and may only occur in the three dimensional forward light-cone (the only case in which ν is allowed to take values below 1). Concerning sufficiency, it has been proved in [11, 8] that P_ν is bounded in L^p_ν at least in the range

$$\bar{p}'_\nu < p < \bar{p}_\nu := \frac{\nu + \frac{2n}{r} - 2}{\frac{n}{r} - 1}. \tag{1.1}$$

In the light-cone setting (ie when $r = 2$), Conjecture 1 is closely related to other deep conjectures for the wave equation. As shown in [10], this implies slight improvements in the range (1.1) for all ν's, and in fact sets completely the conjecture when ν is sufficiently large (see also [19, 20] for the latest results).

In this paper, we shall not improve these boundedness results, but interest ourselves in equivalent formulations of Conjecture 1 and implications in the theory of holomorphic function spaces in T_Ω. Consider the "box operator" of Ω, denoted $\Box = \Delta(\frac{1}{i}\frac{\partial}{\partial x})$, as the differential operator of degree r in $\mathbb{R}^n$ defined by the equality:

$$\Box\,[e^{i(x|\xi)}] = \Delta(\xi)e^{i(x|\xi)}, \quad x, \xi \in \mathbb{R}^n. \tag{1.2}$$

In the rank 1 setting (that is, when $n = 1$ and $\Omega = (0,\infty)$) this corresponds to $-i\frac{d}{dx}$, and in the rank 2 situation (that is, when Ω is the forward light cone in $\mathbb{R}^n$) we have $\Box = -(\partial^2_{x_1} - \partial^2_{x_2} - \ldots - \partial^2_{x_n})/4$, which explains why $\Delta(\frac{\partial}{\partial x})$ is sometimes called the wave operator. We denote by $\Box_z = \Delta(\frac{1}{i}\frac{\partial}{\partial z})$ the corresponding differential operator in $\mathbb{C}^n$ defined replacing x in (1.2) by $z \in \mathbb{C}^n$. Observe, however, that $\Box_z = \Box_x$ when acting on holomorphic functions in T_Ω. To simplify notation, we will write $\Box$ instead of $\Box_z$. Our first result can then be stated as follows.

THEOREM 1.3 *Let $\nu > \frac{n}{r} - 1$. Then, for $p \geq 2$, the Bergman projection P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if there exists a constant C such that, for all $F \in A^p_\nu$ we have*

$$\int\!\!\int_{T_\Omega} |F(x+iy)|^p \, \Delta^{\nu-\frac{n}{r}}(y)\, dx\, dy \;\leq\; C \int\!\!\int_{T_\Omega} \big|\Delta(y)\Box F(x+iy)\big|^p \, \Delta^{\nu-\frac{n}{r}}(y)\, dx\, dy. \tag{1.4}$$

We will refer to (1.4) as *Hardy inequality (for the parameters (p, ν))*, by reference to the one dimensional setting $n = r = 1$, where it is true for all $\nu > 0$ and $1 \leq p < \infty$. More comments on Hardy inequalities for holomorphic functions in T_Ω have been done in [13], where a weaker statement was announced (see also [11]).

We remark that (1.4) is always valid when $1 \leq p \leq 2$, as can be proved, for instance, from an explicit formula for F in terms of $\Box F$ involving the fundamental solution of the Box operator (see [13]). However, in this range (1.4) has no implications in terms of boundedness of Bergman projections. We also remark that the converse inequality,

$$\big\|\Delta(\Im m\cdot)\Box F\big\|_{L^p_\nu} \;\leq\; C\, \big\|F\big\|_{L^p_\nu} \tag{1.5}$$

for $F \in \mathcal{H}(T_\Omega)$, is valid for all $0 < p \leq \infty$ and $\nu \in \mathbb{R}$, and is an easy consequence of the mean value inequality for holomorphic functions (see [11]). We will prove Theorem 1.3 in Section 3, and add more comments on Hardy inequalities.

The second equivalent formulation of Conjecture 1 concerns duality.

THEOREM 1.6 *Let $\nu > \frac{n}{r} - 1$ and $1 < p < \infty$. Then P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if the natural mapping of $A^{p'}_\nu$ into $(A^p_\nu)^*$ is an isomorphism.*

We prove a bit more, if $p > \tilde{p}_\nu'$ then the inclusion $\Phi : A^{p'}_\nu \hookrightarrow (A^p_\nu)^*$ is injective, and hence boundedness of P_ν is actually equivalent to surjectivity of Φ. When $p \geq \tilde{p}_\nu$ these two properties fail, and $(A^p_\nu)^*$ is a space strictly larger than $A^{p'}_\nu$ which we do not know how to identify. When $1 \leq p \leq 2$, however, it is always possible to identify $(A^p_\nu)^*$ as a "Besov space" of analytic functions modulo equivalence classes, which we do in section 4. Equivalence classes appear naturally in this setting since the injectivity of Φ (or equivalently of $\Box|_{A^{p'}_\nu}$) fails when $p < \tilde{p}_\nu'$. We do not know whether in this range Φ or $\Box$ may be surjective, a question not considered before to which we will come back later.

In section 4 we develop the theory of analytic Besov spaces. These arise naturally in an attempt to give a meaning to $(A^p_\nu)^*$ or $P_\nu(L^p_\nu)$ for indices p, ν for which the operator P_ν is unbounded (see eg the one dimensional theory in [29]). In addition, their definition is very closely linked with the validity of Hardy inequalities, and for this reason we take up this matter here, leaving to subsequent works the development of further properties. It is remarkable that one can develop most of this theory without making use of the (conceptually more complicated) *real variable* Besov spaces adapted to the cone, which were introduced in [10].

To be more precise, for $\nu \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define

$$\mathbb{B}^p_\nu(T_\Omega) := \{F \in \mathcal{H}(T_\Omega) \;:\; \Delta^k(\Im m\,.)\Box^k F \in L^p_\nu\} \tag{1.7}$$

for a large enough integer $k \geq k_0(p,\nu)$ to be given later. This definition is similar to the one dimensional setting [16], with the role of complex derivative now played by the operator $\Box$. The best choice of the value $k_0(p,\nu)$ is related to the validity of Hardy inequality for $(p, \nu + pk_0)$, since only in this case we can guarantee the equivalence of norms for different k's. Of course, when k can be taken equal to 0 one has $\mathbb{B}^p_\nu = A^p_\nu$, but in general one must deal with equivalence classes modulo holomorphic functions annihilated by $\Box^k$. This is a new (and sometimes disturbing) feature compared to the theory of analytic Besov spaces in bounded symmetric domains developed by K. Zhu [30]. When $p = \infty$, the analytic Besov space $\mathbb{B}^\infty$ is the usual *Bloch space* (see e.g. [4, 5]).

Among our results we shall prove the following. Here $P^{(k)}_\nu(f)$ denotes the equivalence class $P_\nu(f) + \ker \Box^k$ (defined at least for f in the dense set $L^2_\nu \cap L^p_\mu$).

THEOREM 1.8 *Let $\nu > \frac{n}{r} - 1$, $2 \leq p \leq \infty$ and $k \geq k_0(p,\nu)$. Then*

1.- For every real $\mu \leq \nu$, the operator $P^{(k)}_\nu$ extends continuously from L^p_μ onto $\mathbb{B}^p_\mu$.

2.- The dual space $(A^{p'}_\nu)^*$ identifies with $\mathbb{B}^p_\nu$, under the pairing

$$\langle F, G\rangle_{\nu,k} = \int_{T_\Omega} F(z)\, \Delta^k(\Im m\, z)\, \overline{\Box^k G(z)}\, dV_\nu(z), \quad F \in A^{p'}_\nu, \quad G \in \mathbb{B}^p_\nu.$$

These properties are standard in the Bergman space theory of bounded symmetric domains (see eg [29, 30]), as far as one allows to take k sufficiently large. The point here is to find the smallest number of derivatives in (1.7) so that these hold. As mentioned above, this is a non trivial question directly related with Conjecture 1.

We will be more precise about this point: if Conjecture 1 holds, then Theorem 1.3 implies that $\mathbb{B}^p_\nu$ is independent of k (and Theorem 1.8 is true) whenever

$$k + \tfrac{\nu}{p} > \max\left\{ (\tfrac{n}{r} - 1)\tfrac{1}{p},\ (\tfrac{n}{r} - 1)(1 - \tfrac{2}{p}) - \tfrac{1}{p},\ (\tfrac{n}{r} - 1)(\tfrac{1}{2} - \tfrac{1}{p}) \right\}. \tag{1.9}$$

Thus, one can conjecture that (1.9) defines the smallest integer for which the above properties hold. With the presently known results (ie the boundedness of P_ν in the range (1.1)) we are constrained to consider larger integers, namely numbers k so that

$$k + \tfrac{\nu}{p} > \max\left\{ (\tfrac{n}{r} - 1)\tfrac{1}{p},\ (\tfrac{n}{r} - 1)(1 - \tfrac{2}{p}) \right\}, \tag{1.10}$$

which is the same condition as (1.9) only when $1 \leq p \leq 3$ (i.e., when the maximum in (1.10) is attained at the first number, and a bit more than this in the case of light-cones), or when $p = \infty$. We also observe that the best integer k satisfying (1.10) is at most one unit above the optimal integer for (1.9).

Related with this question one can also consider a weaker property than Hardy's inequality (but apparently as difficult); namely

Question: *Given $1 \leq p \leq \infty$ and $\nu \in \mathbb{R}$, find the smallest $\ell = \ell(p,\nu) \in \mathbb{N}$ so that, for all $m \geq 1$,*

$$\inf_{H \in \mathcal{H}(T_\Omega)\ :\ \Box^{\ell+m} H = 0} \left\| \Delta^\ell \Box^\ell (F + H) \right\|_{L^p_\nu} \lesssim \left\| \Delta^{\ell+m} \Box^{\ell+m} F \right\|_{L^p_\nu}, \tag{1.11}$$

for all holomorphic F for which the right hand side is finite.

When $\ell = 0$, this is equivalent to the surjectivity of $\Box^m : A^p_\nu \to A^p_{\nu+mp}$, that is, whether $\Box^m F = G$ may have some solution $F \in A^p_\nu$ when the datum $G \in A^p_{\nu+mp}$.

Hardy's inequality for $(p, \nu + \ell p)$ easily implies (1.11), which hence holds in the range (1.10) (with k replaced by ℓ). However, we do not know whether the converse may be true. In fact, we do not even know whether (1.9) is a necessary condition for (1.11). Below we shall prove that the integer ℓ at least must satisfy

$$\ell + \tfrac{\nu}{p} > \max\left\{ (\tfrac{n}{r} - 1)\tfrac{1}{p},\ (\tfrac{n}{r} - 1)(\tfrac{1}{2} - \tfrac{1}{p})\ \right\}.$$

We remark that these type of necessary conditions had not been considered at all in previous work. For instance, for the Bloch space, one can ask whether there exist functions $F \in \mathbb{B}^\infty$ so that $\|\Delta^j \Box^j \widetilde{F}\|_\infty = \infty$ for all $j \leq \frac{n}{r} - 1$ and all $\widetilde{F} = F \pmod{\operatorname{Ker} \Box^{k_0}}$ where $k_0 = \lceil \frac{n}{r} - 1 \rceil$; in such case k_0 would really be a critical number of equivalence classes. The classical example $F(z) = \ln(z \cdot \mathbf{e} + i) \in \mathbb{B}^\infty$ only has this property for $j = 0$, and it does not seem easy to produce explicit examples with $j \geq 1$. See however Proposition 4.42 below for the existence of such functions with $j \leq (\frac{n}{r} - 1)/2$.

Returning to the complex Besov spaces $\mathbb{B}^p_\nu$, in section 4.4 we present a real variable characterization in terms of “Littlewood-Paley decompositions” of the cone, as described in [10]. Roughly speaking, functions $F \in \mathbb{B}^p_\nu$ have Shilov boundary values $f = \lim_{\substack{y \to 0 \\ y \in \Omega}} F(x+iy)$ which are distributions in $\mathbb{R}^n$, with Fourier transform supported in $\overline{\Omega}$ and satisfying a growth condition

$$\left\{ \Delta^{-\frac{\nu}{p}}(\xi_j)\, ||f * \psi_j||_p \right\} \in \ell^p,$$

for a suitable partition of unity $\{\psi_j\}$ associated with a lattice set $\{\xi_j\}$ of Ω. Conversely, every such distribution can be extended via Fourier-Laplace transform into a holomorphic function in $\mathbb{B}^p_\nu$. This allows in some cases to improve the value of k for which the elements of the Besov space can be identified with equivalence classes modulo holomorphic functions annihilated by $\Box^k$. In addition, we consider the real version of Bloch spaces (which is new), and use this characterization to prove the necessary conditions for (1.11) alluded above.

Finally, we mention the special family of Besov spaces corresponding to the weight $\nu = -n/r$ in (1.7); that is,

$$\mathbb{B}^p \;=\; \left\{\, F \in \mathcal{H}(T_\Omega) \;:\; \Delta^k(\Im m\,.)\Box^k F \in L^p(d\lambda) \,\right\}.$$

Here $d\lambda = \Delta^{-\frac{2n}{r}}(y) dx\, dy$ denotes the invariant measure under conformal transformations of T_Ω. These are the analog for T_Ω of the Besov spaces introduced by Arazy and Yan in bounded symmetric domains [1, 27, 28]. Special properties of these spaces, such as Möbius invariance and characterizations of (small) Hankel operators will be described in subsequent papers [17, 23, 14].

The paper is structured as follows: in section 2 we present some prerequisites about cones and Bergman kernels. In section 3 we prove Theorems 1.3 and 1.6. In section 4 we introduce Besov and Bloch spaces and prove Theorem 1.8. The real analysis characterization is in §§4.4 and 4.5 and the necessary conditions related with (1.11) are in §4.6. Finally, section 5 contains a brief list of open questions which we could not answer in relation with this topic. Besides Conjecture 1, the main problem that we leave open concerns the question in (1.11).

2 Bergman kernels and reproduction formulas

2.1 Some prerequisites

Below we shall use some invariance properties of determinants and Box operators. To introduce them we need to recall some basic facts about symmetric cones (see the text [15]).

Considering $V = \mathbb{R}^n$ as a Jordan algebra, we denote its unit element by $\mathbf{e}$ (think of the identity matrix in the cone of positive definite symmetric matrices, or the point $\mathbf{e} = (1, \mathbf{0})$ in the forward light cone). Let G be the identity component of the group of invertible linear transformations which leave the cone Ω invariant. It is well known that G acts transitively on Ω, which may be identified with the Riemannian symmetric space G/K, where K is the compact subgroup of elements of G which leave $\mathbf{e}$ invariant. The determinant function is also preserved by G, in such a way that

$$\Delta(gy) = \Delta(g\mathbf{e})\Delta(y) = \mathrm{Det}(g)^{\frac{r}{n}}\Delta(y), \quad \forall\, g \in G,\ y \in \Omega. \tag{2.1}$$

It follows from this formula that an invariant measure in Ω is given by $\Delta(y)^{-\frac{n}{r}}\,dy$. The invariance of the Box operator through the action of G is an easy consequence of its definition and the invariance of the determinant function, namely

$$\Box\big[F(g\cdot)\big] \;=\; \Delta(g\mathbf{e})\,\big[\Box F\big](g\cdot) \;=\; \mathrm{Det}(g)^{\frac{r}{n}}\big[\Box F\big](g\cdot), \quad \forall\, g \in G. \tag{2.2}$$

Another fundamental property is the following [15, p. 125]: for every $\alpha \in \mathbb{R}$ one has the identity in Ω

$$\Box\Delta^{\alpha} = b(\alpha)\Delta^{\alpha-1} \tag{2.3}$$

where $b(\alpha)$ vanishes only for the r values $0, \alpha_0, \cdots (r-1)\alpha_0$, where $\alpha_0 = -\frac{\frac{n}{r}-1}{r-1}$. In particular,

$$\Box\Delta^{-\frac{n}{r}+1}(y) = 0, \quad y \in \Omega. \tag{2.4}$$

2.2 Bergman kernels and Determinant function

The (weighted) Bergman projection P_ν is defined by

$$P_\nu F(z) = \int_{T_\Omega} B_\nu(z,w)F(w)dV_\nu(w),$$

where $B_\nu(z,w) = c_\nu\,\Delta^{-(\nu+\frac{n}{r})}((z-\overline{w})/i)$ is the reproducing kernel of A^2_ν, which we shall call Bergman kernel (see [15]). For simplicity, we have written $dV_\nu(w) := \Delta^{\nu-\frac{n}{r}}(v)du\,dv$, where $w = u + iv$ is an element of T_Ω. Observe from (2.3) that

$$\Box_z^m\left[B_\nu(z-\bar{w})\right] \;=\; c_{\nu,m}\,B_{\nu+m}(z-\bar{w}) \tag{2.5}$$

for a suitable constant $c_{\nu,m}$, and all $m \in \mathbb{N}$. We will need integrability properties of the determinants and Bergman kernels, which are given by the next lemma.

Lemma 2.6 *Let α, ν be real and $p > 0$. Then*

1) for $y \in \Omega$, the integral

$$J_\alpha(y) = \int_{\mathbb{R}^n} \left|\Delta^{-\alpha}(x+iy)\right| dx$$

converges if and only if $\alpha > \frac{2n}{r} - 1$. In this case, $J_\alpha(y) = C_\alpha \Delta^{-\alpha+\frac{n}{r}}(y)$, where C_α is a constant depending only on α.

2) For $u \in \Omega$, the integral

$$\int_\Omega \Delta^{-\alpha}(y+u)\Delta^{\nu-\frac{n}{r}}(y)dy$$

converges if and only if $\nu > \frac{n}{r} - 1$ and $\alpha > \nu + \frac{n}{r} - 1$, in which case equals $c_\alpha \Delta^{\nu-\alpha}(u)$.

3) The function $F(z) = \Delta^{-\alpha}(\frac{z+it}{i})$, with $t \in \Omega$, belongs to A^p_ν if and only if

$$\nu > \frac{n}{r} - 1 \quad \textit{and} \quad \alpha > \frac{1}{p}(\nu + \frac{2n}{r} - 1).$$

In this case,

$$||F||_{A^p_\nu} = C_{\alpha,p}\Delta^{-\alpha+(\nu+\frac{n}{r})\frac{1}{p}}(t).$$

We refer to the literature for the proof [9]. It means in particular, using (2.4), that for $p > \tilde{p}_\nu$ the function $F(z) = \Delta^{-\frac{n}{r}+1}(z+i\mathbf{e}) \in A^p_\nu$ and is annihilated by $\Box$; so, there is no Hardy inequality for such values of p. In this range of p, as mentioned in the introduction, the Bergman projection P_ν is not bounded in L^p_ν, so we have proved easily Theorem 1.3 for $p > \tilde{p}_\nu$. We shall concentrate on the other values of p later on.

Let us now recall the following density properties (see eg [11, 18]).

Lemma 2.7 *Let $1 \le p < \infty$ and $\nu > \frac{n}{r} - 1$. Then, for all $1 \le q \le \infty$ and $\mu > \frac{n}{r} - 1$, the subspace $A^p_\nu \cap A^q_\mu$ is dense in A^p_ν. Moreover, $A^\infty \cap A^q_\mu$ is dense in A^∞ for the weak*-(L^∞, L^1) topology.*

Proof: Let us consider the case $p = \infty$, which is the only new part. If $F \in A^\infty$ the functions $\Delta^{-\alpha}((\varepsilon z + i\mathbf{e})/i)F(z)$ are in $A^p_\mu \cap A^\infty$ for large values of α, and we clearly have the required property when ε tends to 0 by Lebesgue dominated convergence theorem. $\Box$

2.3 Integral operators

For the characterizations of Besov spaces, we shall need some integral estimates involving Bergman kernel functions. We consider the following integral operators

$$T_{\nu,\alpha}F(z) = \Delta^\alpha(\Im m\, z)\int_{T_\Omega} B_{\nu+\alpha}(z,w)F(w)dV_\nu(w), \tag{2.8}$$

and

$$T^+_{\nu,\alpha}F(z) = \Delta^\alpha(\Im m\, z)\int_{T_\Omega} |B_{\nu+\alpha}(z,w)|F(w)dV_\nu(w), \tag{2.9}$$

when these integrals make sense. Observe that $P_\nu = T_{\nu,0}$.

Lemma 2.10 *Let $\alpha, \nu, \mu \in \mathbb{R}$ and $1 \leq p < \infty$. Then the following conditions are equivalent:*

(a) The operator $T^+_{\nu,\alpha}$ is well defined and bounded on $L^p_\mu(T_\Omega)$.

(b) The parameters satisfy $\nu + \alpha > \frac{n}{r} - 1$ and the inequalities

$$\nu p - \mu > (\tfrac{n}{r} - 1)\max\{1, p-1\}, \quad \alpha p + \mu > (\tfrac{n}{r} - 1)\max\{1, p-1\}.$$

Proof: This result is implicit in [12]. For a complete proof see [22]. $\square$

In particular, when $\nu = \mu > \frac{n}{r} - 1$ and when $p > (\mu + \frac{n}{r} - 1)/\mu$, the condition is satisfied for α large enough. We remark that, concerning the operators $T_{\nu,\alpha}$, the sufficient conditions for L^p_μ-boundedness contained in the previous lemma are far from necessary. Indeed, we mentioned this in the introduction for the special case of Bergman projections (i.e., $\alpha = 0$ and $\mu = \nu$), where other methods, which could be generalized to other values of parameters, give additional ranges of boundedness (see Remark 4.37 below).

Lemma 2.11 *For $\alpha, \nu \in \mathbb{R}$, with $\nu > \frac{n}{r} - 1$. Then the operator $T_{\nu,\alpha}$ (resp. $T^+_{\nu,\alpha}$) is bounded in L^∞ if and only if $\alpha > \frac{n}{r} - 1$.*

Proof: This follows easily from part 3) of Lemma 2.6 (see details in [22]). Remark that now we can write T instead of T^+, the condition being also necessary for T. $\square$

2.4 Reproducing formulas

We will make an extensive use of the following "*integration by parts*". For $\nu > \frac{n}{r} - 1$, $1 \leq p \leq \infty$ and $F \in A^p_\nu$, $G \in A^{p'}_\nu$, we have the formula

$$\int_{T_\Omega} F(z)\overline{G}(z)dV_\nu(z) = c_{\nu,m}\int_{T_\Omega} F(z)\overline{\Box^m G}(z)\Delta^m(\Im m\, z)dV_\nu(z). \tag{2.12}$$

Indeed, the formula holds for $p = 2$, where it can be obtained using Plancherel and the Paley-Wiener characterization of A^2_ν (see eg [15]). The general case follows by density, using the fact that $\Box^m G(x+iy)\Delta^m(y)$ is also in $L^{p'}_\nu$ by (1.5). We can now write the following general reproducing formula. In the next proposition, we write c for some constant that depends on the parameters involved.

Proposition 2.13 *Let $\nu > \frac{n}{r} - 1$ and $1 \leq p \leq \infty$. For all $F \in A^p_\nu$ we have the formula*

$$\Box^\ell F(z) = c\int_{T_\Omega} B_{\nu+\ell}(z,w)\Box^m F(w)\Delta^m(\Im m\, w)dV_\nu(w) \tag{2.14}$$

for $m \geq 0$ and ℓ large enough so that $B_{\nu+\ell}(z,\cdot)$ is in $L^{p'}_\nu$. In particular, when $1 \leq p < \tilde{p}_\nu$, the formula is valid with $\ell = 0$.

Proof: We can assume that $m = 0$. If not, we use (2.12). It is true for $p = 2$ and $\ell = 0$ because of the reproducing property of the Bergman projection. Derivation under the integral and (2.5) gives also the case $\ell > 0$. We then use density in general. □

Corollary 2.15 *Let $1 \le p < \tilde{p}_\nu$ and $\nu > \frac{n}{r} - 1$. Then every $F \in A^p_\nu$ can be written as*

$$F(z) = \int_{T_\Omega} B_\nu(z,w)F(w)dV_\nu(w). \tag{2.16}$$

We shall state two more results which can be similarly proved by density and absolute convergence of the involved integrals (together with Lemma 2.6 (3) to verify the statements about the Bergman kernels).

Proposition 2.17 *Let $\nu > \frac{n}{r} - 1$ and $\alpha > \frac{n}{r} - 1$. Then $B_{\nu+\alpha}(\cdot, i\mathbf{e}) \in L^1_\nu$, and for all holomorphic F with $\Delta^\alpha(\Im m\, z)F(z) \in L^\infty$ and all $m \ge 0$ we have*

$$F(z) \,=\, c \int_{T_\Omega} B_{\nu+\alpha}(z,w)\, \Box^m F(w)\, \Delta^{\alpha+m}(\Im m\, w)\, dV_\nu(w). \tag{2.18}$$

Proposition 2.19 *Let $\mu, \nu, \alpha \in \mathbb{R}$ and $1 \le p < \infty$ satisfying*

$$\nu + \alpha > \tfrac{n}{r} - 1, \quad \nu p - \mu > (p-1)(\tfrac{n}{r} - 1) \quad \textit{and} \quad \mu + \alpha p > (p-1)(\tfrac{n}{r} - 1) - \tfrac{n}{r}.$$

Then, $\Delta^{\nu-\mu}(\Im m\, z)B_{\nu+\alpha}(z, i\mathbf{e}) \in L^{p'}_\mu$, and for all holomorphic F with $\Delta^\alpha(\Im m\, z)F(z) \in L^p_\mu$ we have

$$F(z) = \int_{T_\Omega} B_{\nu+\alpha}(z,w)\, F(w)\, \Delta^\alpha(\Im m\, w)\, dV_\nu(w). \tag{2.20}$$

3 Proofs of Theorems 1.3 and 1.6

Proof of Theorem 1.3: Let us first assume that P_ν is bounded, which implies in particular that $p < \tilde{p}_\nu$, that is, $B_\nu(z,\cdot)$ is in $A^{p'}_\nu$. Then the formula

$$F(z) = c\int_{T_\Omega} B_\nu(z,w)\Box F(w)\Delta(\Im m\, w)dV_\nu(w)$$

implies that F is the projection of the function $\Box F(w)\Delta(\Im w) \in L^p_\nu$. The Hardy inequality follows from the continuity of the operator.

Next, consider $2 < p < \infty$ and assume that the inequality (1.4) holds. We can restrict to the range $2 < p \le \tilde{p}_\nu$, since for larger values $p > \tilde{p}_\nu$, as we have seen above, the Box operator is not injective in A^p_ν, and hence Hardy's inequality does not hold.

Our proof uses Hardy's inequality, not only for the Box operator, but for its power $\Box^m$, with m large enough. We shall use the following lemma.

Lemma 3.1 *Let $\nu > \frac{n}{r} - 1$ and $2 \le p \le \tilde{p}_\nu$. Then,*

$$\left\|\Box F\right\|_{L^p_{\nu+p}} \le C \left\|\Box^{m+1} F\right\|_{L^p_{\nu+(m+1)p}}, \quad \forall\, F \in A^p_\nu,\ \forall\, m \ge 1. \tag{3.2}$$

PROOF: Using (2.12) we can write

$$\Box F(z) = c \int_{T_\Omega} B_{\nu+p}(z,w) \Box^m \left(\Box F(w)\right) \Delta^m(\Im m\, w) dV_{\nu+p}(w),$$

since $\Box F \in A^p_{\nu+p}$ and $B_{\nu+p}(\cdot, z) \in A^{p'}_{\nu+p}$. So the inequality (3.2) follows from the fact that the projector $P_{\nu+p}$ is bounded on $L^p_{\nu+p}$ (since the condition on p implies $p < \bar{p}_{\nu+p}$). $\Box$

So our assumption that Hardy's inequality (1.4) holds implies that, for all $F \in A^p_\nu$ and all positive integer m, we have the inequality

$$\iint_{T_\Omega} |F(x+iy)|^p \, \Delta^{\nu-\frac{n}{r}}(y)\, dx\, dy \ \leq\ C \ \iint_{T_\Omega} \left|\Delta^m(y) \Box^m F(x+iy)\right|^p \Delta^{\nu-\frac{n}{r}}(y)\, dx\, dy. \quad (3.3)$$

We want to prove the existence of some constant C such that, for $f \in L^p_\nu \cap L^2_\nu$, we have the inequality

$$\|P_\nu f\|_{A^p_\nu} \leq C \|f\|_{L^p_\nu}.$$

Consider such an f with $\|f\|_{L^p_\nu} = 1$. Call $F := P_\nu f$. By Fatou's Lemma, it is sufficient to prove that the functions $F_\varepsilon(z) := F(z + i\varepsilon\mathbf{e})$, which belong to A^p_ν, have norms uniformly bounded. So, using (3.3), it is sufficient to prove that $\Box^m F_\varepsilon$ is uniformly in $L^p_{\nu+pm}$ for some m, which is a consequence of the fact that $\Box^m F$ itself is in $L^p_{\nu+pm}$ for some m (see eg [18, Corol. 3.9]). To prove this, we use the identity

$$\Box^m F(z) = c \int_{T_\Omega} B_{\nu+m}(z,w) f(w) dV_\nu(w),$$

so that $\|\Box^m F\|_{L^p_{\nu+pm}} = c\, \|T_{\nu,m} f\|_{L^p_\nu}$, and if m is sufficient large we conclude from Lemma 2.10. This finishes the proof of Theorem 1.3. $\Box$

PROOF of Theorem 1.6: We first consider the case $\tilde{p}_\nu\,' < p < \infty$, for which the Bergman kernel $B_\nu(\cdot, w)$ belongs to A^p_ν. So, if F is in $A^{p'}_\nu$ and if the associated linear form $\Phi(F)$, given by

$$\langle \Phi(F), G\rangle = \int_{T_\Omega} G(z)\overline{F(z)} dV_\nu(z)$$

vanishes on A^p_ν, Corollary 2.15 implies that $F = 0$. Thus, $A^{p'}_\nu$ is embedded into the dual of A^p_ν. Assume that this embedding is onto, and hence by the closed graph theorem that it has continuous inverse. Since every $f \in L^{p'}_\nu$ defines an element of $(A^p_\nu)^*$ by $G \mapsto \int_{T_\Omega} G(z)\overline{f(z)} dV_\nu(z)$, by assumption there exists $F \in A^{p'}_\nu$ such that

$$\int_{T_\Omega} G(z)\overline{f(z)} dV_\nu(z) = \int_{T_\Omega} G(z)\overline{F(z)} dV_\nu(z), \quad \forall\, G \in A^p_\nu$$

with $\|F\|_{A^{p'}_\nu} \leq c\|f\|_{L^{p'}_\nu}$. Taking for G the Bergman kernel, we see that F is the projection $P_\nu f$, so that $P_\nu f$ maps L^p_ν continuously into itself.

Conversely, assume that P_ν is bounded in L^p_ν (and, by duality, on $L^{p'}_\nu$). Then we have the identity

$$\int_{T_\Omega} G(z)\overline{f(z)}dV_\nu(z) = \int_{T_\Omega} G(z)\overline{P_\nu f(z)}dV_\nu(z)$$

for all $f \in L^{p'}_\nu$ and $G \in A^p_\nu$. Indeed, use the fact that this equality is valid in L^2_ν, and density. Since every functional $\gamma \in (A^p_\nu)^*$ can be expressed by Hahn-Banach as $G \mapsto \langle G, f\rangle_\nu$ for some $f \in L^{p'}_\nu$ (with $\|f\|_{L^{p'}_\nu} = \|\gamma\|$), the above identity shows that the functional can be obtained from $P_\nu f \in A^{p'}_\nu$. So, under the assumption that P_ν is bounded in L^p_ν, the embedding $\Phi : A^{p'}_\nu \to (A^p_\nu)^*$ is an isomorphism.

It remains to consider the case when $1 \leq p \leq \tilde{p}'_\nu$, where we know that the Bergman projection is not bounded, and hence we want to show that Φ is not an isomorphism. First, it is easy to see that Φ is not injective when $1 \leq p < \tilde{p}'_\nu$. Indeed, in that range we may find a (non-null) function $F \in A^{p'}_\nu$ with $\Box F = 0$. Now, it follows from (2.12) that

$$\int_{T_\Omega} G(z)\overline{F}(z)dV_\nu(z) = c\int_{T_\Omega} G(z)\,\overline{\Box F(z)}\Delta(\Im m\, z)dV_\nu(z), \quad G \in A^p_\nu, \tag{3.4}$$

which implies $\Phi(F) \equiv 0$.

Let us now consider the end-point, $p = \tilde{p}'_\nu$. If F is in $A^{p'}_\nu$ then $\Box F$ is in $A^{p'}_{\nu+p'}$ and, by (3.4), the norm of $\Phi(F)$ is bounded by the norm of $\Box F$ in this space. So, if Φ was an isomorphism, we would have some constant C independent of F such that

$$\|F\|_{A^{p'}_\nu} \leq C\|\Box F\|_{A^{p'}_{\nu+p'}}.$$

This is exactly Hardy inequality, which is not valid for $p' = \tilde{p}_\nu$, concluding the proof of the theorem.

$\Box$

The next corollary, which is implicitly contained in the previous proofs, will be used later on.

Corollary 3.5 *Let $\nu > \frac{n}{r} - 1$ and $1 \leq p < \tilde{p}_\nu$, and assume that the Hardy inequality (1.4) holds for (p, ν). Then, for every positive integer m, the mapping $\Box^m : A^p_\nu \to A^p_{\nu+mp}$ is an isomorphism. In particular, for all $G \in A^p_{\nu+mp}$ the equation $\Box^m F = G$ has a unique solution in A^p_ν. Moreover,*

$$\|F\|_{A^p_\nu} \leq C\,\|G\|_{A^p_{\nu+mp}},$$

for some constant $C > 0$.

Proof: When $2 \leq p < \tilde{p}_\nu$, by the assumption and Lemma 3.1 we have the estimate $\|F\|_{A^p_\nu} \leq C\,\|\Box^m F\|_{A^p_{\nu+mp}}$, for all $F \in A^p_\nu$, so we only need to establish the surjectivity of $\Box^m$. Since by assumption and Theorem 1.3 the Bergman projection P_ν is bounded in L^p_ν, given any $G \in A^p_{\nu+mp}$, the function $F = P_\nu(\Delta^m(\Im m\,\cdot)G)$ belongs to A^p_ν. Moreover, by the reproducing formula (2.20) we have

$$\Box^m F(z) = \int B_{\nu+m}(z,w)\, G(w)\, \Delta(\Im m\, w)^m\, dV_\nu(w) \;=\; c\, G(z),$$

which proves the surjectivity.

For $1 \leq p \leq 2$, the stated result is even simpler; injectivity follows from Proposition 2.13 (with $\ell = 0$) and surjectivity from the explicit formula involving the fundamental solution of $\Box$ (see [13, Prop. 3.1]).

$\Box$

4 Besov spaces of holomorphic functions and duality

Throughout this section, given $m \in \mathbb{N}$, we shall denote

$$\mathcal{N}_m := \{F \in \mathcal{H}(T_\Omega) : \Box^m F = 0\}$$

and set

$$\mathcal{H}_m(T_\Omega) = \mathcal{H}(T_\Omega)/\mathcal{N}_m.$$

For simplicity, we use the following notation for the normalized operator Box operator: we write

$$\Delta^m \Box^m F(z) := \Delta^m(\Im m\, z)\Box^m F(z), \quad z \in T_\Omega. \tag{4.1}$$

For convenience, we shall use the same notations for holomorphic functions and for equivalence classes in $\mathcal{H}_m$. Remark that, for $F \in \mathcal{H}_m$, we can speak of the function $\Box^m F$. Sometimes we shall write $\Box_z^{-m} G$ for the class in $\mathcal{H}_m$ of all $F \in \mathcal{H}(T_\Omega)$ with $\Box^m F = G$. When $G \in \mathcal{H}(T_\Omega)$ this class is non-empty by the standard theory of PDEs with constant coefficients (see eg [25]).

4.1 Definition of $\mathbb{B}^p_\mu(T_\Omega)$

Given $\mu \in \mathbb{R}$ and $1 \leq p < \infty$ we wish to define a Besov space $\mathbb{B}^p_\mu(T_\Omega)$ consisting of holomorphic F so that $\Delta^m \Box^m F \in L^p_\mu$ for sufficiently large m. The following proposition clarifies the dependence of such spaces on the parameter m.

Proposition 4.2 *Let $\mu \in \mathbb{R}$ and $1 \leq p < \infty$, and two integers $0 \leq k \leq m$.*

(i) If $\Delta^k \Box^k F$ is in L^p_μ, then $\Delta^m \Box^m F$ is in L^p_μ and $\|\Delta^m \Box^m F\|_{L^p_\mu} \leq C\|\Delta^k \Box^k F\|_{L^p_\mu}$.

(ii) If $\mu + kp > \frac{n}{r} - 1$ and Hardy's inequality (1.4) *holds for* $(p, \nu = \mu + kp)$, *then $\Delta^m \Box^m F \in L^p_\mu$ implies the existence of $\widetilde{F} \in \mathcal{H}(T_\Omega)$ so that $\Box^m \widetilde{F} = \Box^m F$ and $\|\Delta^k \Box^k \widetilde{F}\|_{L^p_\mu} \leq C\|\Delta^m \Box^m F\|_{L^p_\mu}$. Moreover the function $\widetilde{F}$ is uniquely determined modulo $\mathcal{N}_k$.*

Proof: Assertion (i) follows from (1.5). We focus on assertion (ii). The assumption on Hardy's inequality implies that $\Box^{m-k} : A^p_{\mu+kp} \to A^p_{\mu+mp}$ is an isomorphism, by Proposition 3.5. Thus since $\Box^m F \in A^p_{\mu+mp}$, there is a unique $H \in A^p_{\mu+kp}$ with $\Box^{m-k} H = \Box^m F$. Now we take for $\widetilde{F}$ any holomorphic solution of $\Box^k \widetilde{F} = H$.

$\Box$

Given $\mu \in \mathbb{R}$, $1 \le p < \infty$ and $m \in \mathbb{N}$ we define the space

$$\mathbb{B}^{p,(m)}_{\mu} := \big\{ F \in \mathcal{H}_m(T_\Omega) \ : \ \Delta^m \Box^m F \in L^p_\mu \big\}$$

endowed with the norm $\|F\|_{\mathbb{B}^p_\mu} = \|\Delta^m \Box^m F\|_{L^p_\mu}$. Observe that each element of $\mathbb{B}^{p,(m)}_\mu$ is the equivalence class of all analytic solutions of the equation $\Box^m F = g$, for some $g \in A^p_{\mu+mp}$. Thus, the spaces are null when $\mu + mp \le \frac{n}{r} - 1$. By the previous proposition, when $0 \le k \le m$ and $\mu + kp > \frac{n}{r} - 1$, the natural projection

$$\begin{array}{ccc} \mathbb{B}^{p,(k)}_\mu & \longrightarrow & \mathbb{B}^{p,(m)}_\mu \\ F + \mathcal{N}_k & \longmapsto & F + \mathcal{N}_m \end{array} \tag{4.3}$$

is an isomorphism of Banach spaces, provided Hardy's inequality (1.4) holds for the indices $(p, \nu = \mu + pk)$. This leads us to the following definition.

Definition 4.4 *Given $\mu \in \mathbb{R}$ and $1 \le p < \infty$, we define $\mathbb{B}^p_\mu := \mathbb{B}^{p,(k_0)}_\mu$ where $k_0 = k_0(p,\mu)$ is fixed by*

$$k_0(p,\mu) := \min\big\{ k \ge 0 \ : \ \mu + kp > \tfrac{n}{r} - 1 \ \ \textit{and Hardy inequality holds for } (p, \mu + pk) \ \big\}. \tag{4.5}$$

Observe that $\mathbb{B}^p_\mu = A^p_\mu$ if and only if $k_0(p,\mu) = 0$. When $1 \le p \le 2$ we have $k_0(p,\mu) = \min\{k \ge 0 \ : \ \mu + kp > \frac{n}{r} - 1\}$. For $p > 2$, however, the exact value of $k_0(p,\mu)$ depends on Conjecture 1, and we only have the estimate

$$k_1(p,\mu) \le k_0(p,\mu) \le k_2(p,\mu)$$

where

$$\begin{array}{rcl} k_1(p,\mu) & = & \min\big\{ k \ge 0 \ : \ \mu + kp > \frac{n}{r} - 1 \quad \text{and} \quad p < p_{\mu+kp} \big\} \\ k_2(p,\mu) & = & \min\big\{ k \ge 0 \ : \ \mu + kp > \frac{n}{r} - 1 \quad \text{and} \quad p < \bar{p}_{\mu+kp} \big\} \end{array}$$

A simple arithmetic manipulation shows that $k_1 \le k_2 \le k_1 + 1$, and hence $k_0 \in \{k_1, k_1 + 1\}$. Of course, the conjecture should be $k_0(p,\mu) = k_1(p,\mu)$, and hence we are at most one unit above the best possible integer in the definition of $\mathbb{B}^p_\mu$. Observe also that $k_1(p,\mu)$ and $k_2(p,\mu)$ can also be written as

$$k_1 \ = \ \min\Big\{ k \ge 0 \ : \ k + \tfrac{\mu}{p} > \max\big\{ (\tfrac{n}{r} - 1)\tfrac{1}{p}, \ (\tfrac{n}{r} - 1)(1 - \tfrac{2}{p}) - \tfrac{1}{p}, \ (\tfrac{n}{r} - 1)(\tfrac{1}{2} - \tfrac{1}{p}) \ \big\} \Big\},$$

$$k_2 \ = \ \min\Big\{ k \ge 0 \ : \ k + \tfrac{\mu}{p} > \max\big\{ (\tfrac{n}{r} - 1)\tfrac{1}{p}, \ (\tfrac{n}{r} - 1)(1 - \tfrac{2}{p}) \ \big\} \Big\}.$$

Thus, we have $k_0 = k_1 = k_2$ when $1 \le p \le 3$. In the light-cone setting, the improved results about Conjecture 1 mentioned in the introduction imply $k_0 = k_1$ for $1 \le p < 3 + \varepsilon$ for some $\varepsilon = \varepsilon_{\mu,n} > 0$.

In all cases, we can summarize part of the discussion above in the following proposition.

Proposition 4.6 *Let $1 \leq p < \infty$, $\mu \in \mathbb{R}$ and $k \geq k_0(p,\mu)$. Then*

$$\Box^k : \mathbb{B}^p_\mu \to A^p_{\mu+kp}$$

is an isomorphism of Banach spaces. In particular, $\mathbb{B}^p_\mu$ is an isomorphic copy of $A^p_{\mu+k_0p}$, and when $\mu > \frac{n}{r}-1$ then $\mathbb{B}^p_\mu = A^p_\mu$ for all $1 \leq p < \bar{p}_\mu$.

Finally we define separately the special family

$$\mathbb{B}^p := \mathbb{B}^p_{-n/r} \;=\; \left\{ F \in \mathcal{H}(T_\Omega) \;:\; \Delta^k \Box^k F \in L^p(T_\Omega, d\lambda) \right\},$$

where k is sufficiently large and $d\lambda(z) = \Delta^{-\frac{2n}{r}}(\Im m\, z) dV(z)$, that is the invariant measure under conformal transformations of T_Ω. When $n = r = 1$, $\mathbb{B}^p$ is the analog in the upper half plane of the analytic Besov space studied by Arazy-Fisher-Peetre, Zhu and others [2, 3, 29, 21]. These spaces have also been considered in bounded symmetric domains by Yan (for $p = 2$), Arazy and Zhu [28, 1, 30].

Some remarkable properties of $\mathbb{B}^p$, which have been or will be presented elsewhere, are the following:

(i) $\mathbb{B}^p \hookrightarrow \mathbb{B}^q$ when $p \leq q$. This follows from trivial embeddings of Bergman spaces.

(ii) If $\frac{n}{r} \in \mathbb{N}$, then $\mathbb{B}^p$ is Möbius invariant, ie $\|F \circ \Phi\|_{\mathbb{B}^p} = \|F\|_{\mathbb{B}^p}$, for all conformal bijections Φ of T_Ω, at least when $p > 2 - \frac{r}{n}$; see [17]. This property fails to be true when $\frac{n}{r} \notin \mathbb{N}$, and it is unknown whether it may hold for $1 \leq p \leq 2 - \frac{r}{n}$ (except in the one dimensional setting; [2]).

(iii) For b analytic in T_Ω, the *small Hankel operator* $h_b : A^2 \to A^2$ is defined by $h_b(f) = P(b\overline{f})$. If $1 \leq p < \infty$, then h_b belongs to the Schatten class $\mathcal{S}_p$ if and only if $b \in \mathbb{B}^p$. See [23, 14].

4.2 Properties of $\mathbb{B}^p_\mu$: image of the Bergman operator and duality

Let $\nu > \frac{n}{r} - 1$, $1 \leq p < \infty$ and $\mu \in \mathbb{R}$. When m is large we extend the definition of the Bergman projection P_ν to functions $f \in L^p_\mu$, by letting $P^{(m)}_\nu(f)$ be the equivalence class (in $\mathcal{H}_m$) of all holomorphic solutions of

$$\Box^m F \;=\; c_{\nu,m} \int_{T_\Omega} B_{\nu+m}(\cdot, w) f(w)\, dV_\nu(w).$$

The constant $c_{\nu,m}$ is as in (2.5), so that if $f \in L^2_\nu \cap L^p_\mu$ then $P^{(m)}_\nu(f) = P_\nu(f) + \mathcal{N}_m$, and in this sense we say that $P^{(m)}_\nu$ is an extension of the Bergman projection. Observe that $P^{(m)}_\nu$ is well defined and bounded from L^p_μ into $\mathbb{B}^{p,(m)}_\mu$ if and only if $T_{\nu,m}$ is bounded in L^p_μ, and in particular, by Lemma 2.10, when $p\nu - \mu > \max(1, p-1)(\frac{n}{r} - 1)$ and m is sufficiently large. Moreover, it follows from the reproducing formulas that the operator is onto. Indeed, by Proposition 2.19, every $F \in \mathbb{B}^{p,(m)}_\mu$ satisfies

$$\Box^m F(z) \;=\; \int_{T_\Omega} B_{\nu+m}(z, w)\, \Box^m F(w)\, \Delta(\Im m\, w)^m\, dV_\nu(w)$$

provided m is sufficiently large, from which it follows $F = cP_\nu^{(m)}(\Delta^m \Box^m F)$. Therefore we have shown the following result, which partially establishes part 1 of Theorem 1.8.

Proposition 4.7 *Let $\nu > \frac{n}{r} - 1$, $\mu \in \mathbb{R}$ and $1 \le p < \infty$ so that*

$$p\nu - \mu > \max(1, p-1)(\tfrac{n}{r} - 1). \tag{4.8}$$

If m is sufficiently large (depending only on p and μ) then $P_\nu^{(m)}$ maps L^p_μ boundedly onto $\mathbb{B}_\mu^{p,(m)}$.

Remark 4.9 In this proposition it is enough to consider integers m so that $mp + \mu > \max\{1, p-1\}(\frac{n}{r} - 1)$, since in this case $T^+_{\nu,m}$ is bounded in L^p_μ (by Lemma 2.10). The result continues to hold as long as $T_{\nu,m}$ is bounded in L^p_μ, for which we give a better range of m and p in Proposition 4.35 below. Remark that $\Box^k P_\nu^{(m)} = P_\nu^{(m+k)}$. We could as well speak of the projection P_ν from L^p_μ onto $\mathbb{B}^p_\mu$.

Turning to duality one has the following result.

Proposition 4.10 *Let $\mu \in \mathbb{R}$ and $1 < p < \infty$. For any integers $m_1 \ge k_0(p, \mu)$ and $m_2 \ge k_0(p', \mu)$, the dual space $(\mathbb{B}^p_\mu)^*$ identifies with $\mathbb{B}^{p'}_\mu$ under the integral pairing*

$$\langle F, G\rangle_{\mu,m_1,m_2} = \int_{T_\Omega} \Delta^{m_1}\Box^{m_1}F(z)\overline{\Delta^{m_2}\Box^{m_2}G(z)}\, dV_\mu(z), \quad F \in \mathbb{B}^p_\mu, \quad G \in \mathbb{B}^{p'}_\mu. \tag{4.11}$$

Moreover, modulo a multiplicative constant, the pairing $\langle\cdot,\cdot\rangle_{\mu,m_1,m_2}$ is independent of m_1 and m_2 satisfying these inequalities.

Proof: The last statement of the theorem follows from the formula of integration by parts in (2.12). Thus, we can assume in (4.11) that $m_1 = m_2 = m$, for m as large as desired.

If we denote $\Phi_G(F) = \langle F, G\rangle_{\mu,m,m}$, then it is clear that Φ_G defines an element of $(\mathbb{B}^p_\mu)^*$ and that the correspondence $G \in \mathbb{B}^{p'}_\mu \mapsto \Phi_G$ is linear and bounded. To see the injectivity, consider for each $w \in T_\Omega$ the function $F_w = B_{\mu+m}(\cdot - \bar{w})$, which belongs to $\mathbb{B}^p_\mu$ if m is sufficiently large (by Lemma 2.6). Then Proposition 2.17 gives, for every $G \in \mathbb{B}^{p'}_\mu$, the identity

$$\Phi_G(F_w) \;=\; c\int_{T_\Omega} B_{\mu+2m}(z - \bar{w})\,\overline{\Box^m G(z)}\,\Delta^m(\Im m\, z)\, dV_{\mu+m}(z) \;=\; c\,\overline{\Box^m G(w)},$$

(for large m), from which the injectivity follows easily.

To see the surjectivity, consider $\gamma \in (\mathbb{B}^p_\mu)^*$. Using the isomorphism $\Box^m : \mathbb{B}^p_\mu \to A^p_{\mu+mp}$ (in Proposition 4.6) we can define an element $\widetilde{\gamma} \in (A^p_{\mu+mp})^*$ by $\widetilde{\gamma}(H) = \gamma(\Box^{-m}H)$. The functional $\widetilde{\gamma}$ can be extended to $(L^p_{\mu+mp})^*$ by Hahn-Banach, and therefore there exists a function $g \in L^{p'}_\mu$ so that we can write

$$\widetilde{\gamma}(H) = \int H(z)\,\overline{g(z)}\, dV_{\mu+m}(z), \quad H \in A^p_{\mu+mp}.$$

Consequently for every $F \in \mathbb{B}^p_\mu$

$$\gamma(F) = \widetilde{\gamma}(\Box^m F) = \int \Box^m F(z)\, \overline{g(z)}\, dV_{\mu+m}(z).$$

Next, let $G = P^{(m)}_{\mu+m}(g)$ which for large m defines element of $\mathbb{B}^{p'}_\mu$ (by Proposition 4.7). We claim that $\gamma = \Phi_G$. Indeed, when $F \in \mathbb{B}^p_\mu$

$$\begin{aligned}
\langle F, G\rangle_{\mu,m,m} &= \int \Box^m F(z)\, \overline{\Box^m G(z)}\, dV_{\mu+2m}(z) \\
&= c \int \Box^m F(z) \left[\int B_{\mu+2m}(w,z) \overline{g(w)}\, dV_{\mu+m}(w) \right] dV_{\mu+2m}(z) \\
&= c \int \left[\int B_{\mu+2m}(w,z)\, \Box^m F(z)\, \Delta(\Im m\, z)^m\, dV_{\mu+m}(z) \right] \overline{g(w)}\, dV_{\mu+m}(w) \\
\text{(by Proposition 2.19)} \quad &= c \int \Box^m F(z)\, \overline{g(w)}\, dV_{\mu+m}(w) \;=\; c\,\gamma(F),
\end{aligned}$$

where Fubini's theorem is justified by the boundedness of the operator $T^+_{\mu+m,m}$ in $L^{p'}_\mu$ when m is sufficiently large. This establishes the claim, and completes the proof of the proposition. $\Box$

As a special case we obtain the following, which establishes part 2 of Theorem 1.8.

Corollary 4.12 *Let $\nu > \frac{n}{r} - 1$ and $1 < p \leq 2$. Then, $(A^p_\nu)^*$ identifies with $\mathbb{B}^{p'}_\nu$ under the integral pairing*

$$\langle F, G\rangle_{\nu,m} = \int_{T_\Omega} F(z) \overline{\Delta^m \Box^m G(z)}\, dV_\nu(z), \quad F \in A^p_\nu, \quad G \in \mathbb{B}^{p'}_\nu, \tag{4.13}$$

for any integer $m \geq k_0(p', \nu)$.

Proof: Just observe that in this range $k_0(p,\nu) = 0$ and $\mathbb{B}^p_\nu = A^p_\nu$ (see Proposition 4.6). $\Box$

Remark 4.14 We observe that the duality of Bergman spaces is still open for values of p for which the Hardy inequality is not valid; that is, we do not know any (non trivial) description of the spaces $(A^p_\nu)^*$ for $p \geq p_\nu$.

4.3 The Bloch space $\mathbb{B}^\infty(T_\Omega)$

The definition of analytic Besov space and the properties in previous sections extend in an analogous way to the case $p = \infty$, for which $\mathbb{B}^\infty$ is called *Bloch space*. In fact, the Bloch space in T_Ω was already introduced in [4, 5] and shown to be the dual of $A^1(T_\Omega)$. Here we recall these results, together with some new facts about the required number of equivalence classes.

The following inequality is elementary, and can be obtained from the mean value property of holomorphic functions exactly as in [11, Prop. 6.1], so we omit the proof here.

Lemma 4.15 *Let $\nu \in \mathbb{R}$. Then*

$$\left\|\Delta(\Im m\,\cdot)^{\nu+1}\Box F\right\|_{L^\infty} \leq C \left\|\Delta(\Im m\,\cdot)^{\nu} F\right\|_{L^\infty}, \quad \forall\, F \in \mathcal{H}(T_\Omega). \tag{4.16}$$

For every integer m we define a Bloch type space

$$\mathbb{B}^{\infty,(m)} := \left\{F \in \mathcal{H}_m(T_\Omega) \;:\; \Delta^m \Box^m F \in L^\infty\right\},$$

endowed with the norm $\|F\|_{\mathbb{B}^{\infty,(m)}} = \|\Delta^m \Box^m F\|_\infty$. We simply write $\mathbb{B}^\infty(T_\Omega)$ for the space $\mathbb{B}^{\infty,(m)}$ with $m = \lceil \frac{n}{r} - 1 \rceil$, the smallest integer greater than $\frac{n}{r} - 1$. We have the following property:

Proposition 4.17 *For all integers $m \geq k > \frac{n}{r} - 1$, the natural inclusion of $\mathbb{B}^{\infty,(k)}$ into $\mathbb{B}^{\infty,(m)}$ is an isomorphism of Banach spaces.*

Proof: We may assume $m = k + 1$. By Lemma 4.15

$$\left\|\Delta^{k+1}\Box^{k+1} f\right\|_{L^\infty} \leq C \left\|\Delta^k \Box^k f\right\|_{L^\infty}, \quad f \in \mathbb{B}^{\infty,(k)}.$$

We want to prove the converse inequality, which is the analogue of Hardy's inequality for $p = \infty$, that is,

$$\|\Delta^k \Box^k f\|_\infty \leq C \|\Delta^{k+1}\Box^{k+1} f\|_\infty, \tag{4.18}$$

for all $k > \frac{n}{r} - 1$ and all $f \in \mathcal{H}(T_\Omega)$ for which the left hand side is finite. Choosing $\nu > \frac{n}{r} - 1$, we may use Proposition 2.17 to write

$$\Box^k f \;=\; c \int_{T_\Omega} B_{\nu+k}(\cdot - \bar{w}) \Box^{k+1} f(w)\, \Delta^{k+1}(\Im m\, w)\, dV_\nu(w). \tag{4.19}$$

The inequality (4.18) follows from the fact that $\int_{T_\Omega} |B_{\nu+k}(z - \bar{w})| dV_\nu(w) \leq C \Delta^{-k}(\Im m\, z)$ by Lemma 2.6.

This implies the injectivity of the mapping. Let us finally prove that the mapping is onto. Let $f \in \mathcal{H}(T_\Omega)$ be such that $\Delta^{k+1}\Box^{k+1} f$ is bounded. Then the right hand side of (4.19) defines a holomorphic function, which may be written as $\Box^k g$. We prove as before that $\Delta^k \Box^k g$ is bounded. Moreover, $\Box^{k+1} g = \Box^{k+1} f$, which proves the surjectivity of the mapping.

$\Box$

Remark 4.20 Observe that when $k \leq \frac{n}{r} - 1$ the injectivity of $\mathbb{B}^{\infty,(k)} \to \mathbb{B}^{\infty,(m)}$ fails. Indeed, the function $F(z) = \Delta^{k+1-\frac{n}{r}}(z + i\mathbf{e})$ belongs to $\mathbb{B}^{\infty,(k)}$ and is typically not null in $\mathcal{H}_k$. However, F is zero in $\mathcal{H}_m$ for all $m > k$ since, by (2.3) and (2.4), we have $\Box^{k+1} F(z) = c \Box \Delta^{1-\frac{n}{r}}(z + i\mathbf{e}) = 0$.

Remark 4.21 We do not whether for some $k \leq \frac{n}{r} - 1$ the correspondence $\mathbb{B}^{\infty,(k)} \to \mathbb{B}^{\infty,(m)}$ may be surjective. This question can also be formulated as follows: *Is it possible that every element f of $\mathbb{B}^\infty$ possesses a representative g such that $\Delta^k \Box^k g$ is bounded, with $k \leq \frac{n}{r} - 1$?* This is the analogue of the question in (1.11), which we shall answer partially in section 4.6. It seems to us that this problem has not been considered before in the literature.

We now turn to the boundedness of Bergman operators in L^∞. As we did in §4.2, when $\nu > \frac{n}{r} - 1$ we may extend the definition of the Bergman projection P_ν to L^∞ functions by letting $P_\nu^{(m)} f$ be the equivalence class (in $\mathcal{H}_m$) of all holomorphic solutions of

$$\Box^m F \;=\; c_{\nu,m} \int_{T_\Omega} B_{\nu+m}(\cdot - \bar{w}) f(w)\, dV_\nu(w),$$

To do this it suffices to consider $m > \frac{n}{r} - 1$, since by Lemma 2.11 the above integral is always absolutely convergent and moreover

$$\|P_\nu^{(m)} f\|_{\mathbb{B}^{\infty,(m)}} \;=\; \|T_{\nu,m} f\|_\infty \lesssim \|f\|_\infty.$$

Thus $P_\nu^{(m)}$ maps $L^\infty \to \mathbb{B}^\infty$ boundedly. The mapping is surjective, as every $F \in \mathbb{B}^\infty$ satisfies (by Proposition 2.17)

$$\Box^m F(z) \;=\; \int_{T_\Omega} B_{\nu+m}(z,w)\, \Box^m F(w)\, \Delta(\Im m\, w)^m\, dV_\nu(w)$$

and therefore, $F = cP_\nu^{(m)}(f)$ with $f = \Delta^m \Box^m F \in L^\infty$. Hence we have established the following result.

Proposition 4.22 *When $\nu, m > \frac{n}{r} - 1$, the Bergman projection $P_\nu^{(m)}$ maps $L^\infty(T_\Omega)$ continuously onto $\mathbb{B}^\infty$.*

Concerning duality, we recall the identification of the Bloch space with the dual of the Bergman space A^1_ν.

Theorem 4.23 (Békollé, [5]) *Let $\nu, m > \frac{n}{r} - 1$. Then the dual space $(A^1_\nu)^*$ identifies with the Bloch space $\mathbb{B}^\infty$ under the integral pairing*

$$\langle F, G\rangle_{\nu,m} = \int_{T_\Omega} F(z) \overline{\Delta(\Im m\, z)^m \Box^m G(z)} dV_\nu(z), \quad F \in A^1_\nu,\;\; G \in \mathbb{B}^\infty. \tag{4.24}$$

Moreover, the pairing $\langle \cdot, \cdot\rangle_{\nu,m}$ is independent of $m > \frac{n}{r} - 1$.

The proof is entirely analogous to the one presented in Proposition 4.10, so we omit it. Let now $\mu \in \mathbb{R}$. Since $\Box^m : \mathbb{B}^1_\mu \to A^1_{\mu+m}$ is an isomorphism when $\mu + m > \frac{n}{r} - 1$ (by Proposition 4.6), we obtain as a corollary the following duality statement.

Corollary 4.25 *Let $\mu \in \mathbb{R}$ and let m_1, m_2 be two integers such that $\mu + m_1 > \frac{n}{r} - 1$ and $m_2 > \frac{n}{r} - 1$. Then $(\mathbb{B}^1_\mu)^*$ identifies with the Bloch space $\mathbb{B}^\infty$ under the integral pairing*

$$\langle F, G\rangle_{\mu,m_1,m_2} = \int_{T_\Omega} L_{m_1} F(z) \overline{L_{m_2} G(z)} \Delta^{\mu - \frac{n}{r}}(\Im z) dz, \quad F \in \mathbb{B}^1_\mu,\;\; G \in \mathbb{B}^\infty,$$

where $L_m H(z) = \Delta^m(Imz) \Box^m_z H(z)$. Again, the pairing $\langle \cdot, \cdot\rangle_{\mu,m_1,m_2}$ is independent of m_1, m_2 (modulo a multiplicative constant).

4.4 A real analysis characterization of $\mathbb{B}^p_\mu$

We briefly recall the real variable theory of Besov spaces adapted to the cone that was developed in [10].

Following [10, §3], we consider a *lattice* $\{\xi_j\}$ in Ω and a sequence $\{\psi_j\}$ of Schwartz functions in $\mathbb{R}^n$ such that $\widehat{\psi}_j$ is supported in an invariant ball centered at ξ_j and $\sum_j \widehat{\psi}_j = \chi_\Omega$. In particular, the sets Supp $\widehat{\psi}_j$ have the finite intersection property and the norms $\|\psi_j\|_{L^1(\mathbb{R}^n)}$ are uniformly bounded. Below we denote by $\mathcal{S}'_{\partial\Omega}$ the space of tempered distributions with Fourier transform supported in $\partial\Omega$. Observe that $\Box u = 0$ (in $\mathcal{S}'$) implies Supp $\hat{u} \subset \partial\Omega \cup (-\partial\Omega)$.

Definition 4.26 *Given $\nu \in \mathbb{R}$ and $1 \leq p < \infty$, we define*

$$B^p_\nu := \big\{ f \in \mathcal{S}'(\mathbb{R}^n) \;:\; \mathit{Supp}\ \widehat{f} \subset \overline{\Omega} \quad \mathit{and} \quad ||f||_{B^p_\nu} < \infty \big\} / \mathcal{S}'_{\partial\Omega},$$

where the seminorm is given by

$$||f||_{B^p_\nu} := \Big(\sum_j \Delta^{-\nu}(\xi_j)\, ||f * \psi_j||_p^p \Big)^{\frac{1}{p}}.$$

It can be shown that B^p_ν is a Banach space and the definition is independent on the choice of $\{\xi_j, \psi_j\}$ (see [10, §3.2]). In the 1-dimensional setting B^p_ν coincides with the classical homogeneous Besov space $\dot{B}^{-\nu/p}_{p,p}(\mathbb{R})$ (of distributions with spectrum in $[0,\infty)$, modulo polynomials).

In certain cases one can avoid equivalence classes in Definition 4.26, and this will turn into a representation of $\mathbb{B}^p_\nu$ as a holomorphic function space. We denote by $\mathcal{L}g(z) = (g, e^{i(z|.)})$, $z \in T_\Omega$, the *Fourier-Laplace transform* of a distribution g compactly supported in Ω (which defines an analytic function in T_Ω). For convenience, we write Υ for the set of indices (p, ν) such that

$$\nu > -\tfrac{n}{r} \quad \text{and} \quad 1 \leq p < \tilde{p}_\nu, \quad \text{or} \quad \nu = -\tfrac{n}{r} \quad \text{and} \quad p = \tilde{p}_\nu = 1. \tag{4.27}$$

Then, in [10, Lemmas 3.38 and 3.43] the following result is shown*.

Lemma 4.28 *Let $(p,\nu) \in \Upsilon$. Then if $f \in B^p_\nu$*

(i) the series $\sum_j f * \psi_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution $f^\sharp$;

(ii) the series $\sum_j \mathcal{L}(\widehat{f}\widehat{\psi}_j)(z)$ converges uniformly on compact sets to a holomorphic function in T_Ω, denoted $\mathcal{E}(f)(z)$, which satisfies

$$\Delta(\Im m\, z)^{(\nu+\frac{n}{r})/p}\, |\mathcal{E}(f)(z)| \leq C\, \|f\|_{B^p_\nu}, \quad z \in T_\Omega.$$

In addition, the mappings

$$f \in B^p_\nu \longrightarrow f^\sharp \in \mathcal{S}'(\mathbb{R}^n) \quad \mathit{and} \quad f \in B^p_\nu \longrightarrow \mathcal{E}(f) \in \mathcal{H}(T_\Omega)$$

*The results in [10] are stated only for $\nu > 0$, but remain valid as long as $(p, \nu) \in \Upsilon$.

are continuous and injective, and for every $f \in B^p_\nu$ we have

$$\lim_{\substack{y\to 0\\ y\in\Omega}} \mathcal{E}(f)(\cdot + iy) = f^\sharp \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \text{ and in } \|\cdot\|_{B^p_\nu}.$$

From this lemma we can define an isometric copy of B^p_ν (and hence of $\mathbb{B}^p_\nu$) as a holomorphic function space in $\mathcal{H}(T_\Omega)$:

Definition 4.29 *For $(p,\nu) \in \Upsilon$ we define the holomorphic function space*

$$\mathcal{B}^p_\nu := \{F = \mathcal{E}f : f \in B^p_\nu\},$$

endowed with the norm $||F||_{\mathcal{B}^p_\nu} = ||f||_{B^p_\nu}$.

The following properties hold

(a) $\mathcal{B}^p_\nu = A^p_\nu$ when Hardy's inequality holds for (p,ν), and in particular when $\nu > \frac{n}{r} - 1$ and $1 \le p < \bar{p}_\nu$ (see [10, p. 351]).

(b) $A^p_\nu \hookrightarrow \mathcal{B}^p_\nu$ when $\nu > \frac{n}{r} - 1$ and $1 \le p < \tilde{p}_\nu$. The inclusion is strict in the 3-dimensional light-cone when $\nu < 1$ and $p_\nu \le p < \tilde{p}_\nu$.

(c) $\mathcal{B}^2_0 = H^2(T_\Omega)$ (Hardy space). Moreover, $\Big\{\mathcal{B}^2_\nu = \mathcal{L}\Big(L^2(\Omega; \Delta^{-\nu}(\xi)\, d\xi)\Big)\Big\}_{\nu > -1}$ is the family of spaces introduced by Vergne and Rossi in the study of irreducible representations of the group of conformal transformations of T_Ω (see [26] or [15, Ch. XIII]).

(d) If $(p,\nu) \in \Upsilon$ then $\Box : \mathcal{B}^p_\nu \to \mathcal{B}^p_{\nu+p}$ is an isomorphism of Banach spaces. This is inherited from the corresponding property in the scale B^p_ν (see [10, Th. 1.4]).

(e) If $(p,\nu) \in \Upsilon$ then $\mathbb{B}^p_\nu$ can be identified with $\mathcal{B}^p_\nu$, in the sense that every $F \in \mathbb{B}^p_\nu$ has a (unique) representative $\widetilde{F}$ in $\mathcal{B}^p_\nu$, and moreover $\|F\|_{\mathbb{B}^p_\nu} \approx \|\widetilde{F}\|_{\mathcal{B}^p_\nu}$. To show this, let $m = k_0(p,\nu)$ so that $\Box^m F \in A^p_{\nu+mp} = \mathcal{B}^p_{\nu+mp}$ (by (a)). Then use (d) to define the unique $\widetilde{F} \in \mathcal{B}^p_\nu$ such that $\Box^m \widetilde{F} = \Box^m F$.

The assertion in (e) above gives a representation of $\mathbb{B}^p_\nu$ as a holomorphic function space with no equivalence classes involved. For example, when $\nu = -n/r$, the space $\mathbb{B}^1$ can be represented by the holomorphic function space $\mathcal{B}^1_{-n/r}$, even in the one-dimensional setting.

Using the box operator, this procedure can be easily extended to all indices (p,ν) (not necessarily in Υ), to represent $\mathbb{B}^p_\nu$ with less equivalence classes than $k_0(p,\nu)$. Namely, given $\nu \in \mathbb{R}$ and $1 \le p < \infty$, define

$$k_* = k_*(p,\nu) \;=\; \min\big\{k \in \mathbb{N} \;:\; (p, \nu + kp) \in \Upsilon\big\}. \tag{4.30}$$

Observe that $k_*(p,\nu) \le k_0(p,\nu)$, and the inequality is often strict. In fact,

$$k_*(p,\nu) = \min\big\{k \;:\; k + \tfrac{\nu}{p} > (\tfrac{n}{r} - 1)(1 - \tfrac{2}{p}) - \tfrac{1}{p}\big\}$$

(and $k_*(1,\nu) = \min\{k \;:\; k + \nu \ge -\frac{n}{r}\}$). Then we have the following result.

Proposition 4.31 *Let $\nu \in \mathbb{R}$, $1 \leq p < \infty$ and $k_*(p,\nu)$ defined as in* (4.30)*. Then every $F \in \mathbb{B}^p_\nu$ has a unique representative $\widetilde{F}$, modulo $\mathcal{N}_{k_*}$, such that $\Box^{k_*}\widetilde{F} \in \mathcal{B}^p_{\nu+k_*p}$, and moreover $\|F\|_{\mathbb{B}^p_\nu} \approx \|\Box^{k_*}\widetilde{F}\|_{\mathcal{B}^p_{\nu+k_*p}}$. In particular, $\mathbb{B}^p_\nu$ identifies with the space*

$$\{G \in \mathcal{H}_{k_*} \;:\; \Box^{k_*} G \in \mathcal{B}^p_{\nu+k_*p}\}. \tag{4.32}$$

Proof: Combine the fact that $\mathbb{B}^p_{\nu+k_*p}$ identifies with $\mathcal{B}^p_{\nu+k_*p}$ (by property (e) above), with the trivial isomorphism $\Box^{k_*} : \mathbb{B}^p_\nu \to \mathbb{B}^p_{\nu+k_*p}$. $\Box$

We turn now to the identification between the spaces $\mathbb{B}^p_\nu$ and B^p_ν via boundary values, as asserted in the introduction. When $(p,\nu) \in \Upsilon$ the result is immediate from (e) above.

Corollary 4.33 *Let $(p,\nu) \in \Upsilon$. Then*
(i) if $F \in \mathbb{B}^p_\nu$, there exists $\lim_{\substack{y\to 0\\ y\in\Omega}} \widetilde{F}(\cdot + iy) = f$ in B^p_ν (and $\mathcal{S}'$), for some representative $\widetilde{F}$ of F.
(ii) if $f \in B^p_\nu$, there exists (a unique) $F \in \mathbb{B}^p_\nu$ such that $\lim_{\substack{y\to 0\\ y\in\Omega}} F(\cdot + iy) = f$ in B^p_ν.
In either case

$$\tfrac{1}{c}\,\|f\|_{B^p_\nu} \;\leq\; \|F\|_{\mathbb{B}^p_\nu} \;\leq\; c\,\|f\|_{B^p_\nu}.$$

The inverse mapping in (ii) is defined by the operator $f \mapsto F = \mathcal{E}(f)$. For general parameters p and ν, $\mathcal{E}f$ is no longer defined when $f \in B^p_\nu$, but $\mathcal{E}(\Box^{k_*} f)$ is well-defined and belongs to $\mathcal{B}^p_{\nu+k_*p}$. Thus, using Proposition 4.31, we may consider a new operator $\mathtt{E}$ from B^p_ν into $\mathbb{B}^p_\nu$ by

$$\Box^{k_*}\mathtt{E}f := \mathcal{E}(\Box^{k_*} f).$$

It is easily seen that $\mathtt{E} : B^p_\nu \to \mathbb{B}^p_\nu$ is an isomorphism, which commutes with the Box operator

$$\Box^\ell_z \circ \mathtt{E} = \mathtt{E} \circ \Box^\ell_x, \qquad \forall\, \ell \in \mathbb{N}.$$

Moreover, duality can be expressed through this isomorphism. Recall first that (see [10, §3.2])

$$(B^p_\nu)^* = B^{p'}_\nu$$

whenever the definition of the duality pairing is given by

$$[f,g]_\nu := \sum_j \langle f, \Box^{-\nu} g * \psi_j \rangle, \qquad f \in B^p_\nu, \quad g \in B^{p'}_\nu. \tag{4.34}$$

On the right hand side the brackets stand for the action of the distribution f on the conjugate of the given test function, while $\Box^{-\nu}$ is defined on the Fourier side by the multiplication by $\Delta(\xi)^{-\nu}$. Then, the duality result in Proposition 4.10 can also be obtained from the above discussion, since when $F = \mathtt{E}f \in \mathbb{B}^p_\mu$, $G = \mathtt{E}g \in \mathbb{B}^{p'}_\mu$ and m is large we have

$$\langle F, G \rangle_{\mu,m,m} = c_{m,\mu}\,[f,g]_\mu.$$

Finally, using real variable techniques we are able to improve on the results in Proposition 4.7 concerning the range of p and number m for which there is boundedness of $P^{(m)}_\nu$ from L^p_μ into $\mathbb{B}^p_\mu$. Below we consider P_ν as a densely defined operator in $L^p_\mu \cap L^2_\nu$.

PROPOSITION 4.35 *Let $\nu > \frac{n}{r} - 1$, $\mu \in \mathbb{R}$ and $1 \leq p < \infty$ so that*

$$p\nu - \mu > \max\{p-1, 2-p\}\,(\tfrac{n}{r} - 1). \tag{4.36}$$

If $k_* = k_*(p,\mu)$ is as in (4.30), then $\Box^{k_*} \circ P_\nu$ extends as a bounded surjective mapping from L^p_μ onto $\mathcal{B}^p_{\mu+k_*p}$.

REMARK 4.37 As a special case we obtain that, in the range in (4.36), $P^{(k_0)}_\nu$ maps L^p_μ continuously onto $\mathbb{B}^p_\mu$, which in particular establishes part 1 of Theorem 1.8. Equivalently, the operator $T_{\nu,m}$ in §2.3 is bounded in L^p_μ for all $m \geq k_0(p,\mu)$; see the discussion preceeding Proposition 4.7.

When $\mu = -n/r$ the condition (4.36) produces no restriction in p, and we obtain the following.

COROLLARY 4.38 *For all $\nu > \frac{n}{r} - 1$ and $1 \leq p < \infty$, the operator $P^{(k_0)}_\nu$ maps $L^p(T_\Omega, d\lambda)$ onto $\mathbb{B}^p$. Moreover, P_ν extends boundedly from $L^1(d\lambda)$ onto $\mathcal{B}^1$.*

PROOF of Proposition 4.35: The continuity follows from a similar reasoning as in [10, Prop. 4.28], where the case $\mu = \nu$ was proved. For completeness, we sketch here the modifications of the general case. Given $f \in L^p_\mu \cap L^2_\nu$, since $P_\nu f \in A^2_\nu$ we can write it, by the Paley-Wiener theorem, as $P_\nu f = \mathcal{L}g$, for some $g \in L^2(\Omega, \Delta^{-\nu}(\xi)d\xi)$. We must show that $\Box^{k_*} P_\nu f = \mathcal{L}(\Delta^{k_*} g)$ belongs to $\mathcal{B}^p_{\mu+k_*p}$, or equivalently that the inverse Fourier transform of the distribution $\Delta^{k_*} g$ belongs to the real space $B^p_{\mu+k_*p}$. Arguing by duality as in (4.34), this is equivalent to show that for all smooth φ with compact spectrum in Ω

$$\Big| \langle \Delta^{k_*} g, \Delta^{-\mu-k_*p} \widehat{\varphi} \rangle \Big| \leq C \, \|f\|_{L^p_\mu} \, \|\varphi\|_{B^{p'}_{\mu+k_*p}}.$$

By the Paley-Wiener theorem for Bergman spaces (see eg [15, p.260])

$$\begin{aligned} LHS &= \int_\Omega g(\xi)\, \Delta^{-\mu-k_*(p-1)}(\xi)\, \overline{\widehat{\varphi}(\xi)}\, \tfrac{\Delta^\nu(\xi)}{\Delta^\nu(\xi)}\, d\xi \\ &= \iint_{T_\Omega} P_\nu f(w)\, \overline{\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)(w)}\, dV_\nu(w) \\ \text{(since } P^*_\nu = P_\nu) \quad &= \langle f, \mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\rangle_{dV_\nu} \leq \|f\|_{L^p_\mu} \, \|\Delta^{\nu-\mu}\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\|_{L^{p'}_\mu}. \end{aligned}$$

If $p > 1$ the last norm equals

$$\|\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\|_{L^{p'}_{(\nu-\mu)p'+\mu}}.$$

Under the conditions (4.36) we have $A^{p'}_{(\nu-\mu)p'+\mu} = \mathcal{B}^{p'}_{(\nu-\mu)p'+\mu}$, since Hardy's inequality holds for the corresponding indices. Thus,

$$\|\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\|_{A^{p'}_{(\nu-\mu)p'+\mu}} \approx \|\Box^{\nu-\mu-k_*(p-1)}\varphi\|_{B^{p'}_{(\nu-\mu)p'+\mu}} \lesssim \|\varphi\|_{B^{p'}_{\mu+k_*p}},$$

as we wished to prove. When $p=1$ one must use instead

$$\|\Delta^{\nu-\mu}\mathcal{E}(\Box^{\nu-\mu}\varphi)\|_{L^\infty} \lesssim \|\Box^{\nu-\mu}\varphi\|_{B^\infty_{\nu-\mu}} \approx \|\varphi\|_{B^\infty_0}$$

(see Lemma 4.40 below), and conclude again by duality. The surjectivity of $\Box^{k_*}\circ P_\nu$ follows from the surjectivity of the operator $P_\nu^{(m)} : L^p_\mu \to \mathbb{B}^{p,(m)}_\mu$ for large m in Proposition 4.7, since the spaces $\mathcal{B}^p_{\mu+k_*p}$ and $\mathbb{B}^{p,(m)}_\mu$ are related by isomorphisms. $\Box$

4.5 A real variable characterization of $\mathbb{B}^\infty$

For completeness, we give here the real variable characterization of the Bloch space $\mathbb{B}^\infty$, starting with the definition of the distribution spaces B^∞_ν introduced in [10].

Definition 4.39 *For $\nu\in\mathbb{R}$ we let*

$$||f||_{B^\infty_\nu} = \sup_j \Delta(\xi_j)^{-\nu}||f*\psi_j||_\infty, \quad f\in S'(\mathbb{R}^n),$$

and define the space B^∞_ν by

$$B^\infty_\nu := \big\{f\in\mathcal{S}'(\mathbb{R}^n) \ : \ \textit{Supp}\ \widehat{f}\subset\overline{\Omega} \quad \textit{and} \quad ||f||_{B^\infty_\nu}<\infty\big\}/\mathcal{S}'_{\partial\Omega}.$$

The following result is the analogue of Lemma 4.28 for $p=\infty$. The result was not stated in [10], so we sketch the proof for completeness.

Lemma 4.40 *Let $\nu>\frac{n}{r}-1$ and $f\in B^\infty_\nu$. Then*

(i) $\sum_j f*\psi_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution $f^\sharp$;

(ii) $\sum_j \mathcal{L}(\widehat{f}\widehat{\psi_j})(z)$ converges uniformly on compact sets of T_Ω to a holomorphic function $\mathcal{E}(f)(z)$, which satisfies

$$\Delta(\Im m\, z)^\nu\,|\mathcal{E}(f)(z)|\le C\,\|f\|_{B^\infty_\nu}, \quad z\in T_\Omega.$$

Proof: By duality, (i) is equivalent to $\mathcal{S}(\mathbb{R}^n)\hookrightarrow B^1_{-\nu}$, which in view of [10, Prop 3.16] happens if and only if $\nu>\frac{n}{r}-1$. Concerning (ii) and reasoning as in the proof of [10, Prop 3.43], it suffices to see that $\mathcal{F}^{-1}(e^{-(\mathbf{e}|\cdot)}\chi_\Omega)$ belongs to the space $B^1_{-\nu}$. Using the isomorphism $\Box^{2\nu}$ and the identity $\mathcal{B}^1_\nu=A^1_\nu$ this is equivalent to $\mathcal{L}(\Delta^{2\nu}e^{-(\mathbf{e}|\cdot)}\chi_\Omega)(z)=c\Delta(z+i\mathbf{e})^{-2\nu-\frac{n}{r}}\in A^1_\nu$, which by Lemma 2.6 happens if and only if $\nu>\frac{n}{r}-1$. $\Box$

For simplicity we denote $B^\infty=B^\infty_0$, which can be identified with the Bloch space $\mathbb{B}^\infty$ as follows.

Proposition 4.41 *For all $k>\frac{n}{r}-1$, the correspondence*

$$f\in B^\infty \longmapsto \Box_z^{-k}\big[\mathcal{E}(\Box^k f)\big]\in\mathbb{B}^\infty$$

is an isomorphism of Banach spaces.

Proof: Since $\Box^k f \in B_k^\infty$, by the previous lemma the function $G := \mathcal{E}(\Box^k f)$ is holomorphic in T_Ω and $\Delta^k(\Im m\, z)G(z)$ is bounded. Thus the equivalence class of all F such that $\Box_z^k F = G$ belongs to $\mathbb{B}^\infty$, and the correspondence $f \mapsto F + \mathcal{N}_k$ defines a bounded operator from B^∞ to $\mathbb{B}^\infty$.

On the other hand, whenever $\nu > \frac{n}{r} - 1$ and $H := \mathcal{E}(h)$ is in A^1_ν, so that h belongs to B^1_ν, one has

$$\int_{T_\Omega} H(z)\overline{\Box^k F(z)}\Delta^k(\Im z)dV_\nu(z) = [h, f]_\nu.$$

Using the duality identities $\mathbb{B}^\infty = (A^1_\nu)^*$ (with the above pairing) and $B^\infty = (B^1_\nu)^*$ (with the pairing $[\cdot,\cdot]_\nu$), it follows that the mapping $f \mapsto F$ is an isomorphism, like the mapping $h \mapsto H$.

$\Box$

4.6 Minimum number of equivalence classes: partial results

Here we consider the question raised in (1.11). We look first at $p = \infty$ and its equivalent formulation raised in Remark 4.21, namely the surjectivity of the mapping $\mathbb{B}^{\infty,(k)} \to \mathbb{B}^\infty$ for $k \le \frac{n}{r} - 1$. We prove that it cannot happen at least when $k \le (\frac{n}{r} - 1)/2$.

Proposition 4.42 *Let k be a non negative integer. If, for every $F \in \mathbb{B}^\infty$, there exists $\widetilde{F}$ such that $\Delta^k \Box^k \widetilde{F}$ is bounded and $\Box^m \widetilde{F} = \Box^m F$ for some $m > \frac{n}{r} - 1$, then necessarily $k > \frac{1}{2}(\frac{n}{r} - 1)$.*

Proof: Let $m > \frac{n}{r} - 1$. By the open mapping theorem, if this property is valid, the natural mapping of $\mathbb{B}^{\infty,(k)}$ into $\mathbb{B}^{\infty,(m)}$, which is surjective, defines an isomorphism from the quotient space $\mathbb{B}^{\infty,(k)}/\mathcal{N}_m$ onto $\mathbb{B}^{\infty,(m)}$. So there is some constant C such that, for each $F \in \mathbb{B}^{\infty,(m)}$, there exists some G with $\Box^m G = 0$ and

$$\|F + G\|_{\mathbb{B}^{\infty,(k)}} \le C\|F\|_{\mathbb{B}^{\infty,(m)}}.$$

In particular,

$$|\Box^k F(x + i\mathbf{e}) + \Box^k G(x + i\mathbf{e})| \le C\|F\|_{\mathbb{B}^{\infty,(m)}}.$$

Consider now $F = \mathcal{E}f$ with $\widehat{f} \in C_c^\infty(\Omega)$, so that $\|F\|_{\mathbb{B}^{\infty,(m)}} \le C\|f\|_{B^\infty}$. Since $\Box^k F(x + i\mathbf{e})$ is bounded, the same is valid for $\Box^k G(x + i\mathbf{e})$. So we can speak of the Fourier transform of $\Box^k G(x + i\mathbf{e})$, whose support is in the boundary of Ω. Let φ be a smooth function whose Fourier transform is compactly supported in Ω, and consider its scalar product, in the x variable, with the function $\Box^k F(x + i\mathbf{e}) + \Box^k G(x + i\mathbf{e})$. By the support condition on $\hat{\varphi}$ we must have $\langle \Box^k G(x + i\mathbf{e}), \varphi\rangle = 0$. So, the following inequality, valid for all such F, holds

$$\left|\int_{\mathbb{R}^n} \Box^k F(x + i\mathbf{e})\overline{\varphi(x)}dx\right| \le C\|f\|_{B^\infty} \times \|\varphi\|_1.$$

The last inequality can as well be written as

$$\left|\int_{\mathbb{R}^n} f(x)\overline{T\varphi(x)}dx\right| \le C\|f\|_{B^\infty} \times \|\varphi\|_1,$$

where $\widehat{(T\varphi)}(\xi) = \Delta(\xi)^k e^{-(\mathbf{e}|\xi)}\hat{\varphi}(\xi)$. In view of the duality $(B_0^1)^* = B^\infty$, it is easily seen that this implies the inequality

$$\|T\varphi\|_{B_0^1} \leq C\|\varphi\|_1. \tag{4.43}$$

We want to find a contradiction by choosing specific functions φ. Assume that $\varphi := \varphi_t$ may be written as

$$\varphi_t(x) = \sum_{j\in J} r_j(t) a_j e^{i(x|\xi_j)}\eta(x),$$

where J is a finite set of indices, and η is a smooth function whose Fourier transform is supported in a small ball centered at 0, in such a way that the functions ψ_j can be assumed to be equal to 1 on the support of $\hat{\eta}(\cdot - \xi_j)$, for all $j \in J$. Here $r_k(t)$ stands for the Rademacher function and the parameter t varies in $(0,1)$. Integrating in t and using Khintchine's Inequality, we have

$$\int_0^1 \|T\varphi_t\|_{B^1} dt \leq C\int_0^1 \|\varphi_t\|_1 dt \leq C' \left(\sum_{j\in J} |a_j|^2\right)^{1/2} \|\eta\|_1. \tag{4.44}$$

Let us find a minorant for the left hand side of (4.44). For every choice of t, we have

$$\|T\varphi_t\|_{B^1} = \sum_{j\in J} |a_j| \left\|T(e^{i(\cdot|\xi_j)}\eta)\right\|_1.$$

Let us take for granted the existence of some uniform constants $c_1, c_2 > 0$ such that

$$\left\|T(e^{i(\cdot|\xi_j)}\eta)\right\|_1 = \left\|\mathcal{F}^{-1}[\Delta^k e^{-(\mathbf{e}|\cdot)}\hat{\eta}(\cdot - \xi_j)]\right\|_1 \geq \frac{1}{c_1}\,\Delta(\xi_j)^k\, e^{-c_2(\mathbf{e}|\xi_j)}\,\|\eta\|_1. \tag{4.45}$$

Then, (4.44) leads to the existence of some (different) constant C such that

$$\sum_{j\in J} |a_j| \Delta(\xi_j)^k e^{-c_2(\mathbf{e}|\xi_j)} \leq C\left(\sum_{j\in J} |a_j|^2\right)^{1/2}.$$

We choose $a_j = \Delta(\xi_j)^k e^{-c_2(\mathbf{e}|\xi_j)}$ and find that

$$\sum_{j\in J} \Delta(\xi_j)^{2k} e^{-2c_2(\mathbf{e}|\xi_j)} \leq C^2$$

uniformly when J varies among finite sets of indices. This allows to have the same estimate for the sum over all indices j, that is

$$\sum_j \Delta(\xi_j)^{2k} e^{-2c_2(\mathbf{e}|\xi_j)} < \infty.$$

By [10, Prop. 2.13] this sum behaves as the integral

$$\int_\Omega \Delta(\xi)^{2k} e^{-(\mathbf{e}|\xi)} \frac{d\xi}{\Delta(\xi)^{n/r}},$$

which is finite for $2k > \frac{n}{r} - 1$.

It remains to prove our claim (4.45), which we do by using group action as in [10, (3.47)]. Write $\xi_j = g_j\mathbf{e}$ with $g_j = g_j^* \in G$, and let $\chi_j(\xi) = \chi(g_j^{-1}\xi)$ for some $\chi \in C_c^\infty(\Omega)$ with the property that $\chi_j \equiv 1$ in Supp $\hat\eta(\cdot - \xi_j)$, $\forall\ j \in J$ (which we can do by our choice of η). Consider the function γ_j whose Fourier transform is defined by

$$\widehat{\gamma_j}(\xi) := e^{(\mathbf{e}|\xi)}\Delta(\xi)^{-k}\chi_j(\xi),$$

so that we can write

$$e^{i(\cdot|\xi_j)}\eta = \gamma_j * T\big(e^{i(\cdot|\xi_j)}\eta\big), \quad \forall\ j \in J.$$

Thus, it suffices to show that

$$\|\gamma_j\|_1 \leq c_1\Delta(\xi_j)^{-k}e^{c_2(\mathbf{e}|\xi_j)}. \tag{4.46}$$

Now, a change of variables gives

$$\|\gamma_j\|_1 \ = \ \big\|\,\mathcal{F}^{-1}\big[\,e^{(\mathbf{e}|g_j\xi)}\Delta(g_j\xi)^{-k}\chi(\xi)\,\big]\,\big\|_1 \ = \ \Delta(\xi_j)^{-k}\,\big\|\,\mathcal{F}^{-1}\big[\,e^{(\xi_j|\cdot)}\Delta^{-k}\chi\,\big]\,\big\|_1,$$

where in the last equality we have used (2.1) and $g_j^* = g_j$. The L^1-norm on the right hand side can be controlled by a Schwartz norm of $e^{(\xi_j|\cdot)}\Delta^{-k}\chi$, which leads to (4.46) using the fact that $e^{(\xi_j|\xi)} \leq e^{c_2(\xi_j|\mathbf{e})}$ when $\xi \in$ Supp χ (see eg [10, Lemma 2.9]). $\Box$

We consider now the same problem for $\mathbb{B}^p_\nu$, namely the surjectivity of $\mathbb{B}^{p,(k)}_\mu \to \mathbb{B}^{p,(m)}_\mu$ for some $k < k_0(p,\mu)$. Again, this cannot happen at least if k is small.

Proposition 4.47 *Let $\mu \in \mathbb{R}$ and k be a non negative integer. If, for every $F \in \mathbb{B}^p_\mu$, there exists $\widetilde{F}$ such that $\Delta^k\Box^k\widetilde{F} \in L^p_\mu$ and $\Box^m\widetilde{F} = \Box^m F$ for some $m \geq k_0(p,\mu)$, then necessarily*

$$k + \tfrac{\mu}{p} \ > \ \max\Big\{(\tfrac{n}{r} - 1)\tfrac{1}{p}\,,\,(\tfrac{n}{r} - 1)\,(\tfrac{1}{2} - \tfrac{1}{p})\Big\}. \tag{4.48}$$

Proof: We must clearly have $\mu + kp > \frac{n}{r} - 1$, since otherwise $\Box^k\widetilde{F} \in A^p_{\mu+kp} = \{0\}$, which implies $F = 0$ (mod $\mathcal{N}_m$). We may also assume that $k < k_0(p,\mu)$, since otherwise (4.48) is trivial. In particular, we only need to consider $p > 2$.

The proof is similar to Proposition 4.42 with some small changes. Under the condition in the statement, the inclusion $\mathbb{B}^{p,(k)}_\mu/\mathcal{N}_m \to \mathbb{B}^{p,(m)}_\mu$ is an isomorphism of Banach spaces. Hence, for every smooth f with Fourier transform compactly supported in Ω, the function $F = \mathcal{E}(f)$ belongs to $\mathbb{B}^{p,(m)}_\mu$ and there exists some $G \in \mathcal{H}(T_\Omega)$ with $\Box^m G = 0$ so that

$$\|\Delta^k\Box^k(F+G)\|_{L^p_\mu} \lesssim \|\Delta^m\Box^m F\|_{L^p_\mu}. \tag{4.49}$$

As before, $\Box^k G$ is the Fourier-Laplace transform of some distribution supported in $\partial\Omega$. Thus, for all $\widehat{\varphi} \in C_c^\infty(\Omega)$ we have

$$\begin{aligned} \Big|\int_{\mathbb{R}^n}\Box^k F(x+i\mathbf{e})\varphi(-x)\,dx\Big| \ &= \ \big|\Box^k(F+G)(\cdot + i\mathbf{e}) * \varphi(0)\big| \\ &\leq \ \|\varphi\|_{p'}\,\big\|\Box^k(F+G)(\cdot + i\mathbf{e})\big\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{4.50}$$

Since $\mu + kp > \frac{n}{r} - 1$ we have $\left\|\Box^k(F+G)(\cdot + i\mathbf{e})\right\|_{L^p(\mathbb{R}^n)} \lesssim \left\|\Box^k(F+G)\right\|_{A^p_{\mu+kp}(T_\Omega)}$ (see e.g. [10, Prop. 4.3]). By (4.49) and the results in §4.4, this last quantity is controlled by

$$\|\Box^m F\|_{A^p_{\mu+mp}} \lesssim \|\Box^m f\|_{B^p_{\mu+mp}} \approx \|f\|_{B^p_\mu},$$

since $m \geq k_0(p,\mu)$. Thus, going back to (4.50) we see that

$$\left|\int_{\mathbb{R}^n} f(x)\overline{T\varphi(x)}dx\right| \leq C\|f\|_{B^p_\mu} \times \|\varphi\|_{p'},$$

where as before $\widehat{T\varphi}(\xi) = \Delta^k(\xi)e^{-(\mathbf{e}|\xi)}\widehat{\varphi}(\xi)$. The left hand side can be written as a duality bracket $[f, T_\mu\varphi]_\mu$ by letting $\widehat{T_\mu\varphi}(\xi) = \Delta(\xi)^{k+\mu}e^{-(\mathbf{e}|\xi)}\widehat{\varphi}(\xi)$, and hence we conclude that

$$\|T_\mu\varphi\|_{B^{p'}_\mu} \leq C\|\varphi\|_{p'}\,. \tag{4.51}$$

As before, we choose $\varphi := \varphi_t$ with

$$\varphi_t(x) = \sum_{j\in J} r_j(t)a_j e^{i(x|\xi_j)}\eta(x),$$

where J is a finite set of indices and η is a smooth function with Fourier transform supported in a small ball centered at 0 so that ψ_j can be assumed to be equal to 1 on the support of $\hat{\eta}(\cdot - \xi_j)$, for all $j \in J$. Integrating in t and using Khintchine's inequality we find that

$$\int_0^1 \|T_\mu\varphi_t\|^{p'}_{B^{p'}_\mu}\,dt \leq C\int_0^1 \|\varphi_t\|^{p'}_{p'}\,dt \;\leq\; C'\left(\sum |a_j|^2\right)^{p'/2}\|\eta\|^{p'}_{p'}\,, \tag{4.52}$$

while the left hand side equals

$$\sum_{j\in J}\Delta(\xi_j)^{-\mu}|a_j|^{p'}\|T_\mu(e^{i(\cdot|\xi_j)}\eta)\|^{p'}_{p'}.$$

Arguing as in the proof of (4.45) one finds two constants c_1, c_2 such that

$$c_1\|T_\mu(e^{i(\cdot|\xi_j)}\eta)\|_{p'} \geq \Delta(\xi_j)^{k+\mu}e^{-c_2(\mathbf{e}|\xi_j)}\|\eta\|_{p'}.$$

So, (4.52) links to the existence of some constant C such that

$$\sum_{j\in J}|a_j|^{p'}\Delta(\xi_j)^{kp'+\mu p'-\mu}e^{-c_2(\mathbf{e}|\xi_j)} \leq C\left(\sum_{j\in J}|a_j|^2\right)^{p'/2}.$$

By the duality $\ell^r, \ell^{r'}$ with $r = 2/p'$ (since we assume $p > 2$), we conclude that

$$\sum_j \Delta(\xi_j)^{r'(kp'+\mu(p'-1))}e^{-c_3(\mathbf{e}|\xi_j)} < \infty,$$

since its partial sums are uniformly bounded. As in the previous proof, we conclude by a comparison with the corresponding integral, and find the constraint on parameters in (4.48).

□

REMARK 4.53 In the special case $k = 0$ we obtain, for $\nu > \frac{n}{r} - 1$ and $m \geq k_0(p,\nu)$, that a necessary condition for the operator $\Box^m : A^p_\nu \to A^p_{\nu+mp}$ to be surjective is

$$1 \leq p < \frac{2(\nu + \frac{n}{r} - 1)}{\frac{n}{r} - 1} = \tilde{p}_\nu + \frac{\nu - 1}{\frac{n}{r} - 1}. \tag{4.54}$$

When $\nu \leq 1$ (in the three dimensional light-cone), (4.54) is the same necessary condition given in Conjecture 1. When $\nu > 1$, however, it is a weaker condition.

5 Open Questions

In this section we pose some questions left open in this topic, in addition to Conjecture 1. Most questions concern the spaces A^p_ν for $p \geq \tilde{p}_\nu$, about which we know very little.

(I) *Is the operator $\Box^m : A^p_\nu \to A^p_{\nu+mp}$ onto for some $p \geq \tilde{p}_\nu$ and $m \geq k_0(p,\nu)$?*
Equivalently, given a datum $G \in A^p_{\nu+mp}$, does the equation

$$\Box^m F = G$$

have some solution F belonging to the space $A^p_\nu(T_\Omega)$?
From Remark 4.53 we only have a negative answer when $p \geq \tilde{p}_\nu + (\nu - 1)/(\frac{n}{r} - 1)$.

(II) *Is the operator $\Phi : A^{q'}_\nu \to (A^q_\nu)^*$ onto for some $q \leq \tilde{p}'_\nu$?*

This question is equivalent to (I) for $p = q'$, using the duality property $(A^q_\nu)^* = \mathbb{B}^p_\nu$ in Corollary 4.12.

(III) *Is the Box operator injective on A^p_ν when $p = \tilde{p}_\nu$?*

Injectivity holds when $1 \leq p < \tilde{p}_\nu$ (by Proposition 2.13), and fails when $p > \tilde{p}_\nu$ (by the explicit example $\Delta(z + i\mathbf{e})^{-\frac{n}{r}+1}$). We do not have a conjecture for the endpoint $p = \tilde{p}_\nu$.

(IV) *Is the mapping $\Phi\colon A^{q'}_\nu \to (A^q_\nu)^*$ injective when $q = \tilde{p}'_\nu$?*

This is equivalent to (III). In fact, from (3.4) it easily seen that Ker $\Phi|_{A^{\tilde{p}_\nu}_\nu} = \mathrm{Ker}\Box|_{A^{\tilde{p}_\nu}_\nu}$.

Our next question stresses further the differences between the spaces A^p_ν, depending on whether $p < \tilde{p}_\nu$ are $p \geq \tilde{p}_\nu$:

(V) *Is the space A^p_ν isomorphic to ℓ^p for some $p \geq \tilde{p}_\nu$?*

Recall here that the Bergman spaces A^p_ν are isomorphic to ℓ^p in the one dimensional setting. This can be proved as a consequence of the atomic decomposition (see [21]). In [7], atomic decompositions for A^p_ν are derived when Hardy's inequality holds, and they will be developed by the last author in a forthcoming paper also for the spaces $\mathbb{B}^p_\nu$. We do not know whether A^p_ν may be isomorphic to ℓ^p, or even to $\mathbb{B}^p_\nu$, when both spaces do not coincide, that is, when Hardy's inequality does not hold.

(VI) *Is it span* $\{B_\mu(\cdot, w) \;:\; w \in T_\Omega\}$ *dense in* $A^p_\nu(T_\Omega)$ *for* $p \geq \tilde{p}_\nu$ *and* μ *sufficiently large?*

The validity of this result was wrongly stated in [11, Corollary 5.4] in the light-cone setting. As we show below (see also [11, Lemma 5.1]), the density holds when the projection P_μ is bounded in L^p_ν, but this restricts p to be smaller than $\tilde{p}_\nu$ (since $P^*_\mu = T_{\nu,\mu-\nu}$ must also be bounded in $L^{p'}_\nu$).

Proposition 5.1 *Let* $\nu > \frac{n}{r} - 1$. *Assume that* p *and* μ *are so that* P_μ *extends as a bounded operator in* L^p_ν. *Then* A^p_ν *is the closed linear span of the set* $\{B_\mu(., w), w \in T_\Omega\}$.

Proof: The boundedness of P_μ in L^p_ν already implies that $B_\mu(\cdot, i\mathbf{e}) \in A^p_\nu$. We take for granted the fact that $P^*_\mu = T_{\nu,\,\mu-\nu}$ (with respect to $\langle \cdot, \cdot \rangle_{dV_\nu}$). To establish the proposition it suffices to prove that, for $f \in L^{p'}_\nu$ such that

$$\langle f, B_\mu(\cdot, w)\rangle_\nu = 0, \quad \forall\, w \in T_\Omega, \tag{5.2}$$

we have also $\langle f, F\rangle_\nu = 0$ for all F in a dense subset of A^p_ν. Now (5.2) is the same as $T_{\nu,\,\mu-\nu}(f)(w) = 0$, by definition of this operator. Thus, if $F \in A^p_\nu \cap A^2_\mu$, using the claim above we have

$$\langle f, F\rangle_\nu = \langle f, P_\mu F\rangle_\nu = \langle P^*_\mu(f), F\rangle_\nu = 0.$$

Finally, we establish the claim, that is $P^*_\mu = T_{\nu,\,\mu-\nu}$. For $f, g \in C_c(T_\Omega)$ we have to justify the exchange of order of integration in

$$\begin{aligned} \langle P_\mu(g), f\rangle_\nu &= \int_{T_\Omega} \Big[\int_{T_\Omega} B_\mu(z,w) g(w) dV_\mu(w)\Big] \overline{f(z)}\, dV_\nu(z) \\ &= \int_{T_\Omega} g(w) \Big[\int_{T_\Omega} \overline{B_\mu(w,z) f(z)} dV_\nu(z)\Big]\, dV_\mu(w) \;=\; \langle g, T_{\nu,\,\mu-\nu} f\rangle_\nu. \end{aligned}$$

but this follows from

$$\int_{T_\Omega}\int_{T_\Omega} |B_\mu(z,w)||g(w)|\, dV_\mu(w) |f(z)|\, dV_\nu(z) \leq \big\| T^+_{\mu,0}|g| \big\|_{L^2_\mu} \big\| \Delta^{\nu-\mu}|f| \big\|_{L^2_\mu} < \infty,$$

using the fact that the operator $T^+_{\mu,0}$ with kernel $|B_\mu(z,w)|$ is bounded on L^2_μ. □

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