Möbius invariance of analytic Besov spaces in tube domains over symmetric cones.

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Abstract

Besov spaces of holomorphic functions in tubes over cones have been recently defined by Békollé *et al.* in [9]. In this paper we show that Besov p-seminorms are invariant under conformal transformations of the domain when n/r is an integer, at least in the range $2 - r/n < p \leq \infty$.

1 Introduction

In the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \; : \; y > 0\}$, when $1 < p \leq \infty$, the *analytic Besov p-space*, $\mathbb{B}^p(\mathbb{H})$, consists of all holomorphic functions $f(z)$ so that

$$\|f\|_{\mathbb{B}^p} \;:=\; \Big\| y\, f'(z) \Big\|_{L^p(\mathbb{H}, dz/y^2)} \;<\; \infty.$$

An easy computation shows that these seminorms are Möbius invariant, that is,

$$\|f \circ \Phi\|_{\mathbb{B}^p} \;=\; \|f\|_{\mathbb{B}^p}, \quad \forall\, \Phi \in \text{Aut}\,(\mathbb{H}). \tag{1.1}$$

When $0 < p \leq 1$, Besov spaces are defined by the condition

$$\big\| y^m f^{(m)} \big\|_{L^p(\mathbb{H}, dz/y^2)} < \infty,$$

where m is the smallest integer so that $m > 1/p$. In this case there is also an equivalent seminorm so that (1.1) holds, although the proof is harder (see [4] or [14, Thm 5.18]).

This family of analytic Besov spaces (actually in the unit disc setting) was extensively studied in the 80's by Arazy, Fisher and Peetre [4, 5, 6, 7] (see also the text by K. Zhu [14]). The Möbius invariance of $\mathbb{B}^p$-norms is a relevant property which is related with a number of remarkable results, such as the characterization of (big) Hankel operators belonging to the Schatten

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p-class [7]. Another interesting fact says that all Möbius invariant semi-Banach spaces $\mathbb{X}$ of analytic functions must necessarily satisfy $\mathbb{B}^1 \hookrightarrow \mathbb{X} \hookrightarrow \mathbb{B}^\infty$, while $\mathbb{B}^2$ is the only semi-Hilbert space with such property (see eg [11]).

In higher dimensions, the right setting for these questions is the class of bounded symmetric domains $D \subset \mathbb{C}^n$ (or their unbounded realizations), in which case the rank r of the domain plays a role. When D is the unit ball of $\mathbb{C}^n$ (ie the case of rank 1), invariant Besov spaces of analytic functions are quite well understood; see eg [12] or [16]. For higher ranks, however, the picture is not yet complete. There is a general theory of analytic Besov spaces in bounded symmetric domains developed by K. Zhu [15], but Möbius invariance is not considered there; in fact it is left as an open question in [17, p. 300]. A different approach is taken by Arazy in [1, 2, 3], which in the special case of *bounded symmetric domains of tube type* defines a family of Besov p-spaces which are indeed $\mathrm{Aut}(D)$-invariant when $\frac{n}{r} \in \mathbb{N}$ and $2 - \frac{r}{n} < p \leq \infty$ (see [3, p. 119], or section 4 below). The condition $\frac{n}{r} \in \mathbb{N}$ turns out to be necessary, while the cases $0 < p \leq 2 - r/n$, that require a different definition, seem to be still open.

In this paper we are interested in Möbius invariance of Besov spaces in *tube domains over cones*, that is the unbounded realization of the domains considered by Arazy. More precisely, $T_\Omega = \{z = x + iy \in \mathbb{C}^n \;:\; y \in \Omega\}$, where Ω is an irreducible symmetric cone in $\mathbb{R}^n$. Analytic Besov spaces in T_Ω have recently been introduced in [9] in relation with a difficult and still open problem about boundedness of Bergman projections (see also [8]). Namely, $\mathbb{B}^p(T_\Omega)$ consists of all functions $F \in \mathcal{H}(T_\Omega)$ so that

$$\|F\|_{\mathbb{B}^p(T_\Omega)} = \left\|\Delta(y)^m \,\Box^m F(z)\,\right\|_{L^p(T_\Omega, d\lambda(z))} < \infty, \tag{1.2}$$

where m is a sufficiently large integer, and $d\lambda(z)$, $\Delta(y)$ and $\Box$ denote respectively the $\mathrm{Aut}(T_\Omega)$-invariant measure in T_Ω, the determinant function of the cone, and the generalized wave operator $\Box = \Delta(\frac{\partial}{\partial z})$ (see §2 for precise definitions). One of the non-trivial questions studied in [9] concerns the smallest number of derivatives m so that the $\mathbb{B}^p$-seminorms in (1.2) are all equivalent.

The purpose of this note is to give a proof of the Möbius invariance of $\mathbb{B}^p(T_\Omega)$ spaces, at least in the same range considered by Arazy for bounded domains. Rather than trying to transfer the result from the bounded setting (which does not seem so straightforward to us), we have preferred to give a direct proof in tube domains, based entirely on elementary properties of symmetric cones from the text [10], and independent of the results in [2, 3].

THEOREM 1.3 *Suppose that $\frac{n}{r} \in \mathbb{N}$ and $2 - \frac{r}{n} < p \leq \infty$. Then the holomorphic Besov space $\mathbb{B}^p(T_\Omega)$ is invariant under conformal transformations of the tube domain T_Ω, that is*

$$\|F \circ \Phi\|_{\mathbb{B}^p} = \|F\|_{\mathbb{B}^p}, \quad \forall\, \Phi \in Aut\,(T_\Omega). \tag{1.4}$$

In (1.4) is understood that we use the $\mathbb{B}^p$-seminorm defined in (1.2) with $m = \frac{n}{r}$, which is indeed an admissible exponent when $p > 2 - r/n$ by the results in [9]. In §3 we also give explicit examples showing that $\frac{n}{r} \in \mathbb{N}$ is a necessary condition for (1.4) (see Remark 3.11), but as in the bounded setting we do not know yet whether Möbius invariance may hold in the range $0 < p \leq 2 - r/n$ (except of course in the 1-dimensional case).

The proof of the theorem is mainly based on the explicit formula

$$\Box^{\frac{n}{r}} \big[F \circ \varphi\big](z) \;=\; J_\varphi(z)\, \big[\Box^{\frac{n}{r}} F\big](\varphi(z)), \quad \varphi \in \mathrm{Aut}\ (T_\Omega), \tag{1.5}$$

which in the bounded setting is known as "intertwining formula" of Arazy, ie when $\varphi \in \mathrm{Aut}(D)$ and D denotes the bounded realization of T_Ω (see [2, Theorem 6.4]). We could not find a reference for (1.5) in the unbounded setting of T_Ω, and for this reason we give a proof in Proposition 3.4 below (which is different from Arazy's). We remark that (1.5) is trivial for linear transformations, so the main case becomes $\varphi(z) = -z^{-1}$, for which the identity takes the form

$$\Box^{\frac{n}{r}} \big[F(-z^{-1})\big] \;=\; \Delta^{-\frac{2n}{r}}(z)\, \big[\Box^{\frac{n}{r}} F\big](-z^{-1}).$$

In §4 we show a variant of (1.5) with φ replaced by the *Cayley transform* c, which maps conformally D into T_Ω. As a consequence we obtain that the Cayley transform actually induces an isometry between $\mathbb{B}^p(T_\Omega)$ and the Besov space $\mathbb{B}^p(D)$ of Arazy. This gives a direct passage between the two settings which may be of independent interest; see Theorem 4.1 below.

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2 Definitions

We denote by $T_\Omega = \mathbb{R}^n + i\Omega$ the tube domain in $\mathbb{C}^n$ based on a cone Ω, which we assume irreducible and symmetric with respect to the usual inner product $(\cdot|\cdot)$ in $\mathbb{R}^n$. We write r for the rank of Ω and $\Delta(y)$ for the associated determinant function, as in the text [10].

Consider the (complex) differential operator $\Box = \Delta(\frac{\partial}{\partial z})$ given by the equality:

$$\Box\, [e^{(z|\xi)}] = \Delta(\xi) e^{(z|\xi)}, \quad z, \xi \in \mathbb{C}^n. \tag{2.1}$$

This is the usual derivative $\frac{d}{dz}$ when the rank is 1 (ie, in the upper half plane), and the complex wave operator $\Box = \frac{1}{4}(\partial^2_{z_1} - \partial^2_{z_2} - \ldots - \partial^2_{z_n})$ when $r = 2$. Observe that $\Box = \Delta(\frac{\partial}{\partial x}) = \Delta(\frac{1}{i}\frac{\partial}{\partial y})$ when acting on holomorphic functions in T_Ω.

Definition 2.2 For $1 \leq p \leq \infty$, we say that a holomorphic function $F(z)$ in T_Ω belongs to the *Besov space* $\mathbb{B}^p = \mathbb{B}^p(T_\Omega)$ when

$$\big\| \Delta^m(\Im m \cdot)\, \Box^m F \big\|_{L^p(T_\Omega, d\lambda)} < \infty, \tag{2.3}$$

where $d\lambda = \Delta^{-\frac{2n}{r}}(y)dz$ is the invariant measure under conformal transformations of T_Ω, and m is the smallest integer such that

$$m > \max\left\{ (\tfrac{2n}{r} - 1)\tfrac{1}{p},\ (\tfrac{n}{r} - 1)(1 - \tfrac{1}{p}) + \tfrac{1}{p} \right\}. \tag{2.4}$$

When $p = \infty$, we call $\mathbb{B}^\infty$ the *Bloch space* of T_Ω.

Remark 2.5 It is shown in [9] that different integers m as in (2.4) lead to equivalent seminorms in $\mathbb{B}^p$. In this paper, *we shall only consider the case $m = \frac{n}{r} \in \mathbb{N}$*, that is we set

$$\|F\|_{\mathbb{B}^p} \equiv \left\| \Delta^{\frac{n}{r}}(\Im m\,\cdot)\, \Box^{\frac{n}{r}} F \right\|_{L^p(T_\Omega, d\lambda)}, \tag{2.6}$$

which in view of (2.4) forces the restriction $2 - \frac{r}{n} < p \leq \infty$. Thus, $\mathbb{B}^p(T_\Omega)$ is the semi-Banach space defined by (2.6), which can be made into a Banach space if we consider $\mathbb{B}^p/\ker \Box^{\frac{n}{r}}$.

Some additional notation

As in [10], we consider $V = \mathbb{R}^n$ with the Jordan algebra structure induced by Ω, and denote by $\mathbf{e}$ its identity element. We shall write $G(\Omega)$ for the group of linear invertible transformations of $\mathbb{R}^n$ which leave the cone Ω invariant, and G for its identity component. It is well known that G acts transitively on Ω, which may be identified with the Riemannian symmetric space G/K, where K is the compact subgroup of elements of G which fix $\mathbf{e}$.

Below we shall use the following invariance property of Δ and $\Box$ under $g \in G(\Omega)$:

$$\Delta(gy) = \Delta(g\mathbf{e})\Delta(y), \quad \forall\, y \in \Omega \tag{2.7}$$

(see eg [10, p. 56]) and, for F holomorphic in T_Ω,

$$\Box\big[F(g\cdot)\big] = \Delta(g\mathbf{e})\, \big[\Box F\big](g\cdot). \tag{2.8}$$

The second formula follows from the first and the definition of $\Box$, by writing $F(z)$ as a Fourier-Laplace integral.

3 Results

Let $\mathrm{Aut}(T_\Omega)$ denote the group of conformal transformations of the tube T_Ω. It is well known (see e.g. [10, Theorem X.5.6]) that this group is generated by

(i) *Real translations*: $z \longmapsto z + u$, where $u \in \mathbb{R}^n$;

(ii) *Linear transformations:* $z \longmapsto gz$, where $g \in G(\Omega)$;

(iii) *Inversion:* $z \longmapsto -z^{-1}$,

where z^{-1} is the usual Jordan algebra inverse. The goal of this section is to prove the following:

Theorem 3.1 *Assume n/r is an integer. Then, for all $\Phi \in Aut(T_\Omega)$ and $F \in \mathcal{H}(T_\Omega)$ we have*

$$\left|\Delta^{\frac{n}{r}}(\Im m\, z)\,\big[\Box^{\frac{n}{r}}(F\circ\Phi)\big](z)\right| \;=\; \left|\Delta^{\frac{n}{r}}(\Im m\,(\Phi(z))\,\big[\Box^{\frac{n}{r}}F\big](\Phi(z))\right|, \quad z\in T_\Omega. \tag{3.2}$$

Theorem 1.3 is an immediate corollary of Theorem 3.1 and the definition of Besov norm in (2.6), since

$$\begin{aligned}\|F\circ\Phi\|_{\mathbb{B}^p} &= \left\|\Delta^{\frac{n}{r}}(\Im m\,\cdot)\,\Box^{\frac{n}{r}}[F\circ\Phi]\,\right\|_{L^p(T_\Omega,d\lambda)}\\ &= \left\|\Delta^{\frac{n}{r}}(\Im m\,\Phi(\cdot))\,[\Box^{\frac{n}{r}}F]\circ\Phi]\,\right\|_{L^p(T_\Omega,d\lambda)} = \|F\|_{\mathbb{B}^p}\,,\end{aligned}$$

where in the last equality we have changed variables and used the Aut(T_Ω)-invariance of $d\lambda$.

There are two special cases in which the identity (3.2) is easy to show, even when $\frac{n}{r}$ is replaced by any integer $k\geq 1$; namely, if Φ is a real translation or a linear transformation. Indeed, the first case is trivial since $\Box$ is translation invariant, while for $\Phi(z)=gz$ we have, from (2.7) and (2.8),

$$\Delta^k(y)\,\big[\Box^k(F\circ g)\big](z) \;=\; \Delta^k(gy)\,(\Box^k F)(gz),$$

where $z=x+iy$. Thus, from now on we will assume $\Phi(z)=-z^{-1}$. We also observe that, in this case, the analog of (3.2) with $\frac{n}{r}$ replaced by any other integer does not hold for general F, as can be already seen in the 1 dimensional setting (see also Remark 3.11 below).

Below we shall prove the following identity for $z\in T_\Omega$

$$\Delta^{\frac{n}{r}}(\Im m\, z)\,\Box^{\frac{n}{r}}\big[F(-z^{-1})\big] \;=\; \Delta^{\frac{n}{r}}(\Im m\,(-z^{-1}))\,\big[\Box^{\frac{n}{r}}F\big](-z^{-1})\,|\Delta(z)|^{\frac{2n}{r}}/\Delta(z)^{\frac{2n}{r}}, \tag{3.3}$$

which clearly implies (3.2) for $\Phi(z)=-z^{-1}$. Observe that

$$\begin{aligned}\Delta\big(\Im m\,(-z^{-1})\big) &= \Delta\big(\tfrac{\bar z^{-1}-z^{-1}}{2i}\big)\\ &= \Delta(\bar z)^{-1}\,\Delta(-z)^{-1}\,\Delta\big(\tfrac{\bar z-z}{2i}\big) \;=\; |\Delta(z)|^{-2}\,\Delta(\Im m\, z),\end{aligned}$$

where the second equality is justified in [10, p. 341]. Thus (3.3) can equivalently be formulated as follows.

Proposition 3.4 *If $z\in T_\Omega$ then*

$$\Box^{\frac{n}{r}}\big[F(-z^{-1})\big] \;=\; \Delta^{-\frac{2n}{r}}(z)\,\big[\Box^{\frac{n}{r}}F\big](-z^{-1}). \tag{3.5}$$

It suffices to prove (3.5) when $F(z)$ is a holomorphic polynomial. To do so, we use the decomposition of the vector space $\mathcal{P}$ of such polynomials as the direct sum $\bigoplus_{\mathbf{m}\geq\mathbf{0}}\mathcal{P}_{\mathbf{m}}$ which is described in [10, Ch. XI]. To describe these spaces, we denote by $\Delta_{\mathbf{m}}(y)$, $\mathbf{m}=(m_1,\dots,m_r)\in\mathbb{C}^r$, the *generalized power function of* Ω

$$\Delta_{\mathbf{m}}(y)=\Delta_1^{m_1-m_2}(y)\,\cdots\,\Delta_{r-1}^{m_{r-1}-m_r}(y)\,\Delta_r^{m_r}(y),\quad y\in\Omega,$$

where Δ_k are the principal minors with respect to a fixed Jordan frame $\{c_1, \dots, c_r\}$. This definition extends to $\Delta_{\mathbf{m}}(z/i)$ when $z \in T_\Omega$. By $\mathbf{m} \geq \mathbf{0}$ we mean that m_i are integers so that $m_1 \geq m_2 \geq \dots \geq m_r \geq 0$, in which case $\Delta_{\mathbf{m}}(z/i)$ are holomorphic polynomials. The subspaces $\mathcal{P}_{\mathbf{m}}$ are defined by

$$\mathcal{P}_{\mathbf{m}} = \text{span} \left\{ \Delta_{\mathbf{m}}(g^{-1}z) \ : \ g \in G(\Omega) \right\}.$$

The polynomials in $\mathcal{P}_{\mathbf{m}}$ are homogeneous of degree $|\mathbf{m}| = m_1 + \dots + m_r$.

Thus, we must show (3.5) for all $F \in \mathcal{P}_{\mathbf{m}}$ and $\mathbf{m} \geq \mathbf{0}$, which will be a consequence of the next two lemmas. Below we shall use the following standard notation: given $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ and $k \in \mathbb{N}$,

$$\begin{aligned} \big(\mathbf{s}\big)_k &= \Gamma_\Omega(\mathbf{s}+k)/\Gamma_\Omega(\mathbf{s}) \\ &= \prod_{i=1}^{r} \Big[\big(s_i - \tfrac{(i-1)d}{2}\big)\big(s_i - \tfrac{(i-1)d}{2} + 1\big) \cdots \big(s_i - \tfrac{(i-1)d}{2} + k - 1\big) \Big] \end{aligned}$$

(see [10, p. 129]). Here d is the integer satisfying $\frac{n}{r} = 1 + \frac{(r-1)d}{2}$, which coincides with the dimension of the subspaces $V_{i,j}$, $i < j$, from the Peirce decomposition of V.

Lemma 3.6 *Let $\mathbf{m} \geq \mathbf{0}$ and k a positive integer. Then for all $p \in \mathcal{P}_{\mathbf{m}}$*

$$\Delta(z)^k \, (\Box^k p)(z) \ = \ \big(\mathbf{m} + \tfrac{n}{r} - k\big)_k \, p(z). \tag{3.7}$$

Proof: This is a particular case of [10, Lemma XIV.2.1], but we sketch the proof for completeness. It suffices to prove the result for the generators of $\mathcal{P}_{\mathbf{m}}$, $p(z) = \Delta_{\mathbf{m}}(gz)$ with $g \in G(\Omega)$. Since $\Delta^k(z)\Box^k$ is $G(\Omega)$-invariant (by (2.8)), we may assume that g is the identity. Then the result follows from [10, Prop. VII.1.6].

$\Box$

Lemma 3.8 *Let $\mathbf{m} \geq \mathbf{0}$ and k a positive integer. Then for all $p \in \mathcal{P}_{\mathbf{m}}$*

$$\Delta^k(z) \left[\Box^k (p \circ \Phi)\right](z) \ = \ (-1)^{kr} \, \big(\mathbf{m}\big)_k \, p(\Phi(z)), \tag{3.9}$$

where $\Phi(z) = -z^{-1}$.

Proof: We shall obtain the result from the following identity

$$\int_\Omega e^{-(y|\xi)} p(\xi) \Delta^{\alpha - \frac{n}{r}}(\xi) \, d\xi \ = \ \Gamma_\Omega(\mathbf{m} + \alpha) \, \Delta^{-\alpha}(y) \, p(y^{-1}), \quad y \in \Omega, \tag{3.10}$$

valid for all complex numbers α with $\Re(\alpha) > \frac{n}{r} - 1$. The proof of (3.10) for the generators of $\mathcal{P}_{\mathbf{m}}$ is straightforward and can be found in [10, Lemma XI.2.3]. The identity continues to hold when y is replaced by z/i with $z \in T_\Omega$. Now, apply the operator $\Box^k$ to both sides of (3.10) to obtain

$$\begin{aligned} \Box^k \big[\Delta^{-\alpha}(z/i) \, p(iz^{-1})\big] &= \tfrac{1}{\Gamma_\Omega(\mathbf{m}+\alpha)} \int_\Omega \Box^k_z \big[e^{i(z|\xi)}\big] \, p(\xi) \Delta^{\alpha - \frac{n}{r}}(\xi) \, d\xi \\ &= \tfrac{1}{\Gamma_\Omega(\mathbf{m}+\alpha)} \int_\Omega e^{i(z|\xi)} \, \Delta^k(i\xi) \, p(\xi) \Delta^{\alpha - \frac{n}{r}}(\xi) \, d\xi \\ &= (-1)^{kr} \, \tfrac{\Gamma_\Omega(\mathbf{m}+\alpha+k)}{\Gamma_\Omega(\mathbf{m}+\alpha)} \, \Delta^{-\alpha}(z/i) \, \Delta^{-k}(z) \, p(iz^{-1}), \end{aligned}$$

where the last equality follows from a new application of (3.10). Since $\frac{\Gamma_\Omega(\mathbf{m}+\alpha+k)}{\Gamma_\Omega(\mathbf{m}+\alpha)} = (\mathbf{m}+\alpha)_k$ is a polynomial in α, the last expression is holomorphic in $\mathbb{C}$ (as a function of α), and hence also valid for $\alpha = 0$. But in this case we obtain precisely (3.9).

□

Combining the previous two lemmas with $k = \frac{n}{r} \in \mathbb{N}$ we see that, for $\mathbf{m} \geq \mathbf{0}$ and $p \in \mathcal{P}_{\mathbf{m}}$

$$\begin{aligned} \Delta(z)^{\frac{n}{r}} \, \Box^{\frac{n}{r}} \big[p(-z^{-1})\big] &= (-1)^n \, \big(\mathbf{m}\big)_{\frac{n}{r}} \, p(-z^{-1}) \\ &= (-1)^n \, \Delta(-z^{-1})^{\frac{n}{r}} \, \big[\Box^{\frac{n}{r}} p\big](-z^{-1}), \end{aligned}$$

which is the same as (3.5). This establishes Proposition 3.4, and hence completes the proof of Theorem 3.1.

□

Remark 3.11 A counterexample when $\frac{\mathbf{n}}{\mathbf{r}} \notin \mathbb{N}$. We claim that there are holomorphic functions $F \in \mathcal{H}(T_\Omega)$ with $\Box F = 0$ and $\Box^k[F \circ \Phi] \not\equiv 0$ for all $k \in \mathbb{N}$, where $\Phi(z) = -z^{-1}$. Thus, such functions F are null in $\mathbb{B}^p$, while $F \circ \Phi$ is not, so the seminorms in (2.3) cannot be Möbius invariant for any $0 < p \leq \infty$.

To see this, first observe that $\frac{n}{r} = 1 + \frac{(r-1)d}{2}$ is not an integer only when r is even and d is odd. That is when $\Omega = \Lambda_n$, the light-cone in $\mathbb{R}^n$ with n odd, or when $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$, the cone of positive definite symmetric matrices with r even (see [10, p. 97]). Consider the function

$$F(z) = \varphi_{\mathbf{m}}(z) = \int_K \Delta_{\mathbf{m}}(kz) \, dk,$$

which is the only K-invariant polynomial in $\mathcal{P}_{\mathbf{m}}$ (see e.g. [10, Ch. XI]). Then, by Lemma 3.6,

$$\Box \varphi_{\mathbf{m}}(z) \;=\; \Big[\prod_{i=1}^{r} \big(m_i + \tfrac{n}{r} - 1 - \tfrac{(i-1)d}{2}\big)\Big] \, \varphi_{\mathbf{m}}(z)/\Delta(z),$$

which is equal to 0 if we choose $m_r = 0$. On the other hand, by Lemma 3.8

$$\big[\Box^k(\varphi_{\mathbf{m}} \circ \Phi)\big](z) \;=\; \big(\mathbf{m}\big)_k \, \varphi_{\mathbf{m}}(-z^{-1}) \, \Delta^{-k}(-z),$$

which is a non zero function when $(\mathbf{m})_k \neq 0$. Let us see that we can choose such an index $\mathbf{m}$ in each of the two cases described above. When $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$ and r is even, this happens e.g. if we set $m_1 = \ldots = m_{r-1} = r/2$ and $m_r = 0$. Thus the function $F(z) = \varphi_{\mathbf{m}}(z)$ satisfies $\Box F = 0$ and $\Box^k[F \circ \Phi] \not\equiv 0$ for all $k \in \mathbb{N}$. When $\Omega = \Lambda_n$ with n odd, just choose $F(z) = \varphi_{(1,0)}(z) = z_1$ (see [10, p. 236]).

4 Besov spaces in the bounded realization of T_Ω

We denote by D the bounded symmetric domain of $\mathbb{C}^n$ which is mapped conformally onto T_Ω by the *Cayley transform*

$$c(w) = i(\mathbf{e}+w)(\mathbf{e}-w)^{-1}, \quad w \in D$$

(see [10, p. 190]). The Bergman kernel in D can be written as a constant multiple of $h(z,w)^{-2n/r}$ for some polynomial $h(z,w)$ (holomorphic in z and antiholomorphic in w) such that $h(x,x) = \Delta(\mathbf{e}-x^2)$, $x \in \mathbb{R}^n$ (see eg [10, p.201 and 262]). Denoting $h(w) = h(w,w)$, then

$$d\mu(w) = h(w)^{-2n/r} dw$$

is an $\mathrm{Aut}(D)$-invariant measure in D.

When $\frac{n}{r} \in \mathbb{N}$ and $2 - \frac{r}{n} < p \leq \infty$, Arazy defines Besov spaces in D as follows: $G \in \mathcal{H}(D)$ belongs to $\mathbb{B}^p(D)$ if

$$\|G\|_{\mathbb{B}^p(D)} \equiv \left\| h(w)^{\frac{n}{r}} \left(\Box^{\frac{n}{r}} G\right)(w) \right\|_{L^p(D,d\mu)} < \infty$$

(see [1, 3]). When $p = \infty$ or $p = 2$ one obtains respectively, generalized Bloch and Dirichlet spaces in D, the latter appearing also in the work of Z. Yan [13]. Our main result in this section, which we shall deduce from the identities in §3, is the following.

Theorem 4.1 *Let $\frac{n}{r} \in \mathbb{N}$ and $2 - \frac{r}{n} < p \leq \infty$. Then for every $F \in \mathcal{H}(T_\Omega)$ we have*

$$\left\| \Delta^{\frac{n}{r}}(\Im m \cdot) \Box^{\frac{n}{r}} F \right\|_{L^p(T_\Omega, d\lambda)} = 2^{n(1-\frac{2}{p})} \left\| h^{\frac{n}{r}} \Box^{\frac{n}{r}} \left(F \circ c\right) \right\|_{L^p(D,d\mu)}. \qquad (4.2)$$

In particular, $F \longmapsto F \circ c$ defines an isometric isomorphism between $\mathbb{B}^p(T_\Omega)$ and $\mathbb{B}^p(D)$.

We shall use the following identity, which can be derived easily from (3.5).

Lemma 4.3 *Let $F \in \mathcal{H}(T_\Omega)$. Then*

$$\Box^{\frac{n}{r}} \left[F \circ c\right](w) = (2i)^n \, \Delta(\mathbf{e}-w)^{-\frac{2n}{r}} \left[\Box^{\frac{n}{r}} F\right](c(w)), \quad w \in D. \qquad (4.4)$$

Proof: Define the elementary transformations

$$\tau(z) = z - i\mathbf{e}, \quad d(z) = -2iz, \quad \Phi(z) = -z^{-1},$$

and write $c(w) = -i\mathbf{e} + 2i(e-w)^{-1}$, so that

$$F \circ c(w) = \left[F \circ \tau \circ d \circ \Phi\right](\mathbf{e}-w), \quad w \in D.$$

Using the identity in (3.5), the invariance of $\Box$ under translations, and the trivial property $\Box(G \circ d) = (-2i)^r (\Box G) \circ d$ (since $\Box$ is homogeneous of degree r), we easily obtain (4.4).

□

Remark 4.5 Observe that (4.4) can also be written as

$$\Box^{\frac{n}{r}}\big[F\circ c\big](w) \;=\; J_c(w)\,\big[\Box^{\frac{n}{r}}F\big](c(w)),\quad w\in D,$$

where $J_c(w) = (2i)^n\,\Delta(\mathbf{e}-w)^{-\frac{2n}{r}}$ is the complex jacobian of the Cayley transform; see [10, p.52 and p.278].

We shall also use the following identity, which can be found in the text [10, p. 263]

$$h(w) \;=\; |\Delta(\mathbf{e}-w)|^2\,\Delta\big(\Im m\,[c(w)]\big),\quad w\in D\,. \tag{4.6}$$

Combining these two results, it is clear that for every $w\in D$ we have

$$h(w)^{\frac{n}{r}}\,\big|\Box^{\frac{n}{r}}\big(F\circ c\big)(w)\big| \;=\; 2^n\,\Delta^{\frac{n}{r}}\big(\Im m\,[c(w)]\big)\,\Big|\big(\Box^{\frac{n}{r}}F\big)(c(w))\Big|.$$

From here (4.2) follows easily, since the change of variables $z=c(w)$ transforms $d\lambda(z)$ into $4^n d\,\mu(w)$ (again see [10, p.263]). This proves Theorem 4.1.

□

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