Littlewood-Paley decompositions related to symmetric cones

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Abstract

We obtain a Whitney decomposition of a symmetric cone Ω, analog to that of the positive real line into dyadic intervals $[2^j, 2^{j+1})$. This gives a natural tool for developing a Littlewood-Paley theory for spaces of functions with spectrum in Ω. Such functions extend into holomorphic functions on the tube T_Ω. We consider here the mixed norm Bergman spaces $A_\nu^{p,2}(T_\Omega)$, for which we find a Littlewood-Paley characterization. As a consequence, we obtain optimal results for the boundedness of the Bergman projector P_ν in $L_\nu^{p,2}(T_\Omega)$. When the projector is unbounded, a precise description of $P_\nu(L_\nu^{p,2})$ is also given, as a space of equivalence classes of holomorphic functions in relation with the dual of $A_\nu^{p',2}(T_\Omega)$.

1 Introduction

Let Ω be an irreducible symmetric cone in $\mathbb{R}^n$, and let

$$T_\Omega = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$$

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be the tube domain based on Ω in the complexified vector space $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. Regarding $V = \mathbb{R}^n$ as a Euclidean Jordan algebra, we shall denote

$$r = \mathrm{rank}\ \Omega, \quad \Delta(x) = \det x, \quad (x|y) = \mathrm{tr}(xy), \quad x, y \in V,$$

as in the text [6]. For ν real and $0 < p, q < \infty$ we define the (mixed-norm weighted) Bergman space on the tube $T_\Omega \subset \mathbb{C}^n$ as the space $A_\nu^{p,q}(T_\Omega)$ of holomorphic functions on T_Ω with

$$\|F\|_{A_\nu^{p,q}} = \left[\int_\Omega \left[\int_V |F(x+iy)|^q\, dx \right]^{\frac{p}{q}} \Delta(y)^{\nu - \frac{2n}{r}}\, dy \right]^{\frac{1}{p}} < \infty.$$

When $p = q$ we just write $A_\nu^p = A_\nu^{p,q}$. We shall also use the notation $L_\nu^{p,q}(T_\Omega) = L^{p,q}(T_\Omega, \Delta(y)^{\nu - \frac{2n}{r}}\, dxdy)$. Finally, observe that $A_\nu^{p,q} = \{0\}$ when $\nu \leq \frac{2n}{r} - 1$. In this paper we shall be concerned with the spaces $A_\nu^{p,2}(T_\Omega)$ and $L_\nu^{p,2}(T_\Omega)$, for which the Plancherel formula in the x-variable allows a simpler description. For these particular spaces, we have a complete answer to a main question, which we describe now. Let P_ν be the *Bergman projector*, defined by

$$P_\nu : L_\nu^2(T_\Omega) \longrightarrow A_\nu^2(T_\Omega)$$
$$f \longmapsto P_\nu f(z) = \int\int_{T_\Omega} B_\nu(z - \overline{w}) f(w)\, \Delta(\Im m\ w)^{\nu - \frac{2n}{r}}\, dw.$$

Here $B_\nu(z - \overline{w})$ is the reproducing kernel of $A_\nu^2(T_\Omega)$, with explicit formula

$$B_\nu(z - \overline{w}) = d(\nu)\, \Delta^{-\nu}\left(\frac{z - \overline{w}}{i} \right), \quad z, w \in T_\Omega.$$

In different papers [1], [5], [4], [2],... the first two authors have considered this projector for the special case of the light-cone $(r = 2)$:

$$\Omega_n = \left\{ y = (y_1, y') \in \mathbb{R}^n \ : \ \Delta(y) = y_1^2 - |y'|^2 > 0, \ y_1 > 0 \right\},$$

Let us recall that, even in this case, the problem of finding the exact range of p, q for which P_ν can be boundedly extended from $L_\nu^{p,q}(T_\Omega)$ onto

$A_\nu^{p,q}(T_\Omega)$ is still open. The best range known up to now, for the light cone, was given in [4]. Moreover, it is deduced from the particular case when $q = 2$, and related to a generalized Hardy inequality for the wave operator. In this paper we propose another method than the one in [4] to determine the exact range of boundedness for the spaces $A_\nu^{p,2}(T_\Omega)$. More precisely, we show the following theorem, which extends to general symmetric cones the content of the previous papers, and gives the behavior for the critical index.

THEOREM 1.1 *Let $\nu > \frac{2n}{r} - 1$. Then the Bergman P_ν can be boundedly extended from $L_\nu^{p,2}(T_\Omega)$ onto $A_\nu^{p,2}(T_\Omega)$ if and only if $p_\nu' < p < p_\nu := \frac{2(\nu-1)}{\frac{n}{r}-1}$.*

When $p \geq p_\nu$, we shall in addition describe the range of the operator $P_\nu(L_\nu^{p,2}(T_\Omega))$ as a space of holomorphic functions (or equivalence classes of them) which is strictly larger than $A_\nu^{p,2}(T_\Omega)$. This space can be identified with the dual of $A_\nu^{p',2}(T_\Omega)$, where now $1 < p' < p_\nu'$. We can collect these results in the following

THEOREM 1.2 *Let $\nu > \frac{2n}{r} - 1$ and $p_\nu' < p < \infty$. Then there is a space $\mathcal{C}_\nu^{p,2}(T_\Omega)$ of equivalence classes of holomorphic functions in T_Ω so that:*

1. *P_ν can be extended as a bounded operator from $L_\nu^{p,2}(T_\Omega)$ onto $\mathcal{C}_\nu^{p,2}(T_\Omega)$.*

2. *$(A_\nu^{p',2})^* \equiv \mathcal{C}_\nu^{p,2}$.*

To do all this, we shall use a new idea. We shall exploit the geometry of a symmetric cone to find an appropriate partition of Ω into "frequency blocks" $\{E_j\}$, where the determinant $\Delta(\xi)$ and other related functions of the cone remain almost constant. This partition reduces to dyadic intervals $[2^j, 2^{j+1})$ for the real positive half-line, which corresponds to rank one. In this case, we recover the well-known Littlewood-Paley decomposition. So our construction may be seen as a generalization of it. More precisely, using the reconstruction of holomorphic functions in T_Ω with the *Fourier-Laplace transform*

$$F(z) = \mathcal{L}f(z) = \int_\Omega e^{i\,(z|\xi)} f(\xi)\,d\xi, \quad z \in T_\Omega, \qquad (1.3)$$

4

we shall obtain a new Littlewood-Paley formulation for the norm of the spaces $A_\nu^{p,2}(T_\Omega)$:

THEOREM 1.4 *Let $\nu > \frac{2n}{r} - 1$ and $0 < p < p_\nu = \frac{2(\nu-1)}{n/r-1}$, and let $\{E_j\}_j$ be a Whitney decomposition of Ω. Then, a function F belongs to $A_\nu^{p,2}(T_\Omega)$ if and only if $F = \mathcal{L}f$ for some $f \in L_{\ell oc}^2(\Omega)$ satisfying:*

$$\|f\|_{b_\nu^{p,2}} := \left[\sum_j \frac{\left(\int_{E_j} |f(\xi)|^2 \, d\xi \right)^{\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}} \right]^{\frac{1}{p}} < \infty. \tag{1.5}$$

In this case, there is a constant $c = c(p, \nu) > 0$ such that

$$\frac{1}{c} \|f\|_{b_\nu^{p,2}} \leq \|F\|_{A_\nu^{p,2}} \leq c \|f\|_{b_\nu^{p,2}}. \tag{1.6}$$

We point out that the critical index $p_\nu = 2(\nu - 1)/(\frac{n}{r} - 1)$ is optimal, in the sense that no such characterization exists for $A_\nu^{p,2}(T_\Omega)$ if $p \geq p_\nu$. This is related to the fact that the projectors P_ν cannot be boundedly extended to $L_\nu^{p,2}(T_\Omega)$ when $p \geq p_\nu$. However, the spaces $b_\nu^{p,2}(\Omega)$ can be identified with the duals of $A_\nu^{p',2}$ above the critical index, which allows to prove Theorem 1.2.

We insist on the fact that the use of Plancherel Theorem simplifies the proofs in the case $q = 2$, and allows to get sharp results. Some of this work may be generalized to other values of q with a considerable effort, using a family of $\mathcal{F}L^q$ multipliers related to the Whitney decomposition of the cone. A joint work with F. Ricci is in progress in this direction (see [3]), and provides a Littlewood-Paley description of the spaces $A_\nu^{p,q}(T_\Omega)$ below some critical index. Up to now, these results are not sharp when q is different from 2. Let us also mention that, even for $q = 2$, we do not know how to describe the dual space of $A_\nu^{p,2}(T_\Omega)$ above the critical index p_ν.

2 Whitney decompositions of the cone

The symmetry of Ω, reflected in the corresponding invariant geometry, will play a central role in our analysis. Most of the properties that we state without proof here, as well as the notation used, can be found in [6].

We recall that the vector space $V = \mathbb{R}^n$, containing the cone Ω, can be regarded as a Euclidean Jordan algebra, say with identity element $\mathbf{e}$. The cone Ω then becomes the connected component of the set of invertible elements in V containing $\mathbf{e}$. As usual, the inner product in V will be denoted by $(x|y) = \mathrm{tr}(xy)$.

If G is the identity component of the group of transformations of the cone, $G(\Omega)$, it is also known that there is a subgroup T of G acting simply transitively on Ω. That is, every $y \in \Omega$ can be written uniquely as $y = t\mathbf{e}$, with $t \in T$. This gives an identification $\Omega \equiv T = G/K$, where K is a maximal compact subgroup of G, namely, $K = G \cap O(V) = \{g \in G \,:\, g\mathbf{e} = \mathbf{e}\}$.

Therefore, we can regard $\Omega \equiv G/K$ as a Riemannian manifold with the G-invariant metric defined by

$$\langle \xi, \eta \rangle_y := (t^{-1}\xi | t^{-1}\eta)$$

if $y = t\mathbf{e}$ and ξ, η are tangent vectors at $y \in \Omega$. We shall denote by d the corresponding distance, and by $B_\delta(\xi)$ the ball centered at ξ of radius δ. Note that, for each $g \in G$, the invariance implies $B_\delta(g\xi) = gB_\delta(\xi)$.

Let $\{c_1, \ldots, c_r\}$ be a fixed Jordan frame in V, and $V = \oplus_{1 \le i \le j \le r} V_{i,j}$. Then one can write $T = NA = AN$, where the products are semidirect, N is a nilpotent subgroup, and A is the subgroup of *diagonal matrices*

$$A = \{P(a) \,:\, a = \sum_{i=1}^{r} a_i c_i, \ \ a_i > 0\}.$$

P is the quadratic representation of V. This leads to the classical decompositions $G = NAK$ and $G = KAK$ (see Chapter VI of [6]).

Let us denote by $\Delta_1(x), \ldots, \Delta_r(x)$ the *principal minors* of $x \in V$, with respect to the fixed Jordan frame $\{c_1, \ldots, c_r\}$ (see Chapter VI of [6]). Recall that these are invariant under the group N: $\Delta_k(nx) = \Delta_k(x)$, $n \in N$, $x \in V$. Also, for $a = a_1 c_1 + \ldots + a_r c_r$ have $\Delta_k(P(a)x) = a_1^2 \cdots a_k^2 \, \Delta_k(x)$. In particular, one can write

$$\Omega = \{x \in V \ : \ \Delta_k(x) > 0, \ \ k = 1, \ldots, r\}.$$

The next lemma show us that these quantities remain almost constant within an invariant ball. (see also [4], [2], for partial results for the light cone).

LEMMA 2.1 *If $\delta > 0$, then there is a constant $\gamma = \gamma(\delta, \Omega) > 0$ such that*

$$if \quad d(y, y') \leq \delta_0 \quad \Rightarrow \quad \frac{1}{\gamma} \leq \frac{\Delta_k(y)}{\Delta_k(y')} \leq \gamma, \quad k = 1, \ldots, r.$$

PROOF: By invariance of the metric and the forms Δ_k under N, we may assume $y' = P(a)\mathbf{e}$. Further, since

$$\frac{\Delta_k(y)}{\Delta_k(P(a)\mathbf{e})} = \frac{\Delta_k(P(a)^{-1}y)}{\Delta_k(\mathbf{e})},$$

we may even assume $y' = \mathbf{e}$. Now, the estimations above and below for Δ_k in a ball $\overline{B}_\delta(\mathbf{e})$ follow easily from the continuity of $y \mapsto \Delta_k(y)$, and a compactness argument. $\qquad\square$

We now prove that the quantities $(\xi|y)$ are also almost constant when ξ varies inside an invariant ball.

LEMMA 2.2 *Let $\delta > 0$, There exists $\gamma = \gamma(\Omega, \delta) > 0$ such that, for $y \in \overline{\Omega}$ and $\xi, \xi' \in \Omega$ with $d(\xi, \xi') \leq \delta$, then*

$$\frac{1}{\gamma} \leq \frac{(\xi|y)}{(\xi'|y)} \leq \gamma. \tag{2.3}$$

7

PROOF: By continuity it suffices to show (2.3) for $y \in \Omega$. Using invariance under G (and the fact that $G = G^*$), we may assume that $y = \mathbf{e}$. To show that $(\xi'|y) \leq \gamma(\xi|y)$, let us write $\xi = kP(a)\mathbf{e}$, for $k \in K$ and $a = a_1 c_1 + \ldots + a_r c_r$. Then $\xi' = kP(a)\xi''$, with $\xi'' \in B_\delta(\mathbf{e})$. We have

$$(\xi'|\mathbf{e}) = (P(a)\xi''|\mathbf{e}) \leq \sqrt{r}\|P(a)\|\|\xi''\| \leq \gamma\|P(a)\|,$$

since the euclidean norm is uniformly bounded on the invariant ball $B_\delta(\mathbf{e})$. Now $P(a)$ has eigenvalues a_i^2 and $a_i a_j$, and therefore

$$\|P(a)\| \leq \sum_{i=1}^r a_i^2 = (P(a)\mathbf{e}|\mathbf{e}) = (\xi|\mathbf{e}). \qquad (2.4)$$

$\square$

Finally, we will need to evaluate the volume of an invariant ball. Recall that the invariant measure in Ω is given by

$$\text{meas}\,(B) = \int_B \Delta(y)^{-\frac{n}{r}}\,dy, \quad B \subset \Omega \text{ measurable.}$$

It is well known that small balls for two Riemannian structures have equivalent volume. It follows that, in our context, for all $y \in \Omega$ and $0 < \delta \leq \delta_0$,

$$\text{meas}\,(B_\delta(y)) = \text{meas}\,(B_\delta(\mathbf{e})) \sim \text{Vol}\,(B_\delta(\mathbf{e})) \sim \delta^n.$$

Here, the equivalences denoted by "$\sim$" are modulo constants depending only on Ω and δ_0. We point out that this is not the case for $\delta \gg 1$, since the invariant measure is in general not doubling. We shall use this remark in the proof of the next covering lemma, which is of crucial importance for the rest of the paper.

LEMMA 2.5 : Whitney Decomposition.
 Let $0 < \delta \leq \delta_0$. Then, there exists a sequence of points $\{\xi_j\}_j$ in Ω and a family of disjoint sets $\{E_j\}_j$ which cover Ω, such that

 (i) one has the inclusion $B_{\delta/2}(\xi_j) \subset E_j \subset B_\delta(\xi_j)$;

(ii) The balls in $\{B_j\} = B_\delta(\xi_j)$ have the finite intersection property. That is, there is an integer $N = N(\delta_0, \Omega)$ so that each point in Ω belongs to at most N of these balls.

DEFINITION 2.6 A sequence of points $\{\xi_j\}_j$ in Ω with the properties above will be called a *δ-lattice*, with associated *Whitney decomposition* $\{E_j\}_j$.

PROOF: We take $\{\xi_j\}_j$ a maximal subset of Ω (under inclusion) among those with the property that their elements are distant at least δ from one another. Let us note B'_j the balls $B_{\delta/2}(\xi_j)$. They are pairwise disjoint, while, by maximality, the balls $\{B_j\}_j$ cover Ω. Note also that, necessarily, the set $\{\xi_j\}_j$ is countable.

For the finite overlapping property, if $\xi \in \cap_{i=1}^N B_{j_i}$, then

$$\bigcup_{i=1}^N B'_{j_i} \subset B(\xi, 3\delta/2) \,.$$

But for the invariant measure on Ω we will have

$N \, \text{meas} \, (B(\mathbf{e}, \delta/2)) = \text{meas} \, (\cup_{i=1}^N B'_{j_i}) \leq \text{meas} \, (B(\xi, 3\delta/2)) = \text{meas} \, (B(\mathbf{e}, 3\delta/2)).$

This, and consideration on the volume, allows to conclude. The sets E_j are then constructed by simply taking

$$E_1 = B_1, \quad \ldots \quad , E_j = B_j \setminus E_{j-1}, \quad \ldots \qquad \square$$

REMARK 2.7 If $\{\xi_j\}_j$ is a δ-lattice, then so is $\{\xi_j^{-1}\}_j$. Indeed, this follows from the fact that $y \to y^{-1}$ is an isometry of the cone (see Chapter III of [6]). Therefore, $B_\delta(\xi_j^{-1}) = B_\delta(\xi_j)^{-1}$, and the conditions of Lemma 2.5 hold. Note that we can look at the sets $\{\xi_j\}_j$ and $\{\xi_j^{-1}\}_j$ as a couple of *dual lattices*. We will note $\{E_j^*\}$ the corresponding Whitney (dual) decomposition.

We have that $\text{Vol} \, (E_j) \sim \Delta(\xi_j)^{\frac{n}{r}}$, and $\text{Vol} \, (E_j^*) \sim \Delta(\xi_j)^{-\frac{n}{r}}$. Moreover, all quantities $\Delta_k(\xi)$ and $(\xi|y)$ are almost constant on E_j or E_j^*. Finally, since $(\xi_j^{-1}|\xi_j) = r$, it follows from the previous lemmas that $(y|\xi) \sim 1$ for $y \in E_j^*$ and $\xi \in E_j$.

3   Integrals on Ω

The *generalized power function* in Ω is defined by

$$\Delta_{\mathbf{s}}(x) = \Delta_1^{s_1-s_2}(x)\,\Delta_2^{s_2-s_3}(x)\cdots\Delta_r^{s_r}(x), \quad \mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r, \quad x \in \Omega,$$

where Δ_k are the principal minors with respect to a fixed Jordan frame $\{c_1, \ldots, c_r\}$. Note that, for $x = a_1 c_1 + \ldots + a_r c_r$, then $\Delta_{\mathbf{s}}(x) = a_1^{s_1} \cdots a_r^{s_r}$. The next result is the key in all the discretization steps we shall do in our integrals below. Its proof is a simple consequence of the geometric lemmas from the previous section.

PROPOSITION 3.1 *Let $0 < \delta \leq 1$ be fixed, and $\{\xi_j\}_j$ be a δ-lattice with associated Whitney decomposition $\{E_j\}_j$. Then, for every $\mathbf{s} \in \mathbb{C}^r$, $y \in \overline{\Omega}$, and for any non-negative function f on the cone, we have*

$$\frac{1}{C} \sum_j e^{-\gamma(y|\xi_j)} \Delta_{\mathbf{s}}(\xi_j) \int_{E_j} f(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} \leq \int_\Omega f(\xi) e^{-(y|\xi)} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}$$

$$\leq C \sum_j e^{-\frac{1}{\gamma}(y|\xi_j)} \Delta_{\mathbf{s}}(\xi_j) \int_{E_j} f(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}},$$

where γ is the constant in (2.3) and C depends only (and continuously) on $\mathbf{s}$.

A particular case of the type of integral in the previous proposition is the *gamma function for the cone* Ω, defined as follows:

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-(\xi|\mathbf{e})} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}, \quad \mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r. \qquad (3.2)$$

All the properties we need are well-known, and can be found in Chapter VII of [6]. For instance, this integral converges if and only if $\Re e\, s_j > (j-1)\frac{n/r-1}{r-1}$, for all $j = 1, \ldots, r$, being in this case equal to

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^r \Gamma(s_j - (j-1)\tfrac{n/r-1}{r-1}), \qquad (3.3)$$

where Γ is the classical gamma function on $\mathbb{R}_+$. We note $\Gamma_\Omega(\mathbf{s}) = \Gamma_\Omega(s)$ when $\mathbf{s} = (s, \dots, s)$. We state separately a slight variant of (3.2) which we shall use often below. The proof is a simple application of the invariance under T.

LEMMA 3.4 *For $y \in \Omega$ and $\mathbf{s} = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$ with $\Re e\, s_j > (j-1)\frac{n/r-1}{r-1}$, $j = 1, \dots, r$, then*

$$\int_\Omega e^{-(\xi|y)} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} = \Gamma_\Omega(\mathbf{s})\, \Delta_{\mathbf{s}}(y^{-1}).$$

REMARK 3.5 When $y \in \Omega$, there is a simple expression for $\Delta_{\mathbf{s}}(y^{-1})$. Indeed, taking any rotation $k_0 \in K$ such that $k_0 c_j = c_{r-(j-1)}$, $j = 1, \dots, r$, then
$$\Delta_{\mathbf{s}}(y^{-1}) = \Delta_{\mathbf{s}^*}(k_0 y)^{-1}, \quad \text{where } \mathbf{s}^* = (s_r, \dots, s_1).$$
Note that we can choose k_0 so that $k_0^{-1} = k_0^* = k_0$. In the particular case $\mathbf{s} = (1, \dots, 1)$ we have $\Delta(y^{-1}) = \Delta(y)^{-1}$, by the invariance under rotations of the determinant.

One may use the previous lemma and Plancherel's Theorem to show the following result:

LEMMA 3.6 *Let $\alpha \in \mathbb{R}$, and define*

$$I_\alpha(y) = \int_{\mathbb{R}^n} |\Delta(x + iy)|^{-\alpha}\, dx, \quad y \in \Omega.$$

Then, I_α is finite if and only if $\alpha > \frac{2n}{r} - 1$. In this case, $I_\alpha(y) = c(\alpha)\, \Delta(y)^{-\alpha + \frac{n}{r}}$.

For computing precisely integrals of the above type (like in formula (3.3)) one introduces the *Gauss coordinates* in Ω. They are defined, also in terms of the Peirce decomposition, as follows (see § VI.3 in [6]). Let $d = \dim V_{ij} = 2\frac{n/r-1}{r-1}$ and

$$V^+ = \{ u = \sum_{j=1}^r u_j c_j + \sum_{j<k} u_{jk} \ : \ u_j > 0, \ u_{jk} \in \mathbb{R}^d \}.$$

We then make the change $u \in V^+ \longmapsto x = x(u) \in \Omega$, where $x(u) = \sum_{j=1}^r x_j c_j + \sum_{j<k} x_{jk}$ with

$$x_j = u_j^2 + \tfrac{1}{2} \sum_{k=1}^{j-1} |u_{kj}|^2 \quad \text{and} \quad x_{jk} = u_j u_{jk} + 2 \sum_{\ell=1}^{j-1} u_{\ell j} u_{\ell k}.$$

The main advantage is that now:

$$\operatorname{tr}(x(u)) = \sum_{j=1}^r u_j^2 + \tfrac{1}{2} \sum_{j<k} |u_{jk}|^2 \quad \text{and} \quad \Delta_k(x(u)) = u_1^2 \cdots u_k^2, \quad k = 1, \ldots, r.$$

After computation of the Jacobian, one obtains the following parametrization of the integrals:

PROPOSITION 3.7 : see Th VI.3.9 in [6].
  If f is a non-negative function in Ω, then

$$\int_\Omega f(x) \, dx = 2^r \int_{V^+} f(x(u)) \prod_{j=1}^r u_j^{(r-j)d+1} \, du.$$

With the aid of this result we can compute easily the following integrals. We denote $\Delta_1^*(\xi) = \Delta_{(0,\ldots,0,1)}(\xi) = u_r^2$, if $\xi = \xi(u)$.

LEMMA 3.8 *Let $g_\alpha(\xi) = \frac{e^{-\operatorname{tr}(\xi)}}{\Delta(\xi)(1+|\log \Delta_1^*(\xi)|)^\alpha}$. Then, g_α is integrable if and only if $\alpha > 1$.*

PROOF: Indeed, using the coordinates in the previous proposition

$$
\begin{aligned}
\int_\Omega g_\alpha(x) \, dx &= 2^r \int_{(0,\infty)^r} \frac{e^{-\sum u_j^2}}{(1+2|\log u_r|)^\alpha} \prod_{j=1}^r u_j^{(r-j)d-1} \, du_1 \ldots du_r \int_{\mathbb{R}^d} e^{-\frac{1}{2}|y|^2} \, dy \\
&= 2^r (2\pi)^{\frac{d}{2}} \prod_{j=1}^{r-1} \left[\frac{1}{2} \Gamma\left(\frac{d(r-j)}{2}\right) \right] \int_0^\infty \frac{e^{-u_r^2}}{(1+2|\log u_r|)^\alpha} \frac{du_r}{u_r},
\end{aligned}
$$

which is finite if and only if $\alpha > 1$.

$\square$

4   The proof of Theorem 1.4

With the geometric properties from the previous sections, the proof of our theorem will follow from a standard discretization process. The critical index p_ν will appear in relation with the range of convergence for the gamma integral. The following lemma will be taken for granted (see [4] for a proof in the case of the light-cone, and [7] for general symmetric cones).

LEMMA 4.1 *Let* $0 < p, q < \infty$ *and* $\nu > \frac{2n}{r} - 1$. *Then, the norms (or quasi-norms) of the spaces* $A_\nu^{p,q}(T_\Omega)$ *are complete. Moreover, the intersection* $A_{\nu'}^{p',q'} \cap A_\nu^{p,q}$ *is dense in* $A_\nu^{p,q}(T_\Omega)$ *for any* p', q', ν' *in the range above.*

4.1   The necessity

We shall show that for $F \in A_\nu^{p,2}(T_\Omega)$, then (1.3) and (1.5) hold, even in the case $0 < p < \infty$. We assume first that $F \in A_\nu^2 \cap A_\nu^{p,2}$, so that by the Paley-Wiener characterization of A_ν^2 (see Chapter XIII of [6]) we have $F = \mathcal{L}f$ for some $f \in L^2(\Omega; \Delta(\xi)^{-(\nu - \frac{2n}{r})} d\xi)$. Then, using the Plancherel formula and the lemmas in §2, we obtain

$$
\begin{aligned}
\|F\|_{A_\nu^{p,2}}^p &= \int_\Omega \left[\int_{\mathbb{R}^n} |F(x+iy)|^2 \, dx \right]^{\frac{p}{2}} \Delta(y)^{\nu - \frac{2n}{r}} \, dy \\
&= (2\pi)^{\frac{np}{2}} \int_\Omega \left[\int_\Omega |f(\xi)|^2 e^{-2(y|\xi)} \, d\xi \right]^{\frac{p}{2}} \Delta(y)^{\nu - \frac{2n}{r}} \, dy \\
&\geq c \sum_j \int_{E_j^*} \left[\int_{E_j} |f(\xi)|^2 \, d\xi \right]^{\frac{p}{2}} e^{-p\gamma(y|\xi_j)} \Delta(y)^{\nu - \frac{2n}{r}} \, dy \\
&\geq c(p, \nu, \delta) \sum_j \frac{\left[\int_{E_j} |f(\xi)|^2 \, d\xi \right]^{\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}}.
\end{aligned}
$$

We have used the fact that $\Delta(y)$ is almost constant on E_j^*, and may be replaced by $\Delta(\xi_j)^{-1}$. This shows $\|F\|_{A_\nu^{p,2}} \geq c\|f\|_{b_\nu^{p,2}}$ for $F \in A_\nu^2 \cap A_\nu^{p,2}$.

For general $F \in A_\nu^{p,2}(T_\Omega)$ one proceeds by density. Taking a sequence F_n in $A_\nu^2 \cap A_\nu^{p,2}$, so that $F_n \to F$ in $A_\nu^{p,2}$, we obtain a corresponding sequence f_n so that

$$\sum_j \frac{\left[\int_{E_j} |(f_n - f_m)(\xi)|^2 \, d\xi\right]^{\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}} \leq c \, \|F_n - F_m\|_{A_\nu^{p,2}}^p.$$

This implies the existence of $f \in L_{\ell oc}^2(\Omega)$ so that $\|f - f_n\|_{b_\nu^{p,2}} \to 0$ as $n \to \infty$. Further, by Fatou's lemma we also obtain that $\|e^{-(y|\cdot)}(f - f_n)\|_{L^2(\Omega)} \to 0$, for all $y \in \Omega$, and $\|f\|_{b_\nu^{p,2}} \leq c\|F\|_{A_\nu^{p,2}}$. To see that formula (1.3) holds, note that for every $z = x + iy \in T_\Omega$

$$|\, F(z) - \int_\Omega f(\xi) e^{i(z|\xi)} \, d\xi \,| \leq |F(z) - F_n(z)| \, + \, c_y \|e^{-\frac{1}{2}(y|\cdot)}(f_n - f)\|_{L^2(\Omega)},$$

which goes to 0 as $n \to \infty$. $\qquad\square$

4.2 The sufficiency

For the sufficiency we take a function f on Ω satisfying (1.5). We shall show the following inequality

$$\int_\Omega \left|\, \int_\Omega |f(\xi)|^2 \, e^{-2(y|\xi)} \, d\xi \,\right|^{\frac{p}{2}} \Delta(y)^{\nu - \frac{2n}{r}} \, dy \, \leq \, c(p,\nu) \sum_j \frac{\left[\int_{E_j} |f(\xi)|^2 \, d\xi\right]^{\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}} \,.$$

$$(4.2)$$

From here we see that the integral in (1.3) is absolutely convergent for every $z \in T_\Omega$, and will define a holomorphic function F on T_Ω. Further, (4.2) together with the Plancherel theorem will give $\|F\|_{A_\nu^{p,2}} \leq c\|f\|_{b_\nu^{p,2}}$, completing the proof of the theorem.

We pass to the proof of (4.2). Note this is almost immediate when $0 < p \leq 2$. Indeed, then the power $p/2$ can go inside the sum and we obtain:

$$\int_\Omega \left|\, \int_\Omega |f(\xi)|^2 \, e^{-2(y|\xi)} \, d\xi \,\right|^{\frac{p}{2}} \Delta(y)^{\nu - \frac{2n}{r}} \, dy \leq$$

$$\leq \quad \int_\Omega \sum_j [\int_{E_j} |f(\xi)|^2 d\xi]^{\frac{p}{2}} \, e^{-\frac{p}{\gamma}(y|\xi_j)} \Delta(y)^{\nu - \frac{2n}{r}} \, dy$$

$$= \quad c(p,\nu) \, \Gamma_\Omega(\nu - \frac{n}{r}) \sum_j \frac{[\int_{E_j} |f(\xi)|^2 \, d\xi]^{\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}} \; .$$

We shall assume therefore, that $2 < p < p_\nu$. For simplicity in the notation, we call $q = \frac{p}{2}$ and $q' = \frac{q}{q-1} = \frac{p}{p-2}$. We also take a real multi-index $\mathbf{s} = (s_1, \ldots, s_r)$ whose precise value will be chosen below. Then, an application of Hölder's inequality gives

$$I := \int_\Omega \left| \int_\Omega |f(\xi)|^2 \, e^{-2(y|\xi)} \, d\xi \right|^{\frac{p}{2}} \Delta(y)^{\nu - \frac{2n}{r}} \, dy \; \leq$$

$$\int_\Omega \left(\sum_j [\int_{E_j} |f(\xi)|^2 d\xi]^q \, e^{-\frac{2}{\gamma}(y|\xi_j)} \Delta_{-\mathbf{s}}^q(\xi_j) \right) \left(\sum_j e^{-\frac{2}{\gamma}(y|\xi_j)} \Delta_{\mathbf{s}}^{q'}(\xi_j) \right)^{\frac{q}{q'}} \Delta(y)^{\nu - \frac{2n}{r}} \, dy.$$

Note that, by Proposition 3.1, the sum in the second parenthesis is bounded by:

$$c \int_\Omega e^{-(y|\xi)} \Delta_{\mathbf{s}}^{q'}(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} = c \, \Gamma_\Omega(q'\mathbf{s}) \, \Delta_{q'\mathbf{s}}(y^{-1}) = c(p,\mathbf{s}) \, \Delta_{-q'\mathbf{s}^*}(k_0 y),$$

and is finite whenever $q' s_j > (j-1) \frac{n/r - 1}{r-1}$, for all $j = 1, \ldots, r$. Inserting this expression in the integral above we are led to:

$$\begin{aligned}
I \quad &\leq \quad c \sum_j [\int_{E_j} |f(\xi)|^2 d\xi]^q \, \Delta_{-\mathbf{s}}^q(\xi_j) \int_\Omega e^{-\frac{2}{\gamma}(y|\xi_j)} \Delta_{-q\mathbf{s}^*}(k_0 y) \Delta(y)^{\nu - \frac{2n}{r}} \, dy \\
&= \quad c' \sum_j [\int_{E_j} |f(\xi)|^2 d\xi]^q \, \Delta_{-\mathbf{s}}^q(\xi_j) \int_\Omega e^{-(y|k_0 \xi_j)} \Delta_{-q\mathbf{s}^*}(y) \Delta(y)^{\nu - \frac{n}{r}} \frac{dy}{\Delta(y)^{\frac{n}{r}}} \\
&= \quad c' \, \Gamma_\Omega(\mathbf{t}) \sum_j [\int_{E_j} |f(\xi)|^2 d\xi]^q \, \Delta_{-\mathbf{s}}^q(\xi_j) \, \Delta_{-\mathbf{t}^*}(\xi_j),
\end{aligned}$$

where $\mathbf{t} = -q\mathbf{s}^* + (\nu - \frac{n}{r}, \ldots, \nu - \frac{n}{r})$, and we need the assumption

$$t_j = -q s_{r-(j-1)} + \nu - \frac{n}{r} > (j-1) \frac{n/r - 1}{r-1}, \quad \text{for all } j = 1, \ldots, r.$$

Note that $-\mathbf{t}^* = q\mathbf{s} - \left(\nu - \frac{n}{r}, \ldots, \nu - \frac{n}{r}\right)$, so that if we can find s_j's with the assumptions above the proof of (4.2) will be complete.

Now, for each $j = 1, \ldots, r$, the two assumptions on s_j can be written

$$\frac{1}{q'} \frac{j-1}{r-1} \left(\frac{n}{r} - 1\right) \; < \; s_j \; < \; \frac{1}{q}\left(\nu - \frac{n}{r} - \frac{r-j}{r-1}\left(\frac{n}{r} - 1\right)\right).$$

Using $\frac{1}{q'} = 1 - \frac{1}{q}$, we see that this is only possible if, for each $j = 1, \ldots, r$,

$$\frac{j-1}{r-1}\left(\frac{n}{r} - 1\right) \; < \; \frac{1}{q}\left(\nu - \frac{n}{r}\right) + \frac{1}{q}\left(\frac{2j-r-1}{r-1}\left(\frac{n}{r} - 1\right)\right),$$

or equivalently, if $\nu > \frac{2n}{r} - 1$ and

$$\frac{p}{2} = q < \min_{2 \le j \le r} \frac{\nu - \frac{n}{r} + \frac{2j-r-1}{r-1}\left(\frac{n}{r} - 1\right)}{\frac{j-1}{r-1}\left(\frac{n}{r} - 1\right)} = \frac{\nu - \frac{n}{r} + \frac{n}{r} - 1}{\frac{n}{r} - 1} = \frac{p_\nu}{2}.$$

These are precisely the ranges of p and ν assumed, so the sufficiency is proved.

□

4.3   The sharpness

To prove the sharpness of the theorem, we let $p = p_\nu = 2\frac{\nu-1}{n/r-1}$, and find a positive function g on Ω such that

$$\|g\|_{b^{p,2}_\nu}^p = \sum_j \frac{\left(\int_{E_j} |g(\xi)|^2 \, d\xi\right)^{\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}} < \infty$$

but

$$I(y) = \int_\Omega |g(\xi)|^2 \, e^{-(y|\xi)} \, d\xi = \infty, \quad \forall \, y \in \Omega.$$

Indeed, letting $g(\xi) = e^{-(\xi|\mathbf{e})}[\Delta(\xi)(1 + |\log \Delta_1^*(\xi)|)]^{-\frac{1}{2}}$, the second assertion follows immediately from Lemma 3.8. To see that the series above

converges note that, since $p = 2\frac{\nu-1}{n/r-1} > 2$, then

$$
\begin{aligned}
\|g\|_{b_\nu^{p,2}}^p &\leq c \sum_j \int_{E_j} \frac{e^{-\frac{p}{2}(\xi|\mathbf{e})}}{(1+|\log\Delta_1^*(\xi)|)^{\frac{p}{2}}} \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} \frac{\Delta(\xi_j)^{(\frac{n}{r}-1)\frac{p}{2}}}{\Delta(\xi_j)^{\nu-\frac{n}{r}}} \\
&\leq c' \int_\Omega \frac{e^{-\frac{p}{2}(\xi|\mathbf{e})}}{(1+|\log\Delta_1^*(\xi)|)^{\frac{p}{2}}} \frac{d\xi}{\Delta(\xi)} < \infty.
\end{aligned}
$$

Exactly the same example shows that the statement of the theorem cannot hold for $p \geq p_\nu$.

$\square$

5   The function spaces $b_\nu^{p,2}(\Omega)$ and $\mathcal{C}_\nu^{p,2}(T_\Omega)$

Let $0 < p < \infty$ and $\nu \in \mathbb{R}$. We say that a function $f \in L^2_{\ell oc}(\Omega)$ belongs to the space $b_\nu^{p,2}(\Omega)$ whenever the (quasi)-norm $\|f\|_{b_\nu^{p,2}}$ in (1.5) is finite. We point out that these spaces do not depend on the δ-lattice $\{\xi_j\}_j$ chosen for their definition. Indeed, this is just a simple consequence of the finite intersection property in Lemma 2.5.

In this section we shall exploit the properties of $b_\nu^{p,2}(\Omega)$ to obtain new (and sharp) results for the Bergman spaces $A_\nu^{p,2}(T_\Omega)$. We start by looking at the Bergman projection and its action into $b_\nu^{p,2}(\Omega)$.

5.1   The Bergman projection on $b_\nu^{p,2}(\Omega)$

Let us denote by P_ν the Bergman projection onto A_ν^2 alluded to in the introduction. Recall that the Bergman kernel, which reproduces the space A_ν^2, is given by

$$
B_\nu(z-\overline{w}) = c_\nu' \int_\Omega e^{i(z-\overline{w}|\xi)} \Delta(\xi)^{\nu-\frac{n}{r}} \, d\xi = d(\nu)\, \Delta^{-\nu}((z-\overline{w})/i),
$$

where the last equality follows from Lemma 3.4 (extended to complex values of y). The precise value of the constant $d(\nu)$ is calculated in

Chapter XIII of [6]:

$$d(\nu) = \frac{2^{\nu r}}{(4\pi)^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - \frac{n}{r})}, \quad \text{if } \nu > \frac{2n}{r} - 1.$$

Thus, for $F \in L^2_\nu(T_\Omega)$ we have

$$P_\nu F(x + iy) = \int_V \int_\Omega B_\nu(x - u + i(y + v)) f(u + iv) \, \Delta(v)^{\nu - \frac{2n}{r}} \, dv du. \quad (5.1)$$

Since $P_\nu F \in A^2_\nu(T_\Omega)$, it can be written as the Fourier-Laplace transform of some $f \in b^2_\nu(\Omega) = L^2(\Omega; \Delta(\xi)^{-(\nu - n/r)} \, d\xi)$:

$$P_\nu F(x + iy) = \int_\Omega e^{i(x + iy|\xi)} f(\xi) \, d\xi, \quad x + iy \in T_\Omega. \quad (5.2)$$

A simple Fourier inversion formula in (5.1) and (5.2) gives the following result.

LEMMA 5.3 *Let* $\nu > \frac{2n}{r} - 1$. *The operator* P_ν *regarded from* $L^2_\nu(T_\Omega)$ *into* $b^2_\nu(\Omega)$ *has the form:*

$$F \longmapsto f(\xi) = c_\nu \, \Delta(\xi)^{\nu - \frac{n}{r}} \int_\Omega e^{-(v|\xi)} \hat{F}(\xi, v) \Delta(v)^{\nu - 2\frac{n}{r}} \, dv, \quad \xi \in \Omega. \quad (5.4)$$

In (5.4) we are denoting by $\hat{F}(\xi, v)$ the inverse Fourier transform of F in the x-variable, for $v \in \Omega$ fixed:

$$\hat{F}(\xi, v) = \frac{1}{(2\pi)^n} \int_V e^{-i(x|\xi)} F(x + iv) \, dx.$$

We use equality (5.4) to extend the definition of P_ν as an operator (densely) defined on $L^{p,2}_\nu(T_\Omega)$, and taking values in $b^{p,2}_\nu(\Omega)$. The main result in this section is then the following theorem. As usual $\frac{1}{p} + \frac{1}{p'} = 1$.

THEOREM 5.5 *Let* $\nu > \frac{2n}{r} - 1$ *and* $p'_\nu < p < \infty$. *Then,* P_ν *can be extended as a bounded operator from* $L^{p,2}_\nu(T_\Omega)$ *into* $b^{p,2}_\nu(\Omega)$.

18

PROOF:

It suffices to show the theorem for $2 \leq p < \infty$. Indeed, once this is done, and using the identification $A_\nu^{p,2}(T_\Omega) \equiv b_\nu^{p,2}(\Omega)$ in Theorem 1.4, we shall obtain that $P_\nu \colon L_\nu^{p,2}(T_\Omega) \to A_\nu^{p,2}(T_\Omega)$ is bounded for all $2 \leq p < p_\nu$. Since the projector P_ν is self-adjoint, this range will extend automatically to $p_\nu' < p < p_\nu$. But again, the identification in Theorem 1.4 gives the boundedness of $P_\nu \colon L_\nu^{p,2}(T_\Omega) \to b_\nu^{p,2}(\Omega)$ for $p > p_\nu'$.

Therefore, we consider $2 \leq p < \infty$ and $F \in L_\nu^{p,2}(T_\Omega) \cap L_\nu^2(T_\Omega)$, and look at the function f in (5.4) whose Fourier-Laplace transform equals $P_\nu F$. We shall show that $\|f\|_{b_\nu^{p,2}} \leq c \|F\|_{L_\nu^{p,2}}$. First of all, by the Cauchy-Schwarz's inequality, for all $\xi \in \Omega$ we have

$$
\begin{aligned}
|f(\xi)|^2 &\leq \Delta(\xi)^{2(\nu - \frac{n}{r})} \int_\Omega e^{-(v|\xi)} |\hat{F}(\xi, v)|^2 \Delta(v)^{\nu - \frac{2n}{r}} \, dv \int_\Omega e^{-(v|\xi)} \Delta(v)^{\nu - \frac{2n}{r}} \, dv \\
&= \Gamma_\Omega \left(\nu - \frac{n}{r} \right) \Delta(\xi)^{\nu - \frac{n}{r}} \int_\Omega e^{-(v|\xi)} |\hat{F}(\xi, v)|^2 \Delta(v)^{\nu - \frac{2n}{r}} \, dv.
\end{aligned}
$$

Therefore, calling $A_j(v) = \int_{E_j} |\hat{F}(\xi, v)|^2 \, d\xi$ we can write

$$
\begin{aligned}
\|f\|_{b_\nu^{p,2}}^p &= \sum_j \frac{\left(\int_{E_j} |f(\xi)|^2 \, d\xi \right)^{\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}} \\
&\leq c \, \Gamma_\Omega \left(\nu - \frac{n}{r} \right)^{\frac{p}{2}} \sum_j \frac{\Delta(\xi_j)^{(\nu - \frac{n}{r})\frac{p}{2}}}{\Delta(\xi_j)^{\nu - \frac{n}{r}}} \left(\int_\Omega e^{-\frac{1}{\gamma}(v|\xi_j)} A_j(v) \Delta(v)^{\nu - \frac{2n}{r}} \, dv \right)^{\frac{p}{2}}.
\end{aligned}
$$

Using Hölder's inequality with $\frac{p}{2} \geq 1$ we can majorize the last integral with

$$
\begin{aligned}
\left(\cdots \right)^{\frac{p}{2}} &\leq \int_\Omega A_j(v)^{\frac{p}{2}} \Delta(v)^{\nu - \frac{2n}{r}} \, dv \left(\int_\Omega e^{-\frac{(p/2)'}{\gamma}(v|\xi_j)} \Delta(v)^{\nu - \frac{2n}{r}} \, dv \right)^{\frac{p/2}{(p/2)'}} \\
&= c \, \Gamma_\Omega \left(\nu - \frac{n}{r} \right)^{\frac{p}{2} - 1} \Delta(\xi_j)^{-(\nu - \frac{n}{r})(\frac{p}{2} - 1)} \int_\Omega A_j(v)^{\frac{p}{2}} \Delta(v)^{\nu - \frac{2n}{r}} \, dv.
\end{aligned}
$$

Thus,

$$
\|f\|_{b_\nu^{p,2}}^p \leq c \, \Gamma_\Omega \left(\nu - \frac{n}{r} \right)^{p-1} \sum_j \int_\Omega A_j(v)^{\frac{p}{2}} \Delta(v)^{\nu - \frac{2n}{r}} \, dv
$$

$$\leq\ c'\int_\Omega \left(\sum_j A_j(v)\right)^{\frac{p}{2}} \Delta(v)^{\nu-\frac{2n}{r}}\,dv$$

$$\leq\ c''\int_\Omega \left(\int_V |F(x+iv)|^2\,dx\right)^{\frac{p}{2}} \Delta(v)^{\nu-\frac{2n}{r}}\,dv\ =\ \|F\|^p_{L^{p,2}_\nu}.$$

By density of $L^{p,2}_\nu \cap L^2_\nu$ on $L^{p,2}_\nu(T_\Omega)$ the theorem follows. $\qquad\square$

As an immediate consequence, as we already said, we obtain the boundedness of P_ν on $L^{p,2}_\nu(T_\Omega)$ for $p'_\nu < p < p_\nu$. To finish the proof of Theorem 1.1, it is sufficient to prove the sharpness of this result. It is given at the end of this section.

5.2 Duality in $b^{p,2}_\nu(\Omega)$ and $A^{p,2}_\nu(T_\Omega)$

The space $b^{p,2}_\nu(\Omega)$ can be identified with $\ell^p_w(L^2(E_j))$, where $w = \{w_j\}$ is the weight given by $w_j = \Delta(\xi_j)^{-(\nu-\frac{n}{r})}$, via the correspondence

$$f \in b^{p,2}_\nu(\Omega) \longmapsto \{f|_{E_j}\} \in \ell^p_w(L^2(E_j)).$$

Therefore, we have the following duality result:

LEMMA 5.6 *Let $\nu \in \mathbb{R}$, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $(b^{p,2}_\nu)^* = b^{p',2}_\nu$, with the duality pairing*

$$\langle f, g \rangle = \sum_j \int_{E_j} f(\xi)\,g(\xi)\,\frac{d\xi}{\Delta(\xi_j)^{\nu-\frac{n}{r}}},$$

for $f \in b^{p',2}_\nu$ and $g \in b^{p,2}_\nu$.

A consequence of the previous lemma and Theorem 1.4 is that the spaces $A^{p,2}_\nu(T_\Omega)$ are reflexive whenever $1 < p < p_\nu$. Further, one may identify $(A^{p,2}_\nu)^*$ with $b^{p',2}_\nu$. During the rest of this section we shall find a more explicit expression for $(A^{p,2}_\nu)^*$ as a space of holomorphic functions.

To begin with, we shall give a meaning to the Fourier-Laplace transform $\mathcal{L}f(z)$, $z \in T_\Omega$, for functions $f \in b^{p,2}_\nu$, when p belongs to a certain "good" range.

LEMMA 5.7 *Let $\nu > \frac{2n}{r} - 1$. Then, the integral in (1.3) converges absolutely for all $f \in b_\nu^{p,2}(\Omega)$ whenever $0 < p < \tilde{p}_\nu = \frac{2(\nu-1)}{(\frac{n}{r}-2)_+}$. In this case, $\mathcal{L}f(z)$ defines a holomorphic function on the tube T_Ω.*

PROOF:

It suffices to show that in the range above

$$\int_\Omega e^{-(\mathbf{e}|\xi)}\,|f(\xi)|\,d\xi < \infty. \tag{5.8}$$

Consider first the case when $0 < p \leq 2$. Then

$$
\begin{aligned}
\int_\Omega e^{-(\mathbf{e}|\xi)}\,|f(\xi)|\,d\xi
&\leq \left(\int_\Omega e^{-(\mathbf{e}|\xi)}\,\frac{|f(\xi)|^2}{\Delta(\xi)^\alpha}\,d\xi\right)^{\frac{1}{2}} \left(\int_\Omega e^{-(\mathbf{e}|\xi)}\,\Delta(\xi)^\alpha\,d\xi\right)^{\frac{1}{2}} \\
&\leq c\,\Gamma_\Omega(\alpha + \tfrac{n}{r})^{\frac{1}{2}} \left(\sum_j \frac{(\int_{E_j}|f(\xi)|^2\,d\xi)^{\frac{p}{2}}}{\Delta(\xi_j)^{\alpha\frac{p}{2}}}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Now, choosing $\alpha = \frac{2}{p}(\nu - \frac{n}{r})$ we see that the above expression is finite when $\alpha > -1$, which is always the case if $\nu > \frac{2n}{r} - 1$.

Suppose now that $2 \leq p < \infty$. Then, for some real number α to be chosen below we have

$$
\begin{aligned}
\int_\Omega e^{-(\mathbf{e}|\xi)}\,|f(\xi)|\,d\xi
&\leq c\sum_j \left(\int_{E_j}|f(\xi)|\Delta(\xi)^\alpha\,\frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}\right)e^{-\frac{1}{\gamma}(\mathbf{e}|\xi_j)}\Delta(\xi_j)^{\frac{n}{r}-\alpha} \\
&\leq c'\left[\sum_j\left(\int_{E_j}|f(\xi)|\Delta(\xi)^\alpha\,\frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}\right)^p\right]^{\frac{1}{p}}\left[\sum_j e^{-\frac{p'}{\gamma}(\mathbf{e}|\xi_j)}\Delta(\xi_j)^{p'(\frac{n}{r}-\alpha)}\right]^{\frac{1}{p'}} \\
&\leq c''\left[\sum_j\frac{(\int_{E_j}|f(\xi)|^2\,d\xi)^{\frac{p}{2}}}{\Delta(\xi)^{(\frac{n}{r}-2\alpha)\frac{p}{2}}}\right]^{\frac{1}{p}}\left[\int_\Omega e^{-(\mathbf{e}|\xi)}\Delta(\xi)^{p'(\frac{n}{r}-\alpha)}\,\frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}\right]^{\frac{1}{p'}}.
\end{aligned}
$$

Now, the integral on the right coincides with $\Gamma_\Omega(p'(\frac{n}{r}-\alpha))$, so for the finiteness of the above expression we require

$$\left(\frac{n}{r}-\alpha\right)p' > \frac{n}{r} - 1 \quad \text{and} \quad \left(\frac{n}{r}-2\alpha\right)\frac{p}{2} = \nu - \frac{n}{r}.$$

After a simple manipulation one sees this is equivalent to

$$\frac{1}{2}\frac{n}{r} - 1 < \frac{\nu - 1}{p}. \tag{5.9}$$

When the left hand side is positive we obtain the range $2 \leq p < \frac{2(\nu-1)}{\frac{n}{r}-2}$. When $n \leq 2r$ one easily sees that (5.9) is verified for all $p \geq 2$. $\qquad\square$

REMARK 5.10 The range of p given in the previous lemma is sharp for the absolute convergence of the integral in (1.3). This can be easily verified with the aid of Lemma 3.8, as we did in the last part of the previous section. Note further that $2 \leq p_\nu < \tilde{p}_\nu$.

REMARK 5.11 In addition, the index $\tilde{p}_\nu$ has the remarkable property that:

$$B_\nu(z + i\mathbf{e}) \in L_\nu^{p',2}(T_\Omega) \quad \Longleftrightarrow \quad 0 < p < \tilde{p}_\nu.$$

This can be easily verified with the help of Lemma 3.6. An immediate consequence is that P_ν cannot be boundedly extended to $L_\nu^{p,2}(T_\Omega)$ outside the range $\tilde{p}_\nu' < p < \tilde{p}_\nu$. Indeed, one may test with $F(z) = \chi_{Q(i\mathbf{e})}(z)\Delta(\Im m\, z)^{-\nu+\frac{2n}{r}}$, where $Q(i\mathbf{e})$ is a closed polydisk in Ω centered at $i\mathbf{e}$. Then, the mean value property for (anti)-holomorphic functions gives

$$P_\nu F(z) = c_\nu\, B_\nu(z + i\mathbf{e}), \quad z \in T_\Omega.$$

Since $B_\nu(z + i\mathbf{e})$ only belongs to $L_\nu^{p,2}(T_\Omega)$ when $p > \tilde{p}_\nu'$, we get the condition $p > \tilde{p}_\nu'$. The corresponding condition $p < \tilde{p}_\nu$ follows from the symmetry of P_ν.

We can now define a space of holomorphic functions on T_Ω, whenever $0 < p < \tilde{p}_\nu$:

$$B_\nu^{p,2}(T_\Omega) = \left\{ \mathcal{L}f \; : \; f \in b_\nu^{p,2}(\Omega) \right\},$$

with norm $\|\mathcal{L}f\|_{B_\nu^{p,2}(T_\Omega)} = \|f\|_{b_\nu^{p,2}(\Omega)}$. Since $\mathcal{L} : b_\nu^{p,2}(\Omega) \to \mathcal{H}(T_\Omega)$ is continuous and one-to-one, the expression $\|\mathcal{L}f\|_{B_\nu^{p,2}}$ is actually a (quasi)-norm, and $B_\nu^{p,2}(T_\Omega)$ a (quasi)-Banach space. Further, note that, when

$\frac{n}{r} \leq 2$, then $\tilde{p}_\nu = \infty$, and therefore $B_\nu^{p,2}(T_\Omega) = \mathcal{L}(b_\nu^{p,2})$ is well-defined in the whole range $0 < p < \infty$.

For the cases $\frac{n}{r} > 2$ and when $p \geq \tilde{p}_\nu$, the definition of $B_\nu^{p,2}(T_\Omega)$ must be done in terms of classes of equivalence. We shall denote by $\square$ the differential operator $\square = \Delta(\frac{\partial}{\partial z})$. We normalize it according to the identity:

$$\Delta(\tfrac{\partial}{\partial z})e^{i(z|\mathbf{e})} = \Delta(z)e^{i(z|\mathbf{e})}, \quad z \in T_\Omega.$$

We now let $\ell = \ell(\nu, p)$ be the smallest non-negative integer such that

$$\ell > \frac{n/r - 2}{2}\left(1 - \frac{\tilde{p}_\nu}{p}\right).$$

This choice of ℓ is taken so that $p < \tilde{p}_{\nu+\ell p} = \tilde{p}_\nu + \frac{2\ell p}{n/r-2}$, and therefore, functions in $B_{\nu+\ell p}^{p,2}(T_\Omega)$ can always be defined by means of the Fourier-Laplace transform. Thus, it makes sense to consider as an extension of the previous class the spaces:

$$B_\nu^{p,2}(T_\Omega) := \left\{ F \in \mathcal{H}(T_\Omega) \ : \ \square^\ell F \in B_{\nu+\ell p}^{p,2}(T_\Omega) \right\}.$$

These spaces are not null, as it follows from the existence of solutions to PDE's with constant coefficients in convex domains (see Theorem 9.4 in [8]). In general, however, $\|F\|_{B_\nu^{p,2}} := \|\square^\ell F\|_{B_{\nu+\ell p}^{p,2}}$ is only a semi-norm, so we will have to quotient with the space

$$\mathcal{J}_\ell = \left\{ F \in \mathcal{H}(T_\Omega) \ : \ \square^\ell F = 0 \right\}.$$

We shall denote this new space of classes of equivalence by

$$\mathcal{C}_\nu^{p,2}(T_\Omega) = B_\nu^{p,2}(T_\Omega)/\mathcal{J}_\ell, \quad \text{for } \ell = \ell(\nu, p),$$

and define its norm as $\|F\|_{\mathcal{C}_\nu^{p,2}} := \|\square^\ell F\|_{B_{\nu+\ell p}^{p,2}}$.

We collect the main properties of the spaces $\mathcal{C}_\nu^{p,2}$ in the next proposition.

PROPOSITION 5.12 *Let* $\nu > \frac{2n}{r} - 1$. *Then:*

1. If $0 < p < \tilde{p}_\nu$, then $\mathcal{C}_\nu^{p,2} = B_\nu^{p,2}$. In fact, if $F \in B_\nu^{p,2}$ and for some $m \geq 1$ we have $\square^m F = 0$, then $F = 0$.

2. If $0 < p < \infty$ and $m \geq \ell(\nu, p)$, then

$$\mathcal{I}_m \quad : \quad \mathcal{C}_\nu^{p,2} \longrightarrow b_\nu^{p,2}$$
$$F \longmapsto \quad \Delta^{-m}(\xi)\, \mathcal{L}^{-1}(\square^m F)(\xi)$$

   is an isomorphism of (quasi)-Banach spaces.

3. If $0 < p < \infty$ and $m \geq 0$, then

$$\square^m : \mathcal{C}_\nu^{p,2} \longrightarrow \mathcal{C}_{\nu+mp}^{p,2}$$

   is an isomorphism of (quasi)-Banach spaces.

PROOF:

1. This follows by the Fourier-Laplace representation. Indeed, if $F(z) = \mathcal{L}f(z) \in B_\nu^{p,2}$, then

$$\square^m F(z) = \int_\Omega e^{i(z|\xi)} \Delta^m(\xi)\, f(\xi)\, d\xi$$

   (at least in the sense of distributions). So, if $\square^m F = 0$, it follows that $f = 0$.

2. The result is elementary for $0 < p < \tilde{p}_\nu$, since when $F(z) = \mathcal{L}f(z) \in B_\nu^{p,2}$ we have

$$\Delta^{-m}(\xi)\, \mathcal{L}^{-1}(\square^m F)(\xi) = f(\xi), \quad m \geq 0,$$

   so one just uses the definition of $B_\nu^{p,2} = \mathcal{L}(b_\nu^{p,2})$. The general case $p \geq \tilde{p}_\nu$ can be easily reduced to this one by definition of the spaces $\mathcal{C}_\nu^{p,2}$, and using that

$$g(\xi) \longrightarrow \Delta^m(\xi)\, g(\xi)$$

   is an isomorphism from $b_\nu^{p,2}$ onto $b_{\nu+mp}^{p,2}$.

3. For $m \geq \ell(\nu, p)$, the fact that $\square^m$ is well-defined and injective is an immediate application of 2 above and the definition of the spaces. Let us see that the correspondence is onto. If $F \in B^{p,2}_{\nu+mp}(T_\Omega)$ and $F = \mathcal{L}f$, then $\Delta^{\ell-m}(\xi)f(\xi) \in b^{p,2}_{\nu+\ell p}$. Here we have chosen $\ell = \ell(\nu, p)$, and therefore, $p < \tilde{p}_{\nu+\ell p}$. Thus, we can say that

$$F_1(z) = \mathcal{L}(\Delta^{\ell-m}f)(z) \in B^{p,2}_{\nu+\ell p} \quad \text{and} \quad \square^{m-\ell}F_1(z) = \mathcal{L}f(z) = F(z).$$

Now, we invoke the existence theorem for PDE's to find $G \in \mathcal{H}(T_\Omega)$ so that $\square^\ell G = F_1$. Clearly, $G \in \mathcal{C}^{p,2}_\nu$ and $\square^m G = F$.

Finally, the general case $m \geq 0$, can be easy reduced to the previous one by the definition of the spaces involved.

$\square$

LEMMA 5.13 *Let $\nu > \frac{2n}{r} - 1$ and $0 < p < \infty$. Then*

$$\square \colon A^{p,2}_\nu(T_\Omega) \longrightarrow A^{p,2}_{\nu+p}(T_\Omega) \tag{5.14}$$

is bounded and injective.

PROOF: The boundedness is not difficult to show from the Cauchy integral formula for derivatives. A standard argument can be found in [4] (see Proposition 6.1 there). For the injectivity one uses the same argument we gave in the proof of (1) of the previous proposition.

$\square$

An immediate corollary is then the following:

COROLLARY 5.15 *Let $\nu > \frac{2n}{r} - 1$ and $0 < p < \infty$. Then, $A^{p,2}_\nu(T_\Omega) \hookrightarrow \mathcal{C}^{p,2}_\nu(T_\Omega)$.*

PROOF: In §4.1 we showed the (continuous) inclusion $\mathcal{L}^{-1}(A^{p,2}_\nu) \subset b^{p,2}_\nu$, for *all* $0 < p < \infty$. Thus, at least when $0 < p < \tilde{p}_\nu$ we have $A^{p,2}_\nu(T_\Omega) \hookrightarrow \mathcal{L}(b^{p,2}_\nu) = B^{p,2}_\nu(T_\Omega)$. The general case $0 < p < \infty$, follows immediately from the last lemma and the definition of the spaces $\mathcal{C}^{p,2}_\nu$.

$\square$

We can now state the main result in this section, which concerns the dual of the space $A_\nu^{p,2}(T_\Omega)$, when $1 < p < p_\nu$.

Given $\nu > \frac{2n}{r} - 1$ and $0 < p < \infty$, we let $\gamma(\nu, p)$ be the smallest non-negative integer so that

$$\gamma(\nu, p) > \frac{n/r - 1}{2}\left(1 - \frac{p_\nu}{p}\right).$$

Note that $\gamma(\nu, p)$ is chosen so that $p < p_{\nu + \gamma p}$. In particular, for all $\gamma \geq \gamma(\nu, p)$ we have

$$F \in \mathcal{C}_\nu^{p,2}(T_\Omega) \implies \square^\gamma F \in A_{\nu + \gamma p}^{p,2}(T_\Omega).$$

Indeed, this follows from Theorem 1.4 and Proposition 5.12.

Now we shall consider $1 < p < \infty$, $\gamma \geq \gamma(\nu, p')$ and define, for every $F \in \mathcal{C}_\nu^{p',2}(T_\Omega)$, the functional

$$\Phi_F^\gamma(G) = \int\int_{T_\Omega} \Delta^\gamma(\Im m\, w) \overline{\square^\gamma F(w)}\, G(w)\, \Delta^{\nu - \frac{2n}{r}}(\Im m\, w)\, dw, \quad G \in A_\nu^{p,2}(T_\Omega).$$

Note that with our assumptions $\Delta^\gamma(\Im m\, w)\, \square^\gamma F(w) \in L_\nu^{p',2}(T_\Omega)$, and therefore, Φ_F^γ belongs to $(A_\nu^{p,2})^*$. Further, by Proposition 5.12 we also have

$$\|\Phi^\gamma\| \leq \|\square^\gamma F\|_{A_{\nu + \gamma p}^{p',2}} \leq C\, \|F\|_{\mathcal{C}_\nu^{p',2}}.$$

THEOREM 5.16 *Let $\nu > \frac{2n}{r} - 1$, $1 < p < p_\nu$ and $\gamma \geq \gamma(\nu, p')$. Then*

$$\begin{aligned} \Phi^\gamma \quad &: \quad \mathcal{C}_\nu^{p',2}(T_\Omega) \longrightarrow (A_\nu^{p,2}(T_\Omega))^* \\ F \quad &\longmapsto \quad \Phi_F^\gamma \end{aligned}$$

is an (anti-linear) isomorphism of Banach spaces.

PROOF:

In view of the previous it suffices to show that Φ^γ is surjective. Let us therefore take $\Phi \in (A_\nu^{p,2}(T_\Omega))^*$. Then, we can define

$$\tilde\Phi(g) := \Phi(\mathcal{L}g), \quad g \in b_\nu^{p,2}(\Omega).$$

¿From Theorem 1.4 it follows that $\tilde{\Phi} \in (b_\nu^{p,2})^*$ and $\|\tilde{\Phi}\| \leq c\,\|\Phi\|$ (it is here where we need $1 < p < p_\nu$). Thus, by Lemma 5.6 there exists $\tilde{f}_\gamma \in b_\nu^{p',2}$ such that $\|\tilde{f}_\gamma\|_{b_\nu^{p',2}} = \|\tilde{\Phi}\|$ and

$$\tilde{\Phi}(g) = \sum_j \int_{E_j} \tilde{f}_\gamma(\xi) g(\xi)\, \frac{d\xi}{\Delta(\xi_j)^{\nu - \frac{n}{r}}}, \quad g \in b_\nu^{p,2}. \tag{5.17}$$

We define

$$f_\gamma(\xi) = \overline{\tilde{f}_\gamma(\xi)}\, \Delta^\gamma(\xi) \left(\sum_j \frac{\Delta(\xi)}{\Delta(\xi_j)}\, \chi_{E_j}(\xi) \right)^{\nu - \frac{n}{r}} \in b_{\nu + \gamma p'}^{p',2}(\Omega)$$

Then, letting $F_\gamma(z) = \mathcal{L} f_\gamma(z)$ and $G(z) = \mathcal{L} g(z)$, we can write (5.17) as

$$\begin{aligned} \tilde{\Phi}(g) &= \int_\Omega \Delta^{-\gamma}(\xi)\, \overline{f_\gamma(\xi)}\, g(\xi)\, \frac{d\xi}{\Delta(\xi)^{\nu - \frac{n}{r}}} \\ &= c_0 \int\!\!\int_{T_\Omega} \Delta^\gamma(\Im m\, w)\, \overline{F_\gamma(w)}\, G(w)\, \Delta^{\nu - \frac{2n}{r}}(\Im m\, w)\, dw, \end{aligned}$$

where the constant $c_0 > 0$ appears from Plancherel and Fubini's Theorems. Now, $F_\gamma \in A_{\nu + \gamma p'}^{p',2}$ and therefore by Proposition 5.12 there exists a function (actually a whole class) $F \in \mathcal{C}_\nu^{p',2}$ with $\Box^\gamma F = c_0\, F_\gamma$. This implies

$$\Phi(G) = \tilde{\Phi}(g) = \Phi_F^\gamma(G), \quad \forall\; G = \mathcal{L} g \in A_\nu^{p,2}(T_\Omega),$$

and hence $\Phi = \Phi_F^\gamma$. Further,

$$\begin{aligned} \|F\|_{\mathcal{C}_\nu^{p',2}} &\leq c_1\, \|\Box^\gamma F\|_{A_{\nu + \gamma p'}^{p',2}} \leq c_2\, \|f_\gamma\|_{b_{\nu + \gamma p'}^{p',2}} \\ &\leq c_3\, \|\tilde{f}_\gamma\|_{b_\nu^{p',2}} \leq c_4\, \|\Phi\|, \end{aligned}$$

establishing the theorem.

$\square$

5.3 Extensions of the Bergman projector P_ν

We already proved that the Bergman projector

$$P_\nu F(z) = \int \int_{T_\Omega} B_\nu(z - \overline{w}) \, F(w) \, \Delta(\Im m \, w)^{\nu - \frac{2n}{r}} \, dw, \quad z \in T_\Omega,$$

originally defined in $L^2(T_\Omega)$, can be boundedly extended to $L_\nu^{p,2}(T_\Omega)$ at least when $p'_\nu < p < p_\nu$. In this section we shall show that, as a densely defined and symmetric operator, P_ν does not admit continuous extensions to $L_\nu^{p,2}(T_\Omega)$ for p outside this range.

On the other hand, we proved in Theorem 5.5 that passing to the frequency domain one can regard P_ν as bounded operator taking values into $b_\nu^{p,2}(\Omega)$. The main task in this section is to show that P_ν can also be seen as an operator taking values in the spaces of holomorphic functions $\mathcal{C}_\nu^{p,2}$ we introduced below.

First of all, note that the kernel $B_\nu(z - \overline{\cdot}) \in L_\nu^{p',2}$, for all $p < \tilde{p}_\nu$, and therefore, the integral defining $P_\nu F(z)$, when $F \in L_\nu^{p,2}(T_\Omega)$, converges absolutely. Further, we know from Theorem 5.5 and the identification $B_\nu^{p,2}(T_\Omega) = \mathcal{L}(b_\nu^{p,2})$ that $P_\nu \colon L_\nu^{p,2}(T_\Omega) \to B_\nu^{p,2}$ is bounded when $p'_\nu < p < \tilde{p}_\nu$. Thus, at least in this range, we have obtained a natural bounded extension of P_ν taking values in a space of holomorphic functions.

When $p \geq \tilde{p}_\nu$ there is also a natural extension, but one needs some more care to describe it, since the integral defining $P_\nu F(z)$ is not necessarily convergent. However, if we wish to define $P_\nu F$ as an equivalence class in $\mathcal{C}_\nu^{p,2}$, it suffices to determine $\square^\ell P_\nu F$, when $\ell = \ell(\nu, p)$. The obvious definition, that extends the original projector, is then the following:

$$\square^\ell P_\nu F(z) := c_{\nu + \ell} \int \int_{T_\Omega} B_{\nu + \ell}(z - \overline{w}) \, f(w) \, \Delta(\Im m \, w)^{\nu - \frac{2n}{r}} \, dw, \quad z \in T_\Omega,$$

This integral is absolutely convergent, since our choice of $\ell = \ell(\nu, p)$ guarantees $B_{\nu + \ell}(z + i\mathbf{e}) \in L_\nu^{p',2}(T_\Omega)$ (see Lemma 3.6). Following the same lines as in §5.1 one writes the frequency representation of this operator:

$$\mathcal{L}^{-1}(\square^\ell P_\nu F)(\xi) = c(\nu, \ell) \, \Delta(\xi)^{\nu + \ell - \frac{n}{r}} \int_\Omega e^{-(v|\xi)} \, \hat{F}(\xi, v) \, \Delta(v)^{\nu - 2\frac{n}{r}} \, dv, \quad \xi \in \Omega.$$

28

Then, a new use of Theorem 5.5 gives $\Delta^{-\ell}\mathcal{L}^{-1}(\square^\ell P_\nu F) \in b_\nu^{p,2}$ and

$$\|P_\nu F\|_{\mathcal{C}_\nu^{p,2}} = \|\Delta^{-\ell}\mathcal{L}^{-1}(\square^\ell P_\nu F)\|_{b_\nu^{p,2}} \leq C\,\|F\|_{L_\nu^{p,2}(T_\Omega)}.$$

Thus, the operator $P_\nu\colon L_\nu^{p,2} \to \mathcal{C}_\nu^{p,2}$ is well-defined and bounded when $p_\nu' < p < \infty$.

We claim something more, which we already stated as a theorem in the introduction.

THEOREM 5.18 *Let $\nu > \frac{2n}{r} - 1$ and $p_\nu' < p < \infty$. Then, P_ν can be extended as a bounded operator from $L_\nu^{p,2}(T_\Omega)$ onto $\mathcal{C}_\nu^{p,2}(T_\Omega)$.*

PROOF:

In view of the previous comments, we just need to show that $P_\nu(L_\nu^{p,2}(T_\Omega)) = \mathcal{C}_\nu^{p,2}$. Let $F \in \mathcal{C}_\nu^{p,2} \equiv (A_\nu^{p',2})^*$. Then, for $\gamma = \gamma(\nu,p)$, the functional $G \in A_\nu^{p',2} \mapsto \Phi_F^\gamma(G)$ extends (by the Hahn-Banach Theorem) continuously to $L_\nu^{p',2}(T_\Omega)$. Thus, there exists $\varphi \in L_\nu^{p,2}(T_\Omega)$ so that

$$\begin{aligned}
\Phi_F^\gamma(G) &= \int\!\!\int_{T_\Omega} \Delta^\gamma(\Im m\, w)\,\overline{\square^\gamma F(w)}\,G(w)\,\Delta^{\nu-\frac{2n}{r}}(\Im m\, w)\,dw \\
&= \int\!\!\int_{T_\Omega} \overline{\varphi(w)}\,G(w)\,\Delta^{\nu-\frac{2n}{r}}(\Im m\, w)\,dw.
\end{aligned}$$

Now, for each $z \in T_\Omega$, we test with

$$G_z(w) := B_{\nu+\ell}(w - \overline{z}) \in L_\nu^{p',2}(T_\Omega),$$

to obtain:

$$\begin{aligned}
\square^\ell P_\nu(\varphi)(z) &= c_{\nu+\ell} \int\!\!\int_{T_\Omega} \varphi(w)\,B_{\nu+\ell}(z - \overline{w})\,\Delta^{\nu-\frac{2n}{r}}(\Im m\, w)\,dw \\
&= c_{\nu+\ell} \int\!\!\int_{T_\Omega} \Delta^\gamma(\Im m\, w)\,\square^\gamma F(w)\,B_{\nu+\ell}(z - \overline{w})\,\Delta^{\nu-\frac{2n}{r}}(\Im m\, w)\,dw \\
&= c_{\nu+\ell}' \int_\Omega \int_\Omega (\mathcal{L}^{-1}\square^\gamma F)(\xi)\,e^{i(z|\xi)}\,\Delta(\xi)^{\nu+\ell-\frac{n}{r}}\,e^{-2(v|\xi)}\,d\xi\,\Delta^{\gamma+\nu-\frac{2n}{r}}(v)\,dv \\
&= c \int_\Omega \Delta^{-\gamma+\ell}(\xi)\,(\mathcal{L}^{-1}\square^\gamma F)(\xi)\,e^{i(z|\xi)}\,d\xi,
\end{aligned}$$

where in the middle equality we have used Parseval and the formula for $\mathcal{L}^{-1}B_{\nu+\ell}(\cdot + \overline{z})$. Now, since $\square^\ell F \in B_{\nu+\ell p}^{p,2}$, we must have

$$(\mathcal{L}^{-1}\square^\gamma F)(\xi) = \Delta^{\gamma-\ell}(\xi)\,(\mathcal{L}^{-1}\square^\ell F)(\xi).$$

Thus, it follows that $\square^\ell P_\nu(c^{-1}\,\varphi) = \square^\ell F$, establishing the theorem.

$\square$

This ends the proof of Theorem 1.2. It remains to prove that the Bergman projection is unbounded outside the range $p'_\nu < p < p_\nu$. By Remark 5.11, we already know that it is unbounded for $p \geq \tilde{p}_\nu$. Using the last theorem as well as Proposition 5.12, we also know that, if it is bounded for some p such that $p'_\nu < p < \tilde{p}_\nu$, then $A_\nu^{p,2}(T_\Omega)$ identifies with $B_\nu^{p,2} = \mathcal{L}(b_\nu^{p,2})$. Since Theorem 1.4 is sharp, this means that $p < p_\nu$. The condition $p > p'_\nu$ follows by symmetry.

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