A WEAK 2-WEIGHT PROBLEM FOR THE POISSON-HERMITE SEMIGROUP

G. GARRIGÓS

ABSTRACT. This survey is a slightly extended version of the lecture given by the author at the *VI International Course of Mathematical Analysis in Andalucía* (CIDAMA), in September 2014. Most results form part of the paper [3], written jointly with S. Hartzstein, T. Signes, J.L. Torrea and B. Viviani.

1. INTRODUCTION

Consider the following integral identity

$$e^{-t\sqrt{L}} = \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-\frac{t^2}{4s}}\, e^{-sL}\, \frac{ds}{s^{3/2}}, \quad t > 0 \tag{1.1}$$

valid for all real numbers $L > 0$. If we allow L be the infinitesimal generator of a "heat" semigroup $\{e^{-sL}\}_{s>0}$ in $L^2(\mathbb{R}^d)$, then (1.1) defines, using the terminology in Stein's book [8, Chapter II.2], a *subordinated Poisson semigroup*. Moreover, for suitably "good" functions $f : \mathbb{R}^d \to \mathbb{C}$, the formal *Poisson integral* $u(t,\cdot) = e^{-t\sqrt{L}} f$ solves the partial differential equation

$$u_{tt} = Lu, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \quad \text{with } u(0) = f.$$

A relevant question is then to find, for each operator L, the most general class of functions f for which the Poisson integrals $u(t,x) = e^{-t\sqrt{L}} f(x)$ satisfy

(i) $u(t,x)$ is well-defined and belongs to $C^\infty((0,\infty) \times \mathbb{R}^d)$
(ii) $u(t,x)$ satisfies the pde $u_{tt} = Lu$ in $(0,\infty) \times \mathbb{R}^d$
(iii) There exists $\lim_{t\to 0^+} u(t,x) = f(x)$, for a.e. $x \in \mathbb{R}^d$.

In the classical setting, corresponding to the Laplace operator $L = -\Delta$ in $\mathbb{R}^d$, the largest class of admissible initial data f is the weighted space

$$L^1(\varphi) = \Big\{ f \ : \ \int_{\mathbb{R}^d} |f(x)| \varphi(x)\, dx < \infty \Big\}, \tag{1.2}$$

with $\varphi(x) = (1 + |x|)^{-(d+1)}$, and the assertions (i)-(iii) can easily be proved from the explicit form of the Poisson kernel.

Date: June 21, 2015.
1991 *Mathematics Subject Classification.* 42C10, 35C15, 33C45, 40A10.
Key words and phrases. Hermite semigroup, Poisson integral, weighted inequalities, fractional laplacian.
Author partially supported by grants MTM2010-16518, MTM2011-25377 and MTM2013-40945-P (Spain).

For general operators L, however, the kernel will not be so explicit, and investigating such results requires very precise estimates of the subordinated integrals in (1.1), as well as of the associated maximal operators.

In this work we take up this question for a collection of Hermite operators in $\mathbb{R}^d$

$$L = -\Delta + |x|^2 + m, \quad \text{with } m \geq -d. \tag{1.3}$$

We shall also consider a slightly more general family of partial differential equations:

$$u_{tt} + \tfrac{1-2\nu}{t}\, u_t = Lu, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \quad \text{with } \nu > 0. \tag{1.4}$$

The parameters m and ν allow us to include various interesting cases, which can all be covered with essentially the same estimates. In particular, $m = 0$ corresponds to the usual Hermite operator, while $m = -d$ leads to an L which can be transformed* into the Ornstein-Uhlenbeck operator $-\Delta + 2x \cdot \nabla$. Likewise, the parameter $\nu = 1/2$ in (1.4) gives the usual Poisson equation, while for general ν it leads to a pde appearing in the theory of *fractional laplacians*†.

Our goal in this work is to give the most general conditions on a function $f : \mathbb{R}^d \to \mathbb{C}$ so that a meaningful solution to (1.4) is given by the *Poisson-like integral*

$$P_t f(x) := \tfrac{t^{2\nu}}{4^\nu \Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4u}} \big[e^{-uL} f\big](x) \, \frac{du}{u^{1+\nu}}, \quad t > 0. \tag{1.5}$$

This subordinated integral is slightly more general than (1.1), and we justify its expression in §2 below. In our results, which we are about to state, the following function will play a crucial role

$$\varphi(y) = \begin{cases} \dfrac{e^{-|y|^2/2}}{(1+|y|)^{\frac{d+m}{2}} [\ln(e+|y|)]^{1+\nu}} & \text{if } m > -d \\[2ex] \dfrac{e^{-|y|^2/2}}{[\ln(e+|y|)]^{\nu}} & \text{if } m = -d \end{cases} \tag{1.6}$$

Theorem 1.1. *For every $f \in L^1(\varphi)$ the function $u(t,x) = P_t f(x)$ in* (1.5) *is defined by an absolutely convergent integral such that*

(i) $u(t,x) \in C^\infty((0,\infty) \times \mathbb{R}^d)$
(ii) $u(t,x)$ *satisfies the pde* (1.4)
(iii) *For a.e.* $x \in \mathbb{R}^d$, *it holds* $\lim_{t \to 0^+} u(t,x) = f(x)$.

Conversely, if a function $f \geq 0$ is such that the integral in (1.5) *is finite for some $(t,x) \in (0,\infty) \times \mathbb{R}^d$, then f must necessarily belong to $L^1(\varphi)$.*

---

*Note that $e^{|x|^2/2} L[e^{-|x|^2/2} u] = -\Delta u + 2x \cdot \nabla u + (m+d)u$.

†The fractional operator L^ν can be recovered from (1.4) and $u(0,x) = f(x)$ by the formula $L^\nu f(x) = c_\nu \lim_{t \to 0} t^{1-2\nu} u_t(t,x)$, at least for suitably good f; see [9, Thm 1.1].

In particular, a function with growth as large as $f(y) = e^{\frac{|y|^2}{2}}/[(1+|y|)^d \ln(e+|y|)]$ has nicely convergent Poisson integrals, for all $m \geq -d$ and $\nu > 0$. This is in contrast with the classical case $L = -\Delta$ for which only a mild sublinear growth is allowed; see (1.2). It also illustrates that $L^1(\varphi)$ is strictly larger than the "gaussian" space $L^1(\mathbb{R}^d, e^{-\frac{|y|^2}{2}} dy)$, which was the natural domain for Poisson integrals considered by Muckenhoupt in [6] (in the special case $\nu = 1/2$, $m = -d$ and $d = 1$).

Our second goal is to investigate the following local maximal operators

$$P_a^* f(x) := \sup_{0<t<a} \left|P_t f(x)\right|, \quad \text{with } a > 0 \text{ fixed.} \tag{1.7}$$

These operators arise naturally in the *a.e.*-pointwise convergence of $P_t f(x) \to f(x)$ as $t \to 0$. In fact, the natural strategy to prove such convergence for all f in a Banach space $\mathbb{X}$, is to establish first the result in a dense class, and next prove the boundedness of P_a^* from $\mathbb{X}$ into $L^{p,\infty}(v)$ (or even better into $L^p(v)$) for some weight $v > 0$. It turns out that we can prove Theorem 1.1 without appeal to such maximal operators, but it still makes sense to consider the following

Problem 1. A weak 2-weight problem for the operator P_a^*. *Given $a > 0$ and $1 < p < \infty$, characterize the weights $w(x) > 0$ for which there exists some other weight $v(x) > 0$ such that*

$$P_a^* : L^p(w) \to L^p(v) \quad \textit{boundedly.} \tag{1.8}$$

We named the problem "weak" in contrast with the "strong" (and more difficult) question of characterizing all pairs of weights (w, v) for which (1.8) holds. Such weak 2-weight problems, for various classical operators, were considered in the early 80s by Rubio de Francia [7] and Carleson and Jones [1], who found explicit answers for the Hardy-Littlewood maximal operator and the Hilbert transform.

Our second main result in [3] gives an answer to Problem 1.

Theorem 1.2. *Let $1 < p < \infty$ and $a > 0$ be fixed. Then, for a weight $w(x) > 0$ the condition*

$$\left\|w^{-\frac{1}{p}}\, \varphi\right\|_{L^{p'}(\mathbb{R}^d)} < \infty \tag{1.9}$$

is equivalent to the existence of some other weight $v(x) > 0$ such that (1.8) *holds.*

Condition (1.9) is easily seen to be equivalent to $L^p(w) \subset L^1(\varphi)$. So, the necessity of (1.9) in Theorem 1.2 is a consequence of the last sentence in Theorem 1.1. Concerning the sufficiency of (1.9), we first point out that, from Theorem 1.1 (iii) and abstract results due to Nikishin, there always exists a weight $u(x) > 0$ such that

$$P_a^* : L^p(w) \to L^{p,\infty}(u) \quad \text{boundedly.} \tag{1.10}$$

The main contribution of Theorem 1.2 is to show that the weak-space $L^{p,\infty}(u)$ in (1.10) can be replaced by the strong space $L^p(v)$ (with perhaps another weight v). This is the main difficulty in the 2-weight Problem 1 described above, and requires additional estimates to those needed in Theorem 1.1.

A last question regards the explicit form of the weight $v(x)$, whose existence, under the condition (1.9), is asserted in Theorem 1.2. In [3] we used a non-constructive procedure which nevertheless provided a size estimate. Namely, for every $\sigma < 1$ a weight $v = v_\sigma$ can be chosen such that

$$\big\| v^{-\frac{\sigma}{p}}\,\varphi \big\|_{L^{p'}(\mathbb{R}^d)} < \infty. \tag{1.11}$$

Notice that this is "almost" the same integrability condition that $w(x)$ satisfies. Here we state a new result, which provides an explicit expression for $v(x)$, and recovers in particular the property (1.11). We shall use the following *local Hardy-Littlewood maximal operator*

$$\mathcal{M}^{\rm loc} f(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f(y)|\, \chi_{\{|y|\leq 3\max(|x|,1)\}}\, dy. \tag{1.12}$$

Theorem 1.3. *Let $1 < p < \infty$ be fixed, and let $w(x) > 0$ be a weight satisfying* (1.9)*. Then a family of weights $v(x)$ such that* (1.8) *holds for all $a > 0$ is given explicitly by*

$$v(x) = \Big[\mathcal{M}^{\rm loc}\big(w^{-\frac{p'}{p}}\, e^{-\frac{p'|y|^2}{2}}\big)(x)\Big]^{-\frac{\alpha}{p'/p}}\, e^{-\frac{p|x|^2}{2}}\,(1+|x|)^{-N}, \tag{1.13}$$

provided $\alpha > 1$ and $N > N_0$, for some $N_0 = N_0(\alpha, p, d, m, \nu)$.

We finally remark that, via the elementary identity

$$e^{|x|^2/2} L[e^{-|\cdot|^2/2}u] \;=\; -\Delta u + 2x\cdot\nabla u + (m+d)u \;=:\; \mathcal{O},$$

all the results in this paper admit corresponding statements with L replaced by the Ornstein-Uhlenbek type operator $\mathcal{O}$. These essentially amount to replace the exponentials $e^{-|y|^2/2}$ (as in (1.6) or (1.13)) by the gaussians $e^{-|y|^2}$. We leave the simple verification to the interested reader.

The proof of Theorems 1.1 and 1.2 was given in [3], but we outline the main steps below. Namely, in §2 we justify why the integral formula in (1.5) gives a solution to the pde (1.4). In §3 we state the optimal kernel estimates which are behind these theorems, and outline the proof of Theorem 1.1, slightly modified with respect to [3]. The new results appear in §4, where we solve a weak 2-weight problem for $\mathcal{M}^{\rm loc}$, and present the proof of Theorem 1.3 (which in turn implies Theorem 1.2).

2. The subordinated integral

For $\nu \in \mathbb{R}$, consider the following real-valued function

$$F_\nu(z) := \int_0^\infty e^{-u-\frac{z^2}{4u}}\, u^{\nu-1}\, du, \quad z>0, \tag{2.1}$$

where the integral is absolutely convergent (actually for all $\nu \in \mathbb{C}$ and $\Re e\,(z^2)>0$). This integral is well-known in the theory of special functions, as it appears in the definition of the so-called modified Bessel function of the third kind $K_\nu(z)$. Namely, they are related by

$$F_\nu(z) = 2\,(z/2)^\nu\, K_\nu(z); \tag{2.2}$$

see e.g. [11, p. 183] or [4, p. 119]. In particular, F_ν satisfies the ordinary differential equation

$$F_\nu''(z) + \frac{1-2\nu}{z} F_\nu'(z) \;=\; F_\nu(z). \tag{2.3}$$

We give next the elementary proof of (2.3), which does not depend on the properties of K_ν. Integrating by parts in (2.1) we can write

$$F_\nu(z) = \int_0^\infty e^{-u}\Big(e^{-\frac{z^2}{4u}}\, u^{\nu-1}\Big)'\, du \;=\; \int_0^\infty e^{-u}\big(\tfrac{z^2}{4u^2}+\tfrac{\nu-1}{u}\big)\, e^{-\frac{z^2}{4u}}\, u^{\nu-1}\, du.$$

Taking derivatives inside the integral in (2.1) we also have

$$F_\nu'(z) = \int_0^\infty e^{-u-\frac{z^2}{4u}}\,\big(-\tfrac{z}{2u}\big)\, u^{\nu-1}\, du \quad \text{and} \quad F_\nu''(z) = \int_0^\infty e^{-u-\frac{z^2}{4u}}\,\big(\tfrac{z^2}{4u^2}-\tfrac{1}{2u}\big)\, u^{\nu-1}\, du.$$

From these identities (2.3) follows easily. Moreover we have the following

Lemma 2.1. *Let ν and L be positive real numbers. Then, the function $u(t) = \frac{1}{\Gamma(\nu)} F_\nu(t\sqrt{L})$, $t>0$, satisfies the differential equation*

$$u''(t) + \frac{1-2\nu}{t}\, u'(t) \;=\; Lu(t), \quad \textit{with} \ \lim_{t\to 0^+} u(t) = 1. \tag{2.4}$$

Moreover, the function $u(t)$ can also be written as

$$u(t) = \frac{(t/2)^\nu}{\Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4v}-Lv}\, \frac{dv}{v^{1+\nu}}, \quad t>0. \tag{2.5}$$

Proof: If $\nu>0$, from (2.1) and dominated convergence it follows that

$$\lim_{z\to 0^+} F_\nu(z) = \int_0^\infty e^{-u}u^{\nu-1}\, du \;=\; \Gamma(\nu).$$

It is then straightforward to derive (2.4) from this observation and (2.3). To obtain the integral expression in (2.5), first set $z^2 = t^2 L$ in (2.1), and then change variables $v = \frac{t^2}{4u}$. □

When L is a positive self-adjoint differential operator which generates a semigroup $\{e^{-sL}\}_{s>0}$ in $L^2(\mathbb{R}^d)$, we may then consider the function

$$u(t,x) = \tfrac{(t/2)^{2\nu}}{\Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4v}}\, \big[e^{-vL} f\big](x)\, \frac{dv}{v^{1+\nu}}, \quad t>0.$$

In view of (2.4), this is a natural candidate to solve the pde

$$u_{tt} \,+\, \tfrac{1-2\nu}{t}\, u_t \;=\; Lu, \quad (t,x) \in (0,\infty)\times\mathbb{R}^d, \quad \text{with } u(0,\cdot) = f$$

(and coincides with the definition we used in (1.5) for the Poisson integral $P_t f(x)$ associated with L). Theorem 1.1 will give a rigorous proof of this formal statement, at least for the Hermite operators in (1.3). We refer to [9] for more on this kind of arguments for general operators L.

3. Estimates on the Poisson kernels

Suppose L is the infinitesimal generator of a semigroup of operators $\{h_t = e^{-tL}\}_{t>0}$, in $L^2(\mathbb{R}^d)$, and that these are described by the integrals

$$h_t f(x) = \int_{\mathbb{R}^d} h_t(x,y) f(y)\, dy, \tag{3.1}$$

for suitable positive kernels $h_t(x,y)$. Then, for the family of subordinated operators $\{P_t\}_{t>0}$ defined in (1.5), a formal computation leads to

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y)\, dy,$$

with the corresponding kernels given by the integrals

$$p_t(x,y) = \tfrac{(t/2)^{2\nu}}{\Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4v}}\, h_v(x,y)\, \frac{dv}{v^{1+\nu}}. \tag{3.2}$$

If one is interested in *optimal* estimates for such kernels $p_t(x,y)$, two things become necessary: first, a precise *a priori* knowledge of $h_v(x,y)$, and next a careful analysis of the integrals (3.2).

Such tasks are difficult to carry in full generality, so in this work we have considered the special case of the Hermite operators $L = -\Delta + |x|^2 + m$, for which we can start with an *explicit* expression of the associated heat kernels

$$h_v(x,y) = e^{-vm}\, [2\pi \mathrm{sh}\, 2v]^{-\frac{d}{2}}\, e^{-\frac{|x-y|^2}{2\mathrm{th}\, 2v} - \mathrm{th}\, v\, x\cdot y}, \quad v>0; \tag{3.3}$$

see e.g. [10, (4.3.14)][‡]. To make this expression more manageable it is common to use the new variable $s = \mathrm{th}\,(v)$ (or equivalently $v = \frac{1}{2}\log(\frac{1+s}{1-s})$), which after elementary computations allows us to write

$$h_v(x,y) = \frac{(1-s)^{\frac{m+d}{2}}}{(1+s)^{\frac{m-d}{2}}}\, \frac{e^{-\frac{1}{4}(\frac{|x-y|^2}{s} + s|x+y|^2)}}{(4\pi s)^{\frac{d}{2}}}.$$

Inserting this into the integral (3.2) (with $dv = \frac{ds}{1-s^2}$) one obtains the expression

$$p_t(x,y) = \tfrac{(t/2)^{2\nu}}{(4\pi)^{\frac{d}{2}}\Gamma(\nu)} \int_0^1 \frac{e^{-\frac{t^2}{2\ln\frac{1+s}{1-s}}}\, (1-s)^{\frac{m+d}{2}-1}\, e^{-\frac{1}{4}(\frac{|x-y|^2}{s} + s|x+y|^2)}}{s^{\frac{d}{2}}\, (1+s)^{\frac{m-d}{2}+1}\, \big(\frac{1}{2}\ln\frac{1+s}{1-s}\big)^{1+\nu}}\, ds, \tag{3.4}$$

This will be our starting formula for $p_t(x,y)$, from which we shall derive the necessary estimates needed for Theorems 1.1, 1.2 and 1.3. These are summarized in the next two lemmas.

[‡]Note that $e^{-vL} = e^{-v(-\Delta+|x|^2)}e^{-vm}$, and the case $m=0$ corresponds to the usual Mehler kernel.

The first one gives, for fixed t and x, the *optimal* decay of $y \mapsto p_t(x,y)$ in terms of the function $\varphi(y)$ defined in (1.6). We shall sketch its proof in §3.2 below.

Lemma 3.1. *Given $t > 0$ and $x \in \mathbb{R}^d$, there exist $c_1(t,x) > 0$ and $c_2(t,x) > 0$ such that*

$$c_1(t,x)\,\varphi(y) \;\leq\; p_t(x,y) \;\leq\; c_2(t,x)\,\varphi(y)\;, \quad \forall\, y \in \mathbb{R}^d. \tag{3.5}$$

The second lemma is a refinement of the upper bound in (3.5) with two main advantages: it is uniform in the variable t, and it restricts to the "local part" the singularities of the kernel $p_t(x,y)$. The proof of this more precise lemma is sketched in §3.3.

Lemma 3.2. *Given $x \in \mathbb{R}^d$, the following estimate holds for all $t > 0$ and $y \in \mathbb{R}^d$*

$$p_t(x,y) \leq \frac{C_1(x)\,t^{2\nu}\,e^{-\frac{|y|^2}{2}}}{(t+|x-y|)^{d+2\nu}}\,\chi_{\left\{|y|\leq 3\max\{|x|,1\}\right\}} \;+\; C_2(x)\,t^{2\nu}\,\varphi(y), \tag{3.6}$$

for some positive functions $C_1(x) \lesssim (1+|x|)^{2\nu+d-1}e^{\frac{|x|^2}{2}}$ and $C_2(x) \lesssim 1/\varphi(x)$.

Observe that, as a consequence of (3.6), we obtain the following bound for the maximal operators P^*_a in (1.7)

$$P^*_a f(x) \lesssim C_1(x)\,\mathcal{M}^{\rm loc}\big(fe^{-\frac{|y|^2}{2}}\big)(x) \;+\; C_2(x)\,a^{2\nu}\int_{\mathbb{R}^d}|f(y)|\,\varphi(y)\,dy, \tag{3.7}$$

where $\mathcal{M}^{\rm loc}$ denotes the local Hardy-Littlewood maximal operator defined in (1.12).

3.1. **Proof of Theorem 1.1.** Assuming the lemmas, we can sketch the proof of Theorem 1.1. First of all, it is a direct consequence of Lemma 3.1 that $P_t|f|(x) < \infty$ for some (or all) $t > 0$ and $x \in \mathbb{R}^d$ if and only if $f \in L^1(\varphi)$. This justifies that $f \in L^1(\varphi)$ is the right setting for this problem. Observe also that taking derivatives with respect to t in $p_t(x,y)$ slightly improves the decay of the kernel, and from here it is not difficult to deduce that (i) and (ii) must hold; see the details in [3, Proposition 4.4].

We shall be a bit more precise about the proof of (iii), that is the pointwise convergence

$$\lim_{t\to 0^+} P_t f(x) = f(x), \quad a.e.\, x \in \mathbb{R}^d \tag{3.8}$$

for all $f \in L^1(\varphi)$. We first claim that such convergence holds in the dense set $\mathcal{D} = \text{span}\{h_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$, where $h_{\mathbf{k}}(x)$ denote the d-dimensional Hermite functions (as in [10, p.5]). These are eigenfunctions of $L = -\Delta + |x|^2 + m$ with

$$Lh_{\mathbf{k}} \;=\; (2|\mathbf{k}| + d + m)\,h_{\mathbf{k}}, \quad \text{if } |\mathbf{k}| = k_1 + \ldots + k_d \geq 0\,;$$

see [10, (1.1.28)]. Recall that the operators $h_t = e^{-tL}$ from the Hermite semigroup can be represented in two ways: as in (3.1) with the Mehler kernel (3.3), or equivalently as

$$h_t f = \sum_{\mathbf{k}\in\mathbb{N}^d} e^{-(2|\mathbf{k}|+d+m)t}\,\langle f, h_{\mathbf{k}}\rangle\,h_{\mathbf{k}},$$

at least for $f \in \mathcal{D}$; see [10, (4.1.1)]. From this last formula and the results in §2 one also deduces that

$$P_t f = \tfrac{1}{\Gamma(\nu)} \sum_{\mathbf{k}\in\mathbb{N}^d} F_\nu\big(t\sqrt{2|\mathbf{k}|+d+m}\big)\,\langle f, h_{\mathbf{k}}\rangle\, h_{\mathbf{k}}, \quad f \in \mathcal{D}. \tag{3.9}$$

This clearly implies (3.8) when $f \in \mathcal{D}$.

To extend this convergence to all $f \in L^1(\varphi)$ we shall argue as in [3, Proposition 4.5]. Namely, it suffices to show (3.8) for *a.e.* $|x| \le R$, for every fixed $R \ge 1$. We split $f \in L^1(\varphi)$ by

$$f = f\chi_{\{|y|\le 3R\}} + f\chi_{\{|y|>3R\}} = f_0 + f_1.$$

Using Lemma 3.2 we see that, for every $|x| \le R$

$$\begin{aligned} \big|P_t f_1(x)\big| &\le \int_{|y|>3R} p_t(x,y)\,|f(y)|\,dy \\ &\le C_R\, t^{2\nu} \int_{\mathbb{R}^d} |f(y)|\varphi(y)\,dy \to 0, \quad \text{as } t \to 0^+. \end{aligned} \tag{3.10}$$

On the other hand, Lemma 3.2 (or rather its consequence in (3.7)) also implies that

$$\sup_{0<t\le 1} |P_t f_0(x)| \le C_R\Big(Mf_0(x) + \int |f_0|\varphi\Big), \quad |x| \le R,$$

where M denotes the usual Hardy-Littlewood maximal operator. Since the right-hand side is a bounded operator from $L^1(B_{3R}(0)) \to L^{1,\infty}(B_R(0))$, a classical procedure[§] then gives, from the validity of (3.8) in the dense class $\mathcal{D}$, the existence of $\lim_{t\to 0^+} P_t f_0(x) = f(x)$ for *a.e.* $|x| \le R$. This completes the proof of Theorem 1.1. □

Remark 3.3. Notice that the series representation of $P_t f$ in (3.9) allows us to reformulate (3.8) as a result on pointwise convergence of Hermite expansions by a "summability method" (based on the function F_ν and the parameter m). This is in the same spirit as the Poisson summability for Hermite expansions considered by Muckenhoupt in [6] (in the special case $\nu = 1/2$, $m = -d$ and $d = 1$). Notice, however, that the integral representation of $P_t f(x)$ in (1.5) is much more versatile, as it exists for functions in $f \in L^1(\varphi)$ which may have $\langle f, h_{\mathbf{k}}\rangle = \infty$ for all $\mathbf{k}$.

3.2. **Proof of Lemma 3.1.** For the sake of originality, we shall use a slightly different approach than the one given in [3, Lemma 4.1]. In the Mehler kernel $h_v(x,y)$ we shall consider the "more natural" variable $r = e^{-2v}$ (or equivalently $v = \frac12 \ln \frac1r$), which leads to the formula[¶]

$$h_v(x,y) = \frac{r^{\frac{m+d}{2}}\, e^{-\frac{|x-ry|^2}{1-r^2}}}{[\pi(1-r^2)]^{d/2}}\; e^{\frac{|x|^2-|y|^2}{2}}.$$

[§]See e.g. [2, Theorem 2.2].

[¶]This change of variables is common in the Ornstein-Uhlenbeck setting; see e.g. [6, (3.3)] or [5].

Inserting this into the integral defining $p_t(x,y)$ (with $dv = dr/(2r)$) one obtains the expression

$$p_t(x,y) = \frac{t^{2\nu}}{2^\nu \pi^{\frac{d}{2}} \Gamma(\nu)} \int_0^1 \frac{e^{-\frac{t^2}{2\ln\frac{1}{r}}}\, r^{\frac{m+d}{2}}\, e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}} \left(\ln\frac{1}{r}\right)^{1+\nu}} \frac{dr}{r}\, e^{\frac{|x|^2-|y|^2}{2}}. \tag{3.11}$$

Starting from this formula, we now argue as in [3, Lemma 4.1]. We may assume that $|y| \geq 3\max\{1,|x|\}$ (since in the region $|y| \leq 3\max\{1,|x|\}$ one can bound $p_t(x,y)/\varphi(y)$ above and below by positive functions of t, x).

The main difficulty is to determine the values of r which carry the main contribution of the integral in (3.11). The leading term will be the exponential in the numerator, and as we shall see, it becomes largest when $r \approx |x|/|y|$.

Consider first the region $1/2 < r < 1$. Since $|y| \geq 3|x|$, we have

$$|ry - x| \geq |y|/2 - |x| \geq |y|/6 \quad \Longrightarrow \quad e^{-\frac{|x-ry|^2}{1-r^2}} \leq e^{-\frac{|y|^2}{36}},$$

so the leading exponential becomes quite small in this part. Since we also have

$$\ln\frac{1}{r} \approx 1 - r, \quad r \in [1/2, 1], \tag{3.12}$$

we can estimate the corresponding integral in (3.11) by

$$\begin{aligned} \int_{1/2}^1 \cdots &\lesssim e^{\frac{|x|^2-|y|^2}{2}}\, e^{-\frac{|y|^2}{36}}\, t^{2\nu} \int_{1/2}^1 \frac{e^{-\frac{ct^2}{1-r}}}{(1-r)^{\frac{d}{2}+\nu+1}}\, dr \\ {\scriptstyle [u=\frac{t^2}{1-r}]} &\leq e^{\frac{|x|^2-|y|^2}{2}}\, e^{-\frac{|y|^2}{36}}\, t^{-d} \int_0^\infty e^{-cu}\, u^{\frac{d}{2}+\nu-1}\, du \\ &\lesssim e^{\frac{|x|^2}{2}}\, t^{-d}\, \varphi(y). \end{aligned} \tag{3.13}$$

Suppose now that $0 < r < \frac{1}{2}$. As we shall see, the main contribution occurs here, precisely when $r \approx \frac{|x|}{|y|}$. We first consider the range $2\frac{|x|}{|y|} < r < 1/2$, where we can estimate

$$|x - ry| \geq r|y| - |x| \geq r|y|/2 \quad \Longrightarrow \quad e^{-\frac{|x-ry|^2}{1-r^2}} \leq e^{-\frac{r^2|y|^2}{4}}.$$

Thus

$$\begin{aligned} \int_{2\frac{|x|}{|y|}}^{1/2} \cdots &\lesssim e^{\frac{|x|^2-|y|^2}{2}}\, t^{2\nu} \int_{2\frac{|x|}{|y|}}^{1/2} \frac{r^{\frac{m+d}{2}}\, e^{-\frac{r^2|y|^2}{4}}}{\left(\ln\frac{1}{r}\right)^{1+\nu}} \frac{dr}{r} \\ {\scriptstyle [u=r|y|]} &\leq t^{2\nu}\, e^{\frac{|x|^2-|y|^2}{2}}\, |y|^{-\frac{m+d}{2}} \int_0^{\frac{|y|}{2}} \frac{u^{\frac{m+d}{2}}\, e^{-u^2/4}}{\left(\ln\frac{|y|}{u}\right)^{1+\nu}} \frac{du}{u}. \end{aligned}$$

Note that when $m + d > 0$ the last integral[‖] has its major contribution at $u \approx 1$, so it can be estimated by $1/(\ln|y|)^{1+\nu}$. Thus, overall one obtains

$$\int_{2\frac{|x|}{|y|}}^{1/2} \cdots \lesssim t^{2\nu}\, e^{\frac{|x|^2}{2}}\, \varphi(y). \tag{3.14}$$

[‖]When $m = -d$, the integral still converges and is controlled by $1/(\ln|y|)^\nu$.

Finally, in the range $0 < r < 2\frac{|x|}{|y|}$ we disregard the exponential $e^{-\frac{|x-ry|^2}{1-r^2}}$ in (3.11) to obtain

$$\int_0^{2\frac{|x|}{|y|}} \dots \lesssim t^{2\nu} e^{\frac{|x|^2-|y|^2}{2}} \int_0^{2\frac{|x|}{|y|}} \frac{r^{\frac{m+d}{2}}}{\left(\ln\frac{1}{r}\right)^{1+\nu}} \frac{dr}{r}.$$

This last integral can be estimated by $(|x|/|y|)^{\frac{d+m}{2}}/[\ln(|y|/|x|)]^{1+\nu}$ when** $m+d>0$. Now, using elementary bounds on logarithms (see Lemma 5.1 in the Appendix) one concludes that

$$\int_0^{2\frac{|x|}{|y|}} \dots \lesssim t^{2\nu} e^{\frac{|x|^2}{2}} |x|^{\frac{d+m}{2}} [\ln(|x|+e)]^{1+\nu} \varphi(y)\,. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15) one obtains the upper bound in (3.5).

To establish the lower bound, it suffices to integrate in the range $0 < r \le 2\frac{|x|}{|y|}$. Note that $|y| \ge 3|x|$ also implies $r \le 2/3$, so we obtain

$$|x-ry| \le |x| + r|y| \le 3|x| \quad \Longrightarrow \quad e^{-\frac{|x-ry|^2}{1-r^2}} \ge e^{-18|x|^2}$$

(using $1-r^2 \ge 1/2$). The first exponential in (3.11) can be handled simply by

$$\exp\Big(-\frac{t^2}{2\ln\frac{1}{r}}\Big) \ge \exp\big(-\frac{t^2}{2\ln(3/2)}\big), \quad r \in [0, \tfrac{2}{3}],$$

so all together we conclude that

$$\begin{aligned} p_t(x,y) &\gtrsim t^{2\nu} e^{-ct^2} e^{-18|x|^2-\frac{|y|^2}{2}} \int_0^{2\frac{|x|}{|y|}} \frac{r^{\frac{m+d}{2}}}{\left(\ln\frac{1}{r}\right)^{1+\nu}} \frac{dr}{r} \\ &\approx t^{2\nu} e^{-ct^2} e^{-18|x|^2-\frac{|y|^2}{2}} \frac{(|x|/|y|)^{\frac{m+d}{2}}}{[\ln(|y|/|x|)]^{1+\nu}} \gtrsim c_1(t,x)\,\varphi(y)\,, \end{aligned}$$

for some positive function $c_1(t,x)$. □

3.3. **Proof of Lemma 3.2.** We split the integral defining $p_t(x,y)$, as

$$p_t(x,y) = \int_0^{\frac{1}{2}} \dots + \int_{\frac{1}{2}}^1 \dots \le I_0 + I_1.$$

The singularity of kernel lies in the first piece I_0, and in order to find a good estimate it will be crucial to use the formula in (3.4)††. Suppose we are in the local region $|y| \le 3\max\{|x|, 1\}$. Then, using (3.12), we can estimate I_0 by

$$\begin{aligned} I_0 &\lesssim t^{2\nu} \int_0^{\frac{1}{2}} \frac{e^{-\frac{ct^2+|x-y|^2}{4s}} e^{-\frac{s|x+y|^2}{4}}}{s^{\frac{d}{2}+1+\nu}} ds \\ &\approx \frac{t^{2\nu}}{(ct^2+|x-y|^2)^{\frac{d}{2}+\nu}} \int_{\frac{ct^2+|x-y|^2}{2}}^{\infty} e^{-u} e^{-\frac{(ct^2+|x-y|^2)|x+y|^2}{16u}} u^{\frac{d}{2}+\nu-1} du, \end{aligned} \tag{3.16}$$

**When $m=-d$, the integral is bounded by $1/[\ln(|y|/|x|)]^{\nu}$.

††The formula for $p_t(x,y)$ in (3.11) does not make so explicit the term $|x-y|$ in the leading exponential.

where we have changed variables $u = (ct^2 + |x-y|^2)/(4s)$. In the last integral we can disregard t in the exponential, and overall estimate it crudely by

$$J := \int_0^\infty e^{-u}\, e^{-\frac{(|x-y||x+y|)^2}{16u}}\, u^{\frac{d}{2}+\nu-1}\, du \;=\; F_{\frac{d}{2}+\nu}\big(\tfrac{|x-y||x+y|}{2}\big),$$

where $F_\sigma(z)$ was defined in (2.1). As we noticed in (2.2) we can write

$$F_\sigma(z) \;=\; 2^{1-\sigma}\, z^\sigma\, K_\sigma(z) \;\lesssim\; (1+z)^{\sigma-\frac12} e^{-z}, \quad z>0,$$

by the standard asymptotics of K_σ; see e.g. [4, p. 136]. Thus, we obtain

$$J \lesssim\; (1+|x-y||x+y|)^{\nu+\frac{d-1}{2}}\; e^{-\frac{|x-y||x+y|}{2}}\,.$$

Now, $|x-y||x+y| \;\geq\; |\langle x+y, x-y\rangle| \;\geq -|x|^2+|y|^2$, so in the region $|y|\leq 3\max\{|x|,1\}$ we have

$$J \;\lesssim\; e^{\frac{|x|^2}{2}}\, e^{-\frac{|y|^2}{2}}\, (1+|x+y|\,|x-y|)^{\nu+\frac{d-1}{2}} \;\lesssim\; (1+|x|)^{2\nu+d-1}\, e^{\frac{|x|^2}{2}}\, e^{-\frac{|y|^2}{2}}.$$

Inserting this into (3.16) we obtain the bound for the local part asserted in the statement of the lemma.

The estimate of I_0 when $|y|\geq 3\max\{|x|,1\}$ is much better, of the order $I_0\lesssim t^{2\nu}e^{-(\frac12+\gamma)|y|^2}$ for some $\gamma>0$; see [3, Lemma 4.2] for details. Likewise, from the arguments we already used in Lemma 3.1 one obtains a bound for $I_1\leq C_2(x)t^{2\nu}\varphi(y)$ with $C_2(x)\lesssim 1/\varphi(x)$. We again refer to [3] for details. This completes the proof of Lemma 3.2. □

4. Proof of Theorems 1.2 and 1.3

As we mentioned in the introduction, it suffices to give a proof of Theorem 1.3. That is, assuming $w\in D_p(\varphi)$, by which we mean

$$\|w\|_{D_p(\varphi)} := \;\big\| w^{-\frac1p}\,\varphi\,\big\|_{L^{p'}(\mathbb{R}^d)} < \infty,$$

we must show that the weights $v(x)$ defined in (1.13) are such that P_a^* maps $L^p(w)\to L^p(v)$ boundedly, for all $a>0$. We shall use the bound for P_a^* in (3.7), namely

$$\begin{aligned} P_a^* f(x) \;&\lesssim\; C_1(x)\,\mathcal{M}^{\mathrm{loc}}\big(f e^{-\frac{|y|^2}{2}}\big)(x) \;+\; C_2(x)\,a^{2\nu}\int_{\mathbb{R}^d} |f(y)|\,\varphi(y)\,dy \\ &=\; I(x)\;+\;II(x), \end{aligned} \tag{4.1}$$

with $C_1(x)$ and $C_2(x)$ given explicitly in Lemma 3.2. We treat first the last term, which by Hölder's inequality is bounded by

$$II(x) \;\leq\; C_2(x)\,a^{2\nu}\,\|f\|_{L^p(w)}\,\|w\|_{D_p(\varphi)}\,.$$

So using $C_2(x)=1/\varphi(x)$, we will have

$$\|II\|_{L^p(v)} \;\leq\; a^{2\nu}\,\|w\|_{D_p(\varphi)}\,\|f\|_{L^p(w)}\,\Big[\int_{\mathbb{R}^d} \frac{v(x)}{\varphi(x)^p}\,dx\Big]^{\frac1p} \tag{4.2}$$

with the last integral being a finite expression provided we choose

$$v(x) \;\leq\; v_1(x) := \;\frac{e^{-\frac{p}{2}|x|^2}}{(1+|x|)^M} \tag{4.3}$$

for any $M > N_1 = d + p(d+m)/2$. We remark that the weights $v(x)$ in (1.13) have this property, at least if N is sufficiently large. This is a consequence of the following elementary lemma.

Lemma 4.1. *If $f \in L^1_{\rm loc}(\mathbb{R}^d)$ satisfies $f(x) > 0$, a.e. $x \in \mathbb{R}^d$, then*

$$\mathcal{M}^{\rm loc} f(x) \,\geq\, c_f\,(1+|x|)^{-d}, \quad \forall\, x \in \mathbb{R}^d, \tag{4.4}$$

with $c_f = \frac{1}{|B_1|}\int_{B_1(0)} f > 0$.

Proof: Choosing $r = 1+|x|$, one trivially has $B_1(0) \subset B_r(x) \cap \{|y| \leq 3\max(|x|,1)\}$, so (4.4) is immediate from the definition of $\mathcal{M}^{\rm loc} f(x)$ in (1.12). □

Now, using the lemma, one sees that the weights $v(x)$ defined in (1.13) satisfy

$$v(x) \,\lesssim\, c'_w\,(1+|x|)^{d\alpha(p-1)}\,e^{-\frac{p}{2}|x|^2}(1+|x|)^{-N},$$

hence choosing $N > N_1 + d\alpha(p-1)$ ensures that (4.3) holds.

We now turn to main part $I(x)$ in (4.1). The following proposition will be crucial. The result is new, and the proof is based on arguments due to Carleson and Jones (see [1]).

Proposition 4.2. *Let $1 < p < \infty$ and $w(x) > 0$ such that $w^{-\frac{1}{p-1}} \in L^1_{\rm loc}(\mathbb{R}^d)$. Define*

$$V_\alpha(x) := \frac{\Big[\mathcal{M}^{\rm loc}(w^{-\frac{1}{p-1}})(x)\Big]^{-(p-1)\alpha}}{(1+|x|)^{(p-1)d\alpha}}, \qquad \text{for } \alpha > 1. \tag{4.5}$$

Then

$$\mathcal{M}^{\rm loc} : L^p(w) \to L^p(V_\alpha) \quad \textit{boundedly.}$$

Moreover, given $\sigma < 1$, if we choose $1 < \alpha < 1/\sigma$, then we also have $V_\alpha^{-\frac{\sigma}{p-1}} \in L^1_{\rm loc}(\mathbb{R}^d)$.

Proof: Note that $\mathcal{M}^{\rm loc}(w^{-\frac{1}{p-1}})(x) < \infty$, a.e. $x \in \mathbb{R}^d$, by the assumption $w^{-\frac{1}{p-1}} \in L^1_{\rm loc}$. This, together with Lemma 4.1, imply that

$$0 < V_\alpha(x) \leq c_w, \qquad a.e.\, x \in \mathbb{R}^d.$$

Now call $E_n = \{x \in \mathbb{R}^d \;:\; \mathcal{M}^{\rm loc}(w^{-\frac{1}{p-1}})(x) < 2^n\}$, $n = 0,1,2,\ldots$, and define the operators

$$T_n g(x) := \chi_{E_n}\,\mathcal{M}^{\rm loc}(w^{-\frac{1}{p-1}}g)(x). \tag{4.6}$$

Note that $T_n : L^1(w^{-\frac{1}{p-1}}) \to L^{1,\infty}(\mathbb{R}^d)$, with a uniform bound in n; in fact

$$\Big|\Big\{T_n g(x) > \lambda\Big\}\Big| \leq \Big|\Big\{M(w^{-\frac{1}{p-1}}g)(x) > \lambda\Big\}\Big| \leq \frac{c_0}{\lambda}\int_{\mathbb{R}^d} w^{-\frac{1}{p-1}}|g|. \tag{4.7}$$

Similarly, $T_n : L^\infty(w^{-\frac{1}{p-1}}) \to L^\infty(\mathbb{R}^d)$ with $\|T_n\| \leq 2^n$, since

$$\big\|T_n g\big\|_\infty = \sup_{x \in E_n}\big|\mathcal{M}^{\rm loc}(w^{-\frac{1}{p-1}}g)(x)\big| \,\leq 2^n\,\|g\|_\infty. \tag{4.8}$$

Thus, by Marcinkiewicz interpolation theorem we obtain

$$\int_{E_n} |T_n(g)|^p \,\leq\, c_0\,2^{\frac{np}{p'}}\int_{\mathbb{R}^d}|g|^p\,w^{-\frac{1}{p-1}}. \tag{4.9}$$

Setting $g = fw^{\frac{1}{p-1}}$ in the above inequality, this is the same as

$$\int_{E_n} |\mathcal{M}^{\rm loc}(f)|^p \leq c_0\, 2^{n(p-1)} \int_{\mathbb{R}^d} |f|^p\, w. \tag{4.10}$$

Now, writing $\mathbb{R}^d = E_0 \cup [\cup_{n\geq 1} E_n \setminus E_{n-1}]$, and using

$$V_\alpha(x) \leq c_w \text{ in } E_0, \quad \text{and} \quad V_\alpha(x) \leq 2^{-(n-1)(p-1)\alpha} \text{ if } x \notin E_{n-1},$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{M}^{\rm loc} f|^p\, V_\alpha \;&\leq\; c_w \int_{E_0} |\mathcal{M}^{\rm loc} f|^p + \sum_{n=1}^\infty 2^{-(n-1)(p-1)\alpha} \int_{E_n} |\mathcal{M}^{\rm loc} f|^p \\ \text{(by (4.10))} \;&\leq\; c_w c_0 \int_{\mathbb{R}^d} |f|^p w \,+\, 2^{(p-1)\alpha} \sum_{n=1}^\infty 2^{-n(\alpha-1)(p-1)} \int_{\mathbb{R}^d} |f|^p w \,\lesssim\, \int_{\mathbb{R}^d} |f|^p w,\end{aligned}$$

as we wished to show. Finally note that, when $\alpha\sigma < 1$, the classical Kolmogorov inequality gives

$$V_\alpha^{-\frac{\sigma}{p-1}}(x) \;=\; \Big[\mathcal{M}^{\rm loc}(w^{-\frac{1}{p-1}})(x)\Big]^{\alpha\sigma} (1+|x|)^{d\alpha\sigma} \in L^1_{\rm loc}(\mathbb{R}^d). \qquad \square$$

Remark 4.3. The condition $w^{-\frac{1}{p-1}} \in L^1_{\rm loc}(\mathbb{R}^d)$ actually characterizes the property that $\mathcal{M}^{\rm loc}$ maps $L^p(w) \to L^p(V)$ for some weight $V(x) > 0$. We sketch a proof of the converse in Proposition 5.2 below.

Below we shall need a refinement of Proposition 4.2, which we state now.

Proposition 4.4. *In the conditions of Proposition 4.2, if $w(x)$ additionally satisfies*

$$\int_{\mathbb{R}^d} w^{-\frac{1}{p-1}}(x)\, e^{-a|x|^2}\, dx \;<\; \infty, \quad \forall\, a > 0, \tag{4.11}$$

then, for every $\sigma < 1$ and $1 < \alpha < 1/\sigma$, the weight $V_\alpha(x)$ defined in (4.5) also satisfies

$$\int_{\mathbb{R}^d} V_\alpha^{-\frac{\sigma}{p-1}}(x)\, e^{-b|x|^2}\, dx \;<\; \infty, \quad \forall\, b > 0. \tag{4.12}$$

PROOF: Note that (4.12) is true if we only integrate in $B_1(0)$. So, we shall consider $S_j = \{2^j \leq |x| < 2^{j+1}\}$, $j = 0, 1, 2, \ldots$ Call $s = \alpha\sigma < 1$ and take any $b > 0$. Then

$$\begin{aligned} I \;&=\; \sum_{j=0}^\infty \int_{S_j} \big|\mathcal{M}^{\rm loc}(w^{-\frac{1}{p-1}})(x)\big|^s (1+|x|)^{ds}\, e^{-b|x|^2}\, dx \\ &\lesssim\; \sum_{j=0}^\infty 2^{jds}\, e^{-b4^j} \int_{S_j} \big|\mathcal{M}^{\rm loc}\big(w^{-\frac{1}{p-1}} \chi_{B_{3\cdot 2^{j+1}}(0)}\big)(x)\big|^s\, dx \\ \text{by Kolmogorov ineq} \;&\lesssim\; \sum_{j=0}^\infty 2^{jds}\, e^{-b4^j}\, |S_j|^{1-s} \Big\| M\big(w^{-\frac{1}{p-1}} \chi_{B_{3\cdot 2^{j+1}}(0)}\big)(x)\Big\|^s_{L^{1,\infty}(\mathbb{R}^d)} \\ &\lesssim\; \sum_{j=0}^\infty 2^{jd}\, e^{-b4^j} \Big[\int_{B_{3\cdot 2^{j+1}}(0)} w^{-\frac{1}{p-1}}(x) e^{a|x|^2} e^{-a|x|^2}\, dx\Big]^s \\ &\lesssim\; \sum_{j=0}^\infty 2^{jd}\, e^{-b4^j}\, e^{6^2 as4^j} \Big[\int_{\mathbb{R}^d} w^{-\frac{1}{p-1}}(x) e^{-a|x|^2}\, dx\Big]^s, \end{aligned}$$

and this is finite if we choose $a < b/(36s)$. $\square$

4.1. **Conclusion of the proof of Theorem 1.3.** Suppose now that $w \in D_p(\varphi)$, that is $\int w^{-\frac{1}{p-1}}(x)\varphi(x)^{p'}\,dx < \infty$, for φ as in (1.6). This implies that $W(x) = w(x)e^{\frac{p}{2}|x|^2}$ satisfies

$$\int_{\mathbb{R}^d} W^{-\frac{1}{p-1}}(x)\,e^{-a|x|^2}\,dx = \int_{\mathbb{R}^d} w^{-\frac{1}{p-1}}(x)\,e^{-\frac{p'}{2}|x|^2}\,e^{-a|x|^2}\,dx < \infty, \tag{4.13}$$

for all $a > 0$. Now, by Proposition 4.2, $\mathcal{M}^{\rm loc}$ maps $L^p(W) \to L^p(V_\alpha)$ boundedly, if we set

$$V_\alpha(x) = \frac{\Big[\mathcal{M}^{\rm loc}(W^{-\frac{1}{p-1}})(x)\Big]^{-(p-1)\alpha}}{(1+|x|)^{(p-1)d\alpha}}, \quad \text{with} \quad \alpha > 1.$$

In particular, if $f \in L^p(w)$ and we write $\tilde{f}(y) = f(y)e^{-\frac{|y|^2}{2}} \in L^p(W)$, we have

$$\|\mathcal{M}^{\rm loc}\tilde{f}\|_{L^p(V_\alpha)} \lesssim \|\tilde{f}\|_{L^p(W)} = \|f\|_{L^p(w)}.$$

So, recalling the value of $C_1(x)$ in Lemma 3.2, and setting

$$v(x) \le v_0(x) := \frac{V_\alpha(x)\,e^{-\frac{p}{2}|x|^2}}{(1+|x|)^L}, \tag{4.14}$$

with $L \ge L_1 = (2\nu + d - 1)p$, we see that the term $I(x)$ in (4.1) is controlled by

$$\begin{aligned} \|I(x)\|^p_{L^p(v)} &\le \int_{\mathbb{R}^d} \frac{C_1(x)^p\,e^{-\frac{p}{2}|x|^2}}{(1+|x|)^L}\,\big|\mathcal{M}^{\rm loc}\tilde{f}(x)\big|^p\,V_\alpha(x)\,dx \\ &\lesssim \|f\|^p_{L^p(w)}. \end{aligned} \tag{4.15}$$

So, combining (4.1), (4.2) and (4.15) we have shown that $\|P^*_a f\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$, provided

$$v(x) \le \min\{v_0(x), v_1(x)\},$$

with $v_0(x)$ and $v_1(x)$ defined in (4.14) and (4.3). But this inequality is clearly satisfied by the weights $v(x)$ defined in (1.13), for every $\alpha > 1$, provided that

$$N > (p-1)d\alpha + \max\big\{(2\nu+d-1)p, d + p\tfrac{d+m}{2}\big\} =: N_0.$$

Finally notice that, since (4.13) holds, if $\sigma < 1$ and $1 < \alpha < 1/\sigma$, we can use Proposition 4.4 to obtain

$$\int_{\mathbb{R}^d} v(x)^{-\frac{\sigma}{p-1}}\,\varphi(x)^{p'}\,dx \le \int_{\mathbb{R}^d} V_\alpha(x)^{-\frac{\sigma}{p-1}}\,e^{-(1-\sigma)\frac{p'|x|^2}{2}}\,(1+|x|)^{\frac{\sigma N}{p-1}}\,dx < \infty.$$

This proves that we can choose the weight $v(x)$ satisfying the inequality (1.11), as asserted in the introduction. It also gives a different proof (constructive) of [3, Theorem 1.3]. $\square$

Acknowledgments. I wish to thank S. Hartzstein, T. Signes, J.L. Torrea and B. Viviani for allowing me to use the results in our joint paper [3]. This presentation is strongly influenced by many conversations with them. Special thanks also to F.J. Martín Reyes and all the organizers of the *VI CIDAMA* conference for their hospitality and nice research environment enjoyed during the workshop.

5. Appendix

The following elementary estimates were used in the proof of Lemma 3.1.

Lemma 5.1. *Let x, y be positive real numbers such that $y \ge \lambda \max\{x, 1\}$, for some $\lambda > 1$. Then there exist $c_\lambda, d_\lambda > 0$ such that*

$$\ln \frac{y}{x} \ \ge\ c_\lambda \, \frac{\ln(y+e)}{\ln(x+e)}. \tag{5.1}$$

and

$$\ln \frac{y}{x} \ \le\ d_\lambda \, \ln(y+e)\ln(\frac{1}{x}+e). \tag{5.2}$$

Proof: We first consider (5.1). Since $\ln y/\ln(y+e)$ is bounded above and below when $y \in [\lambda, \infty)$, it suffices to prove the weaker estimate

$$\ln \frac{y}{x} \ \ge\ c'_\lambda \, \frac{\ln y}{\ln(x+e)}. \tag{5.3}$$

Consider first the case $x \le \sqrt{\lambda}$. Then $y \ge \lambda$ implies that $\sqrt{y} \ge x$ and hence

$$\ln \frac{y}{x} \ge \ln \sqrt{y} = \tfrac{1}{2} \ln y \ge \tfrac{1}{2} \frac{\ln y}{\ln(x+e)},$$

since $\ln(x+e) > 1$. This proves (5.3) with $c'_\lambda = 1/2$. Consider now the case $x \ge \sqrt{\lambda}$, and write

$$\ln y = \ln \tfrac{y}{x} \, + \, \ln x. \tag{5.4}$$

Observe that, if $a \ge a_0 > 0$ and $b \ge b_0 > 0$, then

$$\frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b} \le \frac{1}{a_0} + \frac{1}{b_0}. \tag{5.5}$$

So, using this fact in (5.4) we see that

$$\ln y \le \Big(\tfrac{1}{\ln \lambda} + \tfrac{1}{\ln \sqrt{\lambda}} \Big) \ln \frac{y}{x} \, \ln x \ \le\ \tfrac{3}{\ln \lambda} \, \ln \frac{y}{x} \, \ln(x+e),$$

which implies (5.3) with $c'_\lambda = (\ln \lambda)/3$. The proof of (5.2) is similar. If $x \ge 1/\lambda$ then

$$\ln \tfrac{y}{x} \le \ln(\lambda y) \le 2 \ln y \le 2 \ln(y+e) \, \ln(e + \tfrac{1}{x}).$$

If $x \ge 1/\lambda$ then

$$\ln \tfrac{y}{x} = \ln y + \ln \tfrac{1}{x} \le \tfrac{2}{\ln \lambda} \, \ln y \, \ln(\tfrac{1}{x}),$$

using in the last step the inequality (5.5). □

We give a proof of the converse of Proposition 4.2, whose validity we mentioned in Remark 4.3. The arguments are similar to those in [1].

Proposition 5.2. *Let $1 < p < \infty$, and suppose that $w(x) > 0$ is such that $\mathcal{M}^{\mathrm{loc}}$ maps $L^p(w) \to L^p(V)$ for some weight $V(x) > 0$. Then, necessarily, $w^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.*

Proof: Given $\varepsilon > 0$, we set $w_\varepsilon(x) = w(x) + \varepsilon$. Notice that every $f = w_\varepsilon^{-\frac{1}{p-1}} \chi_{B_R(0)}$ belongs to $L^p(w)$ since

$$\int_{\mathbb{R}^d} |f|^p w \leq \int_{B_R(0)} w_\varepsilon^{-p'} \, w_\varepsilon \;\leq\; |B_R(0)| \, \varepsilon^{-(p'-1)} \;<\; \infty.$$

Call $S_0 = B_1(0)$ and $S_j = \{2^{j-1} \leq |x| < 2^j\}$, $j = 1, 2, \ldots$, and consider the $L^p(w)$-functions $f_j = w_\varepsilon^{-\frac{1}{p-1}} \chi_{S_j}$. Then, it is easy to verify from the definition of $\mathcal{M}^{\rm loc}$ that

$$\mathcal{M}^{\rm loc} f_j(x) \gtrsim 2^{-jd} \int_{S_j} w_\varepsilon^{-\frac{1}{p-1}}, \quad \text{if } x \in S_j,$$

for each $j = 0, 1, 2, \ldots$ Indeed, it suffices to average over a ball $B_r(x)$ of radius $r = 2^j + |x|$, and observe that $S_j \subset B_r(x) \cap \{|y| \leq 3\max(|x|, 1)\}$ when $x \in S_j$. Thus, the assumed boundedness of $\mathcal{M}^{\rm loc}$ gives

$$V(S_j)^{\frac{1}{p}} \, 2^{-jd} \int_{S_j} w_\varepsilon^{-\frac{1}{p-1}} \lesssim \big\|\mathcal{M}^{\rm loc} f_j\big\|_{L^p(V)} \;\leq\; C \, \|f_j\|_{L^p(w)} \;\leq\; C \Big[\int_{S_j} w_\varepsilon^{-(p'-1)} \Big]^{\frac{1}{p}},$$

with a constant C independent of ε. Since both sides are finite and $p' - 1 = 1/(p-1)$, we conclude that

$$\Big[\int_{S_j} w_\varepsilon^{-\frac{1}{p-1}} \Big]^{1/p'} \lesssim C \, 2^{jd} / V(S_j)^{\frac{1}{p}},$$

so letting $\varepsilon \to 0$ we obtain that $w^{-\frac{1}{p-1}} \in L^1(S_j)$ for each $j = 0, 1, 2, \ldots$ □

References

[1] L. Carleson and P. Jones, Weighted norm inequalities and a theorem of Koosis. Mittag-Leffler Inst., report n. 2, 1981.

[2] J. Duoandikoetxea, *Fourier Analysis*, Amer. Math. Soc. 2001.

[3] G. Garrigós, S. Hartzstein, T. Signes, J.L. Torrea, B. Viviani, Pointwise convergence to initial data of heat and Laplace equations. To appear in Trans. Amer. Math. Soc.

[4] N. Lebedev, *Special functions and their applications.* Prentice-Hall Inc., 1965.

[5] L. Liu, P. Sjögren, A characterization of the Gaussian Lipschitz space and sharp estimates for the Ornstein-Uhlenbeck Poisson kernel. Preprint, arXiv:1401.4288v2, 2014.

[6] B. Muckenhoupt, Poisson integrals for Hermite and Laguerre expansions. Trans. Amer. Math. Soc. **139** (1969), 231–242.

[7] J.L. Rubio de Francia, Weighted norm inequalities and vector valued inequalities. In *Harmonic analysis (Minneapolis 1981)*, Lecture Notes Math. **908**, Springer 1982, 86–101.

[8] E. Stein. *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory.* Annals of Mathematical Studies **63**. Princeton University Press, 1970.

[9] P. Stinga, J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators. Comm. Partial Differential Equations **35** (11) (2010), 2092–2122.

[10] S. Thangavelu. *Lecture notes on Hermite and Laguerre expansions.* Princeton Univ. Press. Princeton, New Jersey 1993.

[11] G. N. Watson, *Theory of Bessel functions.* Cambridge Univ. Press, New York, 1922.

Gustavo Garrigós, Departamento de Matemáticas, Universidad de Murcia, 30100 Murcia, Spain. *Email*: gustavo.garrigos@um.es