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The characterization of wavelets and related functions and the connectivity of α-localized wavelets on $\mathbb{R}$.

by
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Contents

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Veritas lucet ipsa per se.

Chapter 1

Introduction

This thesis consists of two independent parts, each belonging to the field of what now can be called *wavelet theory*. In the first part (Chapter 2) we give a characterization of wavelets, and functions that generate more general expansions in $L^2(\mathbb{R}^n)$ (like frames or (φ, ψ)-transforms), in terms of two fundamental equations involving their Fourier transforms. In the second part (Chapter 3) we study some topological properties of a particular class of wavelets in $L^2(\mathbb{R})$: the α-localized ones, our main result being a complete description of this set of wavelets in connected components. Finally, in the appendix we present a few new examples of wavelets in $L^2(\mathbb{R})$ satisfying some pathological conditions.

The motivation to work on each of the problems treated in this dissertation arose after reading various articles, attending seminar lectures and having private conversations with different people from the wavelet field. Although most of the results we present are new, the reader need not have a deep knowledge of wavelet theory to follow the reasonings. Familiarity with the notion of wavelet and multiresoltion analysis will help, but to facilitate the task to the non-specialized reader we present in §§2, 3 of Chapter 1 the standard notation and main properties that will be used in the course of this thesis. However, a knowledge of real analysis (say, at the first year graduate level) will be necessary, since tools like the Fourier transform, the Plancherel Theorem or the Lebesgue Differentiation Theorem will be used without further reference. In §1 of Chapter 3 we introduce certain properties of Sobolev spaces, including known and less-known results, that will be used later on. Whenever possible, we give appropriate references to the standard literature.

1 History and motivation

Wavelets, with a slightly more general definition than the one we will use throughout this thesis (see Definition 2.1 below), were introduced in 1981 by the French geophysicist J. Morlet. Initially, they provided an excellent tool in engineerings for studying and analyzing signals. One considers a *signal* to be a function $f \in L^2(\mathbb{R})$ which, in order to be analyzed in full detail, needs to be "compared" with each element, ψ_i, of an appropriate family of functions $\{\psi_i\}_{i\in\mathcal{I}}$. One way of doing so is by taking the inner products (in $L^2(\mathbb{R})$): $(f, \psi_i), i \in \mathcal{I}$.

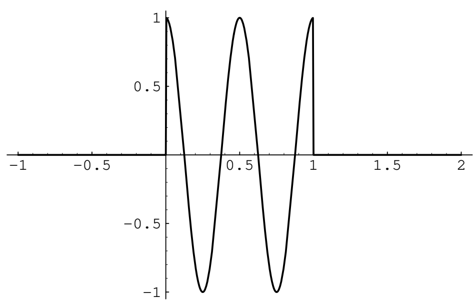
If we want a good and complete description of the signal f, each of the functions ψ_i must have support essentially concentrated in an interval $I_i \subset \mathbb{R}$, but in such a way that the union of all I_i's, $i \in \mathcal{I}$, covers the real line. In some sense, the inner products (f, ψ_i) give a "weighted" mean value of the signal f in the interval I_i. If the family $\{\psi_i\}_{i\in\mathcal{I}}$ is good enough, we can approximately synthesize the original signal f by adding, or "superimposing", the smaller signals $(f, \psi_i)\psi_i$; that is, $f = \sum_{i\in\mathcal{I}}(f, \psi_i)\psi_i$.

Here is an example of a family, $\{g_{k,\ell}\}$, that gives such a reconstruction:

$$g_{k,\ell}(x) = \chi(x-k)\, e^{2\pi i\ell x}, \quad k, \ell \in \mathbb{Z}, \tag{1.1}$$

where $\chi = \chi_{[0,1]}$ is the characteristic function of the interval $[0, 1]$. Indeed, suppose we are given the function:

$$f(x) = \begin{cases} \cos(4\pi x), & \text{if } 0 \leq x \leq 1 \\ 0, & \text{elsewhere,} \end{cases}$$



Then, f represents a signal that "lives" in the interval $[0,1]$ and, consequently, $(f, g_{k,\ell}) = 0$, if $k \neq 0$. On the other hand, when $k = 0$ we have $(f, g_{0,\ell}) = \int_0^1 \cos(4\pi x)\, e^{2\pi i \ell x}\, dx = \frac{1}{2}\delta_{2,|\ell|}$, $\ell \in \mathbb{Z}$, which in particular gives us the reconstruction formula:

$$f = \sum_{k,\ell} (f, g_{k,\ell})\, g_{k,\ell}. \tag{1.2}$$

For this signal, the reconstruction is "perfect", in the sense that the series above consists of just two non-zero terms. Suppose, instead, that we stretch or contract the signal f as in the figures below:

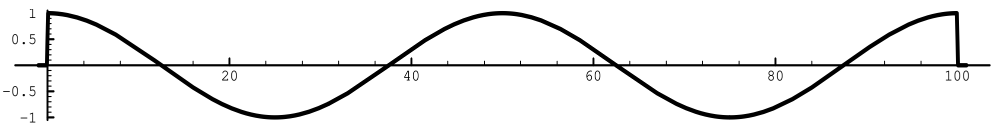


Figure 1.1: Graph of signal f_1.

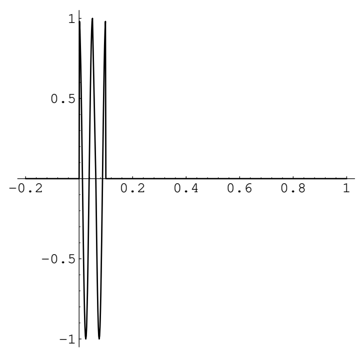
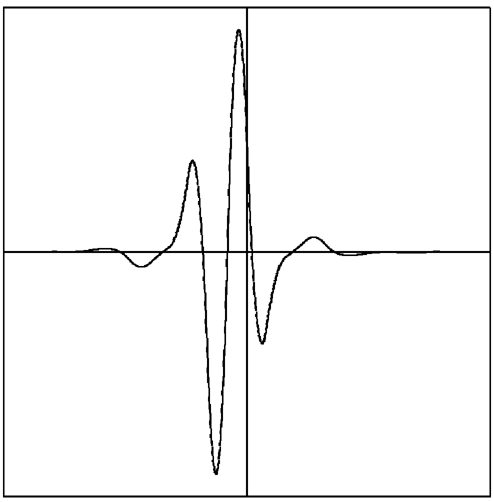


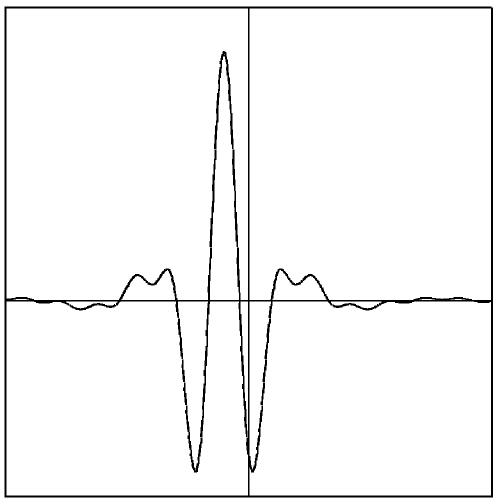
Figure 1.2: Graph of signal f_2.

Then, the analysis of these new signals obtained with the family $\{g_{k,\ell}\}$ is not so accurate this time, the main reason being that the *analyzing windows* $[k, k+1]$ are too "narrow" (for f_1) or too "wide" (for f_2) to provide us with exact information (that is, a finite number of non-zero terms in the series in (1.2)).

A family of functions like $\{g_{k,\ell}\}_{k,\ell}$ in (1.1) is an example of what is called a *windowed Fourier transform.* This type of technique was introduced by D. Gabor in 1946 to study signals that come out in the theory of communication, but for practical purposes has still many limitations. It was not until the eighties that J. Morlet introduced the *wavelet transform* to overcome the difficulties that appeared when analyzing too small or too long signals, as in the examples above. With this new method, the analyzing family, $\{\psi_{j,k}\}$, is created by appropriately stretching and contracting a single function ψ: $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$. In this way, if the support of ψ is concentrated in the interval $I = [0, 1]$ then, the supports of the functions $\psi_{j,k}$ must lie in $I_{j,k} = 2^{-j}(I+k) = [k2^{-j}, (k+1)2^{-j}]$, covering, therefore, the whole real line. This allows us to obtain precise information of all possible signals, like those in the examples above, by comparing them with the functions $\psi_{j,k}$. Suppose we want to identify the high frequencies in a given signal f (for instance, to remove the "noise"). Then, this can be done (if we assume that the generating function ψ has mean zero: $\int_{-\infty}^{\infty} \psi = 0$) by just considering the part of the sum $\sum_{j,k}(f, \psi_{j,k})\psi_{j,k}$ corresponding to large values of j. Indeed, a signal f with not too high frequencies will be almost "flat" in the small intervals $I_{j,k}$ (when j is large), and, therefore, will give coefficients $(f, \psi_{k,j}) \cong 0$. Functions ψ with these properties do exist, and are represented by "small waves" or *wavelets*, like in the figures below.



The Lemarié-Meyer wavelet



The Franklin wavelet

(We wish to thank Steve Xiao for his help in plotting these figures.)

In applications, it is sometimes of interest to have "perfect" reconstructions: $f = \sum_{j,k}(f, \psi_{j,k})\psi_{j,k}$, in the sense that the family $\{\psi_{j,k}\}$ forms an orthonormal basis of $L^2(\mathbb{R})$. Some other times, instead, one will be willing to have some "redundancy" in the family $\{\psi_{j,k}\}$, or even allow the possibility of taking a different "synthesizing" family, $\{\varphi_{j,k}\}$, giving a reconstruction of the type:

$$f = \sum_{j,k}(f, \psi_{j,k})\varphi_{j,k}. \tag{1.3}$$

In this thesis, the term *wavelet* will refer to a function ψ generating an orthonormal basis as above. There is no explicit name for the pairs $\{\varphi, \psi\}$ giving reconstructions as in (1.3), but they include, as particular cases, the terms: biorthogonal wavelet, frame, (φ, ψ)-transform,...

From a mathematical point of view, this type of decompositions are also of great interest. Indeed, an orthonormal basis can always be used to study boundedness of linear operators with a matrix representation in terms of the basis. One of the proofs of the $T1$-theorem, for instance, uses the notion of wavelet to characterize all bounded operators in $L^2(\mathbb{R}^n)$ of Calderón-Zygmund type (see Part II in [DAV]). Moreover, wavelets (and also pairs $\{\varphi, \psi\}$ as above) give rise to bases for a wider range of Banach spaces ($L^p, W^{\alpha,p}, B_p^{\alpha,q}, F_p^{\alpha,q}$, etc...) and provide an excellent tool to analyze some of their fine properties (see [FJW] or Chapter 6 in [HW] and [MEY]).

In this thesis we will be particularly interested in the study of mathematical properties of the set of all wavelets, as a topological subspace of $L^2(\mathbb{R})$. The reader wishing know more about applications, history, or relations of wavelets with other areas of mathematics, can consult any of the books: [DAUB], [DAV], [HUB], [HW], [KAH-LEM], [MEY], ...

The structure of this work is as follows; the rest of Chapter 1 contains an account of definitions, notation, and known results that will be used in the development of next chapters. In Chapter 2, we take up the task of identifying the pairs of functions $\{\varphi, \psi\}$ in $L^2(\mathbb{R}^n)$ that give reconstruction formulas as in (1.3). We prove a characterization theorem in terms of two basic equations, which in many senses extends the already known characterization theorem for wavelets in $L^2(\mathbb{R})$ of Gripenberg and Wang (see Theorem 2.7 below). This part of the work was done in collaboration with M. Frazier, K. Wang and G. Weiss (see [FGWW]). The rest of the chapter, and the appendix, contain applications of this result and constructions of wavelets with apparently unexpected properties. In Chapter 3, we specialize in a particular class of wavelets: the α-localized ones. Those are the wavelets having a decay at infinity rapid enough to belong to the space $L^2((1+|x|^2)^\alpha\, dx)$. We give a detailed account on how to construct them and what their main properties are, when considered as a topological subspace of $L^2((1+|x|^2)^\alpha\, dx)$. In particular, we give a decomposition of the set of all α-localized wavelets in connected components. This part of the work was motivated in an article by A. Bonami, S. Durand and G. Weiss ([BDW]), in which the class of wavelets with polynomial decay was studied. We extend their results to the more general setting of α-localized wavelets and obtain some new and interesting properties not considered by the authors of [BDW] (see §5 in Chapter 3).

2 Definitions and examples

We start this section with a precise definition of what we shall call a wavelet:

Definition 2.1 *An orthonormal wavelet (or simply, a wavelet) is a function* $\psi \in L^2(\mathbb{R})$ *such that* $\{\ \psi_{j,k} \mid j \in \mathbb{Z},\ k \in \mathbb{Z}\ \}$ *is an orthonormal basis for* $L^2(\mathbb{R})$ *, where*

$$\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z}. \tag{2.2}$$

In this case we have the equality

$$f = \sum_{j,k\in\mathbb{Z}} (f, \psi_{j,k})\psi_{j,k}$$

for each $f \in L^2(\mathbb{R})$, where the series converges (unconditionally) in the $L^2(\mathbb{R})$-norm and $(g,h) = \int_{\mathbb{R}} g\overline{h}$.

There many examples of wavelets and different ways of constructing them. In most of these constructions the *Fourier transform* plays an important role. In this thesis, the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ will be given by:

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x}, \qquad \boldsymbol{\xi} \in \mathbb{R}^n. \tag{2.3}$$

For functions $f \in L^2(\mathbb{R}^n)$, $\hat{f}$ will denote the usual extension of the densely defined operator in (2.3) from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. The inverse of this operator acting on a function f will be denoted by $f\check{\ }$.

It is clear that, for any function ψ defined as in (2.2), we have:

$$\widehat{\psi}_{j,k}(\xi) = 2^{-j/2}\, e^{-i2^{-j}k\xi}\, \widehat{\psi}(2^{-j}\xi), \quad j,k \in \mathbb{Z}, \quad \xi \in \mathbb{R}, \tag{2.4}$$

where we write $\widehat{\psi}_{j,k}$, instead of $(\psi_{j,k})\hat{\ }$. We shall use this convention throughout this thesis; that is, the subindices (j,k) on $\widehat{\psi}_{j,k}$ will *always* correspond to translations and dilations in the time domain.

Perhaps, the simplest example of a wavelet is given by the function ψ whose Fourier transform is $\widehat{\psi} = \chi_I$, where the set $I = [-2\pi, -\pi) \cup (\pi, 2\pi]$. This function is sometimes called the *Shannon wavelet.*

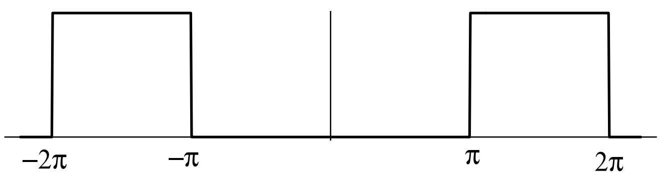


Figure 2.3: Fourier transform of the Shannon wavelet.

Note that, in particular, $\widehat{\psi}_{0,k} = e^{-ik\xi}\chi_I$, $k \in \mathbb{Z}$, which is an orthonormal basis for $L^2(I)$, while $\widehat{\psi}_{j,k} = e^{-ik2^{-j}\xi}\, 2^{-j/2}\chi_{2^jI}$, $k \in \mathbb{Z}$, is an orthonormal basis for $L^2(2^jI)$,

$j \in \mathbb{Z}$. Thus, since $\{2^j I\}$, $j \in \mathbb{Z}$, forms a partition of $\mathbb{R} \setminus \{0\}$, we must have that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$ and, consequently, that ψ is a wavelet.

In general, many other examples of the type $\widehat{\psi} = \chi_K$, for K a (Lebesgue)-measurable subset of $\mathbb{R}$, can be constructed. Again, the geometry of K under translations (by $2k\pi$) and dilations (by 2^j) determines when ψ is a wavelet. Indeed, a necessary and sufficient condition for a set K to give a wavelet $\psi = (\chi_K)\check{}$ is that:

$$\sum_{j\in\mathbb{Z}} \chi_K(2^j\xi) = \sum_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = 1, \quad a.e.\, \xi \in \mathbb{R}, \tag{2.5}$$

or, equivalently, that there is a partition $\{I_\ell\}_{\ell\in\mathbb{Z}}$ of $I = (-2\pi, -\pi] \cup (\pi, 2\pi]$ and two sequences of integers, $\{j_\ell\}$, $\{k_\ell\}$, $\ell \in \mathbb{Z}$, such that we can write the disjoint unions:

$$K = \uplus_{\ell\in\mathbb{Z}} 2^{j_\ell} I_\ell \quad \text{and} \quad I = \uplus_{\ell\in\mathbb{Z}} \{2^{j_\ell} I_\ell + 2k_\ell\pi\} \tag{2.6}$$

For a proof of this property the reader can consult [HW] (see Theorem 2.5 in chapter 7 of [HW]). Alternatively, the equivalence in (2.5) is a particular case of the following theorem of X. Wang (Theorem 1.14 in [WAN]) and G. Gripenberg (Theorem 1 in [GRIP]), which characterizes all wavelets in terms of equations like the ones above:

Theorem 2.7 : G. Gripenberg, X. Wang.

Let $\psi \in L^2(\mathbb{R})$ be such that $\|\psi\|_2 \geq 1$. Then, ψ is a wavelet if and only if

$$\left.\begin{array}{lll} (i) & \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1, & a.e.\ \xi \in \mathbb{R} \\ (ii) & t_q(\xi) \equiv \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi + 2\pi q))} = 0, & a.e.\ \xi \in \mathbb{R},\ \forall q \in 2\mathbb{Z}+1. \end{array}\right\} \tag{2.8}$$

In particular, if $|\widehat{\psi}| = \chi_K$, for some measurable set $K \subset \mathbb{R}$, ψ is a wavelet if and only if

$$\left.\begin{array}{lll} (i) & \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1, & a.e.\ \xi \in \mathbb{R} \\ (ii) & \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + 2k\pi)|^2 = 1, & a.e.\ \xi \in \mathbb{R}. \end{array}\right\} \tag{2.9}$$

In Chapter 2, we prove a more general result that includes Theorem 2.7 as a particular case; namely, we characterize all pairs of functions $\varphi, \psi \in L^2(\mathbb{R}^n)$ that give reconstruction formulas of the type (1.3), by means of simple equations like those in (2.8).

Wavelets ψ such that $|\widehat{\psi}| = \chi_K$ are usually referred to as *MSF*, or *minimally supported frequency wavelets*, since the measure of the support[1] of $\widehat{\psi} = K$ is smallest among all wavelets (and equals 2π). The fact that $|K| = 2\pi$ is an immediate consequence of the Plancherel Theorem and $\|\psi\|_2 = 1$, while the minimality property follows from:

$$2\pi = 2\pi\|\psi\|^2 = \int_{\text{supp }\widehat{\psi}} |\widehat{\psi}|^2 \leq |\text{supp }\widehat{\psi}|,$$

with equality if and only if $|\widehat{\psi}| = \chi_{\text{supp }\widehat{\psi}}$. Note that we have used here that $|\widehat{\psi}| \leq 1$, *a.e.*, whenever ψ is a wavelet (see, e.g., (2.8) (i) above).

In the first part of the appendix, we present a variety of examples of MSF wavelets satisfying very special properties, such as having their Fourier transforms discontinuous at 0, or with unbounded support. These examples seem to be new, and give answers to some questions posed in the wavelet literature (see the appendix for further references).

Finally, in the second part of the appendix we present one last example that shows the independence of equations (2.8) and (2.9). More precisely, we find a function $\psi \in L^2(\mathbb{R})$ whose "amplitude" $|\widehat{\psi}|$ satisfies (2.9), but such that for *no* "phase" $\alpha : \mathbb{R} \to \mathbb{R}$, the function $(e^{i\alpha}|\widehat{\psi}|)^{\check{}}$ is a wavelet. Note that, in view of Theorem 2.7, $|\widehat{\psi}|$ cannot be the characteristic function of a set.

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[1] To motivate the term *MSF*, we are implicitly using in this sentence a definition for the "support" of a measurable function given by supp $f = \{x \in \mathbb{R} \mid f(x) \neq 0\}$. This definition does not agree, in general, with the standard one we will use throughout this thesis (that is, the closure of $\{f \neq 0\}$, see Definition 2.9 in [RUD2]).

3 Multiresolution analyses

In this section we present the basics of multiresolution analyses. Their main properties and notation will be used extensively in Chapter 3, but will be avoided in Chapter 2. We leave to the reader the decission of skipping this section if he or she considers it unnecessary.

There is a standard algorithm for constructing wavelets. It consists of decomposing $L^2(\mathbb{R})$ with an increasing sequence of "resolution spaces", each corresponding to a different dilation level and generated by the integer translations of a single function.

Think, for instance, of the following subspaces of $L^2(\mathbb{R})$:

$$V_j = \{f \in L^2(\mathbb{R}) \mid \text{supp } \hat{f} \subset [-2^j\pi, 2^j\pi]\} =$$

$$= \overline{\text{span}}\,\{(\chi_{[-2^j\pi,2^j\pi]})\check{}\,(\cdot - k) \mid k \in \mathbb{Z}\} = \overline{\text{span}}\,\{\varphi_{j,k} \mid k \in \mathbb{Z}\}, j \in \mathbb{Z},$$

where $\widehat{\varphi} = \chi_{[-\pi,\pi]}$.

In this case, the orthogonal complement of V_j in V_{j+1}: $W_j = V_j^{\perp} \cap V_{j+1}$, is generated by the integer tranlations of the function $(\chi_{[-2^{j+1}\pi,-2^j\pi)\cup(2^j\pi,2^{j+1}\pi]})\check{}$, and since $\cup V_j$ is dense in $L^2(\mathbb{R})$, we have

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j = \oplus_{j\in\mathbb{Z}} \overline{\text{span}}\,\{\psi_{j,k} \mid k \in \mathbb{Z}\},$$

where $\widehat{\psi} = \chi_{[-2\pi,-\pi)\cup(\pi,2\pi]}$. In other words, $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms a complete orthonormal system and, hence, ψ is a wavelet. This is, in fact, the Shannon wavelet defined in the previous section (see Figure 2.3).

This method of constructing wavelets by "resolving" the space $L^2(\mathbb{R})$ with dilations and translations can be described in full generality. We start by defining what we will call a multiresoltion analysis (for details and proofs of the formulas presented here, we refer the reader to Chapter 2 of [HW]).

A *multiresolution analysis* (or MRA) for $L^2(\mathbb{R})$ is a family of closed subspaces of

$L^2(\mathbb{R})$, $\{V_j\}_{j\in\mathbb{Z}}$, such that

$$\left.\begin{array}{ll} (i) & V_j \subset V_{j+1} \\ (ii) & f(x) \in V_j \quad \text{if and only if} \quad f(2x) \in V_{j+1} \\ (iii) & \overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}) \\ (iv) & \text{there exists a function} \quad \varphi \in V_0 \quad \text{such that} \\ & \{\varphi(\cdot - k)\,|\, k \in \mathbb{Z}\} \quad \text{is an orthonormal basis for } V_0. \end{array}\right\} \tag{3.1}$$

Remark 3.2 Any function $\varphi \in L^2(\mathbb{R})$ satifying (3.1) (iv) will be called a *scaling function* for the given MRA. It is a well-known fact that if φ is a scaling function for an MRA $\{V_j\}_{j\in\mathbb{Z}}$, all other possible scaling functions, $\varphi^\sharp$, for the same MRA, are given by $\widehat{\varphi}^\sharp(\xi) = \nu(\xi)\widehat{\varphi}(\xi)$, where $\nu(\xi)$ is a 2π-periodic measurable function such that $|\nu(\xi)| = 1$, *a.e.* $\xi \in \mathbb{R}$ (see, e.g., [HW], Lemma 2.6 in Chapter 2).

Let us assume that $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA. Note that, by (3.1) (i), (ii) and (iv), we must have that $\frac{1}{2}\varphi(\frac{x}{2}) \in V_0$, which in turn implies that there exists a (unique) sequence $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\frac{1}{2}\varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} c_k \varphi(x-k), \tag{3.3}$$

where the (unconditional) convergence of the series is in $L^2(\mathbb{R})$. By taking Fourier transforms of both sides in (3.3), we obtain the *scaling equation*

$$\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi), \quad a.e.\, \xi \in \mathbb{R}, \tag{3.4}$$

where $m_0(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}$, *a.e.* $\xi \in \mathbb{T}$, is a 2π-periodic function in $L^2(\mathbb{T})$ called the *low-pass filter* associated with the scaling function φ, and the coefficients c_k are given by

$$c_k = \int_{\mathbb{R}} \frac{1}{2}\varphi(\frac{x}{2})\overline{\varphi(x-k)}\, dx, \quad k \in \mathbb{Z}. \tag{3.5}$$

For a function $\varphi \in L^2(\mathbb{R})$, it is easy to show that $\{\varphi(\cdot - k)\,|\, k \in \mathbb{Z}\}$ is an orthonormal system if and only if

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2k\pi)|^2 = 1, \quad a.e.\, \xi \in \mathbb{R}. \tag{3.6}$$

(see, e.g., Proposition 1.11, in Chapter 2 of [HW]). In particular, for every scaling function of an MRA, (3.6) must hold. An easy consequence of (3.4) and (3.6) is that the low-pass filter must satisfy

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1, \quad a.e.\, \xi \in \mathbb{T}. \tag{3.7}$$

It is easy to see that condition (3.7) and the 2π-periodicity of m_0 imply that

$$\int_{-\pi}^{\pi} |m_0(\xi)|^2 \, d\xi = 2\pi.$$

Let us see now how to construct wavelets from an MRA. If we let $W_j = V_{j+1} \cap V_j^{\perp}$, then conditions (3.1) (i) and (iii) allow us to write $L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j$, where the direct sum represents an orthogonal decomposition of the space $L^2(\mathbb{R})$. It now follows easily from (3.1) (ii) that each function $\psi \in W_0$ such that $\{\psi(\cdot - k) \,|\, k \in \mathbb{Z}\}$ is an orthonormal basis for W_0 is a wavelet for $L^2(\mathbb{R})$. Functions ψ satisfying this property are completely characterized by

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\, \nu(\xi)\, \overline{m_0(\frac{\xi}{2}+\pi)}\, \widehat{\varphi}(\frac{\xi}{2}), \quad a.e.\, \xi \in \mathbb{R},$$

where φ is a scaling function for the MRA, m_0 is its low-pass filter and ν is any 2π-periodic measurable function such that $|\nu(\xi)| = 1$, $a.e.\, \xi \in \mathbb{R}$ (see Proposition 2.13 in Chapter 2 of [HW]). Any such function ν will be called a *phase* for the wavelet ψ.

It is a well-known fact that not all orthonormal wavelets can be constructed in this way from an MRA. The following proposition tells us when this is possible.

Proposition 3.8 *Let ψ be an orthonormal wavelet for $L^2(\mathbb{R})$. Then, the following five properties are equivalent:*

(I) Let $W_j = \overline{span\{\psi(2^j \cdot - k) \,|\, k \in \mathbb{Z}\}}$ and let $V_j = \oplus_{\ell=-\infty}^{j-1} W_\ell$, for $j \in \mathbb{Z}$ (the direct sum being orthogonal). Then, the family $\{V_j\}_{j\in\mathbb{Z}}$ forms an MRA.

(II) There exists an MRA $\{V_j\}_{j\in\mathbb{Z}}$ such that, if we let $W_0 = V_1 \cap V_0^{\perp}$, then $\{\psi(\cdot - k) \,|\, k \in \mathbb{Z}\}$ is an orthonormal basis for W_0.

(III) There exists a function $\varphi \in L^2(\mathbb{R})$, scaling function of some MRA, and a 2π-periodic measurable function satisfying $|\nu(\xi)| = 1$, a.e. $\xi \in \mathbb{R}$, such that

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\, \nu(\xi)\, \overline{m_0(\frac{\xi}{2} + \pi)}\, \widehat{\varphi}(\frac{\xi}{2}), \quad a.e.\, \xi \in \mathbb{R} \tag{3.9}$$

where m_0 is the low-pass filter associated with φ.

(IV) There exists a function $\varphi \in L^2(\mathbb{R})$, scaling function of some MRA, such that

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\, \overline{m_0(\frac{\xi}{2} + \pi)}\, \widehat{\varphi}(\frac{\xi}{2}), \quad a.e.\, \xi \in \mathbb{R} \tag{3.10}$$

where m_0 is the low-pass filter associated with φ.

(V) $D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi + 2k\pi))|^2 = 1, \quad a.e.\, \xi \in \mathbb{R}.$

A wavelet satisfying any of the five equivalent conditions above is called an MRA wavelet.

We shall not include here a complete proof for Proposition 3.8, but we will give instead a precise reference for each of the statements made.

PROOF:

"(I)⇒(II)"	This implication is trivial.
"(II)⇒(III)"	This implication is included in the statement of Proposition 2.13 in Chapter 2 of [HW].
"(III)⇒(IV)"	This part seems to be new. For a proof of it, see Theorem A in [WUTAM]. (See also Exercise 2.10 in Chapter 2 of [WOJ].)
"(IV)⇒(V)"	This is an easy consequence of the properties of MRAs (see (2.18), Chapter 2 of [HW]).
"(V)⇒(I)"	This part can be found in Theorem 3.2, Chapter 7 of [HW].

□

Remark 3.11 It is not hard to show that under very mild conditions of smoothness and decay at ∞ every wavelet must satisfy equation (V) in Proposition 3.8 above. In fact, if ψ is a wavelet and if $\widehat{\psi} \in C(\mathbb{R})$ and $|\psi(\xi)| \leq C(1+|\xi|^2)^{-\frac{1}{2}-\varepsilon}$, $a.e.\, \xi \in \mathbb{R}$, for some $\varepsilon > 0$, then $D_\psi(\xi) = 1$ for all $\xi \neq 2k\pi$, $k \in \mathbb{Z}$ (see Corollary 3.16 of Chapter 7 of [HW]). These and other sufficient conditions for a wavelet to be associated with an MRA have been studied by P. Auscher and P. G. Lemarié-Rieusset (see [AUS], [LEM] or [KAH-LEM]). We shall go back to this point in §5 of Chapter 3.

One can also characterize, among the functions $\varphi \in L^2(\mathbb{R})$ those that are scaling functions for an MRA. This characterization will be useful in Chapter 3 below.

Proposition 3.12 *A function $\varphi \in L^2(\mathbb{R})$ is a scaling function for an MRA if and only if*

(i) $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+2k\pi)|^2 = 1, \quad a.e.\, \xi \in \mathbb{R}.$

(ii) $\lim_{j\to\infty} |\widehat{\varphi}(2^{-j}\xi)| = 1, \quad a.e.\, \xi \in \mathbb{R}.$

(iii) There exists a 2π-periodic function $m_0 \in L^2(\mathbb{T})$ such that

$$\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi),\ a.e.\, \xi \in \mathbb{R}.$$

For a proof of this result the reader can consult Theorem 5.2 of Chapter 7 of [HW].

Before ending this section, we introduce a special class of MRAs for which one additional assumption is made. Suppose $\alpha > \frac{1}{2}$ is fixed. We say that an MRA $\{V_j\}_{j\in\mathbb{Z}}$, is *localized of degree* α (or α*-localized*) when we can find a scaling function φ satisfying

$$\int_{\mathbb{R}} |\varphi(x)|^2 \,(1+|x|^2)^\alpha \,dx \,< \infty. \tag{3.13}$$

Condition (3.13) can be stated in terms of Sobolev spaces; namely, a function φ satisfies (3.13) if and only if $\widehat{\varphi} \in H^{\alpha}(\mathbb{R})$. Functions φ with these properties are called *α-localized scaling functions.* In Chapter 3 of this thesis we will study in detail the properties of α-localized MRAs by using Sobolev space theory. We shall characterize the low-pass filters associated with them and we shall show that the set of all α-localized scaling functions is arcwise-connected in the topology given by (3.13) above.

Chapter 2

A Characterization of Functions that Generate Wavelet and Related Expansions

The first part of this write-up (§§$1-4$) was done in collaboration with M. Frazier, K. Wang and G. Weiss and has recently appeared in The Journal of Fourier Analysis and Applications (see [FGWW]).

1 Introduction

We remind the reader that an *orthonormal wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that the system:

$$\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z},$$

forms an orthonormal basis for $L^2(\mathbb{R})$. In this case we have the equality

$$f = \sum_{j,k\in\mathbb{Z}} (f, \psi_{j,k})\psi_{j,k} \tag{1.1}$$

for each $f \in L^2(\mathbb{R})$, where the series converges (unconditionally) in the $L^2(\mathbb{R})$-norm. There are several extensions of these notions that have commanded considerable interest during the past decade. If $\mathbb{R}$ is replaced by $\mathbb{R}^n$, n-dimensional Euclidean space, the definition of the family $\{ \psi_{j,\mathbf{k}} \mid j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^n \}$ is

$$\psi_{j,\mathbf{k}}(\mathbf{x}) \equiv 2^{jn/2}\psi(2^j\mathbf{x} - \mathbf{k}), \tag{1.2}$$

where $j \in \mathbb{Z}$ and $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}^n$. This is clearly a natural extension of the 1-dimensional case since the dilations in (1.2) preserve the $L^2(\mathbb{R}^n)$-norm of ψ and the translations must involve points in $\mathbb{R}^n$. The function ψ is then said to be an orthonormal wavelet in $L^2(\mathbb{R}^n)$ if the family defined by (1.2) is an orthonormal basis for this space. It is well-known that many different kinds of wavelets exist in $L^2(\mathbb{R})$. In the higher dimensional case, however, the situation is more complicated. If one imposes some "relatively mild" conditions of smoothness and decrease at infinity on the Fourier transform, it can be shown that a single function $\psi \in L^2(\mathbb{R}^n)$ cannot generate an orthonormal basis by forming the family defined in (1.2) when $n \geq 2$. In this case one needs at least $L = 2^n - 1$ such generating functions $\psi^1, \psi^2, ..., \psi^L \in L^2(\mathbb{R}^n)$ (see [AUS] p.215, [KAH-LEM] p.339, [MEY] p.93). We encounter, therefore, orthonormal bases of the form $\{\psi^\ell_{j,\mathbf{k}}\}$, where $\ell = 1, 2, ..., L$, $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$ with $L \geq 1$. Such bases produce expansions of the form

$$f = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi^\ell_{j,\mathbf{k}}) \psi^\ell_{j,\mathbf{k}} \tag{1.3}$$

for any $f \in L^2(\mathbb{R}^n)$, where the multiple series converges in the norm of $L^2(\mathbb{R}^n)$. Recently, however, several investigators have constructed examples of (single) wavelets in $L^2(\mathbb{R}^n)$ (see [DAI-LAR-SPE], [SOA-WEI]). Thus, the case $L = 1$ in (1.3) is realized in some cases[1]. In general, when $\{\psi^\ell_{j,\mathbf{k}}\}$ is an orthonormal basis we call the family $\Psi = \{\psi^1, ..., \psi^L\}$ a *family of orthonormal wavelets.*

More general representations of functions in $L^2(\mathbb{R}^n)$, sharing the same dyadic dilation and translation structure with these expansions, have been studied and effectively applied. Frazier and Jawerth (see [FJ] and [FJW]) introduced expansions of the form

$$f = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \varphi_{j,\mathbf{k}}) \psi_{j,\mathbf{k}} \tag{1.4}$$

[1] In fact, given any $L \geq 1$, one can find a family of orthonormal wavelets Ψ in $L^2(\mathbb{R}^n)$ consisting of L distinct functions $\psi^1, \ldots, \psi^L$. This follows, essentially, from the arguments in [DAI-LAR-SPE], and was pointed out to us by D. Weiland.

where $\varphi_{j,\mathbf{k}}$, $\psi_{j,\mathbf{k}}$ are defined by equality (1.2) and φ and ψ is an appropriate pair of functions in $L^2(\mathbb{R}^n)$. The convergence of the series in (1.4) is, again, in the $L^2(\mathbb{R}^n)$-norm. More general function spaces and different types of convergence are of great interest; however, in this work we limit our attention to the $L^2(\mathbb{R}^n)$ case.

A feature of expansions of the form given by (1.4) is that one of the functions, φ, provides a system of "analyzing" functions; that is, the needed information about f is obtained by calculating the inner products $(f, \varphi_{j,\mathbf{k}})$. The other function, ψ, provides a "synthesizing" system which enables us to reconstruct f from this information via the series in (1.4). We observe that this situation has much in common with expansions involving frames and those associated with bi-orthogonal wavelets.

Our main purpose is to characterize those pairs of families $\Phi = \{\varphi^1, ..., \varphi^L\}$ and $\Psi = \{\psi^1, ..., \psi^L\}$ in $L^2(\mathbb{R}^n)$ having the property that for each $f \in L^2(\mathbb{R}^n)$

$$f = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \varphi^{\ell}_{j,\mathbf{k}}) \psi^{\ell}_{j,\mathbf{k}}, \tag{1.5}$$

where the convergence of the series on the right is in the norm of $L^2(\mathbb{R}^n)$ or in some weaker sense. We will show that equality (1.5) (or a variant of this representation of f) is, in a sense, equivalent to the fact that the Fourier transforms $\{\widehat{\varphi}^1, ..., \widehat{\varphi}^L\}$, $\{\widehat{\psi}^1, ..., \widehat{\psi}^L\}$ satisfy the following two equations:

$$\left.\begin{array}{lll} (i) & \displaystyle\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \widehat{\varphi}^{\ell}(2^j \boldsymbol{\xi}) \overline{\widehat{\psi}^{\ell}(2^j \boldsymbol{\xi})} = 1, & a.e.\ \boldsymbol{\xi} \in \mathbb{R}^n \\ (ii) & t_{\mathbf{q}}(\boldsymbol{\xi}) \equiv \displaystyle\sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \widehat{\varphi}^{\ell}(2^j \boldsymbol{\xi}) \overline{\widehat{\psi}^{\ell}(2^j(\boldsymbol{\xi} + 2\pi\mathbf{q}))} = 0, & a.e.\ \boldsymbol{\xi} \in \mathbb{R}^n,\ \forall \mathbf{q} \in \mathcal{O}^n \end{array}\right\} \tag{1.6}$$

where $\mathcal{O}^n = \mathbb{Z}^n \setminus (2\mathbb{Z})^n = \{\mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}^n\ :\ \text{at least one component } k_j \text{ is odd}\}$.

We do encounter the problem of determining the meaning of the series in these two equations. It is easy to see that $t_{\mathbf{q}}$ is a well-defined function in $L^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |t_{\mathbf{q}}(\boldsymbol{\xi})|\, d\boldsymbol{\xi} \le \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} |\widehat{\varphi}^{\ell}(2^j \boldsymbol{\xi})|\, |\widehat{\psi}^{\ell}(2^j(\boldsymbol{\xi} + 2\pi\mathbf{q}))|\, d\boldsymbol{\xi} =$$

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}|\widehat{\varphi}^{\ell}(\boldsymbol{\eta})|\,|\widehat{\psi}^{\ell}(\boldsymbol{\eta}+2^{j+1}\pi\mathbf{q})|\,d\boldsymbol{\eta} \le (\sum_{\ell=1}^{L}\|\widehat{\varphi}^{\ell}\|_2\|\widehat{\psi}^{\ell}\|_2)(\sum_{j=0}^{\infty}2^{-jn}) =$$

$$= \frac{2^{2n}\pi^n}{2^n-1}(\sum_{\ell=1}^{L}\|\varphi^{\ell}\|_2\|\psi^{\ell}\|_2) < \infty.$$

On the other hand, the convergence of the series in (1.6) (i) requires a further assumption about the two families Φ and Ψ . In order to explain this matter better we first consider the case when $\Phi = \Psi$. In this case the two equations (1.6) (i) and (ii) have the form

$$\left.\begin{array}{lll}(i) & \displaystyle\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\widehat{\psi}^{\ell}(2^j\boldsymbol{\xi})|^2 = 1, & a.e.\ \boldsymbol{\xi}\in\mathbb{R}^n \\ (ii) & t_{\mathbf{q}}(\boldsymbol{\xi}) \equiv \displaystyle\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}^{\ell}(2^j\boldsymbol{\xi})\overline{\widehat{\psi}^{\ell}(2^j(\boldsymbol{\xi}+2\pi\mathbf{q}))} = 0, & a.e.\ \boldsymbol{\xi}\in\mathbb{R}^n,\ \forall\mathbf{q}\in\mathcal{O}^n\end{array}\right\} \tag{1.7}$$

Since all the summands are non-negative, the convergence of the series in equation (i) is well defined (if we include the possibility that the sum is infinity for some of the $\boldsymbol{\xi}$'s). Thus, in this case the meaning of the series appearing in the two equations is clear.

These equations are very useful for the construction of wavelets as well as explicit, more general solutions that lead to expansions of the form (1.1), (1.3) or (1.4) (see [HW]). We shall also see in the development below that a considerable amount of information is contained in these two equations.

The following simple example of a function that satisfies the two equations illustrates some important features of the general solution. Let us first consider the one-dimensional case for (1.7) with $L = 1$. We choose a non-negative even function b supported in $[-\pi, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \pi]$ such that

$$b^2(\xi) + b^2(\frac{\xi}{2}) = 1, \qquad \xi \in [\frac{\pi}{2}, \pi]. \tag{1.8}$$

It is easy to see that there exist C^∞ functions b with these properties. We let ψ be chosen so that $|\widehat{\psi}| = b$. Equation (1.7) (i) in this case is $\sum_{j\in\mathbb{Z}} |\widehat{\psi}^\ell(2^j\xi)|^2 = 1$. Because of the assumption we made about the support of b, this sum contains at most two non-zero terms and equality (1.8) shows that (1.7) (i) is satisfied by $\widehat{\psi}$. The second equation, (1.7) (ii) , is also clearly satisfied since the two points, $2^j\xi$ and $2^j(\xi + 2\pi q)$, at which the products are evaluated, are at distance from each other that is at least 2π (since $j \geq 0$ and q is odd) which is the diameter of the support of $\widehat{\psi}$.

Thus, it follows from the results we shall prove that equality (1.1) is true for all $f \in L^2(\mathbb{R})$ when ψ is chosen so that $|\widehat{\psi}| = b$. The family $\{\psi_{j,k}\}$, however, is <u>not</u> an orthonormal basis for $L^2(\mathbb{R})$. First of all,

$$\|\psi\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\psi}(\xi)|^2\, d\xi \leq \frac{meas\,(\mathrm{supp}\ \widehat{\psi})}{2\pi} \leq \frac{3}{4} < 1$$

since, as is the case for any function $\widehat{\psi}$ satisfying (1.7) (i) , $|\widehat{\psi}(\xi)| \leq 1$ $a.e.$; moreover, a simple re-normalization cannot convert this system to an orthonormal basis. The discussion in the next section will clarify this matter. It is also clear that a radial version of this example provides us with a similar example in $\mathbb{R}^n$, $n > 1$.

Let us now pass to the precise statement and proof of our principal result when the families Φ and Ψ coincide.

2 The First Theorem

We begin by making some observations about expansions of the type we are considering in a general Hilbert space $\mathcal{H}$ endowed with an inner product $(\cdot,\cdot)$. Let $\mathcal{E} = \{e_j\}$ be a family of vectors in $\mathcal{H}$. For the sake of simplicity we let j range through the natural numbers, $\mathbb{N}$; however, our statements apply when the indexing set is $\mathbb{Z} \times \mathbb{Z}^n$ or $\{1, \ldots, L\} \times \mathbb{Z} \times \mathbb{Z}^n$.

LEMMA 2.1 *Let* $\mathcal{E} = \{e_j\} \subset \mathcal{H}$, $j \in \mathbb{N}$, *then the following two properties are equivalent:*

(i) $\|f\|^2 = (f,f) = \sum_{j=1}^{\infty} |(f,e_j)|^2$ *holds for all* $f \in \mathcal{H}$

(ii) $f = \sum_{j=1}^{\infty}(f,e_j)e_j$, *with convergence in* $\mathcal{H}$, *for all* $f \in \mathcal{H}$.

Moreover, if $\|e_j\| \geq 1$, *for all* $j \in \mathbb{N}$, *(i) or (ii) is equivalent to the fact that* $\mathcal{E}$ *is an orthonormal basis of* $\mathcal{H}$.

The proof of this lemma is elementary and can be found in chapter 7 of [HW] (Theorems (1.7) and (1.8)). A more general version of this result, involving two sets $\mathcal{E} = \{e_j\}$ and $\mathcal{F} = \{f_j\}$ in $\mathcal{H}$, is stated and proved in section 4 (see Lemma 4.25). The following result is also proved in chapter 7 of [HW] (see Lemma (1.10)):

LEMMA 2.2 *Suppose* $\mathcal{E} = \{e_j\} \subset \mathcal{H}$, $j \in \mathbb{N}$, *is a family for which equality* (i) *of Lemma 2.1 holds for all* f *belonging to a dense subset* $\mathcal{D} \subset \mathcal{H}$, *then this equality holds for all* $f \in \mathcal{H}$.

Again, a more general version of Lemma 2.2, involving two systems $\mathcal{E}$ and $\mathcal{F}$, is stated and proved in section 4 (see Lemma 4.23). The following is our main result in case $\Phi = \Psi$:

THEOREM 2.3 *Suppose* $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$, *then*

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} |(f, \psi^\ell_{j,\mathbf{k}})|^2 \tag{2.4}$$

for all $f \in L^2(\mathbb{R}^n)$ *if and only if the functions in* Ψ *satisfy* (1.7) (i) *and* (ii).

Let us make a few observations before embarking on the proof of this theorem:

REMARK 2.5 Because of Lemma 2.1 we see that equality (2.4) for all $f \in L^2(\mathbb{R}^n)$ is equivalent to equality (1.3) for all $f \in L^2(\mathbb{R}^n)$. Thus, Theorem 2.3 gives us the desired characterization of those families $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ for which (1.3) holds for all $f \in L^2(\mathbb{R}^n)$.

REMARK 2.6 The last sentence in Lemma 2.1 leads us to the characterization of all orthonormal wavelets in $L^2(\mathbb{R}^n)$: $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ is a family of orthonormal wavelets in $L^2(\mathbb{R}^n)$ if and only if $\|\psi^1\|_2 = \ldots = \|\psi^L\|_2 = 1$ and this family satisfies (1.7) (i) and (ii) (since, in this case, $\|\psi^\ell_{j,\mathbf{k}}\|_2 = \|\psi^\ell\|_2 \geq 1$ for all $\ell = 1, \ldots, L$, $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$). As we mentioned in Chapter 1, this characterization has been obtained independently by Gustaf Gripenberg [GRIP] and Xihua Wang [WAN], in the case $n = 1$ (see [HW] for an historical account of this matter).

REMARK 2.7 Because of Lemma 2.2 it suffices to show that (1.7) (i) and (ii) imply that (2.4) holds for all f belonging to a dense subset, $\mathcal{D}$, of $L^2(\mathbb{R}^n)$. The dense subset we will choose in our proof is

$$\mathcal{D} = \left\{ f \in L^2(\mathbb{R}^n) \;\middle|\; \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and supp } \hat{f} \text{ is a compact subset of } \mathbb{R}^n \backslash \{\mathbf{0}\} \right\}. \quad (2.8)$$

Unless stated otherwise, therefore, from now on, all functions f that we shall consider will belong to $\mathcal{D}$.

A basic step in the proof of Theorem 2.3 is to decompose the series I on the right in (2.4) into two sums so that

$$(2\pi)^n I = I_0 + I_1, \quad (2.9)$$

where

$$I_0 = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 |\widehat{\psi}^\ell(2^j \boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}$$

and

$$I_1 = \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{p \in \mathbb{Z}} \sum_{\mathbf{q} \in \mathcal{O}^n} \hat{f}(\boldsymbol{\xi} + 2\pi 2^p \mathbf{q}) t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

This is done by first applying the Plancherel theorem to the inner products $(f, \psi^\ell_{j,\mathbf{k}})$, which allows us to obtain

$$(2\pi)^{2n} |(f, \psi^\ell_{j,\mathbf{k}})|^2 = |(\hat{f}, \widehat{\psi}^\ell_{j,\mathbf{k}})|^2 = 2^{-jn} \left| \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) \overline{\widehat{\psi}^\ell(2^{-j}\boldsymbol{\xi})} e^{i2^{-j}\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} \right|^2 =$$

$$= 2^{jn} \left| \int_{\mathbb{R}^n} \hat{f}(2^j\boldsymbol{\xi}) \overline{\widehat{\psi}^\ell(\boldsymbol{\xi})} e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} \right|^2.$$

Here, we have used the fact that the Fourier transform of $\psi^\ell_{j,\mathbf{k}}$ is $\widehat{\psi}^\ell_{j,\mathbf{k}}(\boldsymbol{\xi}) = 2^{-jn/2} \widehat{\psi}^\ell(2^{-j}\boldsymbol{\xi}) e^{-i2^{-j}\mathbf{k}\cdot\boldsymbol{\xi}}$, when $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, $\ell = 1, ..., L$. Thus,

$$(2\pi)^{2n} I = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} 2^{jn} \left| \int_{\mathbb{R}^n} \hat{f}(2^j\boldsymbol{\xi}) \overline{\widehat{\psi}^\ell(\boldsymbol{\xi})} e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} \right|^2.$$

The decomposition (2.9) will be obtained by first using a periodization argument that provides us with the identity

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} \hat{f}(2^j\boldsymbol{\xi}) \overline{\widehat{\psi}^\ell(\boldsymbol{\xi})} e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} \right|^2 =$$

$$= (2\pi)^n \int_{\mathbb{R}^n} \overline{\hat{f}(2^j\boldsymbol{\xi})} \widehat{\psi}^\ell(\boldsymbol{\xi}) \Big\{ \sum_{\mathbf{m} \in \mathbb{Z}^n} \hat{f}(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi)) \overline{\widehat{\psi}^\ell(\boldsymbol{\xi} + 2\mathbf{m}\pi)} \Big\}\, d\boldsymbol{\xi}. \qquad (2.10)$$

Multiplying both sides by 2^{jn} and then summing over $j \in \mathbb{Z}$ and $\ell = 1, \ldots, L$ we obtain a new expression for I. In this last expression we separate the terms with $\mathbf{m} = \mathbf{0}$, obtaining I_0, and the remaining terms, obtaining I_1. Thus, we first establish (2.10); after this we manipulate the expression involving the terms with $\mathbf{m} \neq \mathbf{0}$ and obtain the equality that we need for the definition of I_1.

Let us fix ℓ and j and put $F(\boldsymbol{\xi}) = F^\ell_j(\boldsymbol{\xi}) \equiv \hat{f}(2^j\boldsymbol{\xi}) \overline{\widehat{\psi}^\ell(\boldsymbol{\xi})}$. We remind the reader that $f \in \mathcal{D}$; thus, F is compactly supported in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and belongs to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Moreover,

$$\int_{\mathbb{R}^n} \hat{f}(2^j\boldsymbol{\xi}) \overline{\widehat{\psi}^\ell(\boldsymbol{\xi})} e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \int_{\mathbb{R}^n} F(\boldsymbol{\xi}) e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \hat{F}(-\mathbf{k}). \qquad (2.11)$$

Hence,

$$\hat{F}(-\mathbf{k}) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \int_{\mathbb{T}^n+2\mathbf{m}\pi} F(\boldsymbol{\xi}) e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \int_{\mathbb{T}^n} e^{i\mathbf{k}\xi}\{\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\boldsymbol{\xi}+2\mathbf{m}\pi)\}\, d\boldsymbol{\xi},$$

where $\mathbb{T}^n$ is the n-torus, which we may identify with $\{\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n \,|\, 0 \le \xi_\ell < 2\pi,\ \ell = 1,\ldots,n\}$. The fact that F is compactly supported implies that the series $\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\boldsymbol{\xi}+2\pi\mathbf{m})$ involves only a finite number of terms. This, together with the 2π periodicity of $e^{i\mathbf{k}\cdot\boldsymbol{\xi}}$, justifies the interchange of summation and integration (over $\mathbb{T}^n$) that gives us the last equality. We see, therefore, that the numbers $(2\pi)^{-n}\hat{F}(-\mathbf{k})$ are the Fourier coefficients of the periodic function $\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\boldsymbol{\xi}+2\pi\mathbf{m})$, which is in $L^2(\mathbb{T}^n)$ because the sum only involves a finite number of $\mathbf{m}$'s when $\boldsymbol{\xi}$ is in $\mathbb{T}^n$. Thus, by the Plancherel theorem for Fourier series,

$$\begin{aligned}\frac{1}{(2\pi)^n}\sum_{\mathbf{k}\in\mathbb{Z}^n}|\hat{F}(-\mathbf{k})|^2 &= \int_{\mathbb{T}^n}(\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\boldsymbol{\xi}+2\mathbf{m}\pi))(\sum_{\mathbf{p}\in\mathbb{Z}^n}\overline{F(\boldsymbol{\xi}+2\mathbf{p}\pi)})\, d\boldsymbol{\xi}\\ &= \int_{\mathbb{R}^n}(\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\boldsymbol{\xi}+2\mathbf{m}\pi))\overline{F(\boldsymbol{\xi})}\, d\boldsymbol{\xi}.\end{aligned}$$

Again, the fact that F is compactly supported and the 2π-periodicity of the series, justifies the interchange of summation and integration (over $\mathbb{T}^n$) that gives us the last equality. This last equality, written in terms of f and ψ by using the identity

$$\overline{F(\boldsymbol{\xi})}\sum_{\mathbf{m}\in\mathbb{Z}^n} F(\boldsymbol{\xi}+2\pi\mathbf{m}) = \overline{\hat{f}(2^j\boldsymbol{\xi})\widehat{\psi}^\ell(\boldsymbol{\xi})}\sum_{\mathbf{m}\in\mathbb{Z}^n}\hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi))\overline{\widehat{\psi}^\ell(\boldsymbol{\xi}+2\mathbf{m}\pi)},$$

together with (2.11), gives us (2.12).

Thus,

$$(2\pi)^n I = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} 2^{jn}\int_{\mathbb{R}^n}|\hat{f}(2^j\boldsymbol{\xi})|^2|\widehat{\psi}^\ell(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}+$$

$$+\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} 2^{jn}\int_{\mathbb{R}^n}\overline{\hat{f}(2^j\boldsymbol{\xi})}\widehat{\psi}^\ell(\boldsymbol{\xi})\Big\{\sum_{\mathbf{m}\neq\mathbf{0}}\hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi))\overline{\widehat{\psi}^\ell(\boldsymbol{\xi}+2\mathbf{m}\pi)}\Big\}\, d\boldsymbol{\xi}.$$

The first of these summands is I_0 (after a change of variables $\boldsymbol{\eta} = 2^j\boldsymbol{\xi}$). In order to justify the manipulations that show that the second summand equals I_1, as defined immediately after (2.9), we shall prove the following:

Lemma 2.12 *For every $f \in \mathcal{D}$ and $\psi \in L^2(\mathbb{R}^n)$, then:*

$$\sum_{j\in\mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} |\hat{f}(2^j\boldsymbol{\xi})\widehat{\psi}(\boldsymbol{\xi})| \sum_{\mathbf{m}\neq\mathbf{0}} |\hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi))\widehat{\psi}(\boldsymbol{\xi}+2\mathbf{m}\pi)|\, d\boldsymbol{\xi} < \infty.$$

Remark 2.13 In addition to allowing us to make all the changes in the order of summation and integration that will give us the desired expresion for I_1, this lemma shows that the sum of the squares of the "coefficients" $(f, \psi^{\ell}_{j,\mathbf{k}})$ (which we denoted by I) is finite if and only if $I_0 < \infty$, for each $f \in \mathcal{D}$. By varying f in $\mathcal{D}$ (for example, letting $\hat{f} = \chi_C$, where C is a compact subset of $\mathbb{R}^n \setminus \{\mathbf{0}\}$) we see that $I_0 < \infty$ if and only if $\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} |\widehat{\psi}^{\ell}(2^j\boldsymbol{\xi})|^2$ is locally integrable in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ (that is, integrable over each compact $C \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$). This is, of course, clearly true when (1.7) (i) is valid. As we shall see, this local integrability property will furnish us with an appropiate condition that guarantees the convergence of the series (1.6) (i).

In order to establish Lemma 2.12 we observe that, since

$$2|\widehat{\psi}(\boldsymbol{\xi})|\,|\widehat{\psi}(\boldsymbol{\xi}+2\mathbf{m}\pi)| \leq |\widehat{\psi}(\boldsymbol{\xi})|^2 + |\widehat{\psi}(\boldsymbol{\xi}+2\mathbf{m}\pi)|^2,$$

it suffices to show that

$$\int_{\mathbb{R}^n} \Big\{ \sum_{j\in\mathbb{Z}} \sum_{\mathbf{m}\neq\mathbf{0}} 2^{jn} |\hat{f}(2^j\boldsymbol{\xi})|\,|\hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi))| \Big\} |\widehat{\psi}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} < \infty \tag{2.14}$$

(observe that the sum involving $|\widehat{\psi}(\boldsymbol{\xi}+2\mathbf{m}\pi)|^2$ reduces to (2.14) via the changes of variable $\boldsymbol{\eta} = \boldsymbol{\xi} + 2\mathbf{m}\pi$). But (2.14) is an immediate consequence of

Lemma 2.15 *Suppose $0 < a < b < \infty$, $\hat{f} \in L^\infty(\mathbb{R}^n)$, $supp\hat{f} \subseteq \{\boldsymbol{\xi} : a < |\boldsymbol{\xi}| < b\}$ and $\delta = diam(supp\ \hat{f})$, then*

$$\sigma(\boldsymbol{\xi}) \equiv \sum_{j\in\mathbb{Z}} \sum_{\mathbf{m}\neq\mathbf{0}} 2^{jn} |\hat{f}(2^j\boldsymbol{\xi})|\,|\hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi))| \leq C\delta^n \|\hat{f}\|^2_\infty$$

for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$, where $C = (\frac{3}{2\pi})^n(1+\log_2\frac{b}{a})$.

Proof: If $\delta < 2^j 2\pi$ then at most one of the points $2^j\boldsymbol{\xi}$ and $2^j\boldsymbol{\xi} + 2^j 2\mathbf{m}\pi$ lies in supp $\hat{f}$, since $\mathbf{m}$ is a non-zero n-tuple with integer components. Thus, in the sum defining $\sigma(\boldsymbol{\xi})$ we need only consider $j \le j_0$, where j_0 is the greatest integer satisfying $2^{j_0} \le \frac{\delta}{2\pi}$. We claim that the sum of the terms, in the series defining $\sigma(\boldsymbol{\xi})$, that involves each such j does not exceed $(\frac{3\delta}{2\pi})^n \|\hat{f}\|_\infty^2$. To see this, we first observe that $2^{jn}|\hat{f}(2^j\boldsymbol{\xi})|\,|\hat{f}(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi))| \le 2^{jn}\|\hat{f}\|_\infty^2$. We then observe that, for j and $\boldsymbol{\xi}$ fixed, the number of lattice points $\mathbf{m} \ne \mathbf{0}$ for which $\hat{f}(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi)) \ne 0$ is not larger than $(1 + \frac{2^{-j}\delta}{\pi})^n$. To see this suppose $\mathbf{m}_0$ is a lattice point such that $\hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}_0\pi)) \ne 0$. Then, since $\delta = \text{diam}(\text{supp } f)$, if $\hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi)) \ne 0$, we must have $\delta \ge |2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi) - 2^j(\boldsymbol{\xi} + 2\mathbf{m}_0\pi)| = 2^j|\mathbf{m} - \mathbf{m}_0|2\pi$. Thus, $\mathbf{m}$ must lie within the sphere about $\mathbf{m}_0$ of radius $\frac{2^{-j}\delta}{2\pi}$. This sphere is contained in the n-dimensional "cube" of sidelength $\frac{2^{-j}\delta}{\pi}$ centered at $\mathbf{m}_0$. But the number of lattice points within this cube does not exceed $(1 + \frac{\delta}{\pi}2^{-j})^n$. Putting these estimates together we see that for $j \le j_0$

$$\sum_{\mathbf{m}\ne\mathbf{0}} 2^{jn}|\hat{f}(2^j\boldsymbol{\xi})|\,|\hat{f}(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi))| \le$$

$$(1 + \frac{\delta}{\pi}2^{-j})^n\, 2^{jn}\|\hat{f}\|_\infty^2 = (2^j + \frac{\delta}{\pi})^n\, \|\hat{f}\|_\infty^2 \le (2^{j_0} + \frac{\delta}{\pi})^n\, \|\hat{f}\|_\infty^2 \le \left(\frac{3\delta}{2\pi}\right)^n \|\hat{f}\|_\infty^2.$$

Finally, we observe that for $\hat{f}(2^j\boldsymbol{\xi})$ to be non-zero we must have $a \le 2^j|\boldsymbol{\xi}| \le b$. Thus, j must lie in the interval $[\log_2 \frac{a}{|\xi|}, \log_2 \frac{b}{|\xi|}]$ to produce a non-zero summand in the series defining $\sigma(\boldsymbol{\xi})$. But there are at most $1 + \log_2 \frac{b}{a}$ integers in this interval. Together with the last estimate, this gives us

$$\sigma(\boldsymbol{\xi}) \le (1 + \log_2 \frac{b}{a}) \left(\frac{3}{2\pi}\right)^n \delta^n \|\hat{f}\|_\infty^2.$$

This completes the proof of Lemmas 2.12 and 2.15.

□

We now turn our attention to showing that I_1 has the form announced after equality (2.9). We have shown that

$$(2\pi)^n I =$$

$$= I_0 + \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} 2^{jn}\int_{\mathbb{R}^n} \overline{\hat{f}(2^j\boldsymbol{\xi})}\widehat{\psi}^\ell(\boldsymbol{\xi})\Big\{\sum_{\mathbf{m}\neq\mathbf{0}} \hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi))\overline{\widehat{\psi}^\ell(\boldsymbol{\xi}+2\mathbf{m}\pi)}\Big\}\, d\boldsymbol{\xi}\ . \tag{2.16}$$

If $\mathbf{m} = (m_1, m_2, \ldots, m_n) \neq (0, 0, \ldots, 0) = \mathbf{0}$, then there exists a unique non-negative integer r such that $\mathbf{m} = 2^r\mathbf{q}$ with $\mathbf{q} \in \mathcal{O}^n$. Since, by Lemma 2.12 the integrand in the second summand of (2.16) is absolutely convergent, the following equalities that allow us to isolate the terms involving $t_{\mathbf{q}}(\boldsymbol{\xi})$ are valid and the second term in (2.16) equals

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})}\widehat{\psi}^\ell(2^{-j}\boldsymbol{\xi}) \sum_{\mathbf{m}\neq\mathbf{0}} \hat{f}(\boldsymbol{\xi}+2^j 2\mathbf{m}\pi)\overline{\widehat{\psi}^\ell(2^{-j}\boldsymbol{\xi}+2\mathbf{m}\pi)}\, d\boldsymbol{\xi} =$$
$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})}\widehat{\psi}^\ell(2^{-j}\boldsymbol{\xi}) \sum_{r\geq 0}\sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}(\boldsymbol{\xi}+2^j 2\pi 2^r\mathbf{q})\overline{\widehat{\psi}^\ell(2^{-j}\boldsymbol{\xi}+2\pi 2^r\mathbf{q})}\, d\boldsymbol{\xi} =$$
$$\sum_{\ell=1}^{L}\int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{r\geq 0}\sum_{j\in\mathbb{Z}} \widehat{\psi}^\ell(2^r(2^{-r-j}\boldsymbol{\xi}))\hat{f}(\boldsymbol{\xi}+2^{j+r}2\pi\mathbf{q})\overline{\widehat{\psi}^\ell(2^r(2^{-r-j}\boldsymbol{\xi}+2\mathbf{q}\pi))}\, d\boldsymbol{\xi} =$$
$$\sum_{\ell=1}^{L}\int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{r\geq 0}\sum_{p\in\mathbb{Z}} \widehat{\psi}^\ell(2^r(2^{-p}\boldsymbol{\xi}))\overline{\widehat{\psi}^\ell(2^r(2^{-p}\boldsymbol{\xi}+2\pi\mathbf{q}))}\hat{f}(\boldsymbol{\xi}+2^p 2\pi\mathbf{q})\, d\boldsymbol{\xi} =$$
$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n}\sum_{p\in\mathbb{Z}} \hat{f}(\boldsymbol{\xi}+2^p 2\pi\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

This proves (2.9). It is now clear that the "if" part of Theorem 2.3 is true. Indeed, if the system Ψ satisfies (1.7) (i) and (ii) , then $I_0 = \|\hat{f}\|_2^2$ and $I_1 = 0$. By (2.9) we then have

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n} |(f, \psi^\ell_{j,\mathbf{k}})|^2 = I = (2\pi)^{-n}(I_0 + I_1) = (2\pi)^{-n}\|\hat{f}\|_2^2 + 0 = \|f\|_2^2,$$

which is equality (2.4) for all $f \in \mathcal{D}$. By Lemma 2.2, we then can conclude that (2.4) holds for all $f \in L^2(\mathbb{R})$.

We shall now prove the converse. Let us assume that (2.4) holds for all $f \in \mathcal{D}$. As we explained in Remark 2.13, this implies that

$$\tau(\boldsymbol{\xi}) \equiv \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} |\widehat{\psi}^\ell(2^j\boldsymbol{\xi})|^2$$

is locally integrable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Thus, almost every point in $\mathbb{R}^n$ is a point of differentiability of the integral of τ. Let us choose such a $\boldsymbol{\xi}_0 \neq \mathbf{0}$; that is, if Ω_n is the volume of the unit ball in $\mathbb{R}^n$,

$$\lim_{\delta \to 0+} \frac{1}{\Omega_n \delta^n} \int_{|\boldsymbol{\xi}-\boldsymbol{\xi}_0| \leq \delta} \tau(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \tau(\boldsymbol{\xi}_0). \tag{2.17}$$

Let us fix $\delta > 0$ such that $B_\delta(\boldsymbol{\xi}_0) = \{\boldsymbol{\xi} : |\boldsymbol{\xi}-\boldsymbol{\xi}_0| \leq \delta\} \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ and choose $f_\delta \in \mathcal{D}$ by letting

$$\hat{f}_\delta(\boldsymbol{\xi}) = \frac{1}{\sqrt{\Omega_n \delta^n}}\, \chi_{B_\delta(\boldsymbol{\xi}_0)}(\boldsymbol{\xi}).$$

Using the notation in (2.9), and adding the superscript δ to denote the dependence on this choice of f_δ, we have

$$(2\pi)^n I = (2\pi)^n I^\delta = I_0^\delta + I_1^\delta.$$

Thus, $I^\delta = \|f_\delta\|_2^2 = (2\pi)^{-n} \|\hat{f}_\delta\|_2^2 = (2\pi)^{-n}$ and we have

$$1 = \frac{1}{\Omega_n \delta^n} \int_{B_\delta(\boldsymbol{\xi}_0)} \tau(\boldsymbol{\xi})\, d\boldsymbol{\xi} + I_1^\delta,$$

for every δ small enough. From this we see that if we show that I_1^δ tends to 0 as $\delta \to 0+$, we have $\tau(\boldsymbol{\xi}_0) = 1$ (by (2.17)) and equality (1.7) (i) is satisfied by the system Ψ, since almost all points of $\mathbb{R}^n$ are such $\boldsymbol{\xi}_0$'s.

Arguing as we did when we established Lemma 2.12 we see that $|I_1^\delta|$ is dominated by the sum of two terms:

$$\int_{\mathbb{R}^n} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\mathbf{m} \neq \mathbf{0}} 2^{jn} |\hat{f}_\delta(2^j \boldsymbol{\xi})|\, |\hat{f}_\delta(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi))|\, |\widehat{\psi^\ell}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}$$

and another term in which $\widehat{\psi^\ell}(\boldsymbol{\xi})$ is replaced by $\widehat{\psi^\ell}(\boldsymbol{\xi}+2\mathbf{m}\pi)$ (which, after the change of variables $\boldsymbol{\eta} = \boldsymbol{\xi} + 2\mathbf{m}\pi$, reduces to the first term). Letting ψ denote any one of the L functions in Ψ, it suffices to show, therefore, that

$$I_1^{\delta,\sharp} = \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{\mathbf{m} \neq \mathbf{0}} 2^{jn} |\hat{f}_\delta(2^j \boldsymbol{\xi})|\, |\hat{f}_\delta(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi))|\, |\widehat{\psi}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}$$

tends to 0 as $\delta \to 0+$. The diameter of the support of $\hat{f}_\delta$ is 2δ; hence, since $\mathbf{m} \neq \mathbf{0}$ we must have

$$\hat{f}_\delta(2^j\boldsymbol{\xi})\,\hat{f}_\delta(2^j(\boldsymbol{\xi}+2\mathbf{m}\pi)) = 0$$

if $2^j > \frac{\delta}{\pi}$. Let j_0 be the largest integer such that $2^{j_0} \leq \frac{\delta}{\pi}$; then we need only consider $j \leq j_0$ in the sum defining $I_1^{\delta,\sharp}$. Also, if $\hat{f}_\delta(2^j\boldsymbol{\xi}) \neq 0$ we must have $|2^j\boldsymbol{\xi} - \boldsymbol{\xi}_0| \leq \delta$ and this, in turn, implies $|\boldsymbol{\xi}_0| - \delta \leq 2^j|\boldsymbol{\xi}|$. Since $B_\delta(\boldsymbol{\xi}_0) \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$, we must have $|\boldsymbol{\xi}_0| - \delta > 0$ as well. Hence,

$$|\boldsymbol{\xi}| \geq 2^{-j}(|\boldsymbol{\xi}_0| - \delta) \geq 2^{-j_0}(|\boldsymbol{\xi}_0| - \delta) \geq \frac{\pi}{\delta}(|\boldsymbol{\xi}_0| - \delta) > 0.$$

Thus, applying Lemma 2.15 to f_δ, with $a = |\boldsymbol{\xi}_0| - \delta$, $b = |\boldsymbol{\xi}_0| + \delta$, we have

$$I_1^{\delta,\sharp} \leq \int_{|\boldsymbol{\xi}|\geq(|\boldsymbol{\xi}_0|-\delta)\frac{\pi}{\delta}} \sigma_\delta(\boldsymbol{\xi})\,|\widehat{\psi}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \leq$$

$$\left(\frac{3}{2\pi}\right)^n \left(1 + \log_2 \frac{|\boldsymbol{\xi}_0| + \delta}{|\boldsymbol{\xi}_0| - \delta}\right)(2\delta)^n \|\hat{f}_\delta\|_\infty^2 \int_{|\boldsymbol{\xi}|\geq(|\boldsymbol{\xi}_0|-\delta)\frac{\pi}{\delta}} |\widehat{\psi}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \leq$$

$$\leq \Omega_n^{-1}\left(\frac{3}{\pi}\right)^n \left(1 + \log_2 \frac{|\boldsymbol{\xi}_0| + \delta}{|\boldsymbol{\xi}_0| - \delta}\right) \int_{|\boldsymbol{\xi}|\geq(|\boldsymbol{\xi}_0|-\delta)\frac{\pi}{\delta}} |\widehat{\psi}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}.$$

It is clear that this last expression tends to 0 as $\delta \to 0+$. We can conclude, therefore, that equality (1.7) (i) is satisfied by Ψ. This also shows that $I_1 = 0$ for all $f \in \mathcal{D}$ since I_0 must, then, equal $\|\hat{f}\|_2^2 = (2\pi)^n\|f\|_2^2$ and, thus, $\|f\|_2^2 = I = (2\pi)^{-n}(I_0 + I_1) = \|f\|_2^2 + (2\pi)^{-n}I_1$. That is,

$$0 = I_1 = \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi}$$

for all $f \in \mathcal{D}$. An application of the polarization identity then gives us

$$\int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{g}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi} = 0 \tag{2.18}$$

for all $f, g \in \mathcal{D}$.

Let us fix $\mathbf{q}_0 \in \mathcal{O}^n$ and choose a point $\boldsymbol{\xi}_0$ of differentiability of the integral of $t_{\mathbf{q}_0}$ such that $\boldsymbol{\xi}_0 \neq \mathbf{0} \neq \boldsymbol{\xi}_0 + 2\pi\mathbf{q}_0$. Since $t_{\mathbf{q}_0} \in L^1(\mathbb{R}^n)$ (see the argument that follows

(1.6)) almost all points of $\mathbb{R}^n$ have these properties. We need only consider $\delta > 0$ sufficiently small so that both $B_\delta(\boldsymbol{\xi}_0)$ and $B_\delta(\boldsymbol{\xi}_0 + 2\pi\mathbf{q}_0)$ lie within $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Let f_δ and g_δ in $\mathcal{D}$ be functions such that

$$\hat{f}_\delta(\boldsymbol{\xi}) = \frac{1}{\sqrt{\Omega_n \delta^n}}\,\chi_{B_\delta(\boldsymbol{\xi}_0)}(\boldsymbol{\xi}) \text{ and } \hat{g}_\delta(\boldsymbol{\xi}) = \frac{1}{\sqrt{\Omega_n \delta^n}}\,\chi_{B_\delta(\boldsymbol{\xi}_0+2\pi\mathbf{q}_0)}(\boldsymbol{\xi}).$$

(observe that $\hat{g}_\delta(\boldsymbol{\xi}) = \hat{f}_\delta(\boldsymbol{\xi} - 2\pi\mathbf{q}_0)$). Then

$$\overline{\hat{f}_\delta(\boldsymbol{\xi})}\hat{g}_\delta(\boldsymbol{\xi} + 2\pi\mathbf{q}_0) = \frac{1}{\Omega_n \delta^n}\,\chi_{B_\delta(\boldsymbol{\xi}_0)}(\boldsymbol{\xi})$$

and this allows us to write (2.18) in the form

$$0 = \frac{1}{\Omega_n \delta^n}\int_{B_\delta(\boldsymbol{\xi}_0)} t_{\mathbf{q}_0}(\boldsymbol{\xi})\,d\boldsymbol{\xi} + \sum_{\substack{(p,\mathbf{q})\in\mathbb{Z}\times\mathcal{O}^n\\(p,\mathbf{q})\neq(0,\mathbf{q}_0)}}\int_{\mathbb{R}^n}\overline{\hat{f}_\delta(\boldsymbol{\xi})}\hat{g}_\delta(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi} =$$

$$= \frac{1}{|B_\delta(\boldsymbol{\xi}_0)|}\int_{B_\delta(\boldsymbol{\xi}_0)} t_{\mathbf{q}_0}(\boldsymbol{\xi})\,d\boldsymbol{\xi} + J_\delta.$$

In order to show that equality (1.7) (ii) is satisfied at $\boldsymbol{\xi}_0$ (and, thus, *a.e.*) it suffices to prove that $\lim_{\delta\to 0+} J_\delta = 0$. We therefore examine the sum defining J_δ more closely.

If $\overline{\hat{f}_\delta(\boldsymbol{\xi})}\hat{g}_\delta(\boldsymbol{\xi}+2^p 2\pi\mathbf{q}) \neq 0$ we must have $|\boldsymbol{\xi}-\boldsymbol{\xi}_0| < \delta$ and $|\boldsymbol{\xi}_0+2\pi\mathbf{q}_0-\boldsymbol{\xi}-2^p 2\pi\mathbf{q}| < \delta$. Thus,

$$|\mathbf{q}_0 - 2^p\mathbf{q}| = \frac{1}{2\pi}\,|(\boldsymbol{\xi}_0 + 2\pi\mathbf{q}_0) - (\boldsymbol{\xi} + 2^p 2\pi\mathbf{q}) + (\boldsymbol{\xi} - \boldsymbol{\xi}_0)| < \frac{\delta}{\pi}. \tag{2.19}$$

Since we are interested in $\lim_{\delta\to 0+} J_\delta$ we can assume $\frac{\delta}{\pi} < 1$. If $p > 0$ we must have $|\mathbf{q}_0 - 2^p\mathbf{q}| \geq 1 > \frac{\delta}{\pi}$ since $\mathbf{q}_0 \in \mathcal{O}^n$. If $p = 0$ we must have $|\mathbf{q}_0 - 2^p\mathbf{q}| = |\mathbf{q}_0 - \mathbf{q}| \geq 1 > \frac{\delta}{\pi}$ when $\mathbf{q}_0 \neq \mathbf{q}$. Finally, if $p < 0$ we must have $|\mathbf{q}_0 - 2^p\mathbf{q}| = 2^p|2^{-p}\mathbf{q}_0 - \mathbf{q}| \geq 2^p$ since $\mathbf{q} \in \mathcal{O}^n$. Hence, if j_0 is the largest integer such that $2^{j_0} \leq \frac{\delta}{\pi}$, we have $j_0 < 0$ and

$$J_\delta = \sum_{p\leq j_0}\sum_{\mathbf{q}\in\mathcal{O}^n}\int_{\mathbb{R}^n}\overline{\hat{f}_\delta(\boldsymbol{\xi})}\hat{g}_\delta(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

Making the change of variables $\boldsymbol{\eta} = 2^{-p}\boldsymbol{\xi}$ we obtain

$$|J_\delta| \le \sum_{p\le j_0}\sum_{\mathbf{q}\in\mathcal{O}^n} 2^{pn}\int_{\mathbb{R}^n} |\hat f_\delta(2^p\boldsymbol{\eta})|\,|\hat g_\delta(2^p(\boldsymbol{\eta}+2\pi\mathbf{q}))|\,|t_{\mathbf{q}}(\boldsymbol{\eta})|\,d\boldsymbol{\eta}.$$

Since

$$2|t_{\mathbf{q}}(\boldsymbol{\xi})| \le \sum_{\ell=1}^{L}\Big\{\sum_{m\ge 0}|\widehat{\psi}^\ell(2^m\boldsymbol{\xi})|^2 + \sum_{m\ge 0}|\widehat{\psi}^\ell(2^m(\boldsymbol{\xi}+2\pi\mathbf{q}))|^2\Big\},$$

we can reduce our problem to estimating

$$J_\delta^1 = \sum_{p\le j_0}\sum_{\mathbf{q}\in\mathcal{O}^n} 2^{pn}\int_{\mathbb{R}^n} |\hat f_\delta(2^p\boldsymbol{\xi})|\,|\hat g_\delta(2^p(\boldsymbol{\xi}+2\pi\mathbf{q}))|\Big\{\sum_{m\ge 0}|\widehat{\psi}^\ell(2^m\boldsymbol{\xi})|^2\Big\}\,d\boldsymbol{\xi}.$$

and an analogous term, J_δ^2, in which $\widehat{\psi}^\ell(2^m\boldsymbol{\xi})$ is replaced by $\widehat{\psi}^\ell(2^m(\boldsymbol{\xi}+2\pi\mathbf{q}))$ (this is the same argument we used prior to our introduction of $I_1^{\delta,\sharp}$). Once we have shown that $\lim_{\delta\to 0+} J_\delta^1 = 0$, it will be easy to see how we can modify the argument to obtain $\lim_{\delta\to 0+} J_\delta^2 = 0$. J_δ^1 (and J_δ^2) depends on ℓ; we do not indicate this dependence in the sequel since there are only a finite number of these terms.

Let $T(\boldsymbol{\xi}) = \sum_{m\ge 0}|\widehat{\psi}^\ell(2^m\boldsymbol{\xi})|^2$. The argument in the first section that showed that $t_{\mathbf{q}} \in L^1(\mathbb{R}^n)$ applies here to show $T\in L^1(\mathbb{R}^n)$. Furthermore, since $|B_\delta(\boldsymbol{\xi}_0)|^{1/2}\hat f_\delta = \chi_{B_\delta(\boldsymbol{\xi}_0)}(\boldsymbol{\xi})$ we have

$$J_\delta^1 = \sum_{p\le j_0}\sum_{\mathbf{q}\in\mathcal{O}^n}\frac{2^{pn}}{\sqrt{\Omega_n\delta^n}}\int_{|2^p\boldsymbol{\xi}-\boldsymbol{\xi}_0|<\delta} T(\boldsymbol{\xi})\,|\hat g_\delta(2^p(\boldsymbol{\xi}+2\pi\mathbf{q}))|\,d\boldsymbol{\xi}.$$

If $\hat g_\delta(2^p(\boldsymbol{\xi}+2\pi\mathbf{q})) \ne 0$ we must have $|2^p(\boldsymbol{\xi}+2\pi\mathbf{q}) - (\boldsymbol{\xi}_0+2\pi\mathbf{q}_0)| \le \delta$ and this inequality, together with $|2^p\boldsymbol{\xi}-\boldsymbol{\xi}_0|<\delta$, implies (as in the case (2.19))

$$|\mathbf{q}_0 - 2^p\mathbf{q}| = \frac{1}{2\pi}\,|2^p(\boldsymbol{\xi}+2\pi\mathbf{q}) - (\boldsymbol{\xi}_0+2\pi\mathbf{q}_0) + (\boldsymbol{\xi}_0 - 2^p\boldsymbol{\xi})| \le \frac{\delta}{\pi}. \tag{2.20}$$

In our case $p\le j_0<0$; hence, from (2.20) we have $|2^{-p}\mathbf{q}_0 - \mathbf{q}| \le \frac{2^{-p}\delta}{\pi}$ and $2^{-p}\mathbf{q}_0$ is a lattice point. The number of lattice points $\mathbf{q}$ satisfying this inequality cannot exceed $(1+\frac{2^{-p}2\delta}{\pi})^n$ since they must lie within the n-dimensional cube centered at $2^{-p}\mathbf{q}_0$ of side length at most $\frac{2^{-p}2\delta}{\pi}$. Consequently,

$$\sum_{\mathbf{q}\in\mathcal{O}^n}|\hat g_\delta(2^p(\boldsymbol{\xi}+2\pi\mathbf{q}))| \le (1+\frac{2^{-p}2\delta}{\pi})^n\,\|\hat g_\delta\|_\infty.$$

Using this estimate together with $\|\hat{g}_\delta\|_\infty = \frac{1}{\sqrt{\Omega_n \delta^n}}$ in the above expression for J_δ^1 we obtain

$$\begin{aligned} J_\delta^1 \quad & \leq \sum_{p\leq j_0}\left(2^p+\frac{2\delta}{\pi}\right)^n \frac{1}{\Omega_n\delta^n}\int_{|2^p\boldsymbol{\xi}-\boldsymbol{\xi}_0|<\delta} T(\boldsymbol{\xi})\,d\boldsymbol{\xi} \\ & \leq \sum_{p\leq j_0}\left(2^{j_0}+\frac{2\delta}{\pi}\right)^n \frac{1}{\Omega_n\delta^n}\int_{|2^p\boldsymbol{\xi}-\boldsymbol{\xi}_0|<\delta} T(\boldsymbol{\xi})\,d\boldsymbol{\xi} \\ & \leq \sum_{p\leq j_0}\left(\frac{3}{\pi}\right)^n \frac{1}{\Omega_n}\int_{|2^p\boldsymbol{\xi}-\boldsymbol{\xi}_0|<\delta} T(\boldsymbol{\xi})\,d\boldsymbol{\xi}. \end{aligned}$$

But, $\{\boldsymbol{\xi} : |2^p\boldsymbol{\xi}-\boldsymbol{\xi}_0|<\delta\} \subset \{\boldsymbol{\xi} : 2^{-p}(|\boldsymbol{\xi}_0|-\delta)<|\boldsymbol{\xi}|<2^{-p}(|\boldsymbol{\xi}_0|+\delta)\} \equiv \mathcal{A}_p$. If $3\delta < |\boldsymbol{\xi}_0|$ the sets $\mathcal{A}_p$, $p = j_0,\, j_0-1,\, j_0-2, \ldots$, are mutually disjoint. Thus,

$$\begin{aligned} J_\delta^1 \quad & \leq \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \sum_{p\leq j_0}\int_{\mathcal{A}_p} T(\boldsymbol{\xi})\,d\boldsymbol{\xi} = \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \int_{\cup_{p\leq j_0}\mathcal{A}_p} T(\boldsymbol{\xi})\,d\boldsymbol{\xi} \\ & \leq \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \int_{2^{-j_0}(|\boldsymbol{\xi}_0|-\delta)<|\boldsymbol{\xi}|} T(\boldsymbol{\xi})\,d\boldsymbol{\xi} \leq \frac{1}{\Omega_n}\left(\frac{3}{\pi}\right)^n \int_{\frac{\pi}{\delta}(|\boldsymbol{\xi}_0|-\delta)<|\boldsymbol{\xi}|} T(\boldsymbol{\xi})\,d\boldsymbol{\xi}. \end{aligned}$$

But the last integral tends to 0 as $\delta \to 0+$ since $T \in L^1(\mathbb{R}^n)$. Thus, $J_\delta^1 \to 0$ as $\delta \to 0+$. As mentioned before, a similar argument shows $\lim_{\delta\to 0+} J_\delta^2 = 0$. In fact, the change of variables $\boldsymbol{\eta} = \boldsymbol{\xi} + 2\pi\mathbf{q}$ in the integrals defining J_δ^2 convert this quantity to, essentially, J_δ^1 except that the roles of f_δ and g_δ are interchanged. Because of this we can let the point $\boldsymbol{\xi}_0 + 2\pi\mathbf{q}_0$ play the role of $\boldsymbol{\xi}_0$ in the argument we just gave in order to show that $\lim_{\delta\to 0+} J_\delta^2 = 0$. We thus obtain the desired result $\lim_{\delta\to 0+} J_\delta = 0$ and, therefore, equation (1.7) (ii) is satisfied by the system Ψ almost everywhere. This establishes Theorem 2.3.

$\square$

3 The Second Theorem

We now consider the case when the "analyzing" family $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^L\}$ differs from the "synthesizing" system $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$. As indicated in the first section the result we shall establish "essentially" asserts that equation (1.5) is satisfied for all $f \in L^2(\mathbb{R}^n)$ if and only if equations (1.6) (i) and (ii) are satisfied *a.e.* by Φ and Ψ. There are some basic differences between the general case, however, and the one we just presented in section §2. The local integrability in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ of the expressions $\sum_{j\in\mathbb{Z}} |\widehat{\psi}^\ell(2^j\boldsymbol{\xi})|^2$, first discussed in Remark 2.13, played an important role in our arguments and arose in a natural way from our arguments. We did not, however, need to assume this property for the system Ψ when we announced Theorem 2.3, the principal result in §2. If we do assume that this property holds for both systems Φ and Ψ, the proof that the two equations (1.6) (i) and (ii) are equivalent to the equality

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \varphi^\ell_{j,\mathbf{k}})(\psi^\ell_{j,\mathbf{k}}, f) \tag{3.1}$$

for all $f \in \mathcal{D}$ is essentially the same as the proof we presented for Theorem 2.3. We will be more precise about this below. From this we do obtain a "weak version" of the representation (1.5) for all such f. This is a consequence of the fact that, by polarization, (3.1) implies

$$(f, g) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \varphi^\ell_{j,\mathbf{k}})(\psi^\ell_{j,\mathbf{k}}, g) \tag{3.2}$$

for all f and g in $\mathcal{D}$. We cannot, however, establish the $L^2(\mathbb{R}^n)$-convergence of the series (1.5).

If we do not make the local integrability assumptions for each of the two series

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}^\ell(2^j\boldsymbol{\xi})|^2 \quad \text{and} \quad \sum_{j\in\mathbb{Z}} |\widehat{\varphi}^\ell(2^j\boldsymbol{\xi})|^2, \quad \ell = 1, 2, \ldots, L \tag{3.3}$$

we can still establish the equivalence of (1.6) and some form of (3.1). We will present the arguments for this result in the fourth section. It is also clear from our presentation that the case $L > 1$ offers no more complications than the case $L = 1$; hence, we will not use the upper index ℓ from now on.

We begin by presenting an example that illustrates some of the assertions we have made. Let $I = [-2\pi, -\pi) \cup (\pi, 2\pi]$ and θ be the function satisfying $\hat{\theta} = \chi_I$. Thus, θ is the *Shannon orthonormal wavelet* (see Chapter 1 of this thesis). We then define ψ by $\widehat{\psi}(\xi) = \hat{\theta}(2\xi) = \chi_{[-\pi,-\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]}(\xi)$ or, equivalently, by $\psi(x) = 2^{-1}\theta(2^{-1}x)$. Let φ be the *scaling function* associated with the Shannon wavelet; that is, $\widehat{\varphi} = \chi_{[-\pi,\pi]}$.

PROPOSITION 3.4 *The pair* (φ, ψ) *satisfy equalities* (1.6) (i) *and* (ii) *; that is,*

$$(i) \qquad \sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^j\xi)\overline{\widehat{\psi}(2^j\xi)} = 1 \qquad \text{for all} \quad \xi \neq 0,$$

$$(ii) \qquad t_q(\xi) = \sum_{m=0}^{\infty} \widehat{\varphi}(2^m\xi)\overline{\widehat{\psi}(2^m(\xi + 2\pi q))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R},\ \forall q \in 2\mathbb{Z} + 1.$$

PROOF: Since supp $\widehat{\psi} \subset$ supp $\widehat{\varphi}$ we have $\widehat{\varphi}\,\overline{\widehat{\psi}} = \widehat{\psi}$ (because $\widehat{\psi} = \chi_{\frac{1}{2}I}$ is real-valued) and (i) is an immediate consequence of the fact that $\{2^j I\}$, $j \in \mathbb{Z}$, is partition of $\mathbb{R} \setminus \{0\}$. Equality (ii) follows from the fact that the supports of $\widehat{\varphi}$ and $\widehat{\psi}(\cdot + 2\pi q)$ are disjoint (except for a set of Lebesgue measure zero) since q is an odd integer and, thus, $|q| \geq 1$.

$\square$

PROPOSITION 3.5 $\{\sqrt{2}\,\psi_{j,2\ell}\}$ *and* $\{\sqrt{2}\,\psi_{j,2\ell+1}\}$, $j, \ell \in \mathbb{Z}$ *, are, each, an orthonormal basis for* $L^2(\mathbb{R})$.

PROOF: The identity $\sqrt{2}\,\psi_{j,2\ell} = \theta_{j-1,\ell}$, $j, \ell \in \mathbb{Z}$, and the fact that θ is an orthonormal wavelet show that $\{\sqrt{2}\,\psi_{j,2\ell}\}$, $j, \ell \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$.

To see that the family $\{\sqrt{2}\,\psi_{j,2\ell+1}\}$, $j, \ell \in \mathbb{Z}$, is an orthonormal family we first observe that, since ψ satisfies (1.7) (i) and (ii)[{2}], equality (2.4) is true for all $f \in L^2(\mathbb{R})$; that is,

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |(f, \psi_{j,k})|^2 \tag{3.6}$$

[{2}]The function ψ we are considering now is a particular case of the class of functions ψ satisfying $|\widehat{\psi}| = b$, where b is as in (1.8). Proposition 3.5 clearly shows that a "simple re-normalization" cannot convert the system $\{\psi_{j,k}\}$ to an orthonormal basis, as we indicated at the end of §1.

for all $f \in L^2(\mathbb{R})$. Breaking up the sum on the right into even and odd k's, and using the fact that $\{\sqrt{2}\,\psi_{j,2\ell}\}\; j, \ell \in \mathbb{Z}$ is an orthonormal basis we obtain, by (3.6),

$$\begin{aligned} \|f\|_2^2 &= \sum_{j\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} \{|(f, \psi_{j,2\ell})|^2 + |(f, \psi_{j,2\ell+1})|^2\} \\ &= \frac{1}{2}\,\|f\|_2^2 + \sum_{j\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} |(f, \psi_{j,2\ell+1})|^2. \end{aligned}$$

Hence,

$$\|f\|_2^2 = \sum_{j\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} |(f, \sqrt{2}\,\psi_{j,2\ell+1})|^2 \tag{3.7}$$

for all $f \in L^2(\mathbb{R})$. Since $\|\sqrt{2}\,\psi_{j,2\ell+1}\|_2 = 1$ for all $j, \ell \in \mathbb{Z}$, it follows from Lemma 2.1 and (3.7) that the system $\{\sqrt{2}\,\psi_{j,2\ell+1}\}$, $j, \ell \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$. $\square$

This example illustrates why the unconditional $L^2(\mathbb{R}^n)$-convergence of the series (1.5) is not true in general when the two equations in Proposition 3.4 are satisfied. Let us be more precise. We continue using the functions φ and ψ we just introduced. We observed that $\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1$ for all $\xi \in \mathbb{R} \setminus \{0\}$; however, $\sum_{j\in\mathbb{Z}} |\widehat{\varphi}(2^j\xi)|^2 = \infty$ for all $\xi \in \mathbb{R}$ (in particular this last sum does not define a locally integrable function in $\mathbb{R} \setminus \{0\}$). Thus, we are in the situation described in (3.3) and we shall show that a "weak version" of (1.5) is, indeed, true. The $L^2(\mathbb{R})$-convergence of the series (1.5), however, is not unconditional. To see some of the difficulties we encounter in this case let us choose a function $f \in V_0$, the space generated by the integral translates of the scaling function φ. In fact, let us choose $f = \varphi$. Since $V_0 \subset V_j$ for $j \geq 0$ we have

$$\varphi = \sum_{k\in\mathbb{Z}} (\varphi, \varphi_{j,k})\varphi_{j,k}$$

for each $j \geq 0$. Since $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j

$$\|\varphi\|_2^2 = \sum_{k\in\mathbb{Z}} |(\varphi, \varphi_{j,k})|^2 \tag{3.8}$$

for $j \geq 0$.

Consider the series $\sum_{k\in\mathbb{Z}}(\varphi, \varphi_{j,k})\psi_{j,k}$ for each j. Because of Proposition 3.5, this series is the sum of two orthogonal expansions

$$u_j + v_j = \sum_{\ell\in\mathbb{Z}}(\varphi, \varphi_{j,2\ell})\psi_{j,2\ell} + \sum_{\ell\in\mathbb{Z}}(\varphi, \varphi_{j,2\ell+1})\psi_{j,2\ell+1}$$

and, from (3.8) we have

$$\begin{aligned}\|u_j\|_2^2 + \|v_j\|_2^2 \quad &= \frac{1}{2}\sum_{\ell\in\mathbb{Z}} |(\varphi, \varphi_{j,2\ell})|^2 + \frac{1}{2}\sum_{\ell\in\mathbb{Z}} |(\varphi, \varphi_{j,2\ell+1})|^2 \\ &= \frac{1}{2}\,\|\varphi\|_2^2\end{aligned}$$

for each $j \geq 0$. Hence, for infinitely many j either $\|u_j\|_2^2$ or $\|v_j\|_2^2$ exceeds $\frac{1}{4}\|\varphi\|_2^2 = \frac{1}{4}$. Thus, if, say, $\|u_j\|_2^2 \geq \frac{1}{4}$ infinitely often, then

$$\left\|\sum_{j\geq 0}\sum_{\ell\in\mathbb{Z}}(\varphi, \varphi_{j,2\ell})\psi_{j,2\ell}\right\|_2^2 = \sum_{j\geq 0}\|u_j\|_2^2 = \infty$$

(since $u_j \perp u_{j'}$ if $0 \leq j < j'$). Clearly, then, the series $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(\varphi, \varphi_{j,k})\psi_{j,k}$ cannot converge unconditionally to φ in the $L^2(\mathbb{R})$-norm.

We now turn our attention to the extension of Theorem 2.3 to the case of a pair φ, ψ of "generating functions". We first establish the following

THEOREM 3.9 *Suppose that $\varphi, \psi \in L^2(\mathbb{R}^n)$ are such that the functions defined by the series in (3.3) are locally integrable in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Then φ and ψ satisfy the equations*

$$(i) \quad \sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^j\boldsymbol{\xi})\overline{\widehat{\psi}(2^j\boldsymbol{\xi})} = 1, \qquad \textit{for} \quad a.e.\ \boldsymbol{\xi} \in \mathbb{R}^n,$$

$$(ii) \quad t_{\mathbf{q}}(\boldsymbol{\xi}) = \sum_{m=0}^{\infty} \widehat{\varphi}(2^m\boldsymbol{\xi})\overline{\widehat{\psi}(2^m(\boldsymbol{\xi} + 2\pi\mathbf{q}))} = 0, \quad \textit{for} \quad a.e.\ \boldsymbol{\xi} \in \mathbb{R}^n, \quad \textit{when} \ \mathbf{q} \in \mathcal{O}^n,$$

if and only if

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = \sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f, \varphi_{j,\mathbf{k}})(\psi_{j,\mathbf{k}}, f) \tag{3.10}$$

for all $f \in \mathcal{D}$. The convergence of all these series is absolute and, thus, unconditional.

Since $|(f,\psi_{j,\mathbf{k}})|\,|(\varphi_{j,\mathbf{k}},f)| \le |(f,\psi_{j,\mathbf{k}})|^2 + |(f,\varphi_{j,\mathbf{k}})|^2$, the decomposition (2.9) applied to ψ and φ separately (with $f \in \mathcal{D}$), and the observations made in Remark 2.13, give us the absolute (and unconditional) convergence of the series appearing in Theorem (3.9) (we use, of course, the local integrability of the series in (3.3)).

Let us now indicate which modifications in the argument we presented in §2 are needed to provide a proof of this theorem. Here I denotes the sum on the right in (3.10). We begin by establishing the analog of the decomposition (2.9):

$$(2\pi)^n I = I_0 + I_1, \tag{3.11}$$

where

$$\begin{aligned} I_0 &= \sum_{j\in\mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} |\hat{f}(2^j\boldsymbol{\xi})|^2\, \widehat{\varphi}(\boldsymbol{\xi})\overline{\widehat{\psi}(\boldsymbol{\xi})}\, d\boldsymbol{\xi} \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2\, \widehat{\varphi}(2^{-j}\boldsymbol{\xi})\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})}\, d\boldsymbol{\xi} \end{aligned}$$

and

$$\left.\begin{aligned} I_1 &= \sum_{j\in\mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} \overline{\hat{f}(2^j\boldsymbol{\xi})}\widehat{\varphi}(\boldsymbol{\xi}) \sum_{\mathbf{k}\neq\mathbf{0}} \hat{f}(2^j(\boldsymbol{\xi}+2\mathbf{k}\pi))\overline{\widehat{\psi}(\boldsymbol{\xi}+2\mathbf{k}\pi)}\, d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}(\boldsymbol{\xi}+2\pi 2^p\mathbf{q}) t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\, d\boldsymbol{\xi} \end{aligned}\right\} \tag{3.12}$$

where $t_{\mathbf{q}}$ is defined by (ii) in the statement of Theorem 3.9 and involves both φ and ψ.

The proof of the decomposition (3.11) follows the same line as the one we gave for (2.9). The changes that are needed are obvious: the Plancherel theorem gives us the product

$$\left(\int_{\mathbb{R}^n} \hat{f}(2^j\boldsymbol{\xi})\overline{\widehat{\psi}(\boldsymbol{\xi})} e^{i\mathbf{k}\boldsymbol{\xi}}\, d\boldsymbol{\xi}\right)\overline{\left(\int_{\mathbb{R}^n} \hat{f}(2^j\boldsymbol{\xi})\overline{\widehat{\varphi}(\boldsymbol{\xi})} e^{i\mathbf{k}\boldsymbol{\xi}}\, d\boldsymbol{\xi}\right)}$$

instead of the absolute value squared of the first factor. This leads us to the introduction of the function $G_j(\boldsymbol{\xi}) = \hat{f}(2^j\boldsymbol{\xi})\overline{\widehat{\varphi}(\boldsymbol{\xi})}$ along with $F_j(\boldsymbol{\xi}) = \hat{f}(2^j\boldsymbol{\xi})\overline{\widehat{\psi}(\boldsymbol{\xi})}$. We then "periodize" both F_j and G_j to obtain the equality

$$(2\pi)^{-n} \sum_{\mathbf{k}\in\mathbb{Z}^n} \hat{F}_j(\mathbf{k})\overline{\hat{G}_j(\mathbf{k})} = \int_{\mathbb{R}^n} \Big\{ \sum_{\mathbf{m}\in\mathbb{Z}^n} F_j(\boldsymbol{\xi} + 2\mathbf{m}\pi) \Big\} \overline{G_j(\boldsymbol{\xi})}\, d\boldsymbol{\xi}$$

which leads us to the first expression for I_1 in (3.12) (see the argument preceding Lemma 2.12).

In order to prove that I_1 equals the second expression for I_1 in (3.12) we need to establish the analogs of Lemmas 2.12 and 2.15. This last estimate, of course, does not involve the functions φ and ψ and no change is needed. For the analog of Lemma 2.12 the problem can clearly be reduced to establishing the inequality (2.14).

Having established the decomposition (3.11) it is then immediate that equalities (i) and (ii) of Theorem 3.9 imply (3.10) for all $f \in \mathcal{D}$. We must, therefore, show that the converse is true. Again, the argument we gave in §2 for the corresponding part in (2.4) applies here if we make some simple and natural modifications. In (2.17) we now must choose $\tau(\boldsymbol{\xi}) = \sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^j\boldsymbol{\xi})\overline{\widehat{\psi}(2^j\boldsymbol{\xi})}$. Then, the same choice of $f_\delta \in \mathcal{D}$ and the inequality

$$2|\widehat{\psi}(\boldsymbol{\xi})|\,|\widehat{\varphi}(\boldsymbol{\xi} + 2\mathbf{m}\pi)| \le |\widehat{\psi}(\boldsymbol{\xi})|^2 + |\widehat{\varphi}(\boldsymbol{\xi} + 2\mathbf{m}\pi)|^2,$$

leads us to the equality $1 = \frac{1}{\Omega_n\delta^n} \int_{B_\delta(\boldsymbol{\xi}_0)} \tau(\boldsymbol{\xi})\, d\boldsymbol{\xi} + I_1^\delta$, where $\lim_{\delta\to 0+} I_1^\delta = 0$. When $\boldsymbol{\xi}_0$ is a point of differentiability of the integral of τ we obtain equality (i) in Theorem 3.9 at $\boldsymbol{\xi}_0$. As in §2, this also gives us the equality (2.18) with $t_{\mathbf{q}}(\boldsymbol{\xi})$ as defined in (ii) of Theorem 3.9. Again, we choose f_δ and g_δ as before. The rest of proof given at the end of §2 applies here and this establishes Theorem 3.9.

□

4 Some other results

In the last section we obtained the equivalence between equality (3.10) and the two equations (i) and (ii) announced in Theorem 3.9 provided the two series in (3.3) are locally integrable in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. We also gave an example of two functions $\varphi, \psi \in L^2(\mathbb{R}^n)$ that satisfy (i) and (ii) but $\sum_{j\in\mathbb{Z}} |\widehat{\varphi}(2^j\xi)|^2$ is not locally integrable in $\mathbb{R}\setminus\{0\}$. We left open the question of the validity of (3.10) in this case but did observe that (1.5) cannot be interpreted in terms of unconditional convergence in $L^2(\mathbb{R})$ (which clearly implies (3.10)). In this section we examine other interpretations of (3.10) and its connection with equalities (i) and (ii) without assuming local integrability of the series in (3.3).

Let us first examine the convergence of the series in (3.10) when φ and ψ are functions in $L^2(\mathbb{R}^n)$. Toward this end we establish the following result:

LEMMA 4.1 *Let J be a fixed integer, then the series*

$$S_J = S_J(f) = \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}}, f)$$

converges absolutely for each $f \in \mathcal{D}$ when $\varphi, \psi \in L^2(\mathbb{R}^n)$.

PROOF: By Schwarz's inequality it suffices to show that

$$\sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} |(f, \psi_{j,\mathbf{k}})|^2 < \infty$$

for $f \in \mathcal{D}$. We argue as we did in the proof of the decomposition of (2.9) to obtain

$$(2\pi)^n \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} |(f, \psi_{j,\mathbf{k}})|^2 = \sum_{j\le J} \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 \, |\widehat{\psi}(2^{-j}\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} +$$

$$\sum_{j\le J} 2^{jn} \int_{\mathbb{R}^n} \overline{\hat{f}(2^j\boldsymbol{\xi})\widehat{\psi}}(\boldsymbol{\xi}) \sum_{\mathbf{m}\ne\mathbf{0}} \hat{f}(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi))\overline{\widehat{\psi}(\boldsymbol{\xi} + 2\mathbf{m}\pi)} \, d\boldsymbol{\xi}.$$

Lemma 2.12 assures us that the second summand represents an absolutely convergent series that is integrable. Moreover,

$$\sum_{j\le J} 2^{jn} \int_{\mathbb{R}^n} |\hat{f}(2^j\boldsymbol{\xi})| \, |\widehat{\psi}(\boldsymbol{\xi})| \sum_{\mathbf{m}\ne\mathbf{0}} |\hat{f}(2^j(\boldsymbol{\xi} + 2\mathbf{m}\pi))| \, |\widehat{\psi}(\boldsymbol{\xi} + 2\mathbf{m}\pi)| \, d\boldsymbol{\xi} \le C < \infty$$

where C is independent of J. In order to estimate the first summand we use the fact that $f \in \mathcal{D}$ and, thus, supp $\hat{f}$ lies in an annular region of the form $\{\boldsymbol{\xi} \in \mathbb{R}^n \,|\, 2^{-L}\pi \leq |\boldsymbol{\xi}| \leq 2^L\pi\}$. Hence, we can estimate this first summand as follows:

$$\int_{2^{-L}\pi \leq |\boldsymbol{\xi}| \leq 2^L\pi} |\hat{f}(\boldsymbol{\xi})|^2 \sum_{j \leq J} |\widehat{\psi}(2^{-j}\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \;\leq\; \|\hat{f}\|_\infty^2 \sum_{\ell=-L}^{L-1} \int_{2^\ell\pi \leq |\boldsymbol{\xi}| \leq 2^{\ell+1}\pi} \sum_{j \leq J} |\widehat{\psi}(2^{-j}\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} =$$

$$\|\hat{f}\|_\infty^2 \sum_{\ell=-L}^{L-1} \sum_{j \leq J} 2^{jn} \int_{2^{\ell-j}\pi \leq |\boldsymbol{\eta}| \leq 2^{(\ell-j)+1}\pi} |\widehat{\psi}(\boldsymbol{\eta})|^2 \, d\boldsymbol{\eta} \leq 2^{nJ} \|\hat{f}\|_\infty^2 \sum_{\ell=-L}^{L-1} \int_{2^{\ell-J}\pi \leq |\boldsymbol{\eta}|} |\widehat{\psi}(\boldsymbol{\eta})|^2 \, d\boldsymbol{\eta}$$

$$\leq\; 2^{nJ} 2L \|\hat{f}\|_\infty^2 \|\widehat{\psi}\|_2^2 < \infty$$

and Lemma 4.1 is proved.

$\square$

The double sum on the right of (3.10) corresponds to the expression I in §2. In analogy with (2.9) we shall consider a similar decomposition for the partial sums S_J of this double series:

$$(2\pi)^n S_J = I_0^J + I_1^J \,,\; J \in \mathbb{Z}, \tag{4.2}$$

where

$$I_0^J = \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 \sum_{j \leq J} \widehat{\varphi}(2^{-j}\boldsymbol{\xi}) \overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})} \, d\boldsymbol{\xi}$$

and

$$I_1^J = \sum_{j \leq J} 2^{jn} \int_{\mathbb{R}^n} \overline{\hat{f}(2^j\boldsymbol{\xi})} \widehat{\varphi}(\boldsymbol{\xi}) \sum_{\mathbf{k} \neq \mathbf{0}} \hat{f}(2^j(\boldsymbol{\xi} + 2\mathbf{k}\pi)) \overline{\widehat{\psi}(\boldsymbol{\xi} + 2\mathbf{k}\pi)} \, d\boldsymbol{\xi}.$$

The observations we made that showed how to obtain (3.11) and the first equality in (3.12) are valid here and provide us with (4.2). The argument we made in the proof of Lemma 4.1 shows that $\sum_{j \leq J} |\widehat{\varphi}(2^{-j}\boldsymbol{\xi})| \, |\widehat{\psi}(2^{-j}\boldsymbol{\xi})|$ is locally integrable in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Hence, I_0^J is well-defined for each $f \in \mathcal{D}$. Reasoning as we did at the end of §3, we shall show that I_1^J has an expression involving the functions $t_{\mathbf{q}}$ (as in the term following the second equality in (3.12)); in fact, for J sufficiently large , we have, as in (3.12),

$$I_1^J = \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{p\in\mathbb{Z}} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}(\boldsymbol{\xi} + 2\pi 2^p \mathbf{q}) t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\, d\boldsymbol{\xi} = I_1 \tag{4.3}$$

(the size of J for this to be true depends on the diameter of the support of $\hat{f}$, where $f \in \mathcal{D}$). To see this we repeat the arguments we gave before, but need to take into account that the sum in the index j is limited by J:

$$\begin{aligned}
I_1^J \quad &= \sum_{j\le J} \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \widehat{\varphi}(2^{-j}\boldsymbol{\xi}) \sum_{\mathbf{m}\neq\mathbf{0}} \hat{f}(\boldsymbol{\xi} + 2^j 2\mathbf{m}\pi) \overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi} + 2\mathbf{m}\pi)}\, d\boldsymbol{\xi} \\
&= \sum_{j\le J} \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \widehat{\varphi}(2^{-j}\boldsymbol{\xi}) \sum_{r\ge 0} \sum_{\mathbf{q}\in\mathcal{O}^n} \hat{f}(\boldsymbol{\xi} + 2^{j+r} 2\pi\mathbf{q}) \overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi} + 2\pi 2^r \mathbf{q})}\, d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n} \sum_{r\ge 0} \sum_{j\le J} \widehat{\varphi}(2^r(2^{-(r+j)}\boldsymbol{\xi})) \hat{f}(\boldsymbol{\xi} + 2^{j+r} 2\pi\mathbf{q}) \overline{\widehat{\psi}(2^r(2^{-(r+j)}\boldsymbol{\xi} + 2\mathbf{q}\pi))}\, d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n} \sum_{r\ge 0} \sum_{p\le J+r} \widehat{\varphi}(2^r 2^{-p}\boldsymbol{\xi}) \overline{\widehat{\psi}(2^r(2^{-p}\boldsymbol{\xi} + 2\pi\mathbf{q}))} \hat{f}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q})\, d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n} \sum_{p\le J} \sum_{r\ge 0} \widehat{\varphi}(2^r 2^{-p}\boldsymbol{\xi}) \overline{\widehat{\psi}(2^r(2^{-p}\boldsymbol{\xi} + 2\pi\mathbf{q}))} \hat{f}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q})\, d\boldsymbol{\xi} \\
&+ \int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n} \sum_{p> J} \sum_{r\ge p-J} \widehat{\varphi}(2^r 2^{-p}\boldsymbol{\xi}) \overline{\widehat{\psi}(2^r(2^{-p}\boldsymbol{\xi} + 2\pi\mathbf{q}))} \hat{f}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q})\, d\boldsymbol{\xi}.
\end{aligned}$$

The first summand equals

$$\int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n} \sum_{p\le J} \hat{f}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q}) t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

If the diameter of the support of $\hat{f}$ does not exceed $2^{J+1}2\pi$, then either $\boldsymbol{\xi}$ or $\boldsymbol{\xi}+2^p 2\pi\mathbf{q}$ must lie outside $\operatorname{supp}\, \hat{f}$ if $p > J$ (since $\mathbf{q} \in \mathcal{O}^n$). Thus, $\overline{\hat{f}(\boldsymbol{\xi})}\, \hat{f}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q}) = 0$ and the second term is 0. But, in this case we also have

$$\int_{\mathbb{R}^n} \overline{\hat{f}(\boldsymbol{\xi})} \sum_{\mathbf{q}\in\mathcal{O}^n} \sum_{p> J} \hat{f}(\boldsymbol{\xi} + 2^p 2\pi\mathbf{q}) t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\, d\boldsymbol{\xi} = 0.$$

We have shown, therefore, that (4.3) is true if $J + 2 \ge \log_2\left\{\frac{\operatorname{diam}(\operatorname{supp}\, \hat{f})}{\pi}\right\}$. Thus, together with (4.2), this gives us the equivalence $\lim_{J\to\infty} S_J$ exists if and only if $\lim_{J\to\infty} I_0^J$ exists.

In the present context we have not yet considered the two equations, (i) and (ii), in (3.9); however, these observations can be used to obtain the following version of Theorem 3.9 when we do not assume the local integrability of the series (3.3) in $\mathbb{R}^n \setminus \{\mathbf{0}\}$:

THEOREM 4.4 *Suppose* $\varphi, \psi \in L^2(\mathbb{R}^n)$. *Then*

$$\lim_{J\to\infty} S_J = \lim_{J\to\infty} \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}}, f) = \|f\|_2^2 \tag{4.5}$$

for every $f \in \mathcal{D}$ *if and only if*

$$\left.\begin{array}{ll} (i) & \displaystyle\lim_{J\to\infty} \sum_{j\le J} \widehat{\varphi}(2^{-j}\boldsymbol{\xi})\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})} = \mathbf{1} \\ & \textit{weakly in } L^1(K) \textit{ whenever } K \textit{ is a compact subset of } \mathbb{R}^n \setminus \{\mathbf{0}\}, \\ & \textit{where } \mathbf{1} \textit{ is the constant function that equals 1 on } \mathbb{R}^n, \\ (ii) & t_{\mathbf{q}}(\boldsymbol{\xi}) = \displaystyle\sum_{r=0}^{\infty} \widehat{\varphi}(2^r\boldsymbol{\xi})\overline{\widehat{\psi}(2^r(\boldsymbol{\xi}+2\pi\mathbf{q}))} = 0, \ \textit{for a.e. } \boldsymbol{\xi} \in \mathbb{R}^n, \ \forall\, \mathbf{q} \in \mathcal{O}^n \end{array}\right\} \tag{4.6}$$

PROOF: We first show that (4.6) implies (4.5). Since $t_{\mathbf{q}}(\boldsymbol{\xi}) = 0$ for *a.e.* $\boldsymbol{\xi}$, (4.2) and (4.3) imply

$$S_J(f) = S_J = (2\pi)^{-n} I_0^J = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 \sum_{j\le J} \widehat{\varphi}(2^{-j}\boldsymbol{\xi})\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})}\, d\boldsymbol{\xi}.$$

But, by (i) and the fact that $K = \operatorname{supp} \hat{f}$ is a compact subset of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ we have

$$\lim_{J\to\infty} S_J = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = \|f\|_2^2$$

and, thus, (4.5) is true.

To establish the converse we first show that (4.6) (ii) is a consequence of (4.5). The argument is much like the one we gave for Theorem 2.3. We select a point $\boldsymbol{\xi}_0$ of differentiability of the integral of $t_{\mathbf{q}_0}$ such that neither $\boldsymbol{\xi}_0$ nor $\boldsymbol{\xi}_0 + 2\pi\mathbf{q}_0$ is 0, and $\delta > 0$ such that $B_\delta(\boldsymbol{\xi}_0)$, $B_\delta(\boldsymbol{\xi}_0 + 2\pi\mathbf{q}_0)$ are disjoint balls in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Again we choose f_δ and g_δ such that $\hat{f}_\delta = \frac{1}{\sqrt{\Omega_n\delta^n}}\chi_{B_\delta(\boldsymbol{\xi}_0)}$ and $\hat{g}_\delta = \frac{1}{\sqrt{\Omega_n\delta^n}}\chi_{B_\delta(\boldsymbol{\xi}_0+2\pi\mathbf{q}_0)}$. From

(4.2) and polarization we have, for large J (see the discussion before Theorem 4.4),

$$
\begin{aligned}
&(2\pi)^n \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f_\delta,\psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}},g_\delta)\\
&=\int_{\mathbb{R}^n}\hat f_\delta(\boldsymbol{\xi})\overline{\hat g_\delta(\boldsymbol{\xi})}\sum_{j\le J}\widehat{\varphi}(2^{-j}\boldsymbol{\xi})\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})}\,d\boldsymbol{\xi}\\
&+\int_{\mathbb{R}^n}\overline{\hat g_\delta(\boldsymbol{\xi})}\sum_{p\in\mathbb{Z}}\sum_{\mathbf{q}\in\mathcal{O}^n}\hat f_\delta(\boldsymbol{\xi}+2\pi 2^p\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi}\\
&=\int_{\mathbb{R}^n}\overline{\hat g_\delta(\boldsymbol{\xi})}\sum_{p\in\mathbb{Z}}\sum_{\mathbf{q}\in\mathcal{O}^n}\hat f_\delta(\boldsymbol{\xi}+2\pi 2^p\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi}.
\end{aligned}
$$

since $\hat f_\delta\,\overline{\hat g_\delta}=0$ because $B_\delta(\boldsymbol{\xi}_0)\cap B_\delta(\boldsymbol{\xi}_0+2\pi\mathbf{q}_0)=\emptyset$ (we observe that it suffices to choose J so that $2^{J+2}\pi>\operatorname{diam}(\operatorname{supp}\,(\hat f_\delta+\hat g_\delta)))$. On the other hand, by polarization and (4.5) we have

$$
\lim_{J\to\infty}\sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f_\delta,\psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}},g_\delta)=(f_\delta,g_\delta)=0
$$

(using, again, $\hat f_\delta\,\overline{\hat g_\delta}=0$ and the Plancherel theorem). But we have just shown that for J large enough

$$
(2\pi)^n \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f_\delta,\psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}},g_\delta)=
$$

$$
=\int_{\mathbb{R}^n}\overline{\hat g_\delta(\boldsymbol{\xi})}\sum_{p\in\mathbb{Z}}\sum_{\mathbf{q}\in\mathcal{O}^n}\hat f_\delta(\boldsymbol{\xi}+2\pi 2^p\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi}\equiv A_\delta
$$

and the last expression is independent of J. It follows that $A_\delta=0$.

But the argument that was presented at the end of §2, that showed $\lim_{\delta\to 0+}A_\delta=t_{\mathbf{q}_0}(\boldsymbol{\xi}_0)$, applies here. Consequently $t_{\mathbf{q}_0}(\boldsymbol{\xi}_0)=0$ and (4.6) (ii) is established since almost every point in $\mathbb{R}^n$ is such $\boldsymbol{\xi}_0$, a point of differentiability of the integral of $t_{\mathbf{q}_0}$.

It is now particularly easy to show that (4.6) (i) is also a consequence of (4.5). In fact, if we use (4.2), (4.3) and (4.6) (ii) we have

$$
S_J=(2\pi)^{-n}I_0^J=(2\pi)^{-n}\int_{\mathbb{R}^n}|\hat f(\boldsymbol{\xi})|^2\sum_{j\le J}\widehat{\varphi}(2^{-j}\boldsymbol{\xi})\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})}\,d\boldsymbol{\xi}.
$$

Thus, by (4.5) we have

$$\lim_{J\to\infty}(2\pi)^{-n}\int_K g(\boldsymbol{\xi})\sum_{j\le J}\widehat{\varphi}(2^{-j}\boldsymbol{\xi})\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})}\,d\boldsymbol{\xi} = (2\pi)^{-n}\int_K g(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$

for every $g \ge 0$, $g \in L^\infty(\mathbb{R}^n)$ with supp g a compact subset K of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Writing $g = g_1 + ig_2 = g_1^+ - g_1^- + i(g_2^+ - g_2^-)$ for the general $g \in L^\infty(K)$ we obtain (4.6) (i). □

Remark 4.7 The weak convergence of the sequence in (4.6)(i) can be also expressed as

$$\lim_{J\to\infty}\sum_{j\le J}\widehat{\varphi}(2^{-j}\boldsymbol{\xi})\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})} = \mathbf{1} \quad \text{in } \sigma(L^1_{loc}(\mathbb{R}^n\setminus\{\mathbf{0}\}), L^\infty_c(\mathbb{R}^n\setminus\{\mathbf{0}\})),$$

the weak topology of the Fréchet space $L^1_{loc}(\mathbb{R}^n\setminus\{\mathbf{0}\})$ with respect to its dual $L^\infty_c(\mathbb{R}^n\setminus\{\mathbf{0}\})$. We can also interpret (4.5) as a weak convergence in the "sense of the distributions". More precisely, $\mathcal{D}$ can be considered as a space of *test functions* with the topology: $f_m \to f$ in $\mathcal{D}$, if and only if, there exists a compact set $K \subset \mathbb{R}^n\setminus\{\mathbf{0}\}$ such that supp $\hat{f}_m$, supp $f \subset K$, $\forall m \ge 1$, and $f_m \to f$ in $L^\infty(K)$. Then, for $f \in \mathcal{D}$, we can look at $u_J(f) = \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f,\psi_{j,\mathbf{k}})\varphi_{j,\mathbf{k}}$ as a "distribution" defined by

$$(u_J(f), g) = \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f,\psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}}, g), \quad \text{when } g \in \mathcal{D}.$$

Indeed, it is easy to check that, by the considerations right before Theorem 4.4, u_J is a continuous (conjugate)-linear functional in $\mathcal{D}$. If we denote by $\mathcal{D}^*$ the space of such distributions and by $\sigma(\mathcal{D}^*,\mathcal{D})$ the ω^*-topology defined in $\mathcal{D}^*$, then (4.5) can be written as

$$f = \lim_{J\to\infty}u_J(f) = \lim_{J\to\infty}\sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f,\psi_{j,\mathbf{k}})\varphi_{j,\mathbf{k}} \quad \text{in } \sigma(\mathcal{D}^*,\mathcal{D}) \tag{4.8}$$

whenever $f \in \mathcal{D}$.

This result, Theorem 4.4, applies to the example of the pair of functions, φ and ψ, we introduced in §3. In fact, Proposition 3.4 tells us that, in particular, (4.6)(i)

and (ii) are satisfied by this pair of functions. Moreover, if K is a compact set of $\mathbb{R}^n \setminus \{\mathbf{0}\}$, then (4.6)$(i)$ is a finite sum (in j) when $\boldsymbol{\xi} \in K$. Thus, the pair φ and ψ satisfy (4.5) which, by polarization, is equivalent to

$$\lim_{J\to\infty} \sum_{j\le J} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}}, g) = (f, g)$$

for all $f, g \in \mathcal{D}$ (in this case the series above is again a finite sum over j). This is a weak form of the representation

$$f = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}})\varphi_{j,\mathbf{k}} \tag{4.9}$$

for $f, g \in \mathcal{D}$, as we indicated in (4.8). We have already discussed why we cannot expect (4.9) to be true in $L^2(\mathbb{R}^n)$ as an unconditionally convergent series. Another inconvenient feature of our results is that we have established them only for $f \in \mathcal{D}$. Of course, we assumed very little about φ and ψ besides the equations (4.6)(i) and (ii). We shall end this section with a result that involves the $L^2(\mathbb{R}^n)$-convergence of the series in (4.9) based on a "natural" hypothesis about the systems $\{\varphi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$.

Theorem 4.10 *Suppose* $\varphi, \psi \in L^2(\mathbb{R}^n)$. *Then*

$$f = \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}})\varphi_{j,\mathbf{k}} = \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} (f, \varphi_{j,\mathbf{k}})\psi_{j,\mathbf{k}} \tag{4.11}$$

for all $f \in L^2(\mathbb{R}^n)$ *with both series converging unconditionally in* $L^2(\mathbb{R}^n)$, *is equivalent to the following three properties. There exists a constant* $C > 0$ *such that*

$$\left.\begin{array}{l} \displaystyle\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |(f, \psi_{j,\mathbf{k}})|^2 \le C\|f\|_2^2 \\ \displaystyle\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |(f, \varphi_{j,\mathbf{k}})|^2 \le C\|f\|_2^2 \end{array}\right\} \quad \textit{for all } f \in L^2(\mathbb{R}^n), \tag{4.12}$$

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^j\boldsymbol{\xi})\overline{\widehat{\varphi}(2^j\boldsymbol{\xi})} = 1 \quad \textit{for a.e. } \boldsymbol{\xi} \in \mathbb{R}^n, \tag{4.13}$$

$$t_{\mathbf{q}}(\boldsymbol{\xi}) = \sum_{m=0}^{\infty} \widehat{\varphi}(2^m\boldsymbol{\xi})\overline{\widehat{\psi}(2^m(\boldsymbol{\xi} + 2\pi\mathbf{q}))} = 0 \quad \textit{for a.e. } \boldsymbol{\xi} \in \mathbb{R}^n,\ \forall\, \mathbf{q} \in \mathcal{O}^n, \tag{4.14}$$

where the series in (4.13) and (4.14) are absolutely convergent for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$.

REMARK 4.15 A system $\{e_j\}$ of vectors in a Hilbert space $\mathcal{H}$ such that

$$\sum_j |(f, e_j)|^2 \leq C\|f\|_2^2, \quad \text{for all } f \in \mathcal{H} \tag{4.16}$$

is called a *Bessel sequence*, while the best constant C in (4.16) is said to be the *Bessel bound* of the system. Thus, condition (4.12) asserts that $\{\varphi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$ are each a Bessel sequence. A simple condition that guarantees that ψ generates a Bessel sequence $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$, (see [CHU-SHI]) is the following: *Let θ be any non-negative function on $[0, \infty)$ that is increasing on $[0, 1)$ and decreasing on $[1, \infty)$; suppose, in addition, that*

$$\int_0^\infty \theta(w)\Big(1 + \frac{1}{w}\Big)\, dw < \infty.$$

Then, if $\psi \in L^2(\mathbb{R})$ satisfies $|\widehat{\psi}(w)| \leq \theta(|w|)$ for $w \in \mathbb{R}$, $\{\psi_{j,k}(x)\} = \{a^{j/2}\psi(a^j x - bk)\}$, $j \in \mathbb{Z}, k \in \mathbb{Z}$, is a Bessel sequence whenever $a > 1$ and $b > 0$. An n-dimensional version of this result is easy to obtain. We cite the result in [CHU-SHI] to show that (4.12) is not very restrictive; the couples φ, ψ introduced by Frazier and Jawerth (see [FJ], [FJW]) satisfy these conditions.

REMARK 4.17 It will be shown in the course of the proof of Theorem 4.10 that if (4.12) holds then, there exists a positive constant C such that

$$\sum_{j\in\mathbb{Z}} |\widehat{\varphi}(2^j\boldsymbol{\xi})|^2 \leq C \quad \text{and} \quad \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\boldsymbol{\xi})|^2 \leq C \quad a.e.\ \boldsymbol{\xi} \in \mathbb{R}^n \tag{4.18}$$

In particular, the series in (4.13) is absolutely convergent for $a.e.\ \boldsymbol{\xi}$.

REMARK 4.19 Each of the systems $\{\varphi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$, is a *frame.* In fact, suppose the first inequality in (4.12) is satisfied and

$$\|f\|_2^2 = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}}, f)$$

for all $f \in L^2(\mathbb{R}^n)$ (which follows from (4.11)). Then

$$\begin{aligned}\|f\|_2^2 &= \sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f,\psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}},f)\\ &\le \Big(\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n}|(f,\psi_{j,\mathbf{k}})|^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n}|(f,\varphi_{j,\mathbf{k}})|^2\Big)^{1/2}\\ &\le \sqrt{C}\,\|f\|_2\Big(\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n}|(f,\varphi_{j,\mathbf{k}})|^2\Big)^{1/2}.\end{aligned}$$

Dividing by $\sqrt{C}\|f\|_2$ we obtain

$$\frac{1}{C}\|f\|_2^2 \le \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n}|(f,\varphi_{j,\mathbf{k}})|^2.$$

Thus, together with the second inequality in (4.12) shows that $\{\varphi_{j,\mathbf{k}}\}, j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n$, is a frame.

Proof: We first show that (4.11) implies (4.12), (4.13) and (4.14). We will use the following result for a general Hilbert space $\mathcal{H}$ (see [SIN], Vol I, Lemma (14.9)(b) on page 425):

Lemma 4.20 *Let $\{x_i\}_{i=1}^\infty$ be a system of vectors in $\mathcal{H}$. If $\sum_{i=1}^\infty x_i$ converges unconditionally in $\mathcal{H}$, then*

$$\sum_{i=1}^\infty \|x_i\|^2 \le C = C(\{x_i\}) < \infty.$$

We apply this lemma to the sequence $\{(f,\varphi_{j,\mathbf{k}})\psi_{j,\mathbf{k}}\}, j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n$: $\sum_{j,\mathbf{k}}(f,\varphi_{j,\mathbf{k}})\psi_{j,\mathbf{k}}$ converges unconditionally by (4.11); thus, since φ and ψ are fixed and non-zero in this discussion,

$$\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n}|(f,\varphi_{j,\mathbf{k}})|^2 = \|\psi\|_2^{-2}\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n}\|(f,\varphi_{j,\mathbf{k}})\,\psi_{j,\mathbf{k}}\|_2^2 \le C\,\|\psi\|_2^{-2} = C_f. \qquad (4.21)$$

Consider the linear operator defined on $L^2(\mathbb{R}^n)$ by

$$Tf = \{(f,\varphi_{j,\mathbf{k}})\}, \qquad (j,\mathbf{k})\in\mathbb{Z}\times\mathbb{Z}^n$$

Inequality (4.21) shows that it maps $L^2(\mathbb{R}^n)$ into $\ell^2(\mathbb{Z}\times\mathbb{Z}^n)$. Suppose the pair $(f,\boldsymbol{\ell})$, with $\boldsymbol{\ell}=\{\ell_{j,\mathbf{k}}\}$, is a limit point of its graph. Then there exists a sequence $\{f_m\}$, $m\in\mathbb{N}$, such that $(f_m,Tf_m)\to(f,\boldsymbol{\ell})$ in the graph norm as $m\to\infty$. In particular, $\lim_{m\to\infty}\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n}|(f_m,\varphi_{j,\mathbf{k}})-\ell_{j,\mathbf{k}}|^2=0$ showing that $\ell_{j,\mathbf{k}}=\lim_{m\to\infty}(f_m,\varphi_{j,\mathbf{k}})=(f,\varphi_{j,\mathbf{k}})$ for each $(j,\mathbf{k})\in\mathbb{Z}\times\mathbb{Z}^n$. Thus, the graph of T is closed and, as a consequence, T is bounded: there exists a constant $C>0$, independent of $f\in L^2(\mathbb{R}^n)$, such that

$$\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n}|(f,\varphi_{j,\mathbf{k}})|^2\le C\|f\|_2^2.$$

The same argument shows that

$$\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^n}|(f,\psi_{j,\mathbf{k}})|^2\le C\|f\|_2^2$$

for all $f\in L^2(\mathbb{R}^n)$, and this establishes (4.12).

Let ρ denote either φ or ψ and let us apply the decomposition (2.9) for $f\in\mathcal{D}$ and $t_{\mathbf{q}}(\boldsymbol{\xi})=\sum_{m=0}^{\infty}\hat\rho(2^m\boldsymbol{\xi})\overline{\hat\rho(2^m(\boldsymbol{\xi}+2\pi\mathbf{q}))}$:

$$\begin{aligned}(2\pi)^n\sum_{(j,\mathbf{k})\in\mathbb{Z}\times\mathbb{Z}^n}|(f,\rho_{j,\mathbf{k}})|^2&=\int_{\mathbb{R}^n}|\hat f(\boldsymbol{\xi})|^2\sum_{j\in\mathbb{Z}}|\hat\rho(2^j\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\\&+\int_{\mathbb{R}^n}\overline{\hat f(\boldsymbol{\xi})}\sum_{p\in\mathbb{Z}}\sum_{\mathbf{q}\in\mathcal{O}^n}\hat f(\boldsymbol{\xi}+2\pi 2^p\mathbf{q})t_{\mathbf{q}}(2^{-p}\boldsymbol{\xi})\,d\boldsymbol{\xi}.\end{aligned}$$

By the observation we made in Remark 2.13 and (4.12), we see that $\sum_{j\in\mathbb{Z}}|\hat\rho(2^j\boldsymbol{\xi})|^2=\tau(\boldsymbol{\xi})$ is locally integrable in $\mathbb{R}^n\setminus\{\mathbf{0}\}$. We choose $\boldsymbol{\xi}_0$ to be a point of differentiability of the integral of $\tau(\boldsymbol{\xi})$ and f_δ so that $\hat f_\delta=\frac{1}{\sqrt{\Omega_n\delta^n}}\chi_{B_\delta(\boldsymbol{\xi}_0)}$ (see (2.17) and the sentence that follows). Applying the argument we presented after (2.17) we have

$$\begin{aligned}&\frac{1}{\Omega_n\delta^n}\int_{B_\delta(\boldsymbol{\xi}_0)}\tau(\boldsymbol{\xi})\,d\boldsymbol{\xi}+I_1^\delta=\int_{\mathbb{R}^n}|\hat f_\delta(\boldsymbol{\xi})|^2\tau(\boldsymbol{\xi})\,d\boldsymbol{\xi}+I_1^\delta=(2\pi)^nI^\delta\\&=(2\pi)^n\sum_{(j,\mathbf{k})\in\mathbb{Z}\times\mathbb{Z}^n}|(f_\delta,\rho_{j,\mathbf{k}})|^2\le(2\pi)^nC\|f_\delta\|_2^2=C,\end{aligned}$$

where the inequality follows from (4.12). Letting $\delta\to0+$, since $\lim_{\delta\to0+}I_1^\delta=0$, we obtain $\tau(\boldsymbol{\xi}_0)\le C$. That is (4.18) is satisfied and the series in (4.13) is absolutely convergent for $a.e.\,\boldsymbol{\xi}$. The equality part of (4.13) and equality (4.14) are consequences of Theorem 3.9.

We now turn to the converse. We begin by proving the two extensions of Lemmas 2.1 and 2.2 that involve two systems $\mathcal{E} = \{e_j\}$, $\mathcal{F} = \{f_j\}$, $j \in \mathbb{N}$, of vectors in a Hilbert space $\mathcal{H}$. These are general versions of the systems $\{\varphi_{j,\mathbf{k}}\}$ and $\{\psi_{j,\mathbf{k}}\}$, $j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$, which, for simplicity, we index by the natural numbers $\mathbb{N}$. In this context (4.12) becomes

$$(i) \quad \sum_{i\in\mathbb{N}} |(h,e_i)|^2 \le C\|h\|^2 \qquad (ii) \quad \sum_{i\in\mathbb{N}} |(h,f_i)|^2 \le C\|h\|^2 \tag{4.22}$$

for all $h \in \mathcal{H}$.

LEMMA 4.23 *Suppose $\mathcal{E} = \{e_j\}$ and $\mathcal{F} = \{f_j\}$ satisfy (4.22) and, for all h in a dense subset $\mathcal{D}$ of $\mathcal{H}$*

$$\|h\|^2 = \sum_{i\in\mathbb{N}} (h,e_i)(f_i,h). \tag{4.24}$$

Then equality (4.24) is valid for all $h \in \mathcal{H}$.

PROOF: Let $h \in \mathcal{H}$; (4.22) implies that the series in (4.24) is absolutely convergent. Let $\{h_n\} \subset \mathcal{D}$ be a sequence such that $\|h_n - h\| \to 0$ as $n \to \infty$. Then, by Schwarz's inequality and (4.22),

$$\begin{aligned}\Big|\sum_{j\in\mathbb{N}}(h,e_j)(f_j,h) - \|h\|^2\Big| &\le \Big|\sum_{j\in\mathbb{N}}(h-h_n,e_j)(f_j,h) + (h_n,e_j)(f_j,h-h_n)\Big| \\ +\Big|\,\|h_n\|^2 - \|h\|^2\Big| &\le C\|h-h_n\|\,\|h\| + C\|h_n\|\,\|h-h_n\| + \Big|\,\|h_n\|^2 - \|h\|^2\Big|\end{aligned}$$

which goes to 0 as $n \to \infty$.

$\square$

LEMMA 4.25 *Suppose $\mathcal{E} = \{e_j\}$ and $\mathcal{F} = \{f_j\}$ satisfy (4.22). Then the following two properties are equivalent*

$$(i) \qquad \|h\|^2 = \sum_{j\in\mathbb{N}} (h, e_j)(f_j, h) \quad \textit{for all } f \in \mathcal{H}$$

$$(ii) \qquad h = \sum_{j\in\mathbb{N}} (h, e_j) f_j = \sum_{j\in\mathbb{N}} (h, f_j) e_j \quad \textit{for all } f \in \mathcal{H} \textit{ with convergence in } \mathcal{H}.$$

In this case, the convergence of all the series is unconditional.

PROOF: That (ii) implies (i) is trivial. Let us, then, assume (i). By polarization we have

$$(g, h) = \sum_{j\in\mathbb{N}} (g, e_j)(f_j, h) \quad \text{for all } g, h \in \mathcal{H}. \tag{4.26}$$

If we can show that the partial sums of $\sum_{j=1}^{\infty} (h, e_j) f_j$ (or the second series in (ii)) form a Cauchy sequence, it follows that (ii) must be true. Indeed, if $u = \sum_{j=1}^{\infty} (h, e_j) f_j$ then, using (4.26), we must have

$$(u, g) = \sum_{j=1}^{\infty} (h, e_j)(f_j, g) = (h, g)$$

for all $g \in \mathcal{H}$ and, thus, $u = h$. But

$$\begin{aligned}
&\| \sum_{j=M}^{N} (h, e_j) f_j \| = \sup_{\|g\|=1} | \sum_{j=M}^{N} (h, e_j)(f_j, g) | \leq \\
&\sup_{\|g\|=1} (\sum_{j=M}^{N} |(h, e_j)|^2)^{1/2} (\sum_{j=M}^{N} |(g, f_j)|^2)^{1/2} \leq \\
&\sup_{\|g\|=1} (\sum_{j=M}^{N} |(h, e_j)|^2)^{1/2} C^{1/2} \|g\| = C^{1/2} (\sum_{j=M}^{N} |(h, e_j)|^2)^{1/2},
\end{aligned}$$

where the last inequality is a consequence of (4.22) (ii). Since the series $\sum_{j=1}^{\infty} |(h, e_j)|^2$ is convergent (by (4.22) (i)) we see that the partial sums in question do form a Cauchy sequence.

$\square$

We can now easily finish the proof of Theorem 4.10. Equalities (4.13), (4.14) and inequality (4.12) (which as we indicated implies (4.18)) permit us to apply Theorem 3.9 to obtain (3.10) for all $f \in \mathcal{D}$. An application of Lemma 4.23 then gives us the equality

$$\sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} (f, \psi_{j,\mathbf{k}})(\varphi_{j,\mathbf{k}}, f) = \|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^n)$ and, then, by Lemma 4.25 the desired equalities (4.11).

□

5 Notes and concluding remarks

We devote this section to present some examples and further results related to our our theory.

REMARK 5.1 Theorem 2.3 can be extended to hold in a somewhat more general setting; namely, one can replace the "dyadic" dilations and "integer" translations by "matrix dilations" and "lattice translations". Let Γ be a *lattice* in $\mathbb{R}^n$; that is, $\Gamma = A\mathbb{Z}^n$, where A is a real (non-singular) $n\times n$-matrix. Let us denote by $\Gamma^* = 2\pi(A^*)^{-1}\mathbb{Z}^n$ the *dual lattice* of Γ. Suppose that S is an $n\times n$-matrix with all its eigenvalues having absolute value strictly larger than 1 such that preserves the lattice points; that is, $S(\Gamma) \subset \Gamma$. Let us write, for $\psi \in L^2(\mathbb{R}^n)$,

$$\psi_{j,\boldsymbol{\gamma}}(\mathbf{x}) = \det(S)^{j/2}\psi(S^j\mathbf{x} + \boldsymbol{\gamma}), \quad j \in \mathbb{Z}, \boldsymbol{\gamma} \in \Gamma.$$

Then, Theorem 2.3 holds in this setting when $\psi_{j,\mathbf{k}}$ is replaced by $\psi_{j,\boldsymbol{\gamma}}$ and (1.7) by

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} |\widehat{\psi}^\ell((S^*)^j\boldsymbol{\xi})|^2 = \det(A)$$

$$t_{\boldsymbol{\gamma}^*}(\boldsymbol{\xi}) = \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \widehat{\psi}^\ell((S^*)^j\boldsymbol{\xi})\,\overline{\widehat{\psi}^\ell((S^*)^j(\boldsymbol{\xi}+\boldsymbol{\gamma}^*))} = 0, \quad \boldsymbol{\gamma}^* \in \mathcal{O}^* = \Gamma^* \setminus S^*\Gamma^*.$$

A detailed proof of this fact was recently presented to us by A. Calogero (see [CAL]) and consists of an appropriate (and careful!) modification of the arguments we introduced above. Besides, [CAL] contains some interesting examples on how this situation can be applied.

In a somewhat less general case, A. Ron and Z. Shen obtained a completely different proof of Theorem 2.3 (see, Corollary 5.8 in [RON-SHEN]). Their assumptions include matrix dilations and lattice translations as above, but in their proof a restriction on the function ψ of the type $\widehat{\psi}(\boldsymbol{\xi}) = O((1+|\boldsymbol{\xi}|)^{-(\frac{n}{2}+\varepsilon)})$, when $|\boldsymbol{\xi}| \to \infty$, for some $\varepsilon > 0$, is required. For further details, we refer the reader to [RON-SHEN].

Remark 5.2 Theorems 3.9 and 4.10 are independent of each other. To illustrate this fact we present the following example, covered by the former but not by the latter. For simplicity, we restrict ourselves to the case $n = 1$. Let α be a fixed real number, $0 < \alpha < \frac{1}{2}$, and let us define φ and ψ in $L^2(\mathbb{R})$ by

$$\widehat{\varphi}(\xi) = |\xi - \pi|^{\alpha}\,\chi_{(\pi,2\pi]}(\xi) + |\xi + \pi|^{\alpha}\,\chi_{[-2\pi,-\pi)}(\xi)$$

$$\widehat{\psi}(\xi) = \frac{1}{|\xi - \pi|^{\alpha}}\,\chi_{(\pi,2\pi]}(\xi) + \frac{1}{|\xi + \pi|^{\alpha}}\,\chi_{[-2\pi,-\pi)}(\xi).$$

It is easy to see that with this definition identities (4.13) and (4.14) hold for *a.e.* real number ξ. We claim that the series in (3.3) define both locally integrable functions in $\mathbb{R} \setminus \{0\}$ and, hence, we are under the assumptions of Theorem 3.9. Indeed, if $\ell \in \mathbb{Z}$, then

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \frac{1}{|2^{\ell}|\xi| - \pi|^{2\alpha}} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} |\widehat{\varphi}(2^j\xi)|^2 = |2^{\ell}|\xi| - \pi|^{2\alpha}, \tag{5.3}$$

whenever $2^{-\ell}\pi < |\xi| \le 2^{-\ell+1}\pi$, which clearly establishes our claim.

However, $\{\psi_{j,k}\}$ cannot be a Bessel system, since otherwise (4.18) would imply that the series $\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2$ is a bounded function in $\mathbb{R}$, which clearly is not the case. Thus, we are not under the assumptions of Theorem 4.10 and, consequently, one can find functions $f \in L^2(\mathbb{R})$ such that the series $\sum_{j,k}(f, \psi_{j,k})\varphi_{j,k}$ do not converge unconditionally to f in $L^2(\mathbb{R})$.

This example also shows that the "weak" convergence for functions $f \in \mathcal{D}$ obtained in Theorem 3.9 cannot be improved to unconditional convergence in $L^2(\mathbb{R}^n)$. In Remark 5.12 below, we show how to modify the proof of Theorem 3.9 to obtain convergence for functions in a larger dense set, but still in the weak sense. In Remark 5.15 we show that "weak convergence in $L^2(\mathbb{R})$" does not hold in general, even under stronger assumptions than those in Theorem 3.9.

REMARK 5.4 The families $\{\varphi_{j,k}\}$, $\{\psi_{j,k}\}$ in the example above satisfy the stronger condition of being a pair of *biorthogonal systems* in $L^2(\mathbb{R})$. By this we mean that

$$(\varphi_{j,k}, \psi_{j',k'}) = \delta_{j,j'}\, \delta_{k,k'}, \quad j, j', k, k' \in \mathbb{Z}. \tag{5.5}$$

This is easily seen by just noticing that, $\hat{\rho}_{j,k}(\xi) = 2^{-j/2}\hat{\rho}(2^{-j}\xi)\, e^{-i2^{-j}k\xi}$ (here ρ stands for either φ or ψ) and, therefore,

$$(\varphi_{j,k}, \psi_{j',k'}) = \frac{1}{2\pi}\,(\widehat{\varphi}_{j,k}, \widehat{\psi}_{j',k'}) = 0, \quad \text{if } j \neq j', \quad \text{while}$$

$$(\varphi_{j,k}, \psi_{j,k'}) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\, e^{i(k'-k)\xi}\, d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k'-k)\xi}\, d\xi = \delta_{k,k'}.$$

X. Wang characterized biorthogonal systems of this type in terms of "simple equations" involving φ and ψ (see Lemma 5.12 in [WAN]). More precisely, (5.5) holds if and only if, for *a.e.* $\xi \in \mathbb{R}$, and $j \geq 1$,

$$\left.\begin{array}{ll} \displaystyle\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi + 2k\pi)\,\overline{\widehat{\psi}(\xi + 2k\pi)} & = 1 \\ \displaystyle\sum_{k\in\mathbb{Z}} \widehat{\varphi}(2^j(\xi + 2k\pi))\,\overline{\widehat{\psi}(\xi + 2k\pi)} & = 0 \\ \displaystyle\sum_{k\in\mathbb{Z}} \widehat{\psi}(2^j(\xi + 2k\pi))\,\overline{\widehat{\varphi}(\xi + 2k\pi)} & = 0 \end{array}\right\} \tag{5.6}$$

In his work, Wang was trying to find a characterization of pairs of *biorthogonal Riesz wavelets*{3} by means of "simple equations" involving φ and ψ. He proved that equations (5.6) and (1.6) are necessary conditions and raised the question of whether the converse was also true (see Remark 5-2 in p.104 of [WAN]). Our example above shows that this is not case, the main reason being that the unconditionality of the expansions in (4.11) is more closely related to the fact that $\{\varphi_{j,k}\}$ and $\{\psi_{j,k}\}$ are Bessel sequences, than to the biorthogonality or completeness of the systems.

{3}A function $\psi \in L^2(\mathbb{R})$ is said to be a *Riesz wavelet* if $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an unconditional basis for $L^2(\mathbb{R})$. This notion is a little more general than the term *orthonormal wavelet*.

Remark 5.7 Continuing along with the work of X. Wang, it would be desirable to express the Bessel sequence conditions in (4.12), which a priori are just abstract Hilbert space assumptions, in terms of "simple equations" involving φ and ψ. One such characterization was given by A. Ron and Z. Shen in their excellent paper [RON-SHEN], although the equations in this case turn out not to be so "simple" as one might have expected. More precisely, what they do is the following: given a function $\psi \in L^2(\mathbb{R}^n)$, for every fixed $\boldsymbol{\xi} \in \mathbb{R}^n$ they define the *dual gramian matrix associated to* ψ, $G_\psi(\boldsymbol{\xi})$ in $\ell^2(\mathbb{Z}^n)$, by

$$G_\psi(\boldsymbol{\xi})(\boldsymbol{\ell}, \mathbf{k}) = \sum_{j=0}^{r} \widehat{\psi}(2^{-j}(\boldsymbol{\xi} + 2\pi\boldsymbol{\ell}))\, \overline{\widehat{\psi}(2^{-j}(\boldsymbol{\xi} + 2\pi\mathbf{k}))}, \quad \text{for } \boldsymbol{\ell}, \mathbf{k} \in \mathbb{Z}^n,$$

where the integer $r = r(\boldsymbol{\ell}, \mathbf{k}) \geq 0$ is given by the (unique) decomposition $\boldsymbol{\ell} - \mathbf{k} = 2^r\mathbf{q}$, $\mathbf{q} \in \mathcal{O}^n$, when $\boldsymbol{\ell} - \mathbf{k} \neq \mathbf{0}$, and $r = \infty$, when $\boldsymbol{\ell} = \mathbf{k}$. Then, they prove the following equivalence:

Proposition 5.8 *The system* $\{\psi_{j,\mathbf{k}} \mid j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ *is a Bessel sequence in* $L^2(\mathbb{R}^n)$ *if and only if* $G_\psi(\boldsymbol{\xi})$ *is a bounded operator in* $\ell^2(\mathbb{Z}^n)$ *for a.e.* $\boldsymbol{\xi} \in \mathbb{T}^n$*, and* $C_\psi = \text{ess-}\sup_{\boldsymbol{\xi} \in \mathbb{T}^n} \|G_\psi(\boldsymbol{\xi})\|^2_{\ell^2 \to \ell^2} < \infty$. *When this is the case,* C_ψ *is the best Bessel bound for* $\{\psi_{j,\mathbf{k}}\}$ *(as in (4.16)).*

Using this result and Theorem 4.10 we obtain the following characterization of biorthogonal Riesz wavelets:

Corollary 5.9 *Let* $\varphi, \psi \in L^2(\mathbb{R})$. *Then,* (φ, ψ) *is a pair of biorthogonal Riesz wavelets if and only if* $\|G_\varphi(\xi)\|_{\ell^2 \to \ell^2}, \|G_\psi(\xi)\|_{\ell^2 \to \ell^2} \in L^\infty(\mathbb{T})$ *and equations (1.6), (5.6) hold.*

Note that the same characterization holds in $\mathbb{R}^n$ after taking the n-dimensional analog of (5.6), in which $k \in \mathbb{Z}$ is replaced by $\mathbf{k} \in \mathbb{Z}^n$.

REMARK 5.10 Theorem 4.10 can be restated using the language from the *theory of frames*. We recall the reader that a *frame* in a Hilbert space $\mathcal{H}$ is a system of vectors $\{e_j\}$ for which there exist two constants $0 < A \le B < \infty$ such that

$$A\|x\|^2 \le \sum_j |(x, e_j)|^2 \le B\|x\|^2, \ \forall x \in \mathcal{H}$$

(see, e.g., [DAUB], [HW], or [HAN-LAR]). The main feature of frames is that they provide a "reconstruction formula" of the type

$$x = \sum_j (x, e_j^*) e_j, \quad \text{unconditionally for all } x \in \mathcal{H}, \tag{5.11}$$

when the system $\{e_j^*\}$ is taken to be *the dual frame of* $\{e_j\}$. However, for a given frame $\{e_j\}$, there are many other systems, $\{\tilde{e}_j\}$, that give the same reconstruction formula as in (5.11) when e_j^* is replaced by $\tilde{e}_j$. These systems have been called *alternate dual frames* by D. Han and D. Larson and an extensive study of their properties is given in [HAN-LAR].

Theorem 4.10 above provides a characterization of all the alternate dual frames of the type $\{\varphi_{j,\mathbf{k}}\}$ associated to a given frame $\{\psi_{j,\mathbf{k}}\}$ in $L^2(\mathbb{R}^n)$[{4}]. In fact, by using the equations in the theorem one can construct many examples of such situations. We include one here that is a variation of the example presented at the begining of §3.

Let $\varphi, \psi \in L^2(\mathbb{R})$ be given by:

$$\widehat{\varphi} = \chi_{[-\pi,-\frac{\pi}{4})\cup(\frac{\pi}{4},\pi]}, \quad \widehat{\psi} = \chi_{[-\pi,-\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]}.$$

It is easy to check that these functions satisfy the assumptions of Theorem 4.10[{5}], so that $\{\varphi_{j,k}\}$ and $\{\psi_{j,k}\}$ are frames and

$$f = \sum_{j,k} (f, \varphi_{j,k}) \psi_{j,k}, \quad \text{unconditionally for all } f \in L^2(\mathbb{R}).$$

[{4}]After the write-up of this manuscript we found a similar characterization of pairs of dual frames of the type described above in a new article by A. Ron and Z. Shen. Although their approach is different (following the line of their previous articles), they are led to quite interesting conclusions. For more information we refer the reader to [RON-SHEN2].

[{5}]To show that $\{\varphi_{j,k}\}$,$\{\psi_{j,k}\}$ are Bessel sequences use, e.g., Remark 4.15 or Proposition 3.5.

Thus, $\{\varphi_{j,k}\}$ is an alternate dual frame for $\{\psi_{j,k}\}$, but distinct from *the dual frame of* $\{\psi_{j,k}\}$. In fact, (3.6) above implies that $\{\psi_{j,k}\}$ is a *tight frame*, so the dual is itself.

REMARK 5.12 In the results presented in §§2, 3 and 4, the dense space $\mathcal{D}$ played an important role and arose naturally from our considerations (see Remark 2.13 in §2). It is possible, however, to replace $\mathcal{D}$ by a somewhat larger dense set, $\mathcal{S}$, in such a way that the "delicate steps" in our proofs (mainly, (2.10), Lemma 2.12 and Lemma 4.1) hold in this new setting. One such a choice for a dense set would come after replacing the condition "compact support in $\mathbb{R}^n \setminus \{\mathbf{0}\}$" by "appropriate decay at 0 and ∞". For example, we let $f \in \mathcal{S}$ if $f \in L^2(\mathbb{R}^n)$ and there exist $\varepsilon, \varepsilon' > 0$ and $C = C(\varepsilon, \varepsilon', f) < \infty$ such that

$$\left.\begin{aligned} |\hat{f}(\boldsymbol{\xi})| &\le C\,|\boldsymbol{\xi}|^{\varepsilon}, \quad \text{when} \quad |\boldsymbol{\xi}| \le 1 \\ |\hat{f}(\boldsymbol{\xi})| &\le \frac{C}{|\boldsymbol{\xi}|^{n+\varepsilon'}}, \quad \text{when} \quad |\boldsymbol{\xi}| \ge 1. \end{aligned}\right\} \tag{5.13}$$

If we denote the smallest of the constants $C(\varepsilon, \varepsilon', f)$ by $\|f\|_{\varepsilon,\varepsilon'}$, one can consider $\mathcal{S}$ as a space of "test functions" by introducing the topology: $f_m \to f$ in $\mathcal{S}$ iff $\exists\, \varepsilon, \varepsilon' > 0 \mid \|f_m - f\|_{\varepsilon,\varepsilon'} \to 0$. Then, the statements of Theorems 3.9 and 4.4 can be modified to hold in this new setting. One needs to give an appropriate meaning to the expression $\sum_j \widehat{\varphi}(2^j\boldsymbol{\xi})\overline{\widehat{\psi}(2^j\boldsymbol{\xi})}$, which in general will involve a stronger type of convergence. More precisely, one has the following:

THEOREM 3.9' *Let $\varphi, \psi \in L^2(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 \sum_{j\in\mathbb{Z}} \{|\widehat{\varphi}(2^j\boldsymbol{\xi})|^2 + |\widehat{\psi}(2^j\boldsymbol{\xi})|^2\}\, d\boldsymbol{\xi} < \infty, \quad \textit{for all } f \in \mathcal{S}. \tag{5.14}$$

Then, the pair (φ, ψ) satisfies equations (i) and (ii) in Theorem 3.9 if and only if

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^n} (f, \varphi_{j,\mathbf{k}})(\psi_{j,\mathbf{k}}, f), \quad \textit{for all } f \in \mathcal{S},$$

where the series above converges absolutely.

THEOREM 4.4' *Suppose $\varphi, \psi \in L^2(\mathbb{R}^n)$. Then,*

1. *For each $J \in \mathbb{Z}$ and $f, g \in \mathcal{S}$, the series $S_J(f,g) = \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f,\varphi_{j,\mathbf{k}})(\psi_{j,\mathbf{k}}, f)$ converges absolutely. Moreover, $u_J(f) = \sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f,\varphi_{j,\mathbf{k}})\psi_{j,\mathbf{k}}$ is a distribution in $\mathcal{S}^*$ when defined by $(u_J(f), g) = S_J(f,g)$.*

2. *The following statements are equivalent:*

$$(i) \qquad \text{For } f \in \mathcal{S}, \quad u_J(f) \to f, \quad \text{in } \mathcal{S}^*.$$

$$(ii) \qquad \text{For } f \in \mathcal{S}, \quad \lim_{J\to\infty}\sum_{j\le J}\sum_{\mathbf{k}\in\mathbb{Z}^n}(f,\varphi_{j,\mathbf{k}})(\psi_{j,\mathbf{k}}, f) = \|f\|_2^2.$$

$$(iii) \quad \begin{cases} \displaystyle\lim_{J\to\infty}\int_{\mathbb{R}^n}|\hat{f}(\boldsymbol{\xi})|^2\sum_{j\le J}\widehat{\varphi}(2^{-j}\boldsymbol{\xi})\,\overline{\widehat{\psi}(2^{-j}\boldsymbol{\xi})}\,d\boldsymbol{\xi} = \|\hat{f}\|_2^2, \quad \forall\ f \in \mathcal{S} \\[2ex] t_{\mathbf{q}}(\boldsymbol{\xi}) = 0, \quad a.e.\ \boldsymbol{\xi} \in \mathbb{R}^n,\ \mathbf{q} \in \mathcal{O}^n. \end{cases}$$

The proof of these results is left to be verified by the interested reader.

REMARK 5.15 In the final comment in Remark 5.2 we mentioned the sharpness of Theorem 3.9, in the sense that one cannot replace the (absolute) convergence of $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(f,\varphi_{j,k})(\psi_{j,k},g)$, for functions $f, g \in \mathcal{D}$ (or $\mathcal{S}$), by general functions $f, g \in L^2(\mathbb{R})$. Indeed, we present an example below in which even under the assumptions that $\widehat{\varphi}, \widehat{\psi}$ are compactly supported (so that $\sum_j \widehat{\varphi}(2^j\xi)\overline{\widehat{\psi}(2^j\xi)}$ is a finite series) and (1.6) holds, one can find two functions $f, g \in L^2(\mathbb{R})$ such that $\sum_{k\in\mathbb{Z}}(f,\varphi_{j,k})(\psi_{j,k},g)$ is not an absolutely convergent series.

To be more precise, we find functions $f, g, \varphi, \psi \in L^2(\mathbb{R})$ so that

$$\text{supp } \hat{\rho} \subset [-2\pi, -\pi] \cup [\pi, 2\pi] \equiv I,$$

when $\rho \in \{f, g, \varphi, \psi\}$, $\widehat{\varphi}(\xi)\widehat{\psi}(\xi) = 1$ on I, and $\sum_{k\in\mathbb{Z}}|(f,\psi_{0,k})(\varphi_{0,k},g)| = \infty$. It is easy to check that, with this choice, (1.6) holds. Note that a straightforward application of Plancherel theorem gives us

$$(f, \psi_{0,k}) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\widehat{\psi}(\xi)e^{ik\xi}\, d\xi, \quad k \in \mathbb{Z}.$$

Now, let $F(\xi)$ be a 2π-periodic function in $L^1(I)$ so that $F(\xi) \geq 1$ and has Fourier coefficients $c_k(F) = \frac{1}{\log|k|}$, when $|k| \geq 2$[6]. Then, we can define our functions f, g, φ, ψ by

$$\widehat{\psi} = \hat{f} = F^{1/2}\chi_I, \quad \widehat{\varphi} = F^{-1/2}\chi_I \quad \text{and} \quad \hat{g} = F^{1/2}\chi_{[\pi,2\pi]}.$$

Note that all of them belong to $L^2(\mathbb{R})$. Moreover,

$$(f, \psi_{0,k}) = \frac{1}{2\pi} \int_I F(\xi)e^{ik\xi}\, d\xi = \frac{1}{\log|k|}, \quad \text{when } \; |k| \geq 2,$$

and

$$(g, \varphi_{0,k}) = \frac{1}{2\pi} \int_{\pi}^{2\pi} e^{ik\xi}\, d\xi = \begin{cases} 0, & \text{if } k \neq 0 \text{ is even} \\ 1/2, & \text{if } k = 0 \\ \frac{1}{\pi i k}, & \text{if } k \text{ is odd.} \end{cases}$$

It clearly follows from here that $\sum_{k\in\mathbb{Z}} |(f, \psi_{0,k})(\varphi_{0,k}, g)| = \infty$.

[6]That such a function exists follows, e.g., from Theorem 4.1, in Chapter I of [KATZ].

6 One example: tensor product of wavelets in $\mathbb{R}^2$

We conclude this chapter by applying once again Theorem 2.3 to prove that certain functions are wavelets, providing, for instance, an elementary approach to the tensor product of MRA's in $\mathbb{R}^2$.

Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA and let $\varphi \in L^2(\mathbb{R})$ be a scaling function with associated low-pass filter $m_0 \in L^2(\mathbb{T})$. In particular, the following equations hold for $a.e.\, \xi \in \mathbb{R}$ (see §3 in Chapter 1):

$$\widehat{\varphi}(2\xi) = m_0(\xi)\,\widehat{\varphi}(\xi) \tag{6.1}$$

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1 \tag{6.2}$$

$$\lim_{n\to\infty} |\widehat{\varphi}(2^{-n}\xi)| = 1 \tag{6.3}$$

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2k\pi)|^2 = 1. \tag{6.4}$$

Define a function $\psi \in L^2(\mathbb{R})$ by

$$\widehat{\psi}(\xi) = e^{i\frac{\xi}{2}}\,\overline{m_0(\xi/2+\pi)}\,\widehat{\varphi}(\xi/2), \quad a.e.\, \xi \in \mathbb{R}. \tag{6.5}$$

Then, it is a well-known fact from the theory of MRA's that ψ constructed as above is an orthonormal wavelet for $L^2(\mathbb{R})$. Alternatively, this property can be easily proved by using the results presented in §2 of this chapter. Indeed, by Theorem 2.3, everything reduces to check that equations (1.7) (i) and (ii) hold and $\|\psi\|_2 = 1$. In fact, we show something a little more general, namely, that (6.1), (6.2) and (6.3) imply that $\{\psi_{j,k}\}$ is a tight frame in $L^2(\mathbb{R})$. Indeed, from the first two equations and the definition of ψ we obtain

$$|\widehat{\varphi}(2\xi)|^2 + |\widehat{\psi}(2\xi)|^2 = |\widehat{\varphi}(\xi)|^2, \quad a.e.\, \xi \in \mathbb{R}.$$

Iterating, and using that, for $\varphi \in L^2(\mathbb{R})$, $\lim_{n\to\infty} |\widehat{\varphi}(2^n\xi)| = 0$, $a.e.\, \xi \in \mathbb{R}$ (see, e.g., p. 61 of [HW]), we have the well-known equation:

$$|\widehat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2, \quad a.e. \in \mathbb{R}. \tag{6.6}$$

Then, using (6.3), we obtain:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 &= \lim_{n\to\infty}\sum_{j=-n}^{\infty} |\widehat{\psi}(2^j\xi)|^2 = \lim_{n\to\infty}\sum_{j=1}^{\infty} |\widehat{\psi}(2^j(2^{-(n+1)}\xi))|^2 \\ &= \lim_{n\to\infty} |\widehat{\varphi}(2^{-(n+1)}\xi)|^2 = 1,\end{aligned}$$

and (1.7) (i) holds.

On the other hand, using (6.1), (6.2) and (6.3), we have that, for every $q \in 2\mathbb{Z}+1$:

$$\sum_{j=0}^{n} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+2q\pi))} = -\overline{m_0(\xi/2+\pi)}\, m_0(\xi/2)\, \widehat{\varphi}(\xi/2)\, \overline{\widehat{\varphi}(\xi/2+q\pi)}$$

$$+ \sum_{j=1}^{n} \overline{m_0(2^{j-1}\xi+\pi)}\, \widehat{\varphi}(2^{j-1}\xi)\, m_0(2^{j-1}\xi+\pi)\, \overline{\widehat{\varphi}(2^{j-1}(\xi+2q\pi))}$$

$$\begin{aligned} &= -\widehat{\varphi}(\xi)\,\overline{\widehat{\varphi}(\xi+2q\pi)} + \sum_{j=1}^{n}\{1-|m_0(2^{j-1}\xi)|^2\}\,\widehat{\varphi}(2^{j-1}\xi)\,\overline{\widehat{\varphi}(2^{j-1}\xi+2^jq\pi)} \\ &= \sum_{j=2}^{n} \widehat{\varphi}(2^{j-1}\xi)\,\overline{\widehat{\varphi}(2^{j-1}\xi+2^jq\pi)} - \sum_{j=1}^{n}\widehat{\varphi}(2^j\xi)\,\overline{\widehat{\varphi}(2^j\xi+2^{j+1}q\pi)} \\ &= -\widehat{\varphi}(2^n\xi)\,\overline{\widehat{\varphi}(2^n(\xi+2q\pi))} \to 0, \quad \text{as } n\to\infty,\end{aligned}$$

which shows that $\{\psi_{j,k}\}$ is a tight frame. Note that orthonormality of the system would follow if we assume, in addition, that (6.4) holds (see Remark 2.6). Indeed,

$$\begin{aligned}\int_{\mathbb{R}} |\widehat{\psi}(2\xi)|^2\, d\xi &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} |m_0(\xi+\pi)|^2\, |\widehat{\varphi}(\xi+2k\pi)|^2\, d\xi \\ &= \int_0^{\pi} \{|m_0(\xi+\pi)|^2 + |m_0(\xi)|^2\}\, d\xi = \pi,\end{aligned}$$

and, hence, $\|\psi\|_{L^2(\mathbb{R})} = 1$.

We consider now a somewhat more difficult problem: with the same conditions as above, we shall show that the collection $\{\varphi\otimes\psi, \psi\otimes\varphi, \psi\otimes\psi\}$ is a wavelet family in $L^2(\mathbb{R}^2)$, where the *tensor product* $(f\otimes g)(x,y) = f(x)\,g(y)$, $(x,y)\in\mathbb{R}^2$. One way of proving this is by appropriately constructing the MRA $\{V_j\otimes V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R}^2)$ given by the tensor product of $\{V_j\}$ with itself (see, e.g., §3.2 in [MEY]). However, one can proceed as well by justifying that (1.7) holds in this setting. Indeed, for the first equation, using (6.6) and the 1-dimensional case above, we have:

$$\sum_{j\in\mathbb{Z}} |\widehat{\varphi}(2^j\xi_1)|^2\,|\widehat{\psi}(2^j\xi_2)|^2 + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,|\widehat{\varphi}(2^j\xi_2)|^2 + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,|\widehat{\psi}(2^j\xi_2)|^2 =$$

$$= \sum_{j\in\mathbb{Z}}\{\sum_{k=1}^{\infty} |\widehat{\psi}(2^{j+k}\xi_1)|^2\}\,|\widehat{\psi}(2^j\xi_2)|^2 + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,\{\sum_{k=1}^{\infty}|\widehat{\psi}(2^{j+k}\xi_2)|^2\} + \\ + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,|\widehat{\psi}(2^j\xi_2)|^2$$

$$= \sum_{k=1}^{\infty}\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,|\widehat{\psi}(2^{j-k}\xi_2)|^2 + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,\{\sum_{k=1}^{\infty}|\widehat{\psi}(2^{j+k}\xi_2)|^2\} + \\ + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,|\widehat{\psi}(2^j\xi_2)|^2$$

$$= \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,\{\sum_{k=1}^{\infty}[|\widehat{\psi}(2^{j-k}\xi_2)|^2 + |\widehat{\psi}(2^{j+k}\xi_2)|^2] + |\widehat{\psi}(2^j\xi_2)|^2\}$$

$$= \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi_1)|^2\,\{\sum_{\ell\in\mathbb{Z}} |\widehat{\psi}(2^\ell\xi_2)|^2\} = 1, \quad a.e.\,\boldsymbol{\xi} = (\xi_1,\xi_2)\in\mathbb{R}^2.$$

To show that (1.7) (ii) holds in this case, let $\mathbf{q} = (q_1, q_2) \in \mathcal{O}^2$; without loss of generality, we assume that $q_2 \in 2\mathbb{Z}+1$. Then, for $a.e.\,\boldsymbol{\xi} = (\xi_1,\xi_2)\in\mathbb{R}^2$, we have:

$$\begin{aligned} t_{\mathbf{q}}(\boldsymbol{\xi}) &= \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi_1)\,\widehat{\psi}(2^j\xi_2)\,\overline{\widehat{\psi}(2^j(\xi_1+2q_1\pi))}\,\overline{\widehat{\psi}(2^j(\xi_2+2q_2\pi))} + \\ &\quad \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi_1)\,\widehat{\varphi}(2^j\xi_2)\,\overline{\widehat{\psi}(2^j(\xi_1+2q_1\pi))}\,\overline{\widehat{\varphi}(2^j(\xi_2+2q_2\pi))} + \\ &\quad \sum_{j=0}^{\infty} \widehat{\varphi}(2^j\xi_1)\,\widehat{\psi}(2^j\xi_2)\,\overline{\widehat{\varphi}(2^j(\xi_1+2q_1\pi))}\,\overline{\widehat{\psi}(2^j(\xi_2+2q_2\pi))} \\ &= \sum_{j=0}^{\infty}\{I + II + III\}. \end{aligned}$$

Now,

$$I+II = \widehat{\psi}(2^j\xi_1)\,\overline{\widehat{\psi}}(2^j(\xi_1+2q_1\pi))\,\{e^{i2^{j-1}\xi_2}\,\overline{m_0}(2^{j-1}\xi_2+\pi)\,\widehat{\varphi}(2^{j-1}\xi_2)\,e^{-i2^{j-1}\xi_2}\,e^{-i2^jq_2\pi}\,\times$$

$$\times\, m_0(2^{j-1}\xi_2 + 2^jq_2\pi + \pi)\,\overline{\widehat{\varphi}}(2^{j-1}\xi_2 + 2^jq_2\pi)$$

$$+m_0(2^{j-1}\xi_2)\,\widehat{\varphi}(2^{j-1}\xi_2)\,\overline{m_0}(2^{j-1}\xi_2 + 2^jq_2\pi)\,\overline{\widehat{\varphi}}(2^{j-1}\xi_2 + 2^jq_2\pi)\,\}$$

$$
\begin{aligned}
&= \widehat{\psi}(2^j\xi_1)\,\overline{\widehat{\psi}}(2^j(\xi_1+2q_1\pi))\,\widehat{\varphi}(2^{j-1}\xi_2)\,\overline{\widehat{\varphi}}(2^{j-1}\xi_2+2^jq_2\pi)\times\\
&\quad\times\begin{cases} 0\,, & \text{if } j=0 \text{ (since } q_2\in 2\mathbb{Z}+1)\\ |m_0(2^{j-1}\xi_2+\pi)|^2+|m_0(2^{j-1}\xi_2)|^2=1, & \text{if } j\geq 1.\end{cases}
\end{aligned}
$$

To simplify our expressions, let us denote, for $j \geq 0$,

$$\alpha_j = \widehat{\varphi}(2^j\xi_1)\,\overline{\widehat{\varphi}}(2^j(\xi_1+2q_1\pi))\,\widehat{\varphi}(2^j\xi_2)\,\overline{\widehat{\varphi}}(2^j(\xi_2+2q_2\pi)).$$

Then, using (6.5) in the equation above, we have:

$$I+II=\begin{cases} 0, & \text{if } j=0\\ |m_0(2^{j-1}\xi_1+\pi)|^2\,\alpha_{j-1}, & \text{if } j\geq 1.\end{cases}$$

Thus, we can write $t_{\mathbf{q}}(\boldsymbol{\xi})$ as follows:

$$
\begin{aligned}
t_{\mathbf{q}}(\boldsymbol{\xi}) &= \sum_{j=1}^{\infty}|m_0(2^{j-1}\xi_1+\pi)|^2\,\alpha_{j-1} + \sum_{j=0}^{\infty}\{III\}\\
&= \sum_{j=1}^{\infty}|m_0(2^{j-1}\xi_1+\pi)|^2\,\alpha_{j-1}\\
&\quad -\widehat{\varphi}(\xi_1)\,\overline{\widehat{\varphi}}(\xi_1+2q_1\pi)\,\overline{m_0}(\xi_2/2+\pi)\,\widehat{\varphi}(\xi_2/2)\,m_0(\xi_2/2)\,\overline{\widehat{\varphi}}(\xi_2/2+q_2\pi)\\
&\quad +\sum_{j=1}^{\infty}|m_0(2^{j-1}\xi_1)|^2\,\widehat{\varphi}(2^{j-1}\xi_1)\,\overline{\widehat{\varphi}}(2^{j-1}(\xi_1+2q_1\pi))\times\\
&\qquad\qquad\times|m_0(2^{j-1}\xi_2+\pi)|^2\,\widehat{\varphi}(2^{j-1}\xi_2)\,\overline{\widehat{\varphi}}(2^{j-1}(\xi_2+2q_2\pi))\\
&= \sum_{j=1}^{\infty}|m_0(2^{j-1}\xi_1+\pi)|^2\,\alpha_{j-1}-\alpha_0+\sum_{j=1}^{\infty}|m_0(2^{j-1}\xi_1)|^2\,|m_0(2^{j-1}\xi_2+\pi)|^2\,\alpha_{j-1}\\
&= -\alpha_0+\sum_{j=1}^{\infty}\{|m_0(2^{j-1}\xi_1+\pi)|^2+|m_0(2^{j-1}\xi_1)|^2\,|m_0(2^{j-1}\xi_2+\pi)|^2\}\,\alpha_{j-1}\\
&= -\alpha_0+\sum_{j=1}^{\infty}\{1-|m_0(2^{j-1}\xi_1)|^2\,|m_0(2^{j-1}\xi_2)|^2\}\,\alpha_{j-1}.
\end{aligned}
$$

Since

$$|m_0(2^{j-1}\xi_1)|^2\,|m_0(2^{j-1}\xi_2)|^2\,\alpha_{j-1}=\alpha_j, \quad \text{for } j\geq 1,$$

we have:

$$
\begin{aligned}
t_{\mathbf{q}}(\boldsymbol{\xi}) &= -\alpha_0+\sum_{j=1}^{\infty}(\alpha_{j-1}-\alpha_j)\\
&= \lim_{j\to\infty}\alpha_j = 0,
\end{aligned}
$$

because $\lim_{j\to\infty} \widehat{\varphi}(2^j\xi) = 0,\ a.e.\, \xi \in \mathbb{R}$.

Chapter 3

On the Connectivity of the Set of α-Localized Wavelets

Let α be a real number greater than $1/2$. A function φ is localized of degree α (or α-localized) if $\varphi \in L^2((1+|x|^2)^\alpha\, dx)$. In this chapter we study α-localized wavelets and scaling functions. We show that, with the topology induced by $L^2((1+|x|^2)^\alpha\, dx)$, the topological space of all α-localized scaling functions is homeomorphic to the (topological) subspace of $H^\alpha(\mathbb{T})$ consisting of all the low-pass filters associated with those scaling functions. Moreover, we shall show that this set of low-pass filters is an arcwise-connected infinite dimensional manifold. In §5, we turn to the study of α-localized wavelets. By a theorem of Lemarié-Rieusset, each of these wavelets satisfying an additional but "mild" smoothness condition (say, $\psi \in H^\varepsilon(\mathbb{R})$, for some $\varepsilon > 0$) arises from an MRA with a scaling function having the same localization and smoothness properties. We shall use this theorem, together with our results in §§$2-4$ to find a decomposition of the space of α-localized wavelets in connected components. It turns out that two wavelets belong to the same connected component if and only if their "phases" have the same homotopy degree. In §5.3 we solve the functional equation that determines the phase of an α-localized wavelet and give a formula for the homotopy degree of each phase in terms of the "center of mass" of ψ.

1 Some Properties of Sobolev Spaces

The material we include in this section is a compilation of some known results from Sobolev space theory. More general statements and proofs than the ones we present here can be found in different articles of the classical literature (see, e.g., [ADAM], [STRI] or [TAIB]). Two references that contain the basics properties of Sobolev spaces (either as theorems, or as exercises) are: §6 in Chapter 8 of [FOL], and Chapter V of [STE]. We give complete proofs of the not so well-known results we will use in the sequel.

1.1 Sobolev spaces in $\mathbb{R}$

Let $\alpha > 0$ be fixed. We define the *Sobolev space of degree* α by:

$$H^\alpha(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \;\mid\; \|f\|^2_{H^\alpha(\mathbb{R})} = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, (1 + |\xi|^2)^\alpha \, d\xi \; < \infty \right\} \tag{1.1}$$

Then, $H^\alpha(\mathbb{R})$ is a Hilbert space with the inner product

$$< f, g >_{H^\alpha(\mathbb{R})} = \int_{\mathbb{R}} \hat{f}(\xi) \, \overline{\hat{g}(\xi)} \, (1 + |\xi|^2)^\alpha \, d\xi,$$

and associated norm $\|\cdot\|_{H^\alpha(\mathbb{R})}$ as defined above. We shall denote this norm by $\|f\|_\alpha$, whenever there is no confusion with $\|f\|_2 = \|f\|_{L^2(\mathbb{R})}$.

These spaces can also be defined as the range of certain Bessel potential operators. Let $\alpha > 0$ and let G^α be the tempered distribution given by $\widehat{G}^\alpha(\xi) = (1 + |\xi|^2)^{-\frac{\alpha}{2}}$. Then, it can be shown that $G^\alpha \in L^1(\mathbb{R})$ (see Proposition 2 of §3, in Chapter V of [STE]). We define the *Bessel potential operator of order* $\alpha > 0$ by

$$J^\alpha(f) = G^\alpha * f, \quad f \in L^2(\mathbb{R}). \tag{1.2}$$

Taking Fourier transforms of both sides, this operator can be written as

$$(\widehat{J^\alpha f})(\xi) = (1 + |\xi|^2)^{-\frac{\alpha}{2}} \, \hat{f}(\xi), \quad f \in L^2(\mathbb{R}). \tag{1.3}$$

Now it is easy to see that $f \in H^\alpha(\mathbb{R})$ if and only if there exists a function $g \in L^2(\mathbb{R})$ such that $f = J^\alpha g$, and $\|f\|_{H^\alpha(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}$. If this is the case, then $\hat{g}(\xi) = (1+|\xi|^2)^{\frac{\alpha}{2}}\hat{f}(\xi)$. With this identification, J^α becomes an (isometric) isomorphism of Hilbert spaces

$$\begin{aligned} J^\alpha : \quad & L^2(\mathbb{R}) \longrightarrow H^\alpha(\mathbb{R}) \\ & g \mapsto J^\alpha g = G^\alpha * g \end{aligned}$$

In the same way one can show that J^α is also an (isometric) isomorphism between $H^\beta(\mathbb{R})$ and $H^{\alpha+\beta}(\mathbb{R})$ (see §3.3 of Chapter V of [STE]).

It is possible to define an equivalent norm[{1}] in $H^\alpha(\mathbb{R})$. Suppose that $\alpha = n \in \mathbb{Z}^+$, where here $\mathbb{Z}^+$ denotes the set of all positive integers. Then $f \in H^\alpha(\mathbb{R})$ if and only if $f, f', \ldots, f^{(n)} \in L^2(\mathbb{R})$, where $f', \ldots, f^{(n)}$ denote the derivatives in the sense of distributions of f. Moreover,

$$\|f\|_{H^\alpha(\mathbb{R})} \approx \sum_{k=0}^{n} \|f^{(k)}\|_{L^2(\mathbb{R})} \approx \|f\|_{L^2(\mathbb{R})} + \|f^{(n)}\|_{L^2(\mathbb{R})}. \tag{1.4}$$

where the symbol "$\approx$" means that the two norms are equivalent. This equivalence follows easily from the fact that $(f^{(n)})\hat{}(\xi) = (i\xi)^n \hat{f}(\xi)$ (in $L^2(\mathbb{R})$ and in the sense of distributions). For the non-integer case, if $\alpha = n + \varepsilon$, where $n \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \varepsilon < 1$, we have that $f \in H^\alpha(\mathbb{R})$ if and only if $f, f', \ldots, f^{(n)} \in L^2(\mathbb{R})$, and

$$\omega_\varepsilon(f^{(n)}) = \left[\int_{\mathbb{R}}\int_{\mathbb{R}} |f^{(n)}(x+h) - f^{(n)}(x)|^2 \, dx \, \frac{dh}{|h|^{1+2\varepsilon}}\right]^{\frac{1}{2}} < \infty \tag{1.5}$$

where $f', \ldots, f^{(n)}$ denote the distributional derivatives of f and, again,

$$\|f\|_{H^\alpha(\mathbb{R})} \approx \|f\|_{L^2(\mathbb{R})} + \|f^{(n)}\|_{L^2(\mathbb{R})} + \omega_\varepsilon(f^{(n)}). \tag{1.6}$$

A proof for these norm equivalences can be found in §3.5 of Chapter V of [STE] (see Theorem 3, Lemma 3 and Proposition 4 there).

[{1}] We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a normed space $\mathbb{X}$ are equivalent if there exist two constants $0 < C_1 \le C_2 < \infty$ such that $C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1$, for all $x \in \mathbb{X}$.

We shall denote the space of bounded continuous functions on $\mathbb{R}$ by $C_b(\mathbb{R})$ and its norm by $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. Then, for $0 < \alpha < 1$, we say that a function $f \in C_b(\mathbb{R})$ belongs to $\Lambda^\alpha(\mathbb{R})$ whenever

$$\|f\|_{\Lambda^\alpha(\mathbb{R})} = \|f\|_\infty + \sup_{x\in\mathbb{R}, h\neq 0} \frac{|f(x+h) - f(x)|}{|h|^\alpha} < \infty. \tag{1.7}$$

The spaces $\Lambda^\alpha(\mathbb{R})$ are sometimes called *Lipschitz spaces* (or spaces of Hölder continuous functions), and are related to the Sobolev spaces by the following theorem.

THEOREM 1.8 : Sobolev Imbedding Theorem

(I) Let $0 < \alpha < \beta < \infty$. Then, $H^\beta(\mathbb{R}) \subset H^\alpha(\mathbb{R})$ and there exists a constant $C = C(\alpha, \beta)$ such that

$$\|f\|_{H^\alpha(\mathbb{R})} \leq C \, \|f\|_{H^\beta(\mathbb{R})}, \quad \text{for all } f \in H^\beta(\mathbb{R}). \tag{1.9}$$

(II) Let $\alpha > 1$. Then $f \in H^\alpha(\mathbb{R})$ if and only if $f \in L^2(\mathbb{R})$ and $f' \in H^{\alpha-1}(\mathbb{R})$. Moreover,

$$\|f\|_{H^\alpha(\mathbb{R})} \approx \|f\|_{L^2(\mathbb{R})} + \|f'\|_{H^{\alpha-1}(\mathbb{R})}. \tag{1.10}$$

(III) Let $\frac{1}{2} < \alpha < \frac{3}{2}$. Then, $H^\alpha(\mathbb{R}) \subset \Lambda^{\alpha-\frac{1}{2}}(\mathbb{R})$ and there exists a constant $C = C(\alpha)$ such that

$$\|f\|_{\Lambda^{\alpha-\frac{1}{2}}(\mathbb{R})} \leq C \, \|f\|_{H^\alpha(\mathbb{R})}, \quad \text{for all } f \in H^\alpha(\mathbb{R}). \tag{1.11}$$

PROOF: The first statement is a simple consequence of the boundedness of the inverse of the Bessel potential $(J^{\beta-\alpha})^{-1} : H^\beta(\mathbb{R}) \to H^\alpha(\mathbb{R})$. The second statement follows from the definition of norm given in (1.4) and (1.6). A proof for the third part can be found in [STE] (see § 6.7, in Chapter V) or, in a more general context, in [TAIB] (see Theorem 9).

□

Remark 1.12 Note that, in particular, if $\alpha > \frac{1}{2}$, part (III) of the theorem above tells us that each function $f \in H^\alpha(\mathbb{R})$ is bounded and continuous in $\mathbb{R}$. More generally, if $\alpha > k + \frac{1}{2}$, where $k = 0, 1, 2, \ldots$, then f and all its derivatives up to order k are bounded and continuous in $\mathbb{R}$ (after, maybe, a modification on a set of measure zero). In most of what follows we restrict ourselves to the case $\alpha > \frac{1}{2}$ and, therefore, our functions f are considered to be continuously defined everywhere in $\mathbb{R}$.

Let us now prove a lemma that we shall use later and that gives us another equivalent norm in $H^\alpha(\mathbb{R})$.

Lemma 1.13 : Uniform localization lemma.

Let $\alpha > 0$ be fixed, and let $w \in C_c^\infty(-2\pi, 2\pi)$ (meaning that the support of w is a compact subset of $(-2\pi, 2\pi)$). Then, there exists a constant $C = C(\alpha, w)$ such that

$$\sum_{k\in\mathbb{Z}} \|f(\cdot)\, w(\cdot + 2k\pi)\|_\alpha^2 \le C\, \|f\|_\alpha^2\,, \quad \textit{for all } f \in H^\alpha(\mathbb{R}). \tag{1.14}$$

Proof:

Case 1: $\alpha = n \in \mathbb{Z}^+$.

By the Leibnitz rule, we can write the (distributional) derivatives of $f(\cdot)\, w(\cdot{+}2k\pi)$ as

$$D^{(h)}[f(\cdot)\, w(\cdot + 2k\pi)] = \sum_{m=0}^{h} \binom{h}{m} f^{(m)} w^{(h-m)}(\cdot + 2k\pi), \quad \text{for } 0 \le h \le n.$$

Thus,

$$\sum_{k\in\mathbb{Z}} \|D^{(h)}[f(\cdot)\, w(\cdot + 2k\pi)]\,\|_{L^2(\mathbb{R})}^2 \le$$

$$\le 2^{h+1} \sum_{m=0}^{h} \binom{h}{m}^2 \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |f^{(m)}(x)|^2\, |w^{(h-m)}(x + 2k\pi)|^2\, dx$$

$$= 2^{h+1} \sum_{m=0}^{h} \binom{h}{m}^2 \int_{\mathbb{R}} |f^{(m)}(x)|^2 \,[\sum_{k\in\mathbb{Z}} |w^{(h-m)}(x + 2k\pi)|^2]\, dx$$

$$\le 2^{h+1} \sum_{m=0}^{h} \binom{h}{m}^2 2\, \|w^{(h-m)}\|_\infty^2\, \|f^{(m)}\|_2^2 \le C\, \|f\|_{H^n(\mathbb{R})}^2\,,$$

where the last two inequalities follow from (1.4) and from the fact that the series $\sum_{k\in\mathbb{Z}} |w^{(h-m)}(x+2k\pi)|^2$ has at most two non-zero terms. Thus, summing over h, when $0 \le h \le n$, and using (1.4) again, we obtain

$$\sum_{k\in\mathbb{Z}} \|f(\cdot)\, w(\cdot+2k\pi)\|^2_{H^n(\mathbb{R})} \le C \sum_{h=0}^{n}\sum_{k\in\mathbb{Z}} \|D^{(h)}[f(\cdot)\, w(\cdot+2k\pi)]\,\|^2_{L^2(\mathbb{R})} \le C\, \|f\|^2_{H^n(\mathbb{R})}\,,$$

and this completes the proof of case 1.

Case 2: $\alpha = n+\varepsilon$, where $n\in\mathbb{Z}$ and $0<\varepsilon<1$.

By using the same reasoning as above we have, when $0\le h\le n$,

$$\sum_{k\in\mathbb{Z}} \|D^{(h)}[f(\cdot)\, w(\cdot+2k\pi)]\,\|_2^2 \le C\, \|f\|^2_{H^n(\mathbb{R})} \le C\, \|f\|^2_{H^\alpha(\mathbb{R})}\,. \tag{1.15}$$

Therefore, it is enough to estimate the last term in the definition of the norm given in (1.6), which in this case is

$$\sum_{k\in\mathbb{Z}} \omega_\varepsilon(D^{(n)}[f(\cdot)\, w(\cdot+2k\pi)]\,)^2 \le 2^{n+1}\sum_{k\in\mathbb{Z}}\sum_{h=0}^{n}\binom{n}{h}^2 \omega_\varepsilon(f^{(h)}(\cdot)\, w^{(n-h)}(\cdot+2k\pi)\,)^2.$$

Given a fixed $0\le h\le n$, for the terms on the right hand side of the inequality above, we have

$$\sum_{k\in\mathbb{Z}} \omega_\varepsilon(f^{(h)}(\cdot)\, w^{(n-h)}(\cdot+2k\pi)\,)^2 =$$

$$= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{R}} |f^{(h)}(x+y)\, w^{(n-h)}(x+y+2k\pi) - f^{(h)}(x)\, w^{(n-h)}(x+2k\pi)|^2 \,\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$\le 2\int_{\mathbb{R}}\int_{\mathbb{R}} |f^{(h)}(x+y)|^2\, [\sum_{k\in\mathbb{Z}} |w^{(n-h)}(x+y+2k\pi) - w^{(n-h)}(x+2k\pi)|^2]\,\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$+2\int_{\mathbb{R}}\int_{\mathbb{R}} [\sum_{k\in\mathbb{Z}} |w^{(n-h)}(x+2k\pi)|^2]\, |f^{(h)}(x+y) - f^{(h)}(x)|^2\,\frac{dx\,dy}{|y|^{1+2\varepsilon}} = A+B.$$

Now, B is easy to estimate because at most two terms in $\sum_{k\in\mathbb{Z}} |w^{(n-h)}(x+2k\pi)|^2$ can be non-zero. Thus,

$$B \le 2\, \|w^{(n-h)}\|^2_\infty\, \omega_\varepsilon(f^{(h)})^2 \le C\, \|f\|^2_\alpha\,.$$

To estimate A, we write it as

$$A = 2\int_{\mathbb{R}} |f^{(h)}(x)|^2 \left[\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |w^{(n-h)}(x+2k\pi) - w^{(n-h)}(x+y+2k\pi)|^2 \,\frac{dy}{|y|^{1+2\varepsilon}}\right] dx.$$

We can now consider separately, for each fixed $x \in \mathbb{R}$, the integral inside the brackets in the expression above

$$I_x = \int_{|y|>2\pi} + \int_{|y|\le 2\pi} = I_x^1 + I_x^2,$$

and we obtain

$$I_x^1 \le \int_{|y|>2\pi} 16\,\|w^{(n-h)}\|_\infty^2 \,\frac{dy}{|y|^{1+2\varepsilon}} = C < \infty,$$

because only four terms of $\sum_{k\in\mathbb{Z}} |w^{(n-h)}(x+2k\pi)|^2 + \sum_{k\in\mathbb{Z}} |w^{(n-h)}(x+y+2k\pi)|^2$ can be non-zero. Finally,

$$I_x^2 \le \int_{|y|\le 2\pi} \sum_{k\in\mathbb{Z}} \left| \int_{x+2k\pi}^{x+2k\pi+y} w^{(n-h+1)}(t)\,dt \right|^2 \frac{dy}{|y|^{1+2\varepsilon}}$$

$$\le \int_{|y|\le 2\pi} \left[\sum_{k\in\mathbb{Z}} \sup_{t\in[x+2k\pi,\,x+2k\pi+y]} |w^{(n-h+1)}(t)|^2\right] |y|^2 \frac{dy}{|y|^{1+2\varepsilon}}$$

$$\le 3\,\|w^{(n-h+1)}\|_\infty^2 \int_{|y|\le 2\pi} |y|^{1-2\varepsilon}\,dy = C < \infty, \quad \text{for all } x \in \mathbb{R},$$

where the last inequality follows from the fact that, for fixed $x \in \mathbb{R}$ and $y \in [-2\pi, 2\pi]$, for, at most, three of the k's, the intervals $[x+2k\pi, x+2k\pi+y]$ intersect the support of w. Thus, combining the two inequalities for I_x^1 and I_x^2 we can estimate A by

$$A = 2\int_{\mathbb{R}} |f^{(h)}(x)|^2\, I_x\, dx \le C\,\|f^{(h)}\|_2^2 \le C\,\|f\|_\alpha^2.$$

This implies that $\sum_{k\in\mathbb{Z}} \omega_\varepsilon(D^{(n)}[f(\cdot)\,w(\cdot+2k\pi)]\,)^2 \le A+B \le C\,\|f\|_\alpha^2$, which together with (1.15) gives us (1.14) and completes the proof of case 2.

□

Remark 1.16 This lemma, stated for the case when $\alpha \in \mathbb{Z}^+$, can be found in [MEY] (see Lemma 6 in Chapter 2). For a more general statement and proof see

Theorem 3.1 in Chapter I of [STRI]. If we assume, further, that $0 \leq w \leq 1$, and $w|_{[-\pi,\pi]} \equiv 1$, then it can be shown that the norms involved are equivalent; that is,

$$\frac{1}{C}\,\|f\|_\alpha^2 \leq \sum_{k\in\mathbb{Z}} \|f(\cdot)\,w(\cdot+2k\pi)\|_\alpha^2 \leq C\,\|f\|_\alpha^2.$$

Corollary 1.17 *Let $\alpha > \frac{1}{2}$, and let $f \in H^\alpha(\mathbb{R})$. Then, $\sum_{k\in\mathbb{Z}} |f(x+2k\pi)|^2$ converges uniformly in $[-\pi,\pi]$ and*

$$\|\sum_{k\in\mathbb{Z}} |f(\cdot+2k\pi)|^2\|_\infty \leq C\,\|f\|_\alpha^2,$$

where the constant C depends only on α.

Proof: Let us take a function $w \in C_c^\infty(-2\pi,2\pi)$ such that $0 \leq w \leq 1$, and $w|_{[-\pi,\pi]} \equiv 1$. Then, for every two positive integers N, M, and for every $x \in [-\pi,\pi]$, we have

$$\begin{aligned}\sum_{N\leq|k|\leq N+M} |f(x+2k\pi)|^2 &\leq \sum_{N\leq|k|\leq N+M} \|f(\cdot)\,w(\cdot+2k\pi)\|_\infty^2 \\ &\leq C \sum_{N\leq|k|\leq N+M} \|f(\cdot)\,w(\cdot+2k\pi)\|_\alpha^2 \leq C\,\|f\|_\alpha^2\ ,\end{aligned}$$

where the last two inequatilies follow from (1.11) and (1.14). Thus, we have a uniform Cauchy condition in our series, which implies the desired uniform convergence.

□

As an application to the theory of MRAs that we introduced in Chapter 1, we have the following corollary:

Corollary 1.18 *Suppose that φ is a scaling function for an α-localized MRA (as defined in (3.1) and (3.13) of Chapter 1), where $\alpha > \frac{1}{2}$. Then,*

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+2k\pi)|^2 = 1, \quad \textit{for all } \xi \in \mathbb{R}.$$

Proof: Note that (3.13) of Chapter 1, together with (1.1) above, implies that $\widehat{\varphi} \in H^\alpha(\mathbb{R})$ and, since $\alpha > \frac{1}{2}$, Corollary 1.17 tells us that $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+2k\pi)|^2$ converges uniformly in $[-\pi,\pi]$ to a continuous 2π-periodic function $S(\xi)$. But by equality (3.6) of Chapter 1 we know that this series converges *a.e.* to 1. Thus, $S(\xi) = 1$ for every $\xi \in \mathbb{R}$, and this establishes the corollary.

□

1.2 Sobolev spaces in $\mathbb{T}$

Similar definitions and properties hold in the periodic case. We shall say that a 2π-periodic measurable function f belongs to $L^2(\mathbb{T})$ when

$$\|f\|^2_{L^2(\mathbb{T})} = \int_{-\pi}^{\pi} |f(x)|^2\, dx < \infty$$

Here we identify the 1-dimensional torus with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \cong [-\pi, \pi)$. We know that every function $f \in L^2(\mathbb{T})$ can be expressed in terms of its Fourier series,

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}, \quad a.e.\, x \in \mathbb{T}, \quad \text{where} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx\,, \quad k \in \mathbb{Z}. \tag{1.19}$$

Moreover, the sequence $\{c_k\}_{k\in\mathbb{Z}}$ must satisfy $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$. With this in mind, we define the *Sobolev space in* $\mathbb{T}$ *of degree* $\alpha > 0$ as

$$H^\alpha(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) \;\mid\; \|f\|^2_{H^\alpha(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |c_k|^2 (1 + |k|^2)^\alpha < \infty \right\}, \tag{1.20}$$

where $c_k = c_k(f)$ are the Fourier coefficients of f defined by (1.19). As in the non-periodic case, the spaces $H^\alpha(\mathbb{T})$ are Hilbert spaces, and equivalent norms of the type (1.4) and (1.6) can be found. Indeed, suppose that $\alpha = n \in \mathbb{Z}^+$, then $f \in H^\alpha(\mathbb{T})$ if and only if $f, f', \ldots, f^{(n)} \in L^2(\mathbb{T})$, where $f', \ldots, f^{(n)}$ denote the distributional derivatives (in $\mathbb{T}$) of f. Moreover,

$$\|f\|_{H^\alpha(\mathbb{T})} \approx \sum_{k=0}^{n} \|f^{(k)}\|_{L^2(\mathbb{T})} \approx \|f\|_{L^2(\mathbb{T})} + \|f^{(n)}\|_{L^2(\mathbb{T})}. \tag{1.21}$$

In a similar fashion one deals with the case $\alpha = n + \varepsilon$, where $n \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \varepsilon < 1$. Now we have $f \in H^\alpha(\mathbb{T})$ if and only if $f, f', \ldots, f^{(n)} \in L^2(\mathbb{T})$, and

$$\omega_\varepsilon(f^{(n)}) = \left[\int_{\mathbb{T}}\int_{\mathbb{T}} |f^{(n)}(x+h) - f^{(n)}(x)|^2\, dx\, \frac{dh}{|h|^{1+2\varepsilon}}\right]^{\frac{1}{2}} < \infty. \tag{1.22}$$

As before, we have the equivalence of the norms

$$\|f\|_{H^\alpha(\mathbb{T})} \approx \|f\|_{L^2(\mathbb{T})} + \|f^{(n)}\|_{L^2(\mathbb{T})} + \omega_\varepsilon(f^{(n)}). \tag{1.23}$$

The proof for these norm equivalences follows essentially from the same arguments as in the real line case. For a more detailed discussion on this, the reader can consult Chapter VII of [TAIB]. If we denote by $C(\mathbb{T})$ the space of continuous 2π-periodic functions in $\mathbb{R}$, and by $\|f\|_\infty = \sup_{x\in\mathbb{T}} |f(x)|$ its norm, we can define, in a similar way to (1.7), the 2π-*periodic Lipschitz spaces*. More precisely, if $0 < \alpha < 1$, we say that a function $f \in C(\mathbb{T})$ belongs to $\Lambda^\alpha(\mathbb{T})$ whenever

$$\|f\|_{\Lambda^\alpha(\mathbb{T})} = \|f\|_\infty + \sup_{\substack{x,h\in\mathbb{T}\\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^\alpha} < \infty. \tag{1.24}$$

We now state the periodic version of the Sobolev Imbedding Theorem (Theorem 3.8).

Theorem 1.25 : Sobolev Imbedding Theorem for $\mathbb{T}$.

(I) Let $0 < \alpha < \beta < \infty$. Then, $H^\beta(\mathbb{T}) \subset H^\alpha(\mathbb{T})$ and there exists a constant $C = C(\alpha,\beta)$ such that

$$\|f\|_{H^\alpha(\mathbb{T})} \leq C\,\|f\|_{H^\beta(\mathbb{T})}, \quad \text{for all } f \in H^\beta(\mathbb{T}). \tag{1.26}$$

(II) Let $\alpha > 1$. Then $f \in H^\alpha(\mathbb{T})$ if and only if $f \in L^2(\mathbb{T})$ and $f' \in H^{\alpha-1}(\mathbb{T})$. Moreover,

$$\|f\|_{H^\alpha(\mathbb{T})} \approx \|f\|_{L^2(\mathbb{T})} + \|f'\|_{H^{\alpha-1}(\mathbb{T})}. \tag{1.27}$$

(III) Let $\frac{1}{2} < \alpha < \frac{3}{2}$. Then, $H^\alpha(\mathbb{T}) \subset \Lambda^{\alpha-\frac{1}{2}}(\mathbb{T})$ and there exists a constant $C = C(\alpha)$ such that

$$\|f\|_{\Lambda^{\alpha-\frac{1}{2}}(\mathbb{T})} \leq C\,\|f\|_{H^\alpha(\mathbb{T})}, \quad \text{for all } f \in H^\alpha(\mathbb{T}). \tag{1.28}$$

REMARK 1.29 The same conclusion as in Remark 1.12 applies here. When $\alpha > \frac{1}{2}$, each function $f \in H^\alpha(\mathbb{T})$ is automatically 2π-periodic and continuous in $\mathbb{R}$ (after, maybe, a modification on a null set). Note that, in particular, this implies that f is uniformly continuous. When $\alpha > k + \frac{1}{2}$, f can be extended 2π-periodically to $\mathbb{R}$ in such a way that $f \in C^k(\mathbb{R})$.

It is not hard to show that $H^\alpha(\mathbb{R})$ is an algebra under pointwise multiplication whenever $\alpha > \frac{1}{2}$. That is, if $f, g \in H^\alpha(\mathbb{R})$ then $f \cdot g \in H^\alpha(\mathbb{R})$ (see Exercise 6.11 of Chapter V in [STE] or the proof of Lemma 1.30 below). In the periodic case, the space $H^\alpha(\mathbb{T})$ has the stronger property of being a *Banach algebra*{2}. In particular, the constant function $\mathbf{1}$ belongs to it. This special feature is due to the fact that $\mathbb{T}$ is compact, while $\mathbb{R}$ is not, and has important consequences. We present in the next lemmas a more precise description of what we are saying.

LEMMA 1.30 *Let $\alpha > \frac{1}{2}$. Then, the Sobolev spaces $H^\alpha(\mathbb{T})$ are commutative Banach algebras under pointwise multiplication. That is, we can find a norm $\|\cdot\|_\alpha$ in $H^\alpha(\mathbb{T})$ such that, for every $f, g \in H^\alpha(\mathbb{T})$ then $f \cdot g \in H^\alpha(\mathbb{T})$. Moreover,*

$$\|f \cdot g\|_\alpha \leq \|f\|_\alpha \|g\|_\alpha \quad \textit{and} \quad \|\mathbf{1}\|_\alpha = 1\,. \tag{1.31}$$

PROOF:

It is clear that $\mathbf{1} \in H^\alpha(\mathbb{T})$ for all $\alpha > \frac{1}{2}$. Then, it is enough to show that there exists a constant $C = C(\alpha) > 0$ such that if $f, g \in H^\alpha(\mathbb{T})$ then $f \cdot g \in H^\alpha(\mathbb{T})$ and $\|f \cdot g\|_\alpha \leq C\,\|f\|_\alpha \|g\|_\alpha$. Indeed, once we establish this, the following proposition tells us that we can find an equivalent norm in $H^\alpha(\mathbb{T})$ such that (1.31) holds.

PROPOSITION 1.32 : see Theorem 10.2 of [RUD1]. *Assume that $\mathcal{A}$ is a Banach space as well as a complex algebra with unit element $\mathbf{1} \neq 0$, in which multiplication is left-continuous and right-continuous. Then, there is a norm on $\mathcal{A}$ which induces the same topology as the given one and which makes $\mathcal{A}$ into a Banach algebra.*

{2}We say that a Banach space $\mathbb{X}$ is a Banach algebra if it is an algebra with a unit element $\mathbf{1}$ and if there is a norm $\|\cdot\|$ in $\mathbb{X}$ such that $\|\mathbf{1}\| = 1$ and $\|f \cdot g\| \leq \|f\|\,\|g\|, \quad \forall\, f, g \in \mathbb{X}$.

Note that if $f, g \in H^\alpha(\mathbb{T})$ then $f \cdot g \in L^2(\mathbb{T})$ and $\|f \cdot g\|_2 \leq \|f\|_\infty \|g\|_2$. If we use the defintion of norm given in (1.21) (or (1.23)), we just need to estimate $\|(f \cdot g)^{(n)}\|_2$ (or $\omega_\varepsilon((f \cdot g)^{(n)})$). We shall first show the case in which α is a positive integer.

Case 1: $\alpha = n \in \mathbb{Z}^+$.

$$\|(f \cdot g)^{(n)}\|_2 \leq \sum_{k=0}^{n} \binom{n}{k} \|f^{(k)} g^{(n-k)}\|_2 \leq$$

$$\sum_{k=1}^{n-1} \binom{n}{k} \|f^{(k)}\|_\infty \|g^{(n-k)}\|_2 + \|f\|_\infty \|g^{(n)}\|_2 + \|f^{(n)}\|_2 \|g\|_\infty \leq C \|f\|_\alpha \|g\|_\alpha ,$$

the last inequality following from the Sobolev Imbedding Theorem 1.25.

Case 2: $\alpha = \varepsilon, \quad \frac{1}{2} < \varepsilon < 1$.

$$\omega_\varepsilon(fg) \leq \|f\|_\infty \omega_\varepsilon(g) + \omega_\varepsilon(f) \|g\|_\infty \leq C \|f\|_\alpha \|g\|_\alpha ,$$

the last inequality following from (1.28).

Case 3: $\alpha = n + \varepsilon, \quad$ where $n \in \mathbb{Z}^+,\ 0 < \varepsilon < 1$.

$$\omega_\varepsilon((fg)^{(n)}) \leq \sum_{k=1}^{n-1} \binom{n}{k} \omega_\varepsilon(f^{(k)} g^{(n-k)}) + \omega_\varepsilon(f^{(n)} g) + \omega_\varepsilon(f g^{(n)}) = A + B + C.$$

We can now easily estimate A since for each term for which $1 \leq k \leq n$, Sobolev's Theorem 1.25 implies that

$$\omega_\varepsilon(f^{(k)} g^{(n-k)}) \leq \|f^{(k)}\|_\infty \omega_\varepsilon(g^{(n-k)}) + \|g^{(n-k)}\|_\infty \omega_\varepsilon(f^{(k)}) \leq C \|f\|_\alpha \|g\|_\alpha.$$

In order to estimate B (or C), suppose that $n \geq 2$ or that $\frac{1}{2} < \varepsilon < 1$ (that is, $\alpha > \frac{3}{2}$). Then we have

$$B = \omega_\varepsilon(f^{(n)} g) \leq \|g\|_\infty \omega_\varepsilon(f^{(n)}) + \left[\int_{\mathbb{T}} \int_{\mathbb{T}} |f^{(n)}(x)|^2 \, |g(x) - g(x+y)|^2 \frac{dx\, dy}{|y|^{1+2\varepsilon}}\right]^{\frac{1}{2}}$$

$$\leq \|g\|_\infty \omega_\varepsilon(f^{(n)}) + \|g'\|_\infty \left[\int_{\mathbb{T}} |f^{(n)}(x)|^2 \, dx\right]^{\frac{1}{2}} \left[\int_{-\pi}^{\pi} |y|^2 \frac{dy}{|y|^{1+2\varepsilon}}\right]^{\frac{1}{2}} \leq C \|f\|_\alpha \|g\|_\alpha.$$

Note that in the last inequality we have used that $g' \in H^{\alpha-1} \subset H^{\varepsilon} \subset \Lambda^{\varepsilon-\frac{1}{2}}$, by the Sobolev Imbedding Theorem 1.25.

When $n = 1$ and $0 < \varepsilon \leq \frac{1}{2}$ we cannot use this fact and the following refinement of the Sobolev Imbedding Theorem 1.25 is needed. (See Chapter V of [STE] or Theorem 9' in Chapter VII of [TAIB].)

Proposition 1.33 *Let $\alpha > 0$ and $1 \leq p \leq \infty$. Then if $\alpha > \frac{1}{2} - \frac{1}{p}$ we have that $H^{\alpha}(\mathbb{T}) \subset L^p(\mathbb{T})$ and there exists a constant $C = C(\alpha, p) > 0$ such that*

$$\|f\|_{L^p(\mathbb{T})} \leq C\, \|f\|_{H^{\alpha}(\mathbb{T})}\,, \quad \textit{for all } f \in H^{\alpha}(\mathbb{T}).$$

In our case, let us choose any p such that $\frac{1}{1-\varepsilon} < p < \frac{1}{\frac{1}{2}-\varepsilon}$ (note that this is always possible when $0 < \varepsilon \leq \frac{1}{2}$). Then, we have that $\varepsilon > \frac{1}{2} - \frac{1}{p}$ and, therefore,

$$|g(x+y) - g(x)| \leq \int_x^{x+y} |g'(t)|\, dt \leq \|g'\|_p |y|^{1-\frac{1}{p}} \leq C\, \|g\|_{\alpha} |y|^{1-\frac{1}{p}},$$

because, by Proposition 1.33 and (1.27), $\|g'\|_p \leq C\|g'\|_{H^{\varepsilon}} \leq C\|g\|_{H^{\alpha}}$. If we go back to the estimation of B above we have

$$B \leq \|g\|_{\infty}\omega_{\varepsilon}(f') + C\, \|g\|_{\alpha} \left[\int_{\mathbb{T}} |f'(x)|^2\, dx\right]^{\frac{1}{2}} \left[\int_{-\pi}^{\pi} |y|^{2-\frac{2}{p}} \frac{dy}{|y|^{1+2\varepsilon}}\right]^{\frac{1}{2}} \leq C\, \|f\|_{\alpha}\, \|g\|_{\alpha},$$

because $\int_{-\pi}^{\pi} |y|^{1-\frac{2}{p}-2\varepsilon}\, dy < \infty$ since, by assumption, p was taken so that $p > \frac{1}{1-\varepsilon}$ and, therefore, $1 - 2\varepsilon - \frac{2}{p} > -1$.

$\square$

Remark 1.34 The proof of Lemma 1.30 can be adapted to show that $H^{\alpha}(\mathbb{R})$ is also a Banach algebra (with no unit this time). These are particular cases of a more general result in the context of *multipliers*: see Theorem 2.1 in Chapter III of [STRI].

The following result tells us that $H^{\alpha}(\mathbb{R})$ is an $H^{\alpha}(\mathbb{T})$-module.

Lemma 1.35 *Let $\alpha > \frac{1}{2}$. Then, for $m \in H^{\alpha}(\mathbb{T})$ and $f \in H^{\alpha}(\mathbb{R})$ we have that $m \cdot f \in H^{\alpha}(\mathbb{R})$. Moreover, there exists a constant $C = C(\alpha) > 0$ such that*

$$\|m \cdot f\|_{H^{\alpha}(\mathbb{R})} \leq C\, \|m\|_{H^{\alpha}(\mathbb{T})}\, \|f\|_{H^{\alpha}(\mathbb{R})}.$$

PROOF:

First of all notice that

$$\left[\int_{\mathbb{R}} |m(x)\cdot f(x)|^2\,dx\right]^{\frac{1}{2}} \le \|m\|_\infty \|f\|_{L^2(\mathbb{R})} \le C\,\|m\|_{H^\alpha(\mathbb{T})}\|f\|_{L^2(\mathbb{R})}, \tag{1.36}$$

by the Sobolev Imbedding Theorem 1.25 *(III)*. Now, let us consider different cases to show the rest of the lemma.

Case 1: $\alpha = n \in \mathbb{Z}^+$.

In this case, by the Leibnitz rule, the distributional derivative of order n of $m\cdot f$ can be written as

$$\frac{d^n}{dx^n}(m\cdot f) = \sum_{h=0}^{n}\binom{n}{h}\, m^{(h)} f^{(n-h)}$$

and therefore,

$$\int_{\mathbb{R}}\left|\frac{d^n}{dx^n}(m\cdot f)(x)\right|^2 dx \le 2^n \sum_{h=0}^{n-1}\binom{n}{h}^2 \|m^{(h)}\|_\infty^2 \int_{\mathbb{R}} |f^{(n-h)}(x)|^2\,dx +$$

$$2\int_{\mathbb{T}} |m^{(n)}(x)|^2\, \|\sum_{k\in\mathbb{Z}} |f(\cdot+2k\pi)|^2\|_\infty\,dx \le\ C\,\|m\|_{H^\alpha(\mathbb{T})}^2\,\|f\|_{H^\alpha(\mathbb{R})}^2,$$

where in the last inequality we have used Corollary 1.17. This, together with (1.36), gives us the desired result for $\alpha = n$.

Case 2: $\alpha = n+\varepsilon$, where $0<\varepsilon<1$.

By the definition of norm given in (1.6) and *case 1* above, it is enough to show that

$$\omega_\varepsilon((m\cdot f)^{(n)}) \le C\,\|f\|_{H^\alpha(\mathbb{R})}\|m\|_{H^\alpha(\mathbb{T})}. \tag{1.37}$$

Once more, Leibnitz's rule and the triangle inequality tell us that

$$\omega_\varepsilon((m\cdot f)^{(n)}) \le \sum_{h=0}^{n}\binom{n}{h}\,\omega_\varepsilon(m^{(h)} f^{(n-h)}).$$

Therefore, it is enough to estimate each of the terms $\omega_\varepsilon(m^{(h)} f^{(n-h)})$ separately, when $0\le h\le n$. Fix one such h. As in the proof of Lemma 1.13 we have

$$\omega_\varepsilon(m^{(h)} f^{(n-h)})^2 =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |m^{(h)}(x+y)\, f^{(n-h)}(x+y) - m^{(h)}(x)\, f^{(n-h)}(x)|^2 \,\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$\leq 2 \int_{\mathbb{R}} \int_{\mathbb{R}} |m^{(h)}(x+y)|^2\, |f^{(n-h)}(x+y) - f^{(n-h)}(x)|^2 \,\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$+2 \int_{\mathbb{R}} \int_{\mathbb{R}} |f^{(n-h)}(x)|^2\, |m^{(h)}(x+y) - m^{(h)}(x)|^2 \,\frac{dx\,dy}{|y|^{1+2\varepsilon}} = 2A + 2B.$$

To estimate B we proceed as follows. If $1 \leq h \leq n$ or if $h = 0$ and $\frac{1}{2} < \varepsilon < 1$, the Sobolev Imbedding Theorem 1.25 and Corollary 1.17 imply that

$$B \;\leq\; \|\sum_{k\in\mathbb{Z}} |f^{(n-h)}(\cdot + 2k\pi)|^2\|_\infty \int_{\mathbb{R}} \int_{\mathbb{T}} |m^{(h)}(x+y) - m^{(h)}(x)|^2 \,\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$\leq C\, \|f\|^2_{H^\alpha(\mathbb{R})}\, \|m\|^2_{H^\alpha(\mathbb{T})}.$$

In the case when $h = 0$ and $0 < \varepsilon \leq \frac{1}{2}$ (note that in this case $n \geq 1$) we have that

$$B \;\leq\; 2 \int_{|y|\geq 1} \|m\|^2_\infty \,\frac{dy}{|y|^{1+2\varepsilon}}\, \|f^{(n)}\|^2_{L^2(\mathbb{R})} +$$

$$+\left[\int_{\mathbb{R}} |f^{(n)}(x)|^2\, dx\right] \left[\int_{|y|<1} \|m\|^2_{\Lambda^{\frac{1+\varepsilon}{2}}(\mathbb{T})} |y|^{1+\varepsilon} \frac{dy}{|y|^{1+2\varepsilon}}\right] \leq C\, \|f\|^2_{H^\alpha(\mathbb{R})} \|m\|^2_{H^\alpha(\mathbb{T})},$$

because $H^\alpha(\mathbb{T}) \subset H^{1+\frac{\varepsilon}{2}}(\mathbb{T}) \subset \Lambda^{\frac{1+\varepsilon}{2}}(\mathbb{T})$ (by (1.28)) and the integral $\int_{|y|<1} \frac{dy}{|y|^\varepsilon}$ is finite.

To estimate A we proceed similarly. Suppose that $0 \leq h \leq n-1$ or that $h = n$ and $\frac{1}{2} < \varepsilon < 1$. Then, the Sobolev Imbedding Theorem 1.25 tells us that

$$A \leq \|m^{(h)}\|^2_\infty\, \omega_\varepsilon(f^{(n-h)}) \leq C \|m\|^2_{H^\alpha(\mathbb{T})} \|f\|^2_{H^\alpha(\mathbb{R})}.$$

We have left the case $h = n$ when $0 < \varepsilon \leq \frac{1}{2}$ (and, again, necessarily, $n \geq 1$). If this is the case

$$A = \int_{\mathbb{R}} \int_{\mathbb{R}} |m^{(n)}(x+y)|^2\, |f(x+y) - f(x)|^2 \,\frac{dx\,dy}{|y|^{1+2\varepsilon}} =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{T}} |m^{(n)}(x)|^2 \sum_{k\in\mathbb{Z}} |f(x+y+2k\pi) - f(x+2k\pi)|^2 \,\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$\leq \int_{|y|\geq 1} \|m^{(n)}\|^2_{L^2(\mathbb{T})}\, 4\|\sum_{k\in\mathbb{Z}} |f(\cdot + 2k\pi)|^2\|_\infty \,\frac{dy}{|y|^{1+2\varepsilon}} +$$

$$\int_{|y|<1} \left[\int_{\mathbb{T}} |m^{(n)}(x)|^2\, dx\right] \left[\sum_{k\in\mathbb{Z}} |f(x+y+2k\pi) - f(x+2k\pi)|^2\right] \frac{dy}{|y|^{1+2\varepsilon}} = A_1 + A_2.$$

Then, $A_1 \leq C\,\|m\|^2_{H^\alpha(\mathbb{T})}\|f\|^2_{H^\alpha(\mathbb{R})}$, by Corollary 1.17. To estimate A_2 we will use the same kind of argument as in Lemma 1.13. Let $w \in C_c^\infty(-2\pi, 2\pi)$ be such that $0 \leq w \leq 1$, and $w|_{[-\pi,\pi]} \equiv 1$. Then, for every $x \in \mathbb{T}$

$$\sum_{k\in\mathbb{Z}} |f(x+y+2k\pi) - f(x+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |f(x+y+2k\pi)w(x) - f(x+2k\pi)w(x)|^2 \leq$$

$$2\sum_{k\in\mathbb{Z}}\{|f(x+y+2k\pi)w(x+y) - f(x+2k\pi)w(x)|^2 + |f(x+y+2k\pi)|^2|w(x)-w(x+y)|^2\}.$$

If we multiply both sides of the inequality by $\frac{1}{2|y|^{1+\varepsilon}}$, where $0 < |y| < 1$, we have

$$\frac{1}{2|y|^{1+\varepsilon}} \sum_{k\in\mathbb{Z}} |f(x+y+2k\pi) - f(x+2k\pi)|^2$$

$$\leq \sum_{k\in\mathbb{Z}} \sup_{\substack{z\in\mathbb{R}\\ y\neq 0}} \left|\frac{f(z+y)w(z+y+2k\pi) - f(z)w(z+2k\pi)}{|y|^{\frac{1+\varepsilon}{2}}}\right|^2 +$$

$$+\,\|\sum_{k\in\mathbb{Z}} |f(\cdot+2k\pi)|^2\|_\infty \frac{|y|^2}{|y|^{1+\varepsilon}}\,\|w'\|^2_\infty \leq C \sum_{k\in\mathbb{Z}} \|f(\cdot)\,w(\cdot+2k\pi)\|^2_{\Lambda^{\frac{1+\varepsilon}{2}}(\mathbb{R})}$$

$$+C\,\|\sum_{k\in\mathbb{Z}} |f(\cdot+2k\pi)|^2\|_\infty \leq C\,\|f\|^2_{H^\alpha(\mathbb{R})},$$

where the last inequality follows from $H^\alpha(\mathbb{R}) \subset H^{1+\frac{\varepsilon}{2}}(\mathbb{R}) \subset \Lambda^{\frac{1+\varepsilon}{2}}(\mathbb{R})$, Lemma 1.13 and Corollary 1.17. Now, A_2 becomes:

$$A_2 \leq C \int_{\mathbb{T}} |m^{(n)}(x)|^2\,dx\,\|f\|^2_{H^\alpha(\mathbb{R})} \int_{|y|<1} \frac{dy}{|y|^\varepsilon} \leq C\,\|m\|^2_{H^\alpha(\mathbb{T})}\|f\|^2_{H^\alpha(\mathbb{R})}.$$

This shows (1.37) and completes the proof of the lemma.

□

Note that in the last part of the proof of the previous lemma we have shown the following useful result:

COROLLARY 1.38 *Let $0 < \gamma < 2$. Then, there exists a constant $C = C(\gamma) > 0$ such that for every $f \in H^{\frac{1+\gamma}{2}}(\mathbb{R})$ we have that*

$$\sup_{\substack{x\in\mathbb{R}\\ y\neq 0}} \frac{1}{|y|^\gamma} \sum_{k\in\mathbb{Z}} |f(x+y+2k\pi) - f(x+2k\pi)|^2 \leq C\,\|f\|^2_{H^{\frac{1+\gamma}{2}}(\mathbb{R})}.$$

1.3 A brief review in Banach algebras

Before listing some of the consequences of Lemma 1.30 over $H^\alpha(\mathbb{T})$, we recall a few general facts and definitions about Banach algebras. Suppose that $\mathcal{A}$ is a Banach algebra with unit $\mathbf{1}$. We say that an element $a \in \mathcal{A}$ is *invertible* if there exist a (unique) element in $\mathcal{A}$, denoted by a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = \mathbf{1}$. We define the *spectrum* of $a \in \mathcal{A}$ as

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda\mathbf{1} - a \quad \text{is not invertible in } \mathcal{A}\}. \tag{1.39}$$

It is a well-known fact that $\sigma(a)$ is a non-empty compact set of $\mathbb{C}$.

The *holomorphic functional calculus* in the theory of Banach algebras tells us how to define new elements in $\mathcal{A}$ of the type $\sqrt{a}$ or e^a or, more generally, $f(a)$, from a given element $a \in \mathcal{A}$. Here f is a holomorphic function[{3}] in a suitable domain $\Omega \subset \mathbb{C}$. We state below the main theorem concerning the holomorphic functional calculus of Banach algebras. We shall apply this theorem later to the particular case of $\mathcal{A} = H^\alpha(\mathbb{T})$. We refer the reader to Chapter 10 of [RUD1] for proofs and a more careful treatment on this theory.

Theorem 1.40 *Let $\mathcal{A}$ be a Banach algebra and let $\Omega \subset \mathbb{C}$ be an open set on the complex plane. Then,*

(i) $\mathcal{A}_\Omega = \{a \in \mathcal{A} \mid \sigma(a) \subset \Omega\}$ *is an open set of $\mathcal{A}$.*

(ii) For every holomorphic function $F \in \mathcal{H}(\Omega)$ we can define a continuous map $\widetilde{F}\colon \mathcal{A}_\Omega \longrightarrow \mathcal{A}$, mapping $a \mapsto \widetilde{F}(a)$, such that $\widetilde{F}(\lambda\mathbf{1}) = F(\lambda)\cdot\mathbf{1}$, $\forall\lambda \in \Omega$ and, when we take the functions $F_1(\lambda) \equiv 1$, $F_2(\lambda) \equiv \lambda$, $(\lambda \in \Omega)$, then $\widetilde{F}_1(a) = \mathbf{1}$ and $\widetilde{F}_2(a) = a$, for all $a \in \mathcal{A}_\Omega$. Moreover, $F \in \mathcal{H}(\Omega) \mapsto \widetilde{F}$ is a homomorphism of algebras.

[{3}] We shall denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions in an open set Ω, endowed with its usual locally uniform topology. That is, $f_n \to f$ in $\mathcal{H}(\Omega)$ means that $f_n \to f$ uniformly in compact sets of Ω.

(iii) *If* $(F_n)_{n=1}^{\infty} \subset \mathcal{H}(\Omega)$ *is such that* $F_n \to F$ *in* $\mathcal{H}(\Omega)$*, then* $\widetilde{F}_n(a) \to \widetilde{F}(a)$ *in the topology of* $\mathcal{A}$*, for every* $a \in \mathcal{A}_\Omega$.

(iv) *Suppose* $a \in \mathcal{A}_\Omega$, $F \in \mathcal{H}(\Omega)$, $\Omega_1 \subset \mathbb{C}$ *is an open set containing* $F(\sigma(a)) = \sigma(\widetilde{F}(a))$ *and* $G \in \mathcal{H}(\Omega_1)$. *Let* $H(\lambda) \equiv G(F(\lambda))$ *be defined in* $\Omega_0 = F^{-1}(\Omega_1)$. *Then,* $\widetilde{F}(a) \in \mathcal{A}_{\Omega_1}$ *and* $\widetilde{H}(a) = \widetilde{G}(\widetilde{F}(a))$.

Remark 1.41 In fact, $\widetilde{F}(a)$ in (ii) is given by $\widetilde{F}(a) = \frac{1}{2\pi i}\int_\gamma F(z)\cdot(z\mathbf{1}-a)^{-1}\,dz$ where γ is a contour surrounding $\sigma(a)$ in Ω. (Note that we are taking a vector-valued integral in this case.) When $\mathcal{A}$ is commutative, we can write $\widetilde{F}(a)$ as a power series as well. More precisely, if $F \in \mathcal{H}(\Omega)$, $\lambda_0 \in \Omega$ and if $\delta > 0$ is such that $F(\lambda_0+\lambda) = \sum_{k=0}^{\infty} \frac{F^{(k)}(\lambda_0)}{k!}\,\lambda^k$ when $|\lambda| < \delta$, then for every $a_0 \in \mathcal{A}_\Omega$ there exists an $\varepsilon > 0$ such that when $h \in \mathcal{A}$, $\|h\| < \varepsilon$, we have

$$\widetilde{F}(a_0+h) = \sum_{k=0}^{\infty} \frac{\widetilde{F}^{(k)}(a_0)}{k!}\,h^k, \quad \text{with convergence in the norm of } \mathcal{A}. \tag{1.42}$$

For a proof of these assertions see Theorems 10.27, 10.29 and 10.36 of [RUD1].

Let us go back to the case $\mathcal{A} = H^\alpha(\mathbb{T})$. The following lemma tells us which ones are the invertible elements in $H^\alpha(\mathbb{T})$.

Lemma 1.43 *Let* $\alpha > \frac{1}{2}$. *Then,* $f \in H^\alpha(\mathbb{T})$ *is invertible (in the sense of Banach algebras) if and only if* $f(x) \neq 0$, $\forall x \in \mathbb{T}$. *In this case,* $f^{-1} = 1/f$.

Proof:

Suppose that $f \in H^\alpha(\mathbb{T})$ is invertible. Then, there must exist some element $g \in H^\alpha(\mathbb{T})$ such that $f \cdot g = \mathbf{1}$, that is, $f(x)g(x) = 1$, $\forall x \in \mathbb{T}$. This implies that $f(x) \neq 0$, $\forall x \in \mathbb{T}$ and, therefore, that $g(x) = 1/f(x)$.

Conversely, suppose that $f \in H^\alpha(\mathbb{T})$ and $f(x) \neq 0$, $\forall x \in \mathbb{T}$. Then, since $f \in C(\mathbb{T})$, we must have that $\inf_{x\in\mathbb{T}} |f(x)| = m > 0$. It follows that $1/f \in C(\mathbb{T}) \subset L^2(\mathbb{T})$. In order to show that $1/f \in H^\alpha(\mathbb{T})$ we shall consider different cases.

Case 1: $\frac{1}{2} < \alpha < 1$.

In this case,

$$\omega_\alpha(1/f)^2 = \int_{\mathbb{T}}\int_{\mathbb{T}} \left| \frac{1}{f}(x+h) - \frac{1}{f}(x) \right|^2 \frac{dx\,dh}{|h|^{1+2\alpha}}$$
$$\leq \frac{1}{m^2}\int_{\mathbb{T}}\int_{\mathbb{T}} |f(x+h)-f(x)|^2 \frac{dx\,dh}{|h|^{1+2\alpha}} < \infty$$

which implies (by (1.23)) that $\frac{1}{f} \in H^\alpha(\mathbb{T})$.

Case 2: $\alpha = 1$.

Now we know, by (1.21), that $f, f' \in L^2(\mathbb{T})$. Thus, the function $g = \frac{-f'}{f^2} \in L^2(\mathbb{T})$. Then, it is enough to see that g is the derivative in the $L^2(\mathbb{T})$-sense (and therefore in the sense of the distributions) of $\frac{1}{f}$. Indeed, for $h > 0$

$$\int_{\mathbb{T}} \left| \frac{\frac{1}{f}(x+h) - \frac{1}{f}(x)}{h} - g(x) \right|^2 dx =$$
$$\int_{\mathbb{T}} \left| \frac{f(x)-f(x+h)+hf'(x)}{hf(x)f(x+h)} + \frac{f'(x)(f(x+h)-f(x))}{f(x)^2 f(x+h)} \right|^2 dx \leq$$
$$\frac{2}{m^2}\int_{\mathbb{T}} \left| \frac{f(x+h)-f(x)}{h} - f'(x) \right|^2 dx + \frac{2}{m^3}\,\|f'\|_2 \sup_{x\in\mathbb{T}} |f(x+h)-f(x)|.$$

But the last part of the inequality converges to zero by the (uniform) continuity of f in $\mathbb{T}$ and the definition of $L^2(\mathbb{T})$-derivative. Thus, $(\frac{1}{f})' = g \in L^2(\mathbb{T})$ and, therefore, $1/f \in H^1(\mathbb{T})$. To show the remaining case we first claim the following:

CLAIM *Suppose $f \in H^\alpha(\mathbb{T})$, where $\alpha \geq n$, for some $n = 1, 2, \ldots$. Then, the distributional derivatives of $\frac{1}{f}$ can be written as:*

$$\frac{d^p}{dx^p}(\frac{1}{f}) = \sum_{m=1}^{p} \frac{g_m}{f^{m+1}}, \quad \text{where} \quad g_m \in H^{\alpha-p}(\mathbb{T}),\ (\text{for } p = 1, 2, \ldots, n). \tag{1.44}$$

Proof of claim:

We proceed by induction on p. For $p = 1$ we saw in *case 2* above that the claim holds. In fact, $(\frac{1}{f})' = \frac{-f'}{f^2}$ and $g_1 = -f' \in H^{\alpha-1}(\mathbb{T})$ [4]. Suppose, therefore, that the

[4] For convenience, we consider $H^0(\mathbb{T}) = L^2(\mathbb{T})$.

claim holds for some $p \leq n-1$. Then we must have, by the Leibnitz rule and *case 2* above, that the distributional derivative of order $p+1$ can be written as:

$$\frac{d^{p+1}}{dx^{p+1}}(\frac{1}{f}) = \sum_{m=1}^{p}\left\{\frac{g'_m}{f^{m+1}} - \frac{(m+1)g_m f'}{f^{m+2}}\right\} = \sum_{m=1}^{p+1}\frac{h_m}{f^{m+1}},$$

where, using (1.27) and Lemma 1.30, we see that $h_1 = g'_1 \in H^{\alpha-p-1}(\mathbb{T})$, $h_m = g'_m - mg_{m-1}f' \in H^{\alpha-p-1}(\mathbb{T})$, for $2 \leq m \leq p$ and $h_{p+1} = -(p+1)g_p f' \in H^{\alpha-p}(\mathbb{T}) \subset H^{\alpha-p-1}(\mathbb{T})$. This completes the induction process and the proof of the claim.

□

Let us now continue with the proof of the lemma.

Case 3: $\alpha > 1$.

Suppose that $\alpha = n + \varepsilon$, where $0 < \varepsilon \leq 1$, $n \geq 1$. Then, by (1.44), we can write

$$\frac{d^n}{dx^n}(\frac{1}{f}) = \sum_{m=1}^{n}\frac{g_m}{f^{m+1}}, \quad \text{where } g_m \in H^{\alpha-n}(\mathbb{T}) = H^{\varepsilon}(\mathbb{T}) \text{ for } 1 \leq m \leq n. \tag{1.45}$$

Since, by *cases 1* and *2*, $\frac{1}{f} \in H^{\varepsilon}(\mathbb{T})$, then Lemma 1.30 and (1.45) imply that $\frac{d^n}{dx^n}(\frac{1}{f}) \in H^{\varepsilon}(\mathbb{T})$. Thus, by (1.27) we have that $\frac{1}{f} \in H^{\alpha}(\mathbb{T})$.

□

As a corollary, we give a simple characterization of the spectrum of an element in $H^{\alpha}(\mathbb{T})$.

COROLLARY 1.46 *If* $\alpha > \frac{1}{2}$ *and* $f \in H^{\alpha}(\mathbb{T})$, *then* $\sigma(f) = \{f(x) \mid x \in \mathbb{T}\} \subset \mathbb{C}$.

PROOF:

The proof is an easy consequence of the definition of the spectrum (1.39) and Lemma 1.43. Indeed,

$$\lambda \in \sigma(f) \Leftrightarrow \lambda\mathbf{1} - f \quad \text{is not invertible} \Leftrightarrow \exists\, x \in \mathbb{T} \mid \lambda - f(x) = 0 \Leftrightarrow \lambda \in f(\mathbb{T}).$$

□

Once we have identified the spectra of elements in $H^\alpha(\mathbb{T})$, we can use the functional calculus defined in Theorem 1.40 with some particular examples. For instance, let $F(\lambda) = e^\lambda$, $\lambda \in \mathbb{C}$. Then, we can consider, for $f \in H^\alpha(\mathbb{T})$, the "symbol" $\widetilde{F}(f) = e^f \in H^\alpha(\mathbb{T})$. It is clear, by Remark 1.41, that

$$(e^f)(x) = e^{f(x)}, \quad \text{when } x \in \mathbb{T}.$$

When we consider the case in which $F(\lambda) = \sqrt{\lambda}$, $\lambda \in \Omega$ (where Ω is any simply connected subset of $\mathbb{C}$ that does not contain 0, e.g., $\Omega = \mathbb{C} \setminus (-\infty, 0]$), we can say that $\widetilde{F}(f) = \sqrt{f}$ belongs to $\in H^\alpha(\mathbb{T})$ only when $\sigma(f) = f(\mathbb{T}) \subset \Omega$, in which case we have $(\sqrt{f})(x) = \sqrt{f(x)}$. The following example shows that this is not the case if we take $f \in H^\alpha(\mathbb{T})$ such that $0 \in \sigma(f)$.

EXAMPLE 1

Consider $f(x) = \sin x$, $x \in \mathbb{R}$. Then, f is 2π-periodic and C^∞ and, therefore, $f \in H^\alpha(\mathbb{T})$ for all $\alpha > 0$ (in fact, its Fourier coefficients are $c_{\pm 1} = \pm\frac{1}{2}$ and 0 otherwise). But notice that $\sqrt{f}$ cannot be in any Lipschitz space $\Lambda^\beta(\mathbb{T})$, for any $\beta > \frac{1}{2}$, because

$$\sup_{\substack{x,h\in\mathbb{T}\\ h\neq 0}} \frac{|\sqrt{f(x+h)} - \sqrt{f(x)}|}{|h|^\beta} \geq \frac{\sqrt{\sin h}}{|h|^\beta} \geq \frac{2}{\pi}\, |h|^{\frac{1}{2}-\beta} \to \infty, \quad \textit{as } h \to 0.$$

Thus, by Theorem 3.23 *(III)*, $\sqrt{f(x)} \notin H^\alpha(\mathbb{T})$, for any $\alpha > 1$.

EXAMPLE 2

In this example we show that the assumption $\sigma(f) \subset \Omega$, where Ω is simply connected (and does not contain 0), cannot be dropped either. Indeed, consider $f(x) = e^{ix}$, $x \in \mathbb{R}$. Clearly, $f \in H^\alpha(\mathbb{T})$ for all $\alpha > 0$ and $\sigma(f) = \{z \in \mathbb{C} \mid |z| = 1\}$. But now, $\sqrt{f(x)} = e^{i\frac{x}{2}}$ cannot be in any of the $H^\alpha(\mathbb{T})$ spaces because it is not a 2π-periodic function.

To conclude this section, we show a general result about $H^\alpha(\mathbb{T})$ that we shall need later.

Lemma 1.47 *Let $\alpha > 0$. Then, for every $f \in H^\alpha(\mathbb{T})$ its Fourier series (as defined in (1.19)) converges to f in the $H^\alpha(\mathbb{T})$-norm. That is,*

$$\|f - \sum_{k=-N}^{N} c_k e^{ikx}\|_\alpha \to 0, \quad \textit{as } N \to \infty. \tag{1.48}$$

As a consequence the set $\mathcal{T} = \{\sum_{k=-M}^{N} c_k e^{ikx} \mid M, N \geq 0,\ c_k \in \mathbb{C}\}$ of trigonometric polynomials is dense in $H^\alpha(\mathbb{T})$.

Proof:

By the defintion of the $H^\alpha(\mathbb{T})$-norm given in (1.20), we can write (1.48) as

$$\|f - \sum_{k=-N}^{N} c_k e^{ikx}\|_\alpha^2 = \sum_{|k| \geq N+1} |c_k|^2 (1 + |k|^2)^\alpha \to 0, \quad \text{as } N \to \infty,$$

because $\|f\|_{H^\alpha(\mathbb{T})}^2 = \sum_{k \in \mathbb{Z}} |c_k|^2 (1 + |k|^2)^\alpha < \infty$.

□

1.4 The locally integrable case

We say that $f : \mathbb{R} \to \mathbb{C}$ belongs to $L^2_{loc}(\mathbb{R})$ if for any compact set $K \subset \mathbb{R}$ we have that $\int_K |f(x)|^2 \, dx < \infty$. For $n = 1, 2, \ldots$, we say that $f \in H^n_{loc}(\mathbb{R})$ when $f, f', \ldots, f^{(n)} \in L^2_{loc}(\mathbb{R})$. Given a compact set $K \subset \mathbb{R}$ we will use the following notation

$$\|f\|_{H^n_{loc}, K} = \|f\|_{L^2(K)} + \ldots + \|f^{(n)}\|_{L^2(K)}. \tag{1.49}$$

When $\alpha = n + \varepsilon > 0$, where $0 < \varepsilon < 1$, we say that $f \in H^\alpha_{loc}(\mathbb{R})$ whenever $f, f', \ldots, f^{(n)} \in L^2_{loc}(\mathbb{R})$ and, for every compact set $K \subset \mathbb{R}$, we have

$$\omega_{\varepsilon, K}(f^{(n)}) = \left[\int_K \int_K |f^{(n)}(x+h) - f^{(n)}(x)|^2 \, dx \, \frac{dh}{|h|^{1+2\varepsilon}} \right]^{\frac{1}{2}} < \infty.$$

As before, we let

$$\|f\|_{H^\alpha_{loc}, K} = \|f\|_{L^2(K)} + \ldots + \|f^{(n)}\|_{L^2(K)} + \omega_{\varepsilon, K}(f^{(n)}). \tag{1.50}$$

Note that $H^{\alpha}(\mathbb{R}) \subset H^{\alpha}_{loc}(\mathbb{R})$. In fact, if we take an increasing sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that $\cup_{n=1}^{\infty} K_n = \mathbb{R}$ then, by the Monotone Convergence Theorem, we have

$$\|f\|_{\alpha} = \lim_{n\to\infty} \|f\|_{H^{\alpha}_{loc},K_n}, \quad \forall f \in H^{\alpha}(\mathbb{R}), \tag{1.51}$$

where the norm on the left hand side is the one defined in (1.4) (or (1.6)). This implies that $H^{\alpha}_{loc}(\mathbb{R})$, $\alpha > 0$, are Fréchet spaces with the seminorms defined in (1.49) and (1.50).

The following proposition gives us a more precise relation between $H^{\alpha}(\mathbb{R})$ and $H^{\alpha}_{loc}(\mathbb{R})$.

Proposition 1.52 *Let $\alpha > 0$. Suppose that $f \in H^{\alpha}_{loc}(\mathbb{R})$ and $w \in C_c^{\infty}(\mathbb{R})$, with supp $w \subset [-R, R]$. Let $K = [-2R, 2R]$, then $f \cdot w \in H^{\alpha}(\mathbb{R})$ and we can find a constant $C = C(w, R, \alpha) > 0$ such that*

$$\|f \cdot w\|_{H^{\alpha}(\mathbb{R})} \leq C \, \|f\|_{H^{\alpha}_{loc},K}.$$

Proof:

It is clear that

$$\|f \cdot w\|_{L^2(\mathbb{R})} = \left[\int_{\mathbb{R}} |f(x)w(x)|^2 \, dx\right]^{\frac{1}{2}} \leq \|w\|_{\infty} \, \|f\|_{L^2(-R,R)}.$$

As usual, we divide the rest of the proof into two different cases:

Case 1: $\alpha = k \in \mathbb{Z}^+$.

In order to estimate $\|(f \cdot w)^{(k)}\|_{L^2(\mathbb{R})}$, it is enough to consider each of the terms $\|f^{(k-\ell)} \cdot w^{(\ell)}\|_{L^2(\mathbb{R})}$, separately, where $0 \leq \ell \leq k$. But for any of these terms we have

$$\|f^{(k-\ell)} \cdot w^{(\ell)}\|_{L^2(\mathbb{R})} \leq \|w^{(\ell)}\|_{\infty} \, \|f^{(k-\ell)}\|_{L^2(-R,R)} \leq C \, \|f\|_{H^k_{loc},K}.$$

Case 2: $\alpha = k + \varepsilon$, $0 < \varepsilon < 1$.

In this case we need to estimate $\omega_{\varepsilon}((f \cdot w)^{(k)})$. By using Leibnitz's rule it is enough to consider the terms $\omega_{\varepsilon}(f^{(k-\ell)} \cdot w^{(\ell)})$, $0 \leq \ell \leq k$. We compute $\omega_{\varepsilon}(f \cdot w)$. The estimation for the other terms follows from this if we replace f by $f^{(k-\ell)}$ and w by $w^{(\ell)}$.

$$\omega_\varepsilon(f\cdot w)^2 \le \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x+y)w(x+y)-f(x)w(x)|^2 \frac{dx\,dy}{|y|^{1+2\varepsilon}} =$$

$$\int_{|y|>R}\int_{\mathbb{R}} \quad + \int_{|y|\le R}\int_{\mathbb{R}} \quad = I + II.$$

The first integral is easily bounded by

$$I \le 4\left(\int_{|y|>R}\frac{dy}{|y|^{1+2\varepsilon}}\right)\left(\int_{\mathbb{R}}|f(x)w(x)|^2\,dx\right) \le C\,\|f\|_{H^\alpha_{loc},K}.$$

For the second term, we use the fact that when $|y|\le R$ and $|x|\ge 2R$, then $|x+y|\ge R$ and, therefore, $w(x+y)=w(x)=0$. Then, II is bounded by

$$II \le 2\int_{|y|\le R}\int_{|x|<2R}|f(x)|^2\,|w(x+y)-w(x)|^2\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$+2\int_{|y|\le R}\int_{|x|\le R}|w(x)|^2\,|f(x+y)-f(x)|^2\frac{dx\,dy}{|y|^{1+2\varepsilon}}$$

$$\le 2\,\|w'\|_\infty^2\left(\int_{|y|\le R}\frac{|y|^2}{|y|^{1+2\varepsilon}}\,dy\right)\|f\|^2_{L^2(-2R,2R)} + 2\,\|w\|^2_\infty\,\omega_{\varepsilon,[-R,R]}(f)^2 \le C\,\|f\|^2_{H^\alpha_{loc},K}.$$

□

Let $\alpha>\frac{1}{2}$ and let $0<R_1<R_2<\infty$. Suppose that $w\in C_c^\infty(-R_2,R_2)$ is such that $w|_{[-R_1,R_1]}\equiv 1$. Let us denote by K the compact set $[-2R_2,2R_2]$. Then, for all $f\in H^\alpha_{loc}(\mathbb{R})$, we have that $f\in C([-R_1,R_1])$ and

$$\|f\|_{C([-R_1,R_1])}\le C\,\|f\|_{H^\alpha_{loc},K}. \tag{1.53}$$

This is a consequence of the Sobolev Imbedding Theorem 1.8 and the proposition above. Moreover, if we assume that $\alpha>k+\frac{1}{2}$, where $k=1,2,\ldots$, then, by induction, one can easily see that

$$\|f\|_{C^k(-R_1,R_1)}\le C\,\|f\|_{H^\alpha_{loc},K}. \tag{1.54}$$

We can write all this in a proposition:

Proposition 1.55 *Let $\alpha > k + \frac{1}{2}$, where $k = 0, 1, \ldots$, and suppose that $f_n, f \in H^{\alpha}_{loc}(\mathbb{R})$ and $f_n \to f$ in the topology of $H^{\alpha}_{loc}(\mathbb{R})$. Then, $f_n, f \in C^k(\mathbb{R})$ and, for any $0 \le h \le k$,*

$$D^{(h)} f_n \to D^{(h)} f \quad \textit{uniformly in compact sets of } \mathbb{R}.$$

This concludes our review about Sobolev spaces.

2 The set $\mathcal{S}_\alpha$ of α-localized scaling functions

In this section we present the main properties of the set of α-localized scaling functions and its close relation with the set of associated low-pass filters. We introduce the notation and definitions that will lead, in §3, to show that these two sets are homeomorphic topological spaces.

Fix $\alpha > \frac{1}{2}$. Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is an α-localized MRA as defined in (3.13) of Chapter 1. That is, $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA for which we can find a scaling function φ such that $\widehat{\varphi} \in H^\alpha(\mathbb{R})$. Since we are considering only cases in which $\alpha > \frac{1}{2}$, by Remark 1.12 we may assume that $\widehat{\varphi} \in C(\mathbb{R})$. It is a well-known fact that in an MRA $\{V_j\}_{j\in\mathbb{Z}}$, any scaling function φ such that $|\widehat{\varphi}|$ is continuous at 0 must satisfy that $|\widehat{\varphi}(0)| = 1$ (see Theorem 1.7 in Chapter 2 of [HW]). Unless otherwise specified, from now on, we will assume that all scaling functions from α-localized MRAs satisfy the additional assumption

$$\widehat{\varphi}(0) = 1. \tag{2.1}$$

(Note that we can always do this by multiplying our arbitrary α-localized scaling function φ by the unimodular constant $\overline{\widehat{\varphi}(0)}$, the associated low-pass filter being the same in this case.)

In particular, when this is the case, the low-pass filter associated with φ must satisfy

$$\left.\begin{array}{l} m_0(0) = 1 \\ |m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1, \quad a.e.\,\xi \in \mathbb{R} \end{array}\right\} \tag{2.2}$$

as a consequence of equalities (3.4) and (3.7) in Chapter 1. Moreover, if $\widehat{\varphi} \in H^\alpha(\mathbb{R})$, the just mentioned scaling relation (3.4) suggests that the low-pass filter m_0 will satisfy some sort of "smoothness" condition. We shall show in the next lemma that in fact $m_0 \in H^\alpha(\mathbb{T})$.

Lemma 2.3 *Let $\alpha > \frac{1}{2}$. If φ is an α-localized scaling function and if m_0 is its* *low-pass filter, then* $m_0 \in H^\alpha(\mathbb{T})$ *and*

$$\|m_0\|_{H^\alpha(\mathbb{T})} \leq C \, \|\widehat{\varphi}\|^2_{H^\alpha(\mathbb{R})} \,, \tag{2.4}$$

for a constant $C = C(\alpha) > 0$.

The proof of this lemma is essentially taken from a result by Lemarié-Rieusset (see Lemma 2 in Chapter 3 of [KAH-LEM]).

Proof:

Let $m_0(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi} \in L^2(\mathbb{T})$ be the low-pass filter associated with φ, where the sequence of coefficients $\{c_k\} \in \ell^2(\mathbb{Z})$ is given by (3.5) of Chapter 1. In order to show that $m_0 \in H^\alpha(\mathbb{T})$ we use the definition of the Sobolev space given in (1.20). Then, inequality (2.4) is equivalent to

$$\sum_{k\in\mathbb{Z}} |c_k|^2 (1+|k|^2)^\alpha \leq C \, \|\widehat{\varphi}\|^4_{H^\alpha(\mathbb{R})}. \tag{2.5}$$

To show (2.5) we proceed as follows

$$\sum_{k\in\mathbb{Z}} |c_k|^2 (1+|k|^2)^\alpha = \frac{1}{4} \sum_{k\in\mathbb{Z}} \Big| \int_{\mathbb{R}} \varphi(\frac{x}{2}) \overline{\varphi(x-k)} \, dx \Big|^2 (1+|k|^2)^\alpha \leq$$

$$\leq \frac{1}{2} \sum_{k\in\mathbb{Z}} \left\{ \Big| \int_{|x|>\frac{|k|}{10}} \Big|^2 + \Big| \int_{|x|\leq\frac{|k|}{10}} \Big|^2 \right\} (1+|k|^2)^\alpha \leq$$

$$\frac{1}{2} \sum_{k\in\mathbb{Z}} \left\{ \left[\int_{|x|>\frac{|k|}{10}} |\varphi(\frac{x}{2})|^2 \frac{(1+|k|^2)^\alpha}{(1+|x-k|^2)^\alpha} \, dx \right] \left[\int_{\mathbb{R}} |\varphi(x-k)|^2 \, (1+|x-k|^2)^\alpha \, dx \right] + \right.$$

$$\left. \left[\int_{|x-k|\geq\frac{9}{10}|k|} |\varphi(x-k)|^2 \frac{(1+|k|^2)^\alpha}{(1+|x|^2)^\alpha} \, dx \right] \left[\int_{\mathbb{R}} |\varphi(\frac{x}{2})|^2 \, (1+|x|^2)^\alpha \, dx \right] \right\},$$

where in the last step we have used Schwarz's inequality. Now, if we take into account the fact that whenever $|x| > \frac{|k|}{10}$ then $1+|k|^2 \leq 1+100|x|^2$, and whenever $|x-k| \geq \frac{9}{10}|k|$ then $1+|k|^2 \leq 1+\frac{100}{81}|x-k|^2$, we have

$$\sum_{k\in\mathbb{Z}} |c_k|^2 (1+|k|^2)^\alpha \leq \frac{1}{2} \, \|\widehat{\varphi}\|^2_{H^\alpha(\mathbb{R})} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |\varphi(\frac{x}{2})|^2 \frac{(1+100|x|^2)^\alpha}{(1+|x-k|^2)^\alpha} \, dx +$$

$$2^{2\alpha} \, \|\widehat{\varphi}\|^2_{H^\alpha(\mathbb{R})} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |\varphi(x-k)|^2 \frac{(1+\frac{100}{81}|x-k|^2)^\alpha}{(1+|x|^2)^\alpha} \, dx \leq$$

$$\|\widehat{\varphi}\|^2_{H^\alpha(\mathbb{R})}\left(\sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{(1+|x-k|^2)^\alpha}\right)400^\alpha\int_{\mathbb{R}}|\varphi(x)|^2(1+|x|^2)^\alpha\,dx+$$

$$\left(\frac{400}{81}\right)^\alpha\|\widehat{\varphi}\|^2_{H^\alpha(\mathbb{R})}\left(\sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{(1+|x+k|^2)^\alpha}\right)\int_{\mathbb{R}}|\varphi(x)|^2(1+|x|^2)^\alpha\,dx\le C\,\|\widehat{\varphi}\|^4_{H^\alpha(\mathbb{R})}.$$

This shows (2.5) and establishes the lemma.

□

DEFINITION 2.6 *Let $\alpha>\frac{1}{2}$. We say that a function $\widehat{\varphi}$ belongs to the set $\mathcal{S}_\alpha$ if $\widehat{\varphi}\in H^\alpha(\mathbb{R})$ and φ is a scaling function satisfying (2.1). That is, $\mathcal{S}_\alpha$ is the space of Fourier transforms of all α-localized scaling functions. We will consider $\mathcal{S}_\alpha$ as a topological subspace of $H^\alpha(\mathbb{R})$. Therefore, when we say that $\widehat{\varphi}_n\to\widehat{\varphi}$, in $\mathcal{S}_\alpha$, we mean that $\widehat{\varphi}_n,\widehat{\varphi}\in\mathcal{S}_\alpha$, $n=1,2,\ldots$, and $\widehat{\varphi}_n\to\widehat{\varphi}$, in $H^\alpha(\mathbb{R})$.*

Now, we can write a stronger version of Lemma 2.3.

LEMMA 2.7 *Let $\alpha>\frac{1}{2}$. The mapping $M:\mathcal{S}_\alpha\longrightarrow H^\alpha(\mathbb{T})$ that maps every function $\widehat{\varphi}\in\mathcal{S}_\alpha$ into its corresponding low-pass filter, is continuous.*

PROOF:

We need to show that if $\widehat{\varphi}_n\in\mathcal{S}_\alpha$, $n=0,1,2,\ldots$ and $\widehat{\varphi}_n\to\widehat{\varphi}_0$ in $\mathcal{S}_\alpha$, then $m_n\to m_0$ in $H^\alpha(\mathbb{T})$, where for each $n=0,1,2,\ldots$, $m_n=M(\widehat{\varphi}_n)$ is the low-pass filter corresponding to $\widehat{\varphi}_n$. The proof follows the same line as in Lemma 2.3, once we notice the following:

CLAIM *If $\widehat{\varphi}_1,\widehat{\varphi}_2\in\mathcal{S}_\alpha$ then,*

$$\|M(\widehat{\varphi}_1)-M(\widehat{\varphi}_2)\|_{H^\alpha(\mathbb{T})}\le C\,\|\widehat{\varphi}_1-\widehat{\varphi}_2\|_{H^\alpha(\mathbb{R})}\,(\|\widehat{\varphi}_1\|_{H^\alpha(\mathbb{R})}+\|\widehat{\varphi}_2\|_{H^\alpha(\mathbb{R})}).\tag{2.8}$$

Proof of CLAIM:

Suppose that the Fourier coefficients of $m_1=M(\widehat{\varphi}_1)$ and $m_2=M(\widehat{\varphi}_2)$ are given by $\{c_k^1\}$ and $\{c_k^2\}$, respectively. Then, by formula (3.5) in Chapter 1, they can be written as

$$c_k^j=\int_{\mathbb{R}}\frac{1}{2}\varphi_j(\frac{x}{2})\overline{\varphi_j(x-k)}\,dx,\quad k\in\mathbb{Z},\,j=1,2.$$

It follows that

$$2\,(\sum_{k\in\mathbb{Z}} |c_k^1 - c_k^2|^2\,(1+|k|^2)^\alpha\,)^{\frac{1}{2}} \le \Big[\sum_{k\in\mathbb{Z}} |\int_{\mathbb{R}} \varphi_1(\frac{x}{2})\overline{(\varphi_1-\varphi_2)(x-k)}\,dx|^2\,(1+|k|^2)^\alpha\,\Big]^{\frac{1}{2}} +$$

$$\Big[\sum_{k\in\mathbb{Z}} |\int_{\mathbb{R}} (\varphi_1-\varphi_2)(\frac{x}{2})\overline{\varphi_2(x-k)}\,dx|^2\,(1+|k|^2)^\alpha\Big]^{\frac{1}{2}}.$$

Now, by using Schwarz's inequality we can separate each of the integrals above into the product of two new integrals, each of them involving now only one of the functions φ_j. For the rest of the proof, the same estimations given immediately after (2.5) above show:

$$(\sum_{k\in\mathbb{Z}} |c_k^1 - c_k^2|^2(1+|k|^2)^\alpha\,)^{\frac{1}{2}} \le C\,\|\widehat{\varphi}_1\|_{H^\alpha(\mathbb{R})}\,\|\widehat{\varphi}_1-\widehat{\varphi}_2\|_{H^\alpha(\mathbb{R})} +$$

$$C\,\|\widehat{\varphi}_1-\widehat{\varphi}_2\|_{H^\alpha(\mathbb{R})}\,\|\widehat{\varphi}_2\|_{H^\alpha(\mathbb{R})} = C\,\|\widehat{\varphi}_1-\widehat{\varphi}_2\|_{H^\alpha(\mathbb{R})}\,(\|\widehat{\varphi}_1\|_{H^\alpha(\mathbb{R})} + \|\widehat{\varphi}_2\|_{H^\alpha(\mathbb{R})})$$

and this establishes the claim.

□

In particular, when we use the claim with $\widehat{\varphi}_n$ and $\widehat{\varphi}_0$, we have

$$\|m_n - m_0\|_{H^\alpha(\mathbb{T})} \le C\,\|\widehat{\varphi}_n-\widehat{\varphi}_0\|_{H^\alpha(\mathbb{R})}\,(\|\widehat{\varphi}_n\|_{H^\alpha(\mathbb{R})} + \|\widehat{\varphi}_0\|_{H^\alpha(\mathbb{R})}) \le$$

$$\le C\,(\sup_{n\ge 1}\|\widehat{\varphi}_n\|_{H^\alpha(\mathbb{R})})\,\|\widehat{\varphi}_n-\widehat{\varphi}_0\|_{H^\alpha(\mathbb{R})} \to 0, \quad \text{as } n\to\infty.$$

This shows that $M(\widehat{\varphi}_n) \to M(\widehat{\varphi}_0)$ and completes the proof of the lemma.

□

Let us show now a somewhat interesting property of the topological space $\mathcal{S}_\alpha$.

Lemma 2.9 *Let $\alpha > \frac{1}{2}$. Then, $\mathcal{S}_\alpha$ is closed in $H^\alpha(\mathbb{R})$.*

Proof:

Suppose that $\{\widehat{\varphi}_n\}_{n=1}^\infty$ is a sequence contained in $\mathcal{S}_\alpha$ such that $\widehat{\varphi}_n \to \widehat{\varphi}$ for some $\widehat{\varphi} \in H^\alpha(\mathbb{R})$ (where the convergence is in $H^\alpha(\mathbb{R})$). In order to show that $\widehat{\varphi} \in \mathcal{S}_\alpha$, it is enough to see that φ is a scaling function and that $\widehat{\varphi}(0) = 1$. The latter is a consequence of Sobolev's Theorem 1.8 (III), since $1 = \widehat{\varphi}_n(0) \to \widehat{\varphi}(0)$ and, therefore, $\widehat{\varphi}(0) = 1$. To show the former, we use Proposition 3.12 in Chapter 1. We have already seen that (ii) in the proposition holds. It remains to show equalities (i) and (iii) from the same proposition. Let us establish (i) first. Note that, by Corollary 1.17, for every $\xi \in \mathbb{T}$,

$$\left| (\sum_{k\in\mathbb{Z}} |\widehat{\varphi}_n(\xi + 2k\pi)|^2)^{\frac{1}{2}} - (\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2k\pi)|^2)^{\frac{1}{2}} \right| \leq$$

$$(\sum_{k\in\mathbb{Z}} |(\widehat{\varphi}_n - \widehat{\varphi})(\xi + 2k\pi)|^2)^{\frac{1}{2}} \leq C \, \|\widehat{\varphi}_n - \widehat{\varphi}\|_{H^\alpha(\mathbb{R})} \to 0, \quad \text{as } n \to \infty.$$

But since $\widehat{\varphi}_n \in \mathcal{S}_\alpha$ and, therefore, $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}_n(\xi + 2k\pi)|^2 \equiv 1$, it follows that (i) in Proposition 3.12 must hold. Finally, consider the low-pass filters m_n associated with φ_n. As a consequence of (2.8) in the proof of the previous lemma, it follows that $\{m_n\}_{n=1}^\infty$ is a Cauchy sequence in $H^\alpha(\mathbb{T})$. This implies that there exists a 2π-periodic function $m_0 \in H^\alpha(\mathbb{T})$ such that $m_n \to m_0$ in $H^\alpha(\mathbb{T})$. Then, for each fixed $\xi \in \mathbb{R}$, Sobolev Embedding Theorems 1.8 and 1.25 imply that

$$\widehat{\varphi}(2\xi) = \lim_{n\to\infty} \widehat{\varphi}_n(2\xi) = \lim_{n\to\infty} \widehat{\varphi}_n(\xi) m_n(\xi) = \widehat{\varphi}(\xi) m(\xi).$$

This shows that (iii) in Proposition 3.12 of Chapter 1 holds, and completes the proof of the lemma.

□

We turn now to the matter of identifying the range of the map M in Lemma 2.7; that is, we seek the necessary and sufficient conditions on a function $m_0 \in H^\alpha(\mathbb{T})$ for it to be the low-pass filter of an α-localized scaling function. This question was succesfully answered by A. Cohen in the case $\alpha = \infty$ and extended to $\alpha > \frac{1}{2}$ by P. G. Lemarié-Rieusset. During the rest of this section we present the main ideas of Cohen's construction as well as a detailed proof of Lemarié's result.

Suppose we are given a scaling function φ with associated low-pass filter m_0. There is then an easy way of expressing $\widehat{\varphi}$ in terms of m_0. Indeed, by iterating the scaling equation (3.4) in Chapter 1, we obtain that, for *a.e.* $\xi \in \mathbb{R}$,

$$\widehat{\varphi}(\xi) = m_0(\frac{\xi}{2})\,\widehat{\varphi}(\frac{\xi}{2}) = \prod_{j=1}^{n} m_0(\frac{\xi}{2^j})\,\widehat{\varphi}(\frac{\xi}{2^n}), \quad n = 1, 2, \ldots \tag{2.10}$$

If we assume further that $\lim_{n\to\infty} \widehat{\varphi}(2^{-n}\xi) = 1$, we arrive at the expression

$$\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j}). \tag{2.11}$$

In the general case, however, the infinite product in (2.11) does not always make sense. The following example, taken from [PSK], shows that one can find two distinct scaling functions (corresponding to two distinct MRAs) having the same low-pass filter. Then, both cannot be written simultaneously as in (2.11).

EXAMPLE 1

Let φ be any scaling function for an MRA, with low-pass filter m_0 and with the property that supp $\widehat{\varphi} = \mathbb{R}$ (e.g., $\varphi = \chi_{[-1,0]}$, the *Haar scaling function*, see Example B in Chapter 2 of [HW]). Let $\widehat{\varphi}^{\sharp}(\xi) = \nu(\xi)\widehat{\varphi}(\xi)$, where

$$\nu(\xi) = \begin{cases} 1 & \text{if } \xi \geq 0 \\ -1 & \text{if } \xi < 0. \end{cases}$$

Then, $\widehat{\varphi}^{\sharp}(2\xi) = m_0(\xi)\widehat{\varphi}^{\sharp}(\xi)$, *a.e.* $\xi \in \mathbb{R}$. In particular, Proposition 3.12 of Chapter 1 implies that $\varphi^{\sharp}$ is also a scaling function for some MRA with low-pass filter m_0. But since supp $\widehat{\varphi} = \mathbb{R}$ and ν is not 2π-periodic, by Remark 3.2 in Chapter 1, φ and $\varphi^{\sharp}$ cannot be both scaling functions of the same MRA.

In the case of α-localized MRAs, the continuity at 0 of $\widehat{\varphi}$ (and the assumption $\widehat{\varphi}(0) = 1$) is enough to give a precise meaning to the infinite product in (2.11). In fact, the next lemma shows that this product converges uniformly on compact sets of $\mathbb{R}$.

LEMMA 2.12 *Let $\alpha > \frac{1}{2}$. Suppose that $m_0 \in H^{\alpha}(\mathbb{T})$ and $m_0(0) = 1$. Then, the infinite product $\prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$ converges uniformly on compact sets of $\mathbb{R}$ to a continuous function $\widehat{\varphi}(\xi)$. Moreover, $\widehat{\varphi}(\xi) = 0$ if and only if there exists a $j \geq 1$ such* *that* $m_0(2^{-j}\xi) = 0$. *In particular, if* m_0 *is the low-pass filter of an* α*-localized scaling function* φ, *the formula (2.11) holds for every* $\xi \in \mathbb{R}$.

PROOF:

This lemma is an easy consequence of the following general fact about infinite products:

PROPOSITION 2.13 : See Theorem 15.4 of [RUD2].

Let $\{f_n\}_{n=1}^{\infty}$ *be a sequence of bounded complex-valued functions on a set* $\mathcal{S}$ *such that* $\sum_{n=1}^{\infty} |1 - f_n|$ *converges uniformly in the set* $\mathcal{S}$. *Then,* $\prod_{n=1}^{\infty} f_n \equiv f$ *converges uniformly in* $\mathcal{S}$. *Moreover,*

(*i*) *For* $x \in \mathcal{S}$, $\quad f(x) = 0 \iff f_n(x) = 0$ *for some* $n \geq 1$.

(*ii*) *For every permutation* $\{n_1, n_2, \ldots\}$ *of* $\mathbb{Z}^+$, $f = \prod_{k=1}^{\infty} f_{n_k}$.

In order to prove Lemma 2.12 we just need to verify that $\sum_{k=1}^{\infty} |1 - m_0(2^{-j}\xi)|$ converges uniformly in compact sets of $\mathbb{R}$. Indeed, take a compact set $[-K, K]$ of $\mathbb{R}$, and take $j_0 \in \mathbb{Z}^+$ such that $K \leq 2^{j_0}\pi$. Let $\varepsilon \in (0,1)$ be such that $0 < \varepsilon \leq \alpha - \frac{1}{2}$. Then, by Sobolev's Theorem 1.25, we have that $H^{\alpha}(\mathbb{T}) \subset H^{\varepsilon+\frac{1}{2}}(\mathbb{T}) \subset \Lambda^{\varepsilon}(\mathbb{T})$. Thus, when $j \geq j_0$ and $\xi \in [-K, K]$, we have

$$|1 - m_0(2^{-j}\xi)| = |m_0(0) - m_0(2^{-j}\xi)| \leq (2^{-j}|\xi|)^{\varepsilon} \, \|m_0\|_{\Lambda^{\varepsilon}(\mathbb{T})} \leq C\, 2^{-j\varepsilon} \|m_0\|_{H^{\alpha}(\mathbb{T})}.$$

Now, it follows from the Weierstrass M-test that $\sum_{k=1}^{\infty} |1 - m_0(2^{-j}\xi)|$ converges uniformly in $[-K, K]$, and the lemma is proved.

$\square$

We already know that a necessary condition for a 2π-periodic function m_0 to be a low-pass filter for some MRA is that (2.2) holds. Unfortunately, this condition is not sufficient, even in the smooth case, as the following well-known example shows:

EXAMPLE 2

Let $m_0(\xi) = \frac{1+e^{i3\xi}}{2}$, $\xi \in \mathbb{R}$. Then, $m_0(\xi) = m_H(3\xi)$, where $m_H(\xi) = \frac{1+e^{i\xi}}{2}$ is the *Haar low-pass filter* (see Example B in Chapter 2 of [HW]). It is then clear that m_0 satisfies (2.2). Moreover,

$$\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j}) = \widehat{\varphi}_H(3\xi) = e^{i3\xi/2}\frac{\sin(3\xi/2)}{3\xi/2}, \quad \xi \in \mathbb{R},$$

and, consequently, $\varphi(x) = \frac{1}{3}\varphi_H(\frac{x}{3}) = \frac{1}{3}\chi_{[-3,0]}(x)$. But then, $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}\}$ is not an orthonormal system in $L^2(\mathbb{R})$.

Albert Cohen studied in [COH] the necessary and sufficient conditions for a function $m_0 \in C^{\infty}(\mathbb{T})$ to be the low-pass filter of a *scaling function with polynomial decay.* That is, a scaling function φ of an MRA such that

$$\int_{\mathbb{R}} |\varphi(x)|^2 \, (1 + |x|^2)^N \, dx = c_N < \infty, \quad \forall\, N = 1, 2, \ldots \tag{2.14}$$

In order to avoid pathological behaviors as in the previous example, Cohen introduces an assumption in m_0 that guarantees that $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2k\pi)|^2 \neq 0$ (for $\widehat{\varphi}$ given by (2.11)) and, then, uses (2.2) to show that this series has to be identically 1. The additional premise required is commonly known as *Cohen's condition* and can be written as follows:

$$\left.\begin{array}{l} \text{There exists a compact set } K \subset \mathbb{R} \quad \text{such that} \quad 0 \in \overset{\circ}{K}, \\ \sum\limits_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = 1, \; a.e.\, \xi \in \mathbb{T}, \text{ and } m_0(2^{-j}\xi) \neq 0, \; j \geq 1, \; \xi \in K. \end{array}\right\} \tag{2.15}$$

Remark 2.16 For a measurable set $K \subset \mathbb{R}$, the property that $\sum_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = 1$, $a.e.\, \xi \in \mathbb{T}$, is often called 2π-translation congruency to the torus $\mathbb{T}$, and denoted by $K \sim_{2\pi} \mathbb{T}$. In general, two measurable sets $K, J \subset \mathbb{R}$ are said to be 2π-translation congruent, $K \sim_{2\pi} J$, if there is a partition $\{K_\ell\}_{\ell\in\mathbb{Z}}$ of K, and a sequence of integers $\{k_\ell\}_{\ell\in\mathbb{Z}}$, such that $\{K_\ell + 2k_\ell\pi\}_{\ell\in\mathbb{Z}}$ forms a partition of J. We already encountered this relation in (2.5) of Chapter 1.

After this motivation, we can state a precise version of Cohen's Theorem.

Theorem 2.17 : A. Cohen.

A function m_0 is the low-pass filter of a scaling function with polynomial decay if and only if $m_0 \in C^\infty(\mathbb{T})$ and satisfies (2.2) and (2.15). Moreover, if this is the case, the scaling function φ is given by the infinite product formula (2.11).

A proof of this theorem can be found in [COH] (Theorem 2.1) or [HW] (Theorem 4.23, in Chapter 7).

Some variations of Cohen's Theorem exist in the literature. A more general result, where the polynomial decay condition in φ is replaced by a less restrictive α-localization ($\int_{\mathbb{R}} |\varphi(x)|^2 (1+|x|^2)^\alpha \, dx < \infty$, for $\alpha > \frac{1}{2}$) was shown by P. G. Lemarié-Rieusset (see Theorem 1 in Chapter 4 of [KAH-LEM]). We give a detailed and slightly different proof of this result because it is crucial for what we will be doing during the rest of this thesis.

Theorem 2.18 *A function m_0 is the low-pass filter of an α-localized scaling function if and only if $m_0 \in H^\alpha(\mathbb{T})$ and satisfies (2.2) and (2.15). Moreover, if this is the case, the scaling function φ is given by the infinite product formula (2.11).*

Proof:

Suppose that m_0 is the low-pass filter of an α-localized scaling function φ. Then, Lemma 2.3 tells us that $m_0 \in H^\alpha(\mathbb{T})$, while (2.2) is trivially satisfied. We only need to show that Cohen's condition (2.15) holds. Here we follow the original proof by Cohen. Since $\widehat{\varphi} \in H^\alpha(\mathbb{R})$ (and $\alpha > \frac{1}{2}$), by Corollary 1.18, we must have that $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2k\pi)|^2 = 1$, for all $\xi \in [-\pi, \pi]$. Thus, for each $\xi \in [-\pi,\pi]$ there exists an integer $k = k(\xi)$ such that $\widehat{\varphi}(\xi + 2k(\xi)\pi) \neq 0$. Observe that, since $\widehat{\varphi}(0) = 1$, we must have $k(0) = 0$. The continuity of $\widehat{\varphi}$ implies the existence of an open (bounded) interval V_ξ, containing ξ, and a constant $C_\xi > 0$, such that

$$|\widehat{\varphi}(\eta + 2k(\xi)\pi)|^2 \geq C_\xi, \quad \forall \eta \in V_\xi. \tag{2.19}$$

Then, $[-\pi,\pi] \subset \cup_{\xi\in\mathbb{T}} V_\xi$, and there exists a finite number of points $\xi_0, \xi_1, \ldots, \xi_n \in [-\pi,\pi]$ such that $[-\pi,\pi] \subset \cup_{j=0}^n V_{\xi_j}$. We may assume that $0 \in V_{\xi_0}$. Define now

$$R_0 = V_{\xi_0} \cap [-\pi,\pi] \quad \text{and} \quad R_j = (V_{\xi_j} \cap [-\pi,\pi]) \setminus (\cup_{\ell=0}^{j-1} R_\ell), \quad j = 1,\ldots,n$$

and take

$$K = \cup_{j=0}^n \overline{(R_j + 2k(\xi_j)\pi)}.$$

Then, K is a finite union of compact intervals and $0 \in \mathring{R}_0 \subset \mathring{K}$. Since for every $\xi \in [-\pi,\pi) = \mathbb{T}$ there exists a unique $j \in \{0,1,\ldots,n\}$ such that $\xi \in R_j$, we must have

$$\sum_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = \sum_{j=0}^n \chi_{R_j+2k(\xi_j)\pi}(\xi + 2k(\xi_j)\pi) = \sum_{j=0}^n \chi_{R_j}(\xi) = 1, \quad a.e.\, \xi \in \mathbb{T}.$$

Finally, for all $\xi \in K$, and $j \geq 1$, (2.11) and (2.19) above give us that

$$|m_0(2^{-j}\xi)| \geq |\widehat{\varphi}(\xi)| \geq \min\{C_{\xi_0}, \ldots, C_{\xi_n}\} = C > 0,$$

and (2.15) holds.

The converse is more elaborate. Suppose that $m_0 \in H^\alpha(\mathbb{T})$ and that (2.2) and (2.15) hold. Then, by Lemma 2.12, the infinite product $\prod_{j=1}^\infty m_0(\frac{\xi}{2^j})$ converges uniformly on compact sets and, therefore, the function

$$\widehat{\varphi}(\xi) \equiv \prod_{j=1}^\infty m_0(\frac{\xi}{2^j}), \quad a.e.\, \xi \in \mathbb{R}. \tag{2.20}$$

is well-defined and continuous in $\mathbb{R}$. In particular, the scaling equation (3.4) in Chapter 1 is satisfied and, since $m_0(0) = 1$, we also have $\widehat{\varphi}(0) = 1$. It is a well-known fact, and not very difficult to prove, that whenever $\widehat{\varphi}$ is defined as in (2.20) (provided the infinite product converges, at least, for $a.e.\, \xi \in \mathbb{R}$), then $\widehat{\varphi} \in L^2(\mathbb{R})$ and $\|\widehat{\varphi}\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi}$[5]. Thus, we can assert that φ exists and belongs to $L^2(\mathbb{R})$.

We claim that φ is a scaling function of some MRA. To prove this we only need to check that $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2k\pi)|^2 = 1$, $a.e.\, \xi \in \mathbb{R}$, or, equivalently (as we pointed out

[5] This essentially follows by Fatou's lemma and an iteration of condition (3.7) in Chapter 1 on m_0 (see, e.g., Lemma 6.2.1 in [DAUB] or Proposition 3.9 in Chapter 2 of [HW]).

right before (3.6) of Chapter 1), that $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$. It is here where we need to use condition (2.15). Following Cohen's proof, let K be the compact set in (2.15) and define the "truncated product"

$$\hat{f}_n(\xi) = \chi_{2^n K}(\xi) \prod_{j=1}^{n} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R},\ n = 1, 2, \ldots$$

Note that $f_n \in L^2(\mathbb{R})$, $n = 1, 2, \ldots$, and $\hat{f}_n(\xi) \to \widehat{\varphi}(\xi)$, $\forall \xi \in \mathbb{R}$, because K is compact and $0 \in \overset{\circ}{K}$. It is not difficult to show (see, e.g., the proof of (2.13) in [COH] or (4.11) in Chapter 7 of [HW]) that conditions (3.7) and $K \sim_{2\pi} \mathbb{T}$ imply that $\{f_n(\cdot - \ell) \mid \ell \in \mathbb{Z}\}$ is an orthonormal system for $L^2(\mathbb{R})$ or, equivalently, that

$$\int_{\mathbb{R}} |\hat{f}_n(\xi)|^2 \, e^{i\ell\xi} \, d\xi = 2\pi \, \delta_{0,\ell}, \ \ \ell \in \mathbb{Z}.$$

Since $\widehat{\varphi} \in L^2(\mathbb{R})$, if we could prove the majorization

$$|\hat{f}_n(\xi)| \leq C \, |\widehat{\varphi}(\xi)|, \quad a.e. \, \xi \in \mathbb{R}, \tag{2.21}$$

then, the Dominated Convergence Theorem will let us conclude that

$$\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 \, e^{i\ell\xi} \, d\xi = \lim_{n \to \infty} \int_{\mathbb{R}} |\hat{f}_n(\xi)|^2 \, e^{i\ell\xi} \, d\xi = 2\pi \, \delta_{0,\ell}, \ \ \ell \in \mathbb{Z},$$

and, using Proposition 3.12 in Chapter 1, our claim would be established. Now, to show (2.21) note that, by Cohen's condition, and as a consequence of Lemma 2.12, $\widehat{\varphi}$ is continuous and non-zero in K. Therefore, we can find a positive constant C such that $|\widehat{\varphi}(\xi)| \geq C$, for all $\xi \in K$. Then, given $n = 1, 2, \ldots$

$$|\hat{f}_n(\xi)| = \chi_{2^n K}(\xi) \prod_{j=1}^{n} |m_0(2^{-j}\xi)| \leq \frac{|\widehat{\varphi}(2^{-n}\xi)|}{C} \prod_{j=1}^{n} |m_0(2^{-j}\xi)| = \frac{1}{C} \, |\widehat{\varphi}(\xi)|, \quad \forall \xi \in \mathbb{R}.$$

This establishes (2.21) and proves our claim.

We are now left with the most technical part of the theorem, namely, to show that $\widehat{\varphi} \in H^\alpha(\mathbb{R})$. To do this, we adapt an argument that can be found in Lemarié's book (cf. Theorem 1 in Chapter 4 of [KAH-LEM]), that provides the proof of the following lemma. We point out to the reader that Cohen's condition does not play any role here.

LEMMA 2.22 *If $m_0 \in H^\alpha(\mathbb{T})$ satisfies (2.2), then $\widehat{\varphi}$ defined as in (2.20) belongs to $H^\alpha(\mathbb{R})$. Moreover, there exists a constant $C = C(\alpha) > 0$ such that*

$$\|\widehat{\varphi}\|_{H^\alpha(\mathbb{R})} \leq C \, \|m_0\|^{3+\alpha}_{H^\alpha(\mathbb{T})} \,. \tag{2.23}$$

PROOF:

We already know that $\widehat{\varphi} \in L^2(\mathbb{R})$ and $\|\widehat{\varphi}\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi} = \|m_0\|_{L^2(\mathbb{T})}$. In order to show that $\widehat{\varphi} \in H^\alpha(\mathbb{R})$ we use the definition of Sobolev space given in (1.4) (or (1.6)). Let us fix a function $w \in C_c^\infty(-2\pi, 2\pi)$ such that $0 \leq w \leq 1$ and $w|_{[-\pi,\pi]} \equiv 1$, and define the truncated product

$$P^n w(\xi) = w(2^{-n}\xi) \prod_{j=1}^{n} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R},\ n = 1, 2, \ldots \tag{2.24}$$

Note that $P^n w \in C(\mathbb{R})$ and $P^n w(\xi) \to \widehat{\varphi}(\xi)$, for all $\xi \in \mathbb{R}$. One can consider P as an operator that maps functions f, in an appropriate domain, into $P(f)(\xi) = m_0(\xi/2) f(\xi/2)$, $\xi \in \mathbb{R}$. This notation is going to be extremely useful as the following lemma shows:

LEMMA 2.25 *Let $\mathcal{H}$ be the normed space defined by*

$$\mathcal{H} = \{\, f\colon \mathbb{R} \to \mathbb{C} \text{ measurable} \mid \|f\|_*^2 = \operatorname{ess-sup}_{\xi \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |f(\xi + 2k\pi)|^2 < \infty \,\}. \tag{2.26}$$

Let τ be a 2π-periodic measurable function and let $P = P_\tau$ be the operator defined at $f \in \mathcal{H}$ by

$$(Pf)(\xi) = \tau(\xi/2) f(\xi/2), \quad \xi \in \mathbb{R}. \tag{2.27}$$

Then,

(I) *For every $f \in \mathcal{H}$*

$$\|Pf\|_* \leq \operatorname{ess-sup}_{\xi \in \mathbb{T}} \left[\, |\tau(\xi)|^2 + |\tau(\xi + \pi)|^2 \,\right]^{\frac{1}{2}} \|f\|_*. \tag{2.28}$$

(*II*) *If* $\tau \in L^2(\mathbb{T})$, *then* P *maps* $\mathcal{H}$ *into* $L^2(\mathbb{R})$, *and for every* $f \in \mathcal{H}$

$$\|Pf\|_{L^2(\mathbb{R})} \leq \sqrt{2}\,\|\tau\|_{L^2(\mathbb{T})}\,\|f\|_*. \tag{2.29}$$

(*III*) *If* $\tau \in L^\infty(\mathbb{T})$, *then* P *maps* $L^2(\mathbb{R})$ *into* $L^2(\mathbb{R})$, *and for every* $f \in L^2(\mathbb{R})$

$$\|Pf\|_{L^2(\mathbb{R})} \leq \sqrt{2}\,\|\tau\|_{L^\infty(\mathbb{T})}\,\|f\|_{L^2(\mathbb{R})}. \tag{2.30}$$

Proof:

The proof of this lemma is easy. Let us first establish (I). For $a.e.\,\xi \in \mathbb{R}$ we have

$$\sum_{k\in\mathbb{Z}} |(Pf)(\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |\tau(\frac{\xi}{2} + k\pi)|^2\,|f(\frac{\xi}{2} + k\pi)|^2 =$$

$$(\sum_{k\in\mathbb{Z}} |f(\frac{\xi}{2} + 2k\pi)|^2\,)\,|\tau(\frac{\xi}{2})|^2 + (\sum_{k\in\mathbb{Z}} |f(\frac{\xi}{2} + \pi + 2k\pi)|^2\,)\,|\tau(\frac{\xi}{2} + \pi)|^2 \leq$$

$$\|f\|_*^2\,(\,|\tau(\frac{\xi}{2})|^2 + |\tau(\frac{\xi}{2} + \pi)|^2\,) \leq \|f\|_*^2\,\text{ess-}\sup_{\eta\in\mathbb{T}}(\,|\tau(\eta)|^2 + |\tau(\eta + \pi)|^2\,).$$

This shows (2.28). To obtain the second inequality, note that

$$\int_{\mathbb{R}} |(Pf)(\xi)|^2\,d\xi = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} |\tau(\frac{\xi}{2} + k\pi)|^2\,|f(\frac{\xi}{2} + k\pi)|^2\,d\xi$$

$$= \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} |\tau(\frac{\xi}{2})|^2\,|f(\frac{\xi}{2} + 2k\pi)|^2\,d\xi + \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} |\tau(\frac{\xi}{2} + \pi)|^2\,|f(\frac{\xi}{2} + \pi + 2k\pi)|^2\,d\xi$$

$$\leq\; 2\,\|\tau\|_{L^2(\mathbb{T})}^2\,\|f\|_*^2,$$

which is (2.29), Finally, (III) is even simpler because

$$\int_{\mathbb{R}} |(Pf)(\xi)|^2\,d\xi = \int_{\mathbb{R}} |\tau(\frac{\xi}{2})|^2\,|f(\frac{\xi}{2})|^2\,d\xi \leq 2\,\|\tau\|_{L^\infty(\mathbb{T})}^2\,\|f\|_{L^2(\mathbb{R})}^2.$$

This proves (2.30) and establishes Lemma 2.25.

□

We return to the proof of Lemma 2.22. We shall consider different cases.

<u>*Case 1*</u>: $\frac{1}{2} < \alpha < 1$.

We need to show that

$$\omega_\alpha(\widehat{\varphi}) = \left[\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\varphi}(\xi+\eta) - \widehat{\varphi}(\xi)|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} \leq C\, \|m_0\|_{H^\alpha(\mathbb{T})}.$$

By Fatou's Lemma and the fact that $P^n w(\xi) \to \widehat{\varphi}(\xi)$, for all $\xi \in \mathbb{R}$, it is enough to see that

$$\sup_{n\geq 1} \omega_\alpha(P^n w) = \sup_{n\geq 1}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} |P^n w(\xi) - P^n w(\xi+\eta)|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} \leq C\, \|m_0\|_{H^\alpha(\mathbb{T})}. \quad (2.31)$$

Let $n \geq 1$, then:

$$\omega_\alpha(P^n w) \leq$$

$$\sum_{j=1}^{n}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \prod_{\ell=1}^{j-1} |m_0(\frac{\xi+\eta}{2^\ell})|^2\, |m_0(\frac{\xi}{2^j}) - m_0(\frac{\xi+\eta}{2^j})|^2 \prod_{\ell=j+1}^{n} |m_0(\frac{\xi}{2^\ell})|^2\, |w(\frac{\xi}{2^n})|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}}$$

$$+\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \prod_{\ell=1}^{n} |m_0(\frac{\xi+\eta}{2^\ell})|^2\, |w(\frac{\xi}{2^n}) - w(\frac{\xi+\eta}{2^n})|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} = \sum_{j=1}^{n} A_j + B.$$

The products $\prod |m_0(\frac{\xi+\eta}{2^\ell})|^2$ can all be majorized by 1 in both A_j and B. To deal with the rest of the integrands we treat each case separately. After a change of variables, B becomes

$$B \leq 2^{-n(\alpha-\frac{1}{2})}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} |w(\xi) - w(\xi+\eta)|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} \leq 2^{-n(\alpha-\frac{1}{2})} \|w\|_{H^\alpha(\mathbb{R})},$$

which is bounded because w is a C^∞ function with compact support (and, therefore, belongs to all the Sobolev spaces $H^\alpha(\mathbb{R})$). To estimate the A_j's, we change variables first and, then, periodize the integral with respect to ξ. Thus

$$A_j \leq 2^{-j(\alpha-\frac{1}{2})}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} |m_0(\xi) - m_0(\xi+\eta)|^2 \prod_{\ell=1}^{n-j} |m_0(\frac{\xi}{2^\ell})|^2\, |w(\frac{\xi}{2^{n-j}})|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} =$$

$$2^{-j(\alpha-\frac{1}{2})}\left[\int_{\mathbb{R}}\int_{\mathbb{T}} |m_0(\xi) - m_0(\xi+\eta)|^2 \left(\sum_{k\in\mathbb{Z}} |P^{n-j} w(\xi+2k\pi)|^2\right) \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} \leq$$

$$2^{-j(\alpha-\frac{1}{2})} \left\| P^{n-j}(w) \right\|_* \left\{ \omega_\alpha(m_0)^2 + 4 \left\| m_0 \right\|^2_{L^2(\mathbb{T})} \int_{|\eta|\geq\pi} \frac{d\eta}{|\eta|^{1+2\alpha}} \right\}^{\frac{1}{2}} \leq$$
$$\leq C\, 2^{-j(\alpha-\frac{1}{2})} \left\| m_0 \right\|_{H^\alpha(\mathbb{T})},$$

since, by (2.28) and (2.2), $\|P^{n-j}(w)\|_* \leq \|w\|_*$, for any $0 \leq j \leq n < \infty$, and $\|w\|^2_* = \|\sum_{k\in\mathbb{Z}} |w(\cdot + 2k\pi)|^2\|_\infty \leq 2$ because the series has at most two non-vanishing terms.

Now, using that $\alpha > \frac{1}{2}$, and summing up the A_j's, we obtain

$$\omega_\alpha(P^n w) \leq C \sum_{j=1}^{n} 2^{-j(\alpha-\frac{1}{2})} \left\| m_0 \right\|_{H^\alpha(\mathbb{T})} + \left\| w \right\|_{H^\alpha(\mathbb{R})} 2^{-n(\alpha-\frac{1}{2})} \leq C' \left\| m_0 \right\|_{H^\alpha(\mathbb{T})},$$

as long as we take the constant C' on the right hand side large enough (so that $\|w\|_{H^\alpha(\mathbb{R})} \leq \sqrt{2}\,\frac{C'}{\sqrt{2}} = \|m_0\|_{L^2(\mathbb{T})}\,\frac{C'}{\sqrt{2}}$). This establishes (2.31) and completes the proof of case 1.

<u>*Case 2*</u>: $\alpha = k \in \mathbb{Z}^+$.

We need to show that $D^{(k)}(\widehat{\varphi}) \in L^2(\mathbb{R})$ and that

$$\|D^{(k)}(\widehat{\varphi})\|_{L^2(\mathbb{R})} \leq C \left\| m_0 \right\|^{k+1}_{H^k(\mathbb{T})}. \tag{2.32}$$

In order to do so, we will first find a formula for $D^{(k)}(\widehat{\varphi})$. Given $0 \leq h \leq k$, we introduce the following notation:

$$m_h(\xi) = \left(\frac{d^h m_0}{d\xi^h}\right)(\xi), \quad a.e.\, \xi \in \mathbb{R}, \tag{2.33}$$

where the derivatives are taken in the sense of distributions. Note that the m_h's are 2π-periodic functions ($a.e.$ in $\mathbb{R}$) and that, when $0 \leq h \leq k-1$, they are actually in $C^{k-h}(\mathbb{T})$ (see Remark 1.29, after Theorem 1.25).

Lemma 2.34 *Let $k \in \mathbb{Z}^+$. If $m_0 \in H^k(\mathbb{T})$ satisfies (2.2) and $\widehat{\varphi}$ is defined by (2.20), we can write the k^{th} distributional derivative of $\widehat{\varphi}$, $D^{(k)}(\widehat{\varphi})$, as:*

$$D^{(k)}(\widehat{\varphi})(\xi) = \sum_{L=1}^{\infty} \sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}^+)^k \\ \sup_{1\leq i\leq k} \ell_i = L}} \left(\frac{1}{2}\right)^{\sum_{i=1}^k \ell_i} \prod_{j=1}^{L} m_{\sum_{i=1}^k \delta_{j,\ell_i}}(2^{-j}\xi)\, \widehat{\varphi}(2^{-L}\xi), \tag{2.35}$$

where the series converges absolutely for $a.e.\, \xi \in \mathbb{R}$.

Proof: We briefly explain the notation in (2.35) for the k^{th}-derivative of the infinite product $\prod_{j\geq 1} m_0(2^{-j}\xi)$. For a fixed $L \geq 1$ and a given multi-index $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_k)$ we spread the k derivatives among the L first terms of the infinite product, where each ℓ_i indicates that the ℓ_i^{th}-term is differentiated one time. The condition $\sup_{1\leq i\leq k} \ell_i = L$ guarantees that the L^{th}-term is always considered and, therefore, that there are no repetitions in the process. However, and since all the derivatives are taken in the sense of distributions, a more rigorous proof is required. We first show that the series in (2.35) converges absolutely $a.e.$ Take a compact set $[-K, K]$ of $\mathbb{R}$, where $K > 0$. First, we shall show that

$$\sum_{L=1}^{\infty} \sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}^+)^k \\ \sup_{1\leq i\leq k}\ell_i = L}} \left(\frac{1}{2}\right)^{\sum_{i=1}^k \ell_i} \int_{[-K,K]} \left| \prod_{j=1}^{L} m_{\sum_{i=1}^k \delta_{j,\ell_i}}(2^{-j}\xi) \right| |\widehat{\varphi}(2^{-L}\xi)| \, d\xi < \infty. \quad (2.36)$$

This would imply, by Tonelli's Theorem, that the series in (2.35) converges absolutely for $a.e.\,\xi \in [-K, K]$ (and also in $L^1([-K, K])$) to a function in $L^1([-K, K])$. Given a multi-index $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_k) \in (\mathbb{Z}^+)^k$, and $j \geq 1$, we denote

$$\varepsilon_j = \varepsilon_j(\boldsymbol{\ell}) = \sum_{i=1}^{k} \delta_{j,\ell_i}. \quad (2.37)$$

Fix $L \geq 1$ and consider a multi-index $\boldsymbol{\ell} \in (\mathbb{Z}^+)^k$ such that $\sup_{1\leq i\leq k} \ell_i = L$, but $\boldsymbol{\ell} \neq (L, \ldots, L)$. In particular, we have that $\varepsilon_j < k$, and at most k of the ε_j's are non-zero, for $j = 1, \ldots, L$ (that is, the k derivatives of the infinite product are not all concentrated in just one of the terms $m_0(2^{-j}\xi)$). Then, using $|m_0|, |\widehat{\varphi}| \leq 1$ and $\|m_h\|_\infty \leq C \|m_0\|_{H^k(\mathbb{T})}$, $1 \leq h \leq k-1$, we obtain that

$$\left| \prod_{j=1}^{L} m_{\varepsilon_j}(2^{-j}\xi) \right| |\widehat{\varphi}(2^{-L}\xi)| \leq C \|m_0\|_{H^k(\mathbb{T})}^k.$$

If, on the contrary, we choose $\boldsymbol{\ell} = (L, \ldots, L)$ (that is, all the derivatives are concentrated in the term $m_0(2^{-L}\xi)$), then, the obvious estimates $|m_0|, |\widehat{\varphi}| \leq 1$ and Hölder's inequality give us:

$$\int_{[-K,K]} \left| \prod_{j=1}^{L-1} m_0(2^{-j}\xi) \right| |m_k(2^{-L}\xi)| \, |\widehat{\varphi}(2^{-L}\xi)| \, d\xi \leq \int_{[-K,K]} |m_k(2^{-L}\xi)| \, d\xi \leq$$

$$(2K)^{\frac{1}{2}}\left[\int_{[-K,K]}|m_k(2^{-L}\xi)|^2\,d\xi\right]^{\frac{1}{2}}\leq(2K)^{\frac{1}{2}}\,2^{L/2}\left[\int_{[-K,K]}|m_k(\eta)|^2\,d\eta\right]^{\frac{1}{2}}\leq$$

$$\leq(2K)^{\frac{1}{2}}\,2^{L/2}\,(K+2)^{\frac{1}{2}}\,\|m_k\|_{L^2(\mathbb{T})}=C_K\,\|m_0\|_{H^k(\mathbb{T})}\,2^{L/2}.$$

Now, (2.36) becomes

$$\sum_{L=1}^{\infty}\sum_{\sup\ell_i=L}\left(\frac{1}{2}\right)^{\sum_{i=1}^k\ell_i}\int_{[-K,K]}\left|\prod_{j=1}^{L}m_{\varepsilon_j}(2^{-j}\xi)\right||\hat{\varphi}(2^{-L}\xi)|\,d\xi\leq C\sum_{L=1}^{\infty}\frac{L^k}{2^{L/2}}<\infty.$$

This shows that the series in (2.35) converges absolutely, for $a.e.\,\xi\in\mathbb{R}$, (and also in $L^1_{loc}(\mathbb{R})$) to a locally integrable function.

Now, take $w\in C_c^\infty(-2\pi,2\pi)$ such that $0\leq w\leq 1$ and $w|_{[-\pi,\pi]}\equiv 1$, and consider $P^n w(\xi)$ as in (2.24). Note that $P^n w(\xi)\to\hat{\varphi}(\xi)$ uniformly in compact sets of $\mathbb{R}$. Indeed, if $[-K,K]\subset\mathbb{R}$ and n_0 is large enough, so that $2^{n_0}\pi>K$, then for all $n\geq n_0$ and $\xi\in[-K,K]$ we have that

$$P^n w(\xi)=w(2^{-n}\xi)\,\prod_{j=1}^{n}m_0(2^{-j}\xi)=\prod_{j=1}^{n}m_0(2^{-j}\xi),$$

and we know from Lemma 2.12 that $\prod_{j=1}^n m_0(2^{-j}\xi)$ converges uniformly to $\hat{\varphi}$ in $[-K,K]$. In particular, we have that $P^n w\to\hat{\varphi}$ converges in the sense of distributions. Thus, we must also have that $D^{(k)}(P^n w)$ converges to $D^{(k)}(\hat{\varphi})$ in the sense of distributions. Therefore, to complete the proof of the lemma it is enough to see that for every compact set $[-K,K]\subset\mathbb{R}$, the sequence $D^{(k)}(P^n w)$ converges in $L^1([-K,K])$ to the series in (2.35) (which we showed above was a function in $L^1([-K,K])$). So, fix such a compact set and take n_0 large enough, so that $2^{n_0}\pi>K$. Then, for all $n\geq n_0$ and $\xi\in[-K,K]$ we have that $2^{-n}\xi<\pi$ and, therefore,

$$\begin{aligned}D^{(k)}(P^n w)(\xi)&=\sum_{L=1}^{n}\sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}^+)^k\\ \sup\ell_i=L}}\left(\frac{1}{2}\right)^{\sum_{i=1}^k\ell_i}\prod_{j=1}^{n}m_{\varepsilon_j}(2^{-j}\xi)\,w(2^{-n}\xi)\;+0\\&=\sum_{L=1}^{n}\sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}^+)^k\\ \sup\ell_i=L}}\left(\frac{1}{2}\right)^{\sum_{i=1}^k\ell_i}\prod_{j=1}^{L}m_{\varepsilon_j}(2^{-j}\xi)\;\prod_{j=L+1}^{n}m_0(2^{-j}\xi).\end{aligned}$$

On the other hand, let us denote the partial sums of the series in (2.35) by

$$T_n(\xi) = \sum_{L=1}^{n} \sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}^+)^k \\ \sup \ell_i = L}} \left(\frac{1}{2}\right)^{\sum_{i=1}^k \ell_i} \prod_{j=1}^{L} m_{\varepsilon_j}(2^{-j}\xi)\, \widehat{\varphi}(2^{-L}\xi), \quad n \geq n_0, \ \xi \in \mathbb{R}.$$

Let $\varepsilon > 0$. Then, there exists a positive integer $n_1 \geq n_0$ such that for every $n \geq n_1$,

$$\sup_{\xi\in[-K,K]} \left| \prod_{j=1}^{n} m_0(2^{-j}\xi) - \widehat{\varphi}(\xi) \right| < \varepsilon.$$

Let $n_2 \geq n_1$ be such that $\sum_{L=N}^{M} \frac{L^k}{2^{L/2}} < \varepsilon$, for every $M \geq N \geq n_2$. We claim that

$$\int_{[-K,K]} |D^{(k)}(P^n w)(\xi) - T_n(\xi)|\, d\xi < C_K\, \varepsilon, \quad \text{for all } n \geq 2n_2, \tag{2.38}$$

where C_K is a constant that does not depend on n or ε. Indeed,

$$\int_{[-K,K]} |D^{(k)}(P^n w)(\xi) - T_n(\xi)|\, d\xi \leq$$

$$\leq \sum_{L=1}^{n} \frac{1}{2^L} \left\{ \int_{[-K,K]} |m_k(2^{-L}\xi)| \left| \prod_{j=L+1}^{n} m_0(2^{-j}\xi) - \widehat{\varphi}(2^{-L}\xi) \right| d\xi + \right.$$

$$\left. + \sum_{\substack{\boldsymbol{\ell}\neq(L,\ldots,L) \\ \sup \ell_i = L}} \int_{[-K,K]} C\, \|m_0\|^k_{H^k(\mathbb{T})} \left| \prod_{j=L+1}^{n} m_0(2^{-j}\xi) - \widehat{\varphi}(2^{-L}\xi) \right| d\xi \right\}$$

$$\leq C \sum_{L=1}^{n} \frac{1}{2^L} \left\{ 2^L \int_{[-2^{-L}K, 2^{-L}K]} |m_k(\eta)| \left| \prod_{j=1}^{n-L} m_0(2^{-j}\eta) - \widehat{\varphi}(\eta) \right| d\eta + \right.$$

$$\left. + L^k\, 2^L \int_{[-2^{-L}K, 2^{-L}K]} \left| \prod_{j=1}^{n-L} m_0(2^{-j}\eta) - \widehat{\varphi}(\eta) \right| d\eta \right\} = C \left[\sum_{L=1}^{n/2} \quad + \sum_{L=n/2}^{n} \quad \right]$$

$$\leq C \sum_{L=1}^{n/2} \left\{ \varepsilon\, (2K2^{-L})^{\frac{1}{2}} \left[\int_{[-K,K]} |m_k(\eta)|^2\, d\eta \right]^{\frac{1}{2}} + L^k\, \varepsilon\, 2K\, 2^{-L} \right\}$$

$$+ C \sum_{L=n/2}^{n} \left\{ 2\, (2K2^{-L})^{\frac{1}{2}} \left[\int_{[-K,K]} |m_k(\eta)|^2\, d\eta \right] + 2\, L^k\, 2K\, 2^{-L} \right\} \Bigg\}$$

$$\leq C_K \left(\sum_{L=1}^{\infty} \frac{L^k}{2^{L/2}} \right) \varepsilon + C_K \sum_{L=n/2}^{n} \frac{L^k}{2^{L/2}} < C_K\, \varepsilon.$$

This shows (2.38) and since T_n converges in $L^1([-K,K])$ to the series in (2.35), we have completed the proof of Lemma 2.34. $\square$

Now, (2.32) will follow right away from Lemma 2.34. Suppose we are given a multi-index $\boldsymbol{\ell} \in (\mathbb{Z}^+)^k$, and a positive integer $j \geq 1$, and let $\varepsilon_j = \varepsilon_j(\boldsymbol{\ell})$ denote the same number as in (2.37). We define the operators $P_{\varepsilon_j} = P_{m_{\varepsilon_j}}$ as in (2.27). Note that, by case 1 above, $\widehat{\varphi}$ belongs, say, to $H^{\frac{3}{4}}(\mathbb{R})$. Thus, by Corollary 1.17, $\widehat{\varphi}$ also belongs to the space $\mathcal{H}$ defined in (2.26). Moreover,

$$\|\widehat{\varphi}\|_* \leq C\,\|\widehat{\varphi}\|_{H^{\frac{3}{4}}(\mathbb{R})} \leq C\,\|m_0\|_{H^{\frac{3}{4}}(\mathbb{T})} \leq C\,\|m_0\|_{H^k(\mathbb{T})}.$$

We can use Lemma 2.25 (II) and (III) to obtain:

$$\begin{aligned}
\|D^{(k)}(\widehat{\varphi})\|_{L^2(\mathbb{R})} &\leq \sum_{L=1}^{\infty} \sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}^+)^k \\ \sup \ell_i = L}} \left(\frac{1}{2}\right)^{\sum_{i=1}^k \ell_i} \|P_{\varepsilon_1}\cdots P_{\varepsilon_L}\widehat{\varphi}\|_{L^2(\mathbb{R})} \\
&\leq \sum_{L=1}^{\infty} \frac{1}{2^L}\left\{ \|P_0\cdots P_0 P_k\widehat{\varphi}\|_{L^2(\mathbb{R})} + \sum_{\substack{\boldsymbol{\ell}\neq(L,\dots,L) \\ \sup \ell_i = L}} \|P_{\varepsilon_1}\cdots P_{\varepsilon_L}\widehat{\varphi}\|_{L^2(\mathbb{R})} \right\} \\
&\leq \sum_{L=1}^{\infty} \frac{1}{2^L}\left\{ 2^{L/2}\sqrt{2}\,\|m_k\|_{L^2(\mathbb{T})}\,\|\widehat{\varphi}\|_* + L^k\,2^{L/2}\,C\,\|m_0\|^k_{H^\alpha(\mathbb{T})}\,\|\widehat{\varphi}\|_{L^2(\mathbb{R})} \right\} \\
&\leq C\left(\sum_{L=1}^{\infty} \frac{L^k}{2^{L/2}}\right)\|m_0\|^{k+1}_{H^\alpha(\mathbb{T})}.
\end{aligned}$$

This shows (2.32) and completes the proof of case 2.

<u>*Case 3*</u>: $\alpha = k+\varepsilon, \quad 0<\varepsilon<1,\ k\in\mathbb{Z}^+$.

We have to show that

$$\omega_\varepsilon(D^{(k)}\widehat{\varphi}) = \left[\int_{\mathbb{R}}\int_{\mathbb{R}} |D^{(k)}\widehat{\varphi}(\xi+\eta) - D^{(k)}\widehat{\varphi}(\xi)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\right]^{\frac{1}{2}} \leq C\,\|m_0\|^{\alpha+3}_{H^\alpha(\mathbb{T})}. \tag{2.39}$$

First of all, note that by using (2.35) we can write

$$\begin{aligned}
&D^{(k)}\widehat{\varphi}(\xi) - D^{(k)}\widehat{\varphi}(\xi+\eta) = \\
&\sum_{L=1}^{\infty} \sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}^+)^k \\ \sup \ell_i = L}} \left(\frac{1}{2}\right)^{\sum_{i=1}^k \ell_i} \left[\sum_{h=1}^{L}\left\{\prod_{j=1}^{h-1} m_{\varepsilon_j}(2^{-j}(\xi+\eta))\,(m_{\varepsilon_h}(2^{-h}\xi) - m_{\varepsilon_h}(2^{-h}(\xi+\eta)))\,\times\right.\right. \\
&\qquad\qquad \left.\times \prod_{j=h+1}^{L} m_{\varepsilon_j}(2^{-j}\xi)\,\widehat{\varphi}(2^{-L}\xi)\right\} +
\end{aligned}$$

$$+ \prod_{j=1}^{L} m_{\varepsilon_j}(2^{-j}(\xi+\eta))\,(\widehat{\varphi}(2^{-L}\xi) - \widehat{\varphi}(2^{-L}(\xi+\eta)))\Big].$$

Now we repeat the argument in case 1 with some minor modifications to obtain:

$$\omega_\varepsilon(D^{(k)}\widehat{\varphi}) \le$$

$$\sum_{L=1}^{\infty} \frac{1}{2^L}\Big\{\sum_{h=1}^{L}\Big[2^{-h\varepsilon}\,2^{L/2}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} |m_0(2^{L-h}\xi) - m_0(2^{L-h}\xi+\eta)|^2\,|m_k(\xi)|^2\,|\widehat{\varphi}(\xi)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}$$

$$+L^k\,2^{-h(\varepsilon-\frac{1}{2})}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}\prod_{j=1}^{h-1}\|m_{\varepsilon_j}\|_\infty^2\,|m_{\varepsilon_h}(\xi) - m_{\varepsilon_h}(\xi+\eta)|^2\,|P_{\varepsilon_{h+1}}\cdots P_{\varepsilon_L}\widehat{\varphi}(\xi)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}\Big]$$

$$+2^{-L(\varepsilon-\frac{1}{2})}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}|m_k(\xi)|^2\,|\widehat{\varphi}(\xi)-\widehat{\varphi}(\xi+\eta)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}$$

$$+\,L^k\,2^{-L(\varepsilon-\frac{1}{2})}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}\prod_{j=1}^{L}\|m_{\varepsilon_1}\|_\infty^2\,|\widehat{\varphi}(\xi) - \widehat{\varphi}(\xi+\eta)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}\Big\}$$

$$\le \sum_{L=1}^{\infty}\frac{1}{2^L}\Big\{\sum_{h=1}^{L}2^{-h\varepsilon}\Big[2^{L/2}\Big(\int_{|\eta|>1}4\,\frac{d\eta}{|\eta|^{1+2\varepsilon}} + \|m_0\|^2_{\Lambda^{\frac{1}{2}+\frac{\varepsilon}{2}}(\mathbb{T})}\int_{|\eta|\le 1}|\eta|^{1+\varepsilon}\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}\times$$

$$\times\Big(\int_{\mathbb{T}}|m_k(\xi)|^2\,d\xi\Big)^{\frac{1}{2}}\,\|\widehat{\varphi}\|_*$$

$$+\,C\,L^k\,2^{-h(\varepsilon-\frac{1}{2})}\,2^{k/2}\,\|m_0\|^k_{H^k(\mathbb{T})}\,\|\widehat{\varphi}\|_*\Big(\omega_\varepsilon(m_{\varepsilon_h}) + \int_{|\eta|>\pi}\frac{d\eta}{|\eta|^{1+2\varepsilon}}\Big)\Big]$$

$$+\,2^{-L(\varepsilon-\frac{1}{2})}\Big(\int_{\mathbb{T}}|m_k(\xi)|^2\,d\xi\Big)^{\frac{1}{2}}\Big(4\,\|\widehat{\varphi}\|_*^2\int_{|\eta|>1}\frac{d\eta}{|\eta|^{1+2\varepsilon}} + \int_{|\eta|<1}\|\widehat{\varphi}(\cdot+\eta)-\widehat{\varphi}(\cdot)\|_*^2\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}$$

$$+\,C\,L^k\,2^{-L(\varepsilon-\frac{1}{2})}\,\|m_0\|^k_{H^k(\mathbb{T})}\Big(4\,\|\widehat{\varphi}\|_*^2\int_{|\eta|>1}\frac{d\eta}{|\eta|^{1+2\varepsilon}} + \int_{|\eta|<1}\|\widehat{\varphi}(\cdot+\eta)-\widehat{\varphi}(\cdot)\|_*^2\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}\Big\}$$

$$\le\; C\,\|m_0\|_\alpha^{k+2}\,\sum_{L=1}^{\infty}\frac{L^k}{2^{L/2}} \;+\; C\,\|m_0\|_\alpha^k\Big(\sum_{L=1}^{\infty}\frac{L^k}{2^{L/2}}\Big)\Big(\int_{|\eta|<1}\|\widehat{\varphi}(\cdot+\eta) - \widehat{\varphi}(\cdot)\|_*^2\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}\Big)^{\frac{1}{2}}.$$

We only need to estimate

$$\int_{|\eta|<1}\|\widehat{\varphi}(\cdot+\eta) - \widehat{\varphi}(\cdot)\|_*^2\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}. \tag{2.40}$$

Suppose that $0<\varepsilon<\frac{1}{2}$. Then, by Corollary 1.38 with $\gamma=1$, we have that

$$\|\widehat{\varphi}(\cdot+\eta) - \widehat{\varphi}(\cdot)\|_*^2 \le\; C\,|\eta|\,\|\widehat{\varphi}\|^2_{H^1(\mathbb{R})}.$$

Introducing this last estimation in (2.40), we obtain

$$\int_{|\eta|<1} \|\widehat{\varphi}(\cdot+\eta)-\widehat{\varphi}(\cdot)\|_*^2 \frac{d\eta}{|\eta|^{1+2\varepsilon}} \le C\,\|\widehat{\varphi}\|^2_{H^1(\mathbb{R})} \int_{|\eta|<1} \frac{d\eta}{|\eta|^{2\varepsilon}} \le C\,\|m_0\|^4_{H^\alpha(\mathbb{T})}.$$

This shows (2.39) when $0<\varepsilon<\frac{1}{2}$.

Suppose now that $\frac{1}{2} \le \varepsilon < 1$. Then, we can use Corollary 1.38 with γ such that $2\varepsilon<\gamma<2$. This gives us

$$\int_{|\eta|<1} \|\widehat{\varphi}(\cdot+\eta)-\widehat{\varphi}(\cdot)\|_*^2 \frac{d\eta}{|\eta|^{1+2\varepsilon}} \le C\,\|\widehat{\varphi}\|^2_{H^{\frac{1+\gamma}{2}}(\mathbb{R})} \int_{|\eta|<1} \frac{d\eta}{|\eta|^{1-(\gamma-2\varepsilon)}} \le C\,\|m_0\|^6_{H^\alpha(\mathbb{T})},$$

because $1<\frac{1+\gamma}{2}<\frac{3}{2}$ and we may use the case above. This shows (2.39) when $\frac{1}{2} \le \varepsilon < 1$ and completes the proof of Lemma 2.22 and, with this, the proof of Theorem 2.18.

□ □

Theorem 2.18 completely characterizes the set of functions in $H^\alpha(\mathbb{T})$ that are low-pass filters of α-localized scaling functions for $L^2(\mathbb{R})$. That is, we have found the range of the mapping M of Lemma 2.7. For convenience, we use the following notation:

Definition 2.41 *Let $\alpha>\frac{1}{2}$. We say that a 2π-periodic function m_0 belongs to the set $\mathcal{E}_\alpha$ if $m_0 \in H^\alpha(\mathbb{T})$ and satisfies (2.2) and Cohen's condition (2.15). We will consider $\mathcal{E}_\alpha$ as a topological subspace of $H^\alpha(\mathbb{T})$. That is, when we say that $m_n \to m_0$ in $\mathcal{E}_\alpha$, we mean that $m_n, m_0 \in \mathcal{E}_\alpha$, $n=1,2,\dots$, and $m_n \to m_0$ in $H^\alpha(\mathbb{T})$.*

Now, we can restate Theorem 2.18 in the following way.

Corollary 2.42 *Let $\alpha>\frac{1}{2}$. The mapping $M\colon \mathcal{S}_\alpha \to \mathcal{E}_\alpha$ defined in Lemma 2.7 is a bijection with inverse $N\colon \mathcal{E}_\alpha \to \mathcal{S}_\alpha$ given by*

$$N(m_0)(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R}. \tag{2.43}$$

We proved in Lemma 2.7 that M is continuous. Moreover, by (2.4) and (2.23), M and N are also *bounded* mappings. That is, they map bounded sets in $\mathcal{S}_\alpha$ into bounded sets in $\mathcal{E}_\alpha$, and conversely. The main question is the following: Can we conclude that N is also continuous or, equivalently, that M is open? Is M a homeomorphism of topological spaces? The answer is yes, although the proof is not obvious, as we show it in the next section (see Corollary 3.32 below). The most important consequence of this property is that the topology of the set of α-localized scaling functions is completely described by the topology of the much simpler topological space $\mathcal{E}_\alpha$. In particular, in §4 we shall show that $\mathcal{E}_\alpha$ satisfies a manifold-like condition and, further, that it is an arcwise connected topological space.

3 Convergence in the sets of filters $\mathcal{F}_\alpha$ and $\mathcal{E}_\alpha$

In this section we prove one of the main results of this dissertation; namely, that the map $M\colon \mathcal{S}_\alpha \to \mathcal{E}_\alpha$, defined in Lemma 2.7, is a homeomorphism of topological spaces. For this, we study how the convergence of a sequence of filters in $\mathcal{E}_\alpha$ will give us convergence in the sequence of scaling functions they generate in the space $\mathcal{S}_\alpha$. We start with a new definition.

DEFINITION 3.1 *Let $\alpha > \frac{1}{2}$. We say that a 2π-periodic function F belongs to the set $\mathcal{F}_\alpha$ if $F \in H^\alpha(\mathbb{T})$ and satisfies (2.2). As before, $\mathcal{F}_\alpha$ is considered a topological subspace of $H^\alpha(\mathbb{T})$.*

Note that, in particular, $\mathcal{F}_\alpha$ is a closed subset of $H^\alpha(\mathbb{T})$. This is just a consequence of Sobolev's Imbedding Theorem 1.25. Note, further, that $\mathcal{E}_\alpha \subset \mathcal{F}_\alpha$, and the functions in $\mathcal{F}_\alpha$ do not necessarily satisfy Cohen's condition (2.15). Indeed, Example 2 in §4 shows that $F(\xi) = \frac{1+e^{i3\xi}}{2} \in \mathcal{F}_\alpha$, but $F \notin \mathcal{E}_\alpha$. In fact, one can see that the set $\mathcal{E}_\alpha$ is not closed in $\mathcal{F}_\alpha$. The following example, taken from [BDW] clarifies this point.

EXAMPLE 1

Consider the family of functions

$$m_{(a,c)}(\xi) = \frac{1+e^{i\xi}}{2}\,(a + be^{i\xi} + ce^{i2\xi}), \quad \xi \in \mathbb{R}, \tag{3.2}$$

where a, b, c are real numbers such that

$$b = 1 - a - c \quad \text{and} \quad a^2 + c^2 = a + c. \tag{3.3}$$

This last condition can be interpreted as (a,c) being in a circle $C = \{(a,c) \in \mathbb{R}^2 \mid (a-\frac{1}{2})^2 + (c-\frac{1}{2})^2 = \frac{1}{2}\}$ (see Figure 3.1, below).

An easy computation shows that (2.2) holds for any of the filters $m_{(a,c)}$, when (a,b,c) is taken as in (3.3), and, therefore, that $m_{(a,c)} \in \mathcal{F}_\alpha$ for all such pairs (a,c).

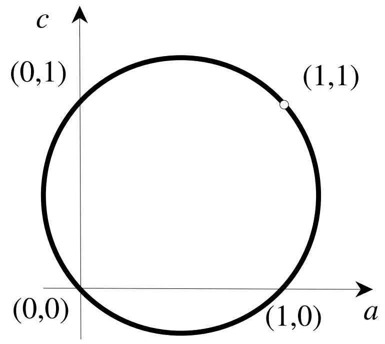

Figure 3.1: Circle C of low-pass filters in Example 1.

Note that when $(a, c) = (1, 1)$ we have

$$m_{(1,1)}(\xi) = \frac{1 + e^{i3\xi}}{2}$$

$$\textbf{CLAIM :} \quad m_{(a,c)} \in \mathcal{E}_\alpha, \quad \text{for all} \quad (a, c) \in C \setminus \{(1, 1)\}. \tag{3.4}$$

PROOF:

We need to show that Cohen's condition (2.15) holds for each of the $m_{(a,c)}$'s, when $(a, c) \in C \setminus \{(1, 1)\}$. Take the compact set $K = [-\pi, \pi]$. It is enough to show that

$$a + be^{i\xi} + ce^{i2\xi} \neq 0, \quad \text{for } \xi \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

when $b = 1 - a - c$ and $(a, c) \in C \setminus \{(1, 1)\}$. First of all note that, if $a = 0$, then c is either 0 or 1, but in both cases $be^{i\xi} + ce^{i2\xi} \neq 0$, $\xi \in \mathbb{R}$. Therefore, we need only consider the case when $a \neq 0$. Suppose that there exists a $\xi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $a + be^{i\xi} + ce^{i2\xi} = 0$. Then, $\xi \neq 0$ (otherwise, it would contradict $a + b + c = 1$). Thus, the roots of the polynomial $a + bx + cx^2$ must be $e^{i\xi}$ and $e^{-i\xi}$, given by the formulas:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

respectively. But then we have

$$1 = e^{i\xi} \cdot e^{-i\xi} = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{4ac}{4a^2} = \frac{c}{a},$$

which implies that $a = c$. Since $(a,c) \in C \setminus \{(1,1)\}$, necessarily $a = c = 0$, and this contradicts our assumption and shows the claim.

□

Note that $m_{(a,c)} \to m_{(1,1)}$ in $\mathcal{F}_\alpha$, when $(a,c) \to (1,1)$, $(a,c) \in C$. Indeed, since $m_{(a,c)}$ are all trigonometric polynomials, the convergence in $H^\alpha(\mathbb{T})$ (with the norm defined in (1.20)) follows from the convergence of each of the non-zero Fourier coefficients of $m_{(a,c)}$ to the corresponding coefficient of $m_{(1,1)}$, or, equivalently, from $(a,c) \to (1,1)$ in $\mathbb{R}^2$. This shows that $\mathcal{E}_\alpha$ is not closed in $\mathcal{F}_\alpha$.

On the other hand, one can show that $\mathcal{E}_\alpha$ is an open subset of $\mathcal{F}_\alpha$. This actually follows from the next lemma.

Lemma 3.5 *Let $\alpha > \frac{1}{2}$ and choose any $\beta \in (0,1)$ such that $\beta \le \alpha - \frac{1}{2}$. Suppose that $m_0 \in \mathcal{F}_\alpha$ and that K is a compact set on $\mathbb{R}$, and let $\varepsilon > 0$. Then, there exists a $\delta = \delta(\beta, \varepsilon, K) > 0$ such that for every $F \in \mathcal{F}_\alpha$ with $\|F - m_0\|_{\Lambda^\beta(\mathbb{T})} < \delta$, then*

$$\left|\prod_{j=1}^{\infty} F(2^{-j}\xi) - \prod_{j=1}^{\infty} m_0(2^{-j}\xi)\right| < \varepsilon, \quad \textit{for all } \xi \in K.$$

In particular, if $F_n \to m_0$ in $\mathcal{F}_\alpha$, then $\prod_{j=1}^\infty F_n(2^{-j}\xi) \to \prod_{j=1}^\infty m_0(2^{-j}\xi)$ uniformly on compact sets of $\mathbb{R}$.

Proof:

Let us denote by

$$\widehat{\varphi}_F(\xi) = \prod_{j=1}^{\infty} F(2^{-j}\xi) \quad \text{and} \quad \widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R}. \tag{3.6}$$

By Lemma 2.12, both infinite products converge uniformly on compact sets and represent continuous functions in $\mathbb{R}$. Note that, for every $\xi \in \mathbb{R}$,

$$\widehat{\varphi}_F(\xi) - \widehat{\varphi}(\xi) = \sum_{\ell=1}^{\infty} [\prod_{j=1}^{\ell-1} m_0(2^{-j}\xi)]\,(F(2^{-\ell}\xi) - m_0(2^{-\ell}\xi))\,\widehat{\varphi}_F(2^{-\ell}\xi), \tag{3.7}$$

where the series converges uniformly and absolutely. Indeed, the partial sums of the series satisfy

$$\sum_{\ell=1}^{N}\Big[\prod_{j=1}^{\ell-1} m_0(2^{-j}\xi)\Big]\,(F(2^{-\ell}\xi)-m_0(2^{-\ell}\xi))\,\widehat{\varphi}_F(2^{-\ell}\xi)=$$

$$=\widehat{\varphi}_F(\xi)-\Big[\prod_{j=1}^{N} m_0(2^{-j}\xi)\Big]\,\widehat{\varphi}_F(2^{-N}\xi)\to\widehat{\varphi}_F(\xi)-\widehat{\varphi}(\xi),\quad \text{as } N\to\infty,$$

the convergence in the last step following from Lemma 2.12 and the continuity of $\widehat{\varphi}_F$ at $\xi=0$. Suppose now that the compact set $K\subset[-M,M]$, for some $M>0$, and take an integer $j_0\geq 0$ such that $2^{-j_0}M\in[-\pi,\pi]$. Then, for all $\xi\in K$, and since $F(0)=m_0(0)=1$

$$|\widehat{\varphi}_F(\xi)-\widehat{\varphi}(\xi)|\leq\sum_{\ell=1}^{\infty}|F(2^{-\ell}\xi)-m_0(2^{-\ell}\xi)|$$

$$\leq\sum_{\ell=1}^{j_0}\|F-m_0\|_\infty+\sum_{\ell=j_0+1}^{\infty}|2^{-\ell}\xi|^\beta\,\|F-m_0\|_{\Lambda^\beta(\mathbb{T})}.$$

Then, if we take $\delta<\frac{\varepsilon}{2j_0}$ and $\delta<\frac{\varepsilon(2^\beta-1)}{2(2M)^\beta}$, we have $|\widehat{\varphi}_F(\xi)-\widehat{\varphi}(\xi)|<\varepsilon$.

□

THEOREM 3.8 *Let $\alpha>\frac{1}{2}$. Then, $\mathcal{E}_\alpha$ is an open subset of $\mathcal{F}_\alpha$.*

PROOF:

Suppose that $m_0\in\mathcal{E}_\alpha$ and let K be a compact set as in (2.15). Then, by Lemma 2.12, $\widehat{\varphi}(\xi)=\prod_{j=1}^{\infty}m_0(2^{-j}\xi)\neq 0$, when $\xi\in K$ and, therefore, there exists a positive number $\varepsilon>0$ such that $\min_{\xi\in K}|\widehat{\varphi}(\xi)|>2\varepsilon$. Given such K and ε, the previous lemma and Sobolev's Imbedding Theorem 1.25 tell us that we can find a $\delta>0$ such that, whenever $F\in\mathcal{F}_\alpha$ and $\|F-m_0\|_{H^\alpha(\mathbb{T})}<\delta$, then,

$$\left|\widehat{\varphi}(\xi)-\prod_{j=1}^{\infty}F(2^{-j}\xi)\right|<\varepsilon,\quad\text{for all } \xi\in K.$$

In particular, for each such function F, $|\prod_{j=1}^{\infty}F(2^{-j}\xi)|>\varepsilon$, when $\xi\in K$ and, therefore, F satisfies Cohen's condition (2.15) (with the same compact set K as m_0) and $F\in\mathcal{E}_\alpha$. □

From Lemma 3.5 we deduced that if $F_n \to m_0$ in $\mathcal{F}_\alpha$, then $\prod_{j=1}^{\infty} F_n(2^{-j}\xi) \to \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$ uniformly on compact sets of $\mathbb{R}$. This local convergence on compacta can actually be improved as the next theorem shows. The appropriate spaces in this case are the local Sobolev spaces $H^\alpha_{loc}(\mathbb{R})$ defined in §1.4.

Theorem 3.9 *Let $\alpha > \frac{1}{2}$. Suppose that $F_n, m_0 \in \mathcal{F}_\alpha$, $n = 1, 2, \ldots$, and that*

$$\widehat{\varphi}_n(\xi) = \prod_{j=1}^{\infty} F_n(2^{-j}\xi) \quad \textit{and} \quad \widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R}. \tag{3.10}$$

Then, if $F_n \to m_0$ in $\mathcal{F}_\alpha$, we have that $\widehat{\varphi}_n \to \widehat{\varphi}$ in $H^\alpha_{loc}(\mathbb{R})$.

Remark 3.11 Note that in this case we cannot expect to have $\widehat{\varphi}_n \to \widehat{\varphi}$ in $H^\alpha(\mathbb{R})$. Indeed, we can take a sequence $\{F_n\} \subset \mathcal{E}_\alpha$ such that $F_n \to m_0$ in $\mathcal{F}_\alpha$, but $m_0 \notin \mathcal{E}_\alpha$ (like in Example 1 above). In this case, $\widehat{\varphi}$ cannot be in $\mathcal{S}_\alpha$ and, since $\mathcal{S}_\alpha$ is closed (Lemma 2.9), $\widehat{\varphi}_n$ *cannot* converge to $\widehat{\varphi}$ in $H^\alpha(\mathbb{R})$.

Remark 3.12 Note that Theorem 3.9, together with Proposition 1.55, imply that, if $\alpha > k + \frac{1}{2}$ (for some $k \in \mathbb{Z}^+ \cup \{0\}$), we have

$$D^{(h)}\widehat{\varphi}_n \to D^{(h)}\widehat{\varphi}, \quad \text{for } 0 \le h \le k,$$

where the convergence is uniform in compact sets of $\mathbb{R}$. In particular, if we allow the case $\alpha = \infty$ (by replacing $H^\alpha(\mathbb{T})$ by the Fréchet space $C^\infty(\mathbb{T})$), then the uniform convergence on compact sets mentioned above holds for all the derivatives ($h = 0, 1, 2, \ldots$). This particular case, with a different proof, was first shown by A. Bonami, S. Durand and G. Weiss (see Proposition 2.4 of [BDW]).

Proof: First of all, notice that, by Lemmas 2.12 and 2.22, the infinite products in (3.10) converge uniformly and represent functions in $H^\alpha(\mathbb{R})$. Therefore, $\widehat{\varphi}_n, \widehat{\varphi} \in H^\alpha_{loc}(\mathbb{R})$. Let K now be a compact set in $\mathbb{R}$. We need to show that $\widehat{\varphi}_n \to \widehat{\varphi}$ in $\|\cdot\|_{H^\alpha_{loc}, K}$. Without loss of generality we will assume that $K = [-M, M]$, for some $M > \pi$. By Lemma 3.5 we know that $\widehat{\varphi}_n \to \widehat{\varphi}$ uniformly in K. Then,

$$\|\widehat{\varphi}_n - \widehat{\varphi}\|_{L^2(K)} \to 0, \quad \text{as } n \to \infty. \tag{3.13}$$

We divide the rest of the proof into the usual three cases.

<u>*Case 1*</u>: $\frac{1}{2} < \alpha < 1$.

We shall show that

$$\omega_{\alpha,K}(\widehat{\varphi}_n - \widehat{\varphi}) = \left[\int_K \int_K |(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - (\widehat{\varphi}_n - \widehat{\varphi})(\xi + \eta)|^2 \frac{d\xi \, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} \to 0, \qquad (3.14)$$

as n approaches infinity.

In the same way as in (3.7) we can write, for $\xi, \eta \in \mathbb{R}$:

$$\widehat{\varphi}(\xi) - \widehat{\varphi}(\xi+\eta) = \sum_{\ell=1}^{\infty} \prod_{j=1}^{\ell-1} m_0(2^{-j}(\xi+\eta)) \left[m_0(2^{-\ell}\xi) - m_0(2^{-\ell}(\xi+\eta))\right] \widehat{\varphi}(2^{-\ell}\xi), \quad (3.15)$$

where the series converges absolutely.

By using (3.15) with $\widehat{\varphi}_n$ and $\widehat{\varphi}$ and subtracting both quantities, we obtain

$$\begin{aligned}
&(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - (\widehat{\varphi}_n - \widehat{\varphi})(\xi + \eta) = \\
&= \sum_{\ell=1}^{\infty} \Bigg\{ \sum_{h=1}^{\ell-1} \prod_{j=1}^{h-1} m_0(2^{-j}(\xi+\eta)) \left[F_n(2^{-h}(\xi+\eta)) - m_0(2^{-h}(\xi+\eta))\right] \prod_{j=h+1}^{\ell-1} F_n(2^{-j}(\xi+\eta)) \times \\
&\qquad \times \left[F_n(2^{-\ell}\xi) - F_n(2^{-\ell}(\xi+\eta))\right] \widehat{\varphi}_n(2^{-\ell}\xi) \Bigg\} + \\
&\quad + \sum_{\ell=1}^{\infty} \prod_{j=1}^{\ell-1} m_0(2^{-j}(\xi+\eta)) \left[(F_n - m_0)(2^{-\ell}\xi) - (F_n - m_0)(2^{-\ell}(\xi+\eta))\right] \widehat{\varphi}_n(2^{-\ell}\xi) \\
&\quad + \sum_{\ell=1}^{\infty} \prod_{j=1}^{\ell-1} m_0(2^{-j}(\xi+\eta)) \left[m_0(2^{-\ell}\xi) - m_0(2^{-\ell}(\xi+\eta))\right] \left[\widehat{\varphi}_n(2^{-\ell}\xi) - \widehat{\varphi}(2^{-\ell}\xi)\right] = \\
&\qquad = A + B + C. \qquad (3.16)
\end{aligned}$$

Using this decomposition, our estimation reduces to:

$$\omega_{\alpha,K}(\widehat{\varphi}_n - \widehat{\varphi}) \le \omega_{\alpha,K}(A) + \omega_{\alpha,K}(B) + \omega_{\alpha,K}(C).$$

Consider first $\omega_{\alpha,K}(A)$. By using the notation in Lemma 2.25 we have

$$\omega_{\alpha,K}(A) \le \sum_{\ell=1}^{\infty} \sum_{h=1}^{\ell-1} \|F_n - m_0\|_\infty \, 2^{-\ell(\alpha - \frac{1}{2})} \, \|\widehat{\varphi}_n \cdot \chi_K\|_* \left[\int_K \int_{\mathbb{T}} |F_n(\xi) - F_n(\xi+\eta)|^2 \frac{d\xi \, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}}.$$

Now,

$$\|\widehat{\varphi}_n \cdot \chi_K\|_*^2 = \sup_{\xi\in\mathbb{T}} \sum_{k\in\mathbb{Z}} |(\widehat{\varphi}_n \cdot \chi_K)(\xi + 2k\pi)|^2 \leq \left(\frac{M}{\pi} + 2\right) \|\widehat{\varphi}_n\|_\infty^2 \leq M + 2,$$

and there is a positive constant C such that $\sup_{n\geq 1}\|F_n\|_{H^\alpha(\mathbb{T})} \leq C < \infty$. Thus,

$$\omega_{\alpha,K}(A) \leq \sqrt{M+2}\left[\int_{|\eta|>\pi} 4\,\frac{d\eta}{|\eta|^{1+2\alpha}} + \omega_\alpha(F_n)^2\right]^{\frac{1}{2}} \|F_n - m_0\|_\infty \sum_{\ell=1}^{\infty} \frac{\ell-1}{2^{\ell(\alpha-\frac{1}{2})}}$$
$$\leq C\,\|F_n - m_0\|_{H^\alpha(\mathbb{T})} \to 0, \quad \text{as } n \to \infty.$$

For the second term we obtain

$$\omega_{\alpha,K}(B) \leq \sum_{\ell=1}^{\infty} 2^{-\ell(\alpha-\frac{1}{2})}\,\|\widehat{\varphi}_n \cdot \chi_K\|_* \left[\int_{|\eta|>\pi} 4\|F_n - m_0\|_\infty^2 \frac{d\eta}{|\eta|^{1+2\alpha}} + \omega_\alpha(F_n - m_0)^2\right]^{\frac{1}{2}}$$
$$\leq C\,\|F_n - m_0\|_{H^\alpha(\mathbb{T})} \to 0, \quad \text{as } n \to \infty.$$

Finally, the third term is bounded by

$$\omega_{\alpha,K}(C) \leq \sum_{\ell=1}^{\infty} 2^{-\ell(\alpha-\frac{1}{2})}\,\|(\widehat{\varphi}_n - \widehat{\varphi}) \cdot \chi_K\|_* \left[\int_{|\eta|>\pi} 4\,\frac{d\eta}{|\eta|^{1+2\alpha}} + \omega_\alpha(m_0)^2\right]^{\frac{1}{2}}$$
$$\leq C\,\sqrt{M+2}\,\sup_{\xi\in[-M,M]} |\widehat{\varphi}_n(\xi) - \widehat{\varphi}(\xi)| \to 0, \quad \text{as } n \to \infty,$$

where the convergence to 0 follows from Lemma 3.5. This shows (3.14) and completes the proof of case 1.

<u>*Case 2*</u>: $\alpha = k \in \mathbb{Z}^+$.

We need to show that:

$$\|D^{(k)}\widehat{\varphi}_n - D^{(k)}\widehat{\varphi}\|_{L^2(K)} \to 0, \quad \text{as } n \to \infty. \tag{3.17}$$

By using Lemma 2.34 and the notation in (2.37), we can write

$$D^{(k)}\widehat{\varphi}_n(\xi) - D^{(k)}\widehat{\varphi}(\xi) = \sum_{L=1}^{\infty}\Bigg\{ \sum_{\substack{\sup \ell_i = L \\ \boldsymbol{\ell}\neq(L,\ldots,L)}} \left(\frac{1}{2}\right)^{\sum_{i=1}^{k}\ell_i} \times$$

$$\times\Bigg[\sum_{h=1}^{L}\prod_{j=1}^{h-1} m_{\varepsilon_j}(2^{-j}\xi)\,[F_{n,\varepsilon_h}(2^{-h}\xi) - m_{\varepsilon_h}(2^{-h}\xi)] \prod_{j=h+1}^{L} F_{n,\varepsilon_j}(2^{-j}\xi)\,\widehat{\varphi}_n(2^{-L}\xi) +$$

$$+\prod_{j=1}^{L} m_{\varepsilon_j}(2^{-j}\xi)\,(\widehat{\varphi}_n(2^{-L}\xi)-\widehat{\varphi}(2^{-L}\xi))\Big]+$$

$$+\Big(\frac{1}{2}\Big)^{kL}\Big[\sum_{h=1}^{L-1}\prod_{j=1}^{h-1} m_0(2^{-j}\xi)\,[(F_n-m_0)(2^{-h}\xi)]\prod_{j=h+1}^{L-1} F_n(2^{-j}\xi)\,{F_n}^{(k)}(2^{-L}\xi)\,\widehat{\varphi}_n(2^{-L}\xi)$$

$$+\prod_{j=1}^{L-1} m_0(2^{-j}\xi)\,({F_n}^{(k)}(2^{-L}\xi)-{m_0}^{(k)}(2^{-L}\xi))\,\widehat{\varphi}_n(2^{-L}\xi)+$$

$$+\prod_{j=1}^{L-1} m_0(2^{-j}\xi)\,{m_0}^{(k)}(2^{-L}\xi)\,[(\widehat{\varphi}_n-\widehat{\varphi})(2^{-L}\xi)]\Big]\Big\}$$

$$=\Big\{\,[\,I+II\,]\,+\,[\,III+IV+V\,]\Big\}. \tag{3.18}$$

Then, if we estimate the $L^2(K)$-norm of each of these terms separately, we obtain

$$\|I\|_{L^2(K)}\le$$

$$\sum_{L=1}^{\infty}\sum_{\substack{\sup \ell_i=L\\ \boldsymbol{\ell}\neq(L,\ldots,L)}}\frac{1}{2^L}\sum_{h=1}^{L}2^{L/2}\prod_{j=1}^{h-1}\|m_{\varepsilon_j}\|_\infty\,\|F_{n,\varepsilon_h}-m_{\varepsilon_h}\|_\infty\prod_{j=h+1}^{L}\|F_{n,\varepsilon_j}\|_\infty\,\|\widehat{\varphi}_n\|_{L^2(K)}$$

$$\le\,C\,\|m_0\|^k_{H^\alpha(\mathbb{T})}\,\|F_n-m_0\|_{H^\alpha(\mathbb{T})}\,\|F_n\|^k_{H^\alpha(\mathbb{T})}\sum_{L=1}^{\infty}\frac{L^{k+1}}{2^{L/2}}\le\,C\,\|F_n-m_0\|_{H^\alpha(\mathbb{T})}\to 0,$$

as n approaches ∞. The second term is treated similarly,

$$\|II\|_{L^2(K)}\le\sum_{L=1}^{\infty}\frac{L^{k+1}}{2^{L/2}}\,\|m_0\|^k_{H^\alpha(\mathbb{T})}\,\|\widehat{\varphi}_n-\widehat{\varphi}\|_{L^2(K)}\to 0,\quad\text{as } n\to\infty,$$

where the convergence to 0 follows from (3.13). For the other three terms there is a minor modification in which we introduce the notation in Lemma 2.25

$$\|III\|_{L^2(K)}\le\sum_{L=1}^{\infty}\Big(\frac{1}{2}\Big)^{kL}\sum_{h=1}^{L-1}2^{L/2}\,\|F_n-m_0\|_\infty\,\|\widehat{\varphi}_n\cdot\chi_K\|_*\,\|{F_n}^{(k)}\|_{L^2(\mathbb{T})}$$

$$\le\,C\,\|F_n-m_0\|_{H^\alpha(\mathbb{T})}\to 0,\quad\text{as } n\to\infty.$$

$$\|IV\|_{L^2(K)}\le\sum_{L=1}^{\infty}\Big(\frac{1}{2}\Big)^{kL}\,2^{L/2}\,\|\widehat{\varphi}_n\cdot\chi_K\|_*\,\|{F_n}^{(k)}-{m_0}^{(k)}\|_{L^2(\mathbb{T})}$$

$$\le\,C\,\|F_n-m_0\|_{H^\alpha(\mathbb{T})}\to 0,\quad\text{as } n\to\infty.$$

$$\|V\|_{L^2(K)} \leq \sum_{L=1}^{\infty} \left(\frac{1}{2}\right)^{kL} 2^{L/2} \|(\widehat{\varphi}_n - \widehat{\varphi}) \cdot \chi_K\|_* \|{m_0}^{(k)}\|_{L^2(\mathbb{T})}$$
$$\leq C\sqrt{M+2} \sup_{\xi \in K} |\widehat{\varphi}_n(\xi) - \widehat{\varphi}(\xi)| \to 0, \quad \text{as } n \to \infty.$$

This shows (3.17) and completes the proof of case 2.

<u>*Case 3*</u>: $\alpha = k + \varepsilon, \quad k \in \mathbb{Z}^+, \ 0 < \varepsilon < 1.$

This case is a little more tedious to write, mainly because of the cumbersome notation needed; however, all the estimations follow from the same type of arguments that we used previously. We have to show that

$$\omega_{\varepsilon,K}(D^{(k)}\widehat{\varphi}_n - D^{(k)}\widehat{\varphi}) =$$
$$\left[\int_K \int_K |D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi + \eta)|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}} \to 0, \quad \text{as } n \to \infty. \tag{3.19}$$

By using (2.35), the notation in (2.37), and the same kind of decomposition as in (3.15), we can write:

$$D^{(k)}\widehat{\varphi}(\xi) - D^{(k)}\widehat{\varphi}(\xi + \eta) =$$
$$\sum_{L=1}^{\infty} \sum_{\sup \ell_i = L} \left(\frac{1}{2}\right)^{\sum_{i=1}^{k} \ell_i} \left[\sum_{p=1}^{L} \left\{\prod_{j=1}^{p-1} m_{\varepsilon_j}(2^{-j}(\xi + \eta)) \left[m_{\varepsilon_p}(2^{-p}\xi) - m_{\varepsilon_p}(2^{-p}(\xi + \eta))\right] \times\right.\right.$$
$$\left. \times \prod_{j=p+1}^{L} m_{\varepsilon_j}(2^{-j}\xi)\, \widehat{\varphi}(2^{-L}\xi) \right\} +$$
$$\left. + \prod_{j=1}^{L} m_{\varepsilon_j}(2^{-j}(\xi + \eta)) \left[\widehat{\varphi}(2^{-L}\xi) - \widehat{\varphi}(2^{-L}(\xi + \eta))\right]\right].$$

Now, using this representation with $\widehat{\varphi}$ and $\widehat{\varphi}_n$ and subtracting both quantities, we obtain

$$D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi + \eta) = \sum_{L=1}^{\infty} \sum_{\sup \ell_i = L} \left(\frac{1}{2}\right)^{\sum_{i=1}^{k} \ell_i} \times$$
$$\left\{ \sum_{p=1}^{L} \left[\sum_{q=1}^{p-1} \left\{ \prod_{j=1}^{q-1} m_{\varepsilon_j}(2^{-j}(\xi + \eta)) \left[F_{n,\varepsilon_q}(2^{-q}(\xi + \eta)) - m_{\varepsilon_q}(2^{-q}(\xi + \eta))\right] \times \right.\right.\right.$$

$$\times \prod_{j=q+1}^{p-1} F_{n,\varepsilon_j}(2^{-j}(\xi+\eta))\,[F_{n,\varepsilon_p}(2^{-p}\xi) - F_{n,\varepsilon_p}(2^{-p}(\xi+\eta))] \prod_{j=p+1}^{L} F_{n,\varepsilon_j}(2^{-j}\xi)\,\widehat{\varphi}_n(2^{-L}\xi)\Big\}$$

$$+\prod_{j=1}^{p-1} m_{\varepsilon_j}(2^{-j}(\xi+\eta))\,[(F_{n,\varepsilon_p} - m_{\varepsilon_p})(2^{-p}\xi) - (F_{n,\varepsilon_p} - m_{\varepsilon_p})(2^{-p}(\xi+\eta))] \times$$

$$\times \prod_{j=p+1}^{L} F_{n,\varepsilon_j}(2^{-j}\xi)\,\widehat{\varphi}_n(2^{-L}\xi)| +$$

$$+\sum_{q=p+1}^{L}\Big\{\prod_{j=1}^{p-1} m_{\varepsilon_j}(2^{-j}(\xi+\eta))\,[m_{\varepsilon_p}(2^{-p}\xi) - m_{\varepsilon_p}(2^{-p}(\xi+\eta))] \prod_{j=p+1}^{q-1} m_{\varepsilon_j}(2^{-j}\xi) \times$$

$$\times [F_{n,\varepsilon_q}(2^{-q}\xi) - m_{\varepsilon_q}(2^{-q}\xi)] \prod_{j=q+1}^{L} F_{n,\varepsilon_j}(2^{-j}\xi)\,\widehat{\varphi}_n(2^{-L}\xi)\Big\}+$$

$$+\prod_{j=1}^{p-1} m_{\varepsilon_j}(2^{-j}(\xi+\eta))\,[m_{\varepsilon_p}(2^{-p}\xi) - m_{\varepsilon_p}(2^{-p}(\xi+\eta))] \prod_{j=p+1}^{L} m_{\varepsilon_j}(2^{-j}\xi)\,(\widehat{\varphi}_n - \widehat{\varphi})(2^{-L}\xi)\Big]+$$

$$+\sum_{q=1}^{L}\Big[\prod_{j=1}^{q-1} m_{\varepsilon_j}(2^{-j}(\xi+\eta))\,[F_{n,\varepsilon_q}(2^{-q}(\xi+\eta)) - m_{\varepsilon_q}(2^{-q}(\xi+\eta))] \prod_{j=q+1}^{L} F_{n,\varepsilon_j}(2^{-j}(\xi+\eta)) \times$$

$$\times [\widehat{\varphi}_n(2^{-L}\xi) - \widehat{\varphi}_n(2^{-L}(\xi+\eta))]\Big]+$$

$$+\prod_{j=1}^{L} m_{\varepsilon_j}(2^{-j}(\xi+\eta))\,[(\widehat{\varphi}_n - \widehat{\varphi})(2^{-L}\xi) - (\widehat{\varphi}_n - \widehat{\varphi})(2^{-L}(\xi+\eta))]\Big\} =$$

$$= \Big[\sum_{L=1}^{\infty}\sum_{\boldsymbol{\ell}\neq(L,\dots,L)}\Big] + \Big[\sum_{L=1}^{\infty}\sum_{\boldsymbol{\ell}=(L,\dots,L)}\Big] = [\,A_1+\ldots+A_6\,] + [\,B_1+\ldots+B_6\,] = A+B. \tag{3.20}$$

The estimates needed to show that $\omega_{\varepsilon,K}(A) \to 0$, when $n \to \infty$, are essentially the same as in the previous cases: we isolate the factor that contains the difference $\|F_n - m_0\|_\infty$ (or, more generally, $\|F_n - m_0\|_{H^\alpha(\mathbb{T})}$), while we show that the rest of the double integral $\int_K\int_K \frac{d\xi\, d\eta}{|\eta|^{1+2\varepsilon}}$ is bounded (by dealing with any of the increments $G(\xi+\eta) - G(\xi)$, where $G = F_{n,\varepsilon_j}, m_{\varepsilon_j}, \widehat{\varphi}, \widehat{\varphi}_n$). The calculations for $\omega_{\varepsilon,K}(B) \to 0$ follow the same pattern, the main difference with A being that the $D^{(k)}G$'s have to be treated with $\|\cdot\|_{L^2(\mathbb{T})}$-norms rather than $\|\cdot\|_\infty$. The cases B_1, B_5, and B_6 contain the essential features of these modifications. For completeness, we include the proof of all the cases, but we encourage the reader to convince him or herself of the validity of these arguments by just checking the non-obvious ones.

$$\omega_{\varepsilon,K}(A_1) \leq$$

$$\leq \sum_{L=1}^{\infty} \frac{L^k}{2^L} \sum_{p=1}^{L} (p-1)\, \|m_0\|_\alpha^k \, \|F_n - m_0\|_\alpha \, \|F_n\|_\alpha^k \, 2^{-p(\varepsilon-\frac{1}{2})} \, \|P_{F_{n,\varepsilon_{p+1}}} \cdots P_{F_{n,\varepsilon_L}}(\widehat{\varphi}_n \chi_K)\|_* \times$$

$$\times \left[\int_{|\eta|>\pi} 4\|F_{n,\varepsilon_p}\|_\infty^2 \, \frac{d\eta}{|\eta|^{1+2\varepsilon}} \, + \, \omega_\varepsilon(F_{n,\varepsilon_p})^2 \right]^{\frac{1}{2}} \leq$$

$$\leq \, C \sum_{L=1}^{\infty} \frac{L^{k+2}}{2^{L/2}} \, 2^{k/2} \, \|\widehat{\varphi}_n \chi_K\|_* \, \|F_n - m_0\|_\alpha \to 0, \quad \text{as } n \to \infty.$$

$$\omega_{\varepsilon,K}(A_2) \leq \sum_{L=1}^{\infty} \frac{L^k}{2^L} \sum_{p=1}^{L} \|m_0\|_\alpha^k \, 2^{-p(\varepsilon-\frac{1}{2})} \, \|P_{F_{n,\varepsilon_{p+1}}} \cdots P_{F_{n,\varepsilon_L}}(\widehat{\varphi}_n \chi_K)\|_* \times$$

$$\times \left[\int_{|\eta|>\pi} 4\|F_{n,\varepsilon_p} - m_{\varepsilon_p}\|_\infty^2 \, \frac{d\eta}{|\eta|^{1+2\varepsilon}} \, + \, \omega_\varepsilon(F_{n,\varepsilon_p} - m_{\varepsilon_p})^2 \right]^{\frac{1}{2}} \leq$$

$$\leq \, C \sum_{L=1}^{\infty} \frac{L^{k+1}}{2^{L/2}} \, 2^{k/2} \, \|\widehat{\varphi}_n \chi_K\|_* \, \|F_n - m_0\|_\alpha \to 0, \quad \text{as } n \to \infty.$$

$$\omega_{\varepsilon,K}(A_3) \leq \sum_{L=1}^{\infty} \frac{L^k}{2^L} \sum_{p=1}^{L} \sum_{q=p+1}^{L} \|m_0\|_\alpha^k \, 2^{-p(\varepsilon-\frac{1}{2})} \, 2^{k/2} \, \|F_{n,\varepsilon_q} - m_{\varepsilon_q}\|_\infty \, \|F_n\|_\alpha^k \, \|\widehat{\varphi}_n \chi_K\|_* \times$$

$$\times \left[\int_{|\eta|>\pi} 4\|m_{\varepsilon_p}\|_\infty^2 \, \frac{d\eta}{|\eta|^{1+2\varepsilon}} \, + \, \omega_\varepsilon(m_{\varepsilon_p})^2 \right]^{\frac{1}{2}} \leq$$

$$\leq \, C \sum_{L=1}^{\infty} \frac{L^{k+2}}{2^{L/2}} \, \|F_n - m_0\|_\alpha \to 0, \quad \text{as } n \to \infty.$$

$$\omega_{\varepsilon,K}(A_4) \leq \sum_{L=1}^{\infty} \frac{L^k}{2^L} \sum_{p=1}^{L} \|m_0\|_\alpha^k \, 2^{-p(\varepsilon-\frac{1}{2})} \, 2^{k/2} \, \|(\widehat{\varphi}_n - \widehat{\varphi})\chi_K\|_* \times$$

$$\times \left[\int_{|\eta|>\pi} 4\|m_{\varepsilon_p}\|_\infty^2 \, \frac{d\eta}{|\eta|^{1+2\varepsilon}} \, + \, \omega_\varepsilon(m_{\varepsilon_p})^2 \right]^{\frac{1}{2}} \leq$$

$$\leq \, C \sup_{\xi \in K} |(\widehat{\varphi}_n - \widehat{\varphi})(\xi)| \, \to 0, \quad \text{as } n \to \infty.$$

$$\omega_{\varepsilon,K}(A_5) \leq \sum_{L=1}^{\infty} \frac{L^k}{2^L} \sum_{q=1}^{L} \|m_0\|_\alpha^k \, 2^{-L(\varepsilon-\frac{1}{2})} \, \|F_{n,\varepsilon_q} - m_{\varepsilon_q}\|_\infty \, \|F_n\|_\alpha^k \, \omega_{\varepsilon,K}(\widehat{\varphi}_n)$$

$$\leq \, C \, \|F_n - m_0\|_\alpha \to 0, \quad \text{as } n \to \infty,$$

where here we have used that, by (2.23), $\omega_{\varepsilon,K}(\widehat{\varphi}_n) \leq \, C' \, \|F_n\|_\alpha^{3+\alpha} \leq \, C$. Finally,

$$\omega_{\varepsilon,K}(A_6) \leq \sum_{L=1}^{\infty} \frac{L^k}{2^L} \, \|m_0\|_\alpha^k \, 2^{-L(\varepsilon-\frac{1}{2})} \, \omega_{\varepsilon,K}(\widehat{\varphi}_n - \widehat{\varphi}) \leq \, C \, \omega_{\varepsilon,K}(\widehat{\varphi}_n - \widehat{\varphi}) \to 0, \quad \text{as } n \to \infty,$$

where the last convergence to 0 follows from (3.14), in case 1 above. This shows that $\omega_{\varepsilon,K}(A) \to 0$, as $n \to \infty$. We proceed similarly to estimate the terms corresponding to $\omega_{\varepsilon,K}(B)$.

$$\omega_{\varepsilon,K}(B_1) \leq \sum_{L=1}^{\infty} \frac{1}{2^L} \sum_{p=1}^{L-1} (p-1) \, \|F_n - m_0\|_\infty \, 2^{-p(\varepsilon-\frac{1}{2})} \, \times$$

$$\times \left[\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_K(2^{-L+p}\xi) \, |F_n(\xi) - F_n(\xi+\eta)|^2 \, |{F_n}^{(k)}(2^{-L+p}\xi) \, \widehat{\varphi}_n(2^{-L+p}\xi)|^2 \, \frac{d\xi \, d\eta}{|\eta|^{1+2\varepsilon}} \right]^{\frac{1}{2}} +$$

$$+ \sum_{L=1}^{\infty} \frac{L-1}{2^L} \, \|F_n - m_0\|_\infty 2^{-L(\varepsilon-\frac{1}{2})} \|\widehat{\varphi}_n \chi_K\|_* \left[\int_{|\eta|>\pi} 4 \|{F_n}^{(k)}\|_{L^2(\mathbb{T})}^2 \, \frac{d\eta}{|\eta|^{1+2\varepsilon}} + \omega_\varepsilon({F_n}^{(k)})^2 \right]^{\frac{1}{2}}.$$

To treat the double integral we use a Lipschitz condition on F_n and, then, periodize over ξ. Let $\theta \in (0, \frac{1}{2})$ be such that $0 < \varepsilon + \theta < 1$. Then, if $\frac{1}{2} \leq \varepsilon < 1$, we have $F_n \in H^{\frac{3}{2}}(\mathbb{T}) \subset \Lambda^{\varepsilon+\theta}(\mathbb{T})$ and, by (1.28),

$$|F_n(\xi) - F_n(\xi+\eta)|^2 \leq \, C \, \|F_n\|_\alpha^2 \, |\eta|^{2\varepsilon+2\theta}.$$

Suppose, on the contrary, that $0 < \varepsilon < \frac{1}{2}$. Then, $F_n \in H^{1+\varepsilon}(\mathbb{T}) \subset \Lambda^{\frac{1}{2}+\varepsilon}(\mathbb{T})$ and

$$|F_n(\xi) - F_n(\xi+\eta)|^2 \leq \, C \, \|F_n\|_\alpha^2 \, |\eta|^{1+2\varepsilon}.$$

Therefore, when $|\eta| \leq 1$, we always have

$$|F_n(\xi) - F_n(\xi+\eta)|^2 \leq \, C \, \|F_n\|_\alpha^2 \, |\eta|^{2\varepsilon+2\theta}, \quad \xi \in \mathbb{R}. \tag{3.21}$$

The double integral, then, becomes

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_K(2^{-L+p}\xi)\,|F_n(\xi)-F_n(\xi+\eta)|^2\,|F_n{}^{(k)}(2^{-L+p}\xi)\,\widehat{\varphi}_n(2^{-L+p}\xi)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}} \le$$

$$C\,2^{L-p}\left[\int_{\mathbb{R}}|F_n{}^{(k)}(\xi)|^2\,|\widehat{\varphi}_n(\xi)|^2\,\chi_K(\xi)\,d\xi\right]\left[\int_{|\eta|>1}4\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}+\int_{|\eta|\le 1}\frac{d\eta}{|\eta|^{1-2\theta}}\right]$$

$$\le\; C\,2^{L-p}\,\|\widehat{\varphi}_n\chi_K\|_*^2\,\|F_n{}^{(k)}\|_{L^2(\mathbb{T})}^2.$$

Thus,

$$\omega_{\varepsilon,K}(B_1)\le\; C\sum_{L=1}^{\infty}\frac{L^2}{2^{L/2}}\,\|\widehat{\varphi}_n\chi_K\|_*\,\|F_n\|_\alpha\,\|F_n-m_0\|_\infty+\,C\sum_{L=1}^{\infty}\frac{L}{2^{L/2}}\,\|F_n-m_0\|_\infty\,\to 0,$$

as n approaches ∞. In a similar way, we deal with B_2, B_3 and B_4. Observe that (3.21) is crucial in these three estimations.

$$\omega_{\varepsilon,K}(B_2)\le\; C\sum_{L=1}^{\infty}\frac{1}{2^L}\sum_{p=1}^{L-1}2^{-p(\varepsilon-\frac12)}\,2^{\frac12(L-p)}\left[\int_{\mathbb{R}}|F_n{}^{(k)}(\xi)|^2\;|\widehat{\varphi}_n(\xi)|^2\,\chi_K(\xi)\,d\xi\right]^{\frac12}\times$$

$$\times\left[\int_{|\eta|>1}4\|F_n-m_0\|_\infty^2\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}+\int_{|\eta|\le 1}\|F_n-m_0\|_\alpha^2\,\frac{d\eta}{|\eta|^{1-2\theta}}\right]^{\frac12}+$$

$$+\,C\sum_{L=1}^{\infty}\frac{2^{-L(\varepsilon-\frac12)}}{2^L}\,\|\widehat{\varphi}_n\chi_K\|_*\left[\int_{|\eta|>\pi}4\|F_n{}^{(k)}-m_0{}^{(k)}\|_{L^2(\mathbb{T})}^2\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}+\omega_\varepsilon(F_n{}^{(k)}-m_0{}^{(k)})^2\right]^{\frac12}$$

$$\le\; C\,\|F_n-m_0\|_\alpha\to 0,\quad \text{as } n\to\infty.$$

$$\omega_{\varepsilon,K}(B_3)\le\sum_{L=1}^{\infty}\frac{1}{2^L}\sum_{p=1}^{L}2^{-p(\varepsilon-\frac12)}\left\{\sum_{q=p+1}^{L-1}\|F_n-m_0\|_\infty\times\right.$$

$$\times\quad\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|m_0(\xi)-m_0(\xi+\eta)|^2\,|F_n{}^{(k)}(2^{-L+p}\xi)\,(\chi_K\widehat{\varphi}_n)(2^{-L+p}\xi)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\right]^{\frac12}+$$

$$\left.+\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|m_0(\xi)-m_0(\xi+\eta)|^2\,|(F_n{}^{(k)}-m_0{}^{(k)})(2^{-L+p}\xi)|^2\,|(\chi_K\widehat{\varphi}_n)(2^{-L+p}\xi)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\right]^{\frac12}\right\}$$

$$\le\; C\sum_{L=1}^{\infty}\frac{L^2}{2^{L/2}}\left\{\|F_n-m_0\|_\infty+\;\|\widehat{\varphi}_n\chi_K\|_*\,\|F_n{}^{(k)}-m_0{}^{(k)}\|_{L^2(\mathbb{T})}\right\}\to 0,\quad\text{as } n\to\infty.$$

$$\omega_{\varepsilon,K}(B_4)\le$$

$$\sum_{L=1}^{\infty} \frac{1}{2^L} \sum_{p=1}^{L-1} 2^{-p(\varepsilon-\frac{1}{2})} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} |m_0(\xi)-m_0(\xi+\eta)|^2 \, |[F_n{}^{(k)} \chi_K(\widehat{\varphi}_n-\widehat{\varphi})](2^{-L+p}\xi)|^2 \, \frac{d\xi \, d\eta}{|\eta|^{1+2\varepsilon}} \right]^{\frac{1}{2}}$$

$$+ \sum_{L=1}^{\infty} \frac{1}{2^L} 2^{-L(\varepsilon-\frac{1}{2})} \, \|(\widehat{\varphi}_n - \widehat{\varphi})\chi_K\|_* \left[\int_{|\eta|>\pi} 4\|m_0{}^{(k)}\|^2_{L^2(\mathbb{T})} \frac{d\eta}{|\eta|^{1+2\varepsilon}} + \omega_\varepsilon(m_0{}^{(k)})^2 \right]^{\frac{1}{2}} \leq$$

$$\leq C \sum_{L=1}^{\infty} \frac{L}{2^L} \left\{ \|F_n{}^{(k)}\|_{L^2(\mathbb{T})} \, \|(\widehat{\varphi}_n-\widehat{\varphi})\chi_K\|_* + \|m_0\|_\alpha \, \|(\widehat{\varphi}_n-\widehat{\varphi})\chi_K\|_* \right\} \to 0, \quad \text{as } n \to \infty.$$

$$\omega_{\varepsilon,K}(B_5) \leq$$

$$\sum_{L=1}^{\infty} \frac{1}{2^L} \sum_{q=1}^{L-1} 2^{-L(\varepsilon-\frac{1}{2})} \, \|F_n - m_0\|_\infty \left[\int_{\mathbb{R}} \int_{\mathbb{R}} |F_n{}^{(k)}(\xi)|^2 \, \chi_{2K}(\xi) \, |\widehat{\varphi}_n(\xi)-\widehat{\varphi}_n(\xi+\eta)|^2 \, \frac{d\xi \, d\eta}{|\eta|^{1+2\varepsilon}} \right]^{\frac{1}{2}}$$

$$+ \sum_{L=1}^{\infty} \frac{1}{2^L} 2^{-L(\varepsilon-\frac{1}{2})} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} |F_n{}^{(k)}(\xi) - m_0{}^{(k)}(\xi)|^2 \, \chi_{2K}(\xi) \, |\widehat{\varphi}_n(\xi) - \widehat{\varphi}_n(\xi+\eta)|^2 \, \frac{d\xi \, d\eta}{|\eta|^{1+2\varepsilon}} \right]^{\frac{1}{2}}.$$

To estimate the double integrals above we need to use this time a Lipschitz condition on $\widehat{\varphi}_n$. Let us denote by $G(\xi)$ any of the 2π-periodic functions $F_n{}^{(k)}$ or $F_n{}^{(k)} - m_0{}^{(k)}$. Then,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |G(\xi)|^2 \, \chi_{2K}(\xi) \, |\widehat{\varphi}_n(\xi) - \widehat{\varphi}_n(\xi+\eta)|^2 \, \frac{d\xi \, d\eta}{|\eta|^{1+2\varepsilon}} = \int_{|\eta|>1} \int_{\xi \in \mathbb{R}} \quad + \int_{|\eta|\leq 1} \int_{\xi \in \mathbb{R}} \quad .$$

The first double integral is easily bounded by

$$\int_{|\eta|>1} \int_{\mathbb{R}} |G(\xi)|^2 \, \chi_{2K}(\xi) \, |\widehat{\varphi}_n(\xi) - \widehat{\varphi}_n(\xi+\eta)|^2 \, \frac{d\xi \, d\eta}{|\eta|^{1+2\varepsilon}} \leq$$

$$\leq 4 \, \|\widehat{\varphi}_n\|^2_* \, \|G\|^2_{L^2(\mathbb{T})} \left[\int_{|\eta|>1} \frac{d\eta}{|\eta|^{1+2\varepsilon}} \right] \leq C \, \|G\|^2_{L^2(\mathbb{T})}.$$

For the second integral we proceed as follows. Let $\widetilde{K} = [-4M-1, 4M+1]$, and let $w \in C_c^\infty(\widetilde{K})$ be such that $0 \leq w \leq 1$ and $w|_{[-2M-1,2M+1]} \equiv 1$. Then, for $|\eta| \leq 1$, we have

$$\|\chi_{2K}(\cdot)(\widehat{\varphi}_n(\cdot) - \widehat{\varphi}_n(\cdot+\eta))\|^2_* = \sup_{\xi \in \mathbb{T}} \sum_{k \in \mathbb{Z}} |\chi_{2K}(\xi + 2k\pi) \, (\widehat{\varphi}_n(\xi+2k\pi) - \widehat{\varphi}_n(\xi+2k\pi+\eta))|^2$$

$$\leq \|(w\widehat{\varphi}_n)(\cdot) - (w\widehat{\varphi}_n)(\cdot+\eta)\|^2_*. \tag{3.22}$$

As in the last part of the proof of Lemma 2.22, we consider two cases. When $0<\varepsilon<\frac{1}{2}$, by Corollary 1.38 and Proposition 1.52 we have

$$\|(w\widehat{\varphi}_n)(\cdot)-(w\widehat{\varphi}_n)(\cdot+\eta)\|_*^2 \leq C\,|\eta|\,\|w\widehat{\varphi}_n\|^2_{H^1(\mathbb{R})} \leq C\,|\eta|\,\|\widehat{\varphi}_n\|^2_{H^1_{loc},2\widetilde{K}} \leq C\,|\eta|.$$

Then,

$$\int_{|\eta|\leq 1}\int_{\mathbb{R}} |G(\xi)|^2\,\chi_{2K}(\xi)\,|\widehat{\varphi}_n(\xi)-\widehat{\varphi}_n(\xi+\eta)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}} \leq$$
$$\leq C\,\|G\|^2_{L^2(\mathbb{T})}\left[\int_{|\eta|\leq 1}|\eta|\,\frac{d\eta}{|\eta|^{1+2\varepsilon}}\right] \leq C\,\|G\|^2_{L^2(\mathbb{T})}.$$

In this case,

$$\omega_{\varepsilon,K}(B_5) \leq C\,\|F_n-m_0\|_\alpha \to 0, \quad \text{as } n\to\infty.$$

When $\frac{1}{2}\leq\varepsilon<1$, again, Corollary 1.38 and Proposition 1.52 imply that, if we take $\gamma\in(2\varepsilon,2)$, then

$$\|(w\widehat{\varphi}_n)(\cdot)-(w\widehat{\varphi}_n)(\cdot+\eta)\|_*^2 \leq C\,|\eta|^\gamma\,\|w\widehat{\varphi}_n\|^2_{H^{\frac{1+\gamma}{2}}(\mathbb{R})}$$
$$\leq C\,|\eta|^\gamma\,\|\widehat{\varphi}_n\|^2_{H_{loc}^{\frac{1+\gamma}{2}},2\widetilde{K}} \leq C\,|\eta|^\gamma.$$

Therefore,

$$\int_{|\eta|\leq 1}\int_{\mathbb{R}} |G(\xi)|^2\,\chi_{2K}(\xi)\,|\widehat{\varphi}_n(\xi)-\widehat{\varphi}_n(\xi+\eta)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}} \leq$$
$$\leq C\,\|G\|^2_{L^2(\mathbb{T})}\left[\int_{|\eta|\leq 1}\frac{d\eta}{|\eta|^{1-(\gamma-2\varepsilon)}}\right] \leq C\,\|G\|^2_{L^2(\mathbb{T})},$$

which in turn shows again that

$$\omega_{\varepsilon,K}(B_5) \leq C\,\|F_n-m_0\|_\alpha \to 0, \quad \text{as } n\to\infty.$$

Finally, let us consider B_6.

$$\omega_{\varepsilon,K}(B_6) \leq$$
$$\leq \sum_{L=1}^{\infty}\frac{1}{2^L}\,2^{-L(\varepsilon-\frac{1}{2})}\left[\int_K\int_{\mathbb{R}} |{m_0}^{(k)}(\xi)|^2\,\chi_{2K}(\xi)\,|(\widehat{\varphi}_n-\widehat{\varphi})(\xi)-(\widehat{\varphi}_n-\widehat{\varphi})(\xi+\eta)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\varepsilon}}\right]^{\frac{1}{2}}.$$

Here the double integral is estimated in the same way as for B_5. We split the integral with respect to η in two parts, one when $|\eta| \leq 1$ and the other when $|\eta| > 1$. The last case is always the easiest one:

$$\int_{\substack{\eta\in K\\|\eta|>1}} \int_{\mathbb{R}} |{m_0}^{(k)}(\xi)|^2 \,\chi_{2K}(\xi)\, |(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - (\widehat{\varphi}_n - \widehat{\varphi})(\xi+\eta)|^2 \,\frac{d\xi\, d\eta}{|\eta|^{1+2\varepsilon}} \leq$$

$$2\int_{1<|\eta|\leq M+1} \int_{\mathbb{R}} |{m_0}^{(k)}(\xi)|^2 \left(\Big|[\chi_{2K}(\widehat{\varphi}_n - \widehat{\varphi})](\xi)\Big|^2 + \Big|[\chi_{\widetilde{K}}(\widehat{\varphi}_n - \widehat{\varphi})](\xi+\eta)\Big|^2 \right) \frac{d\xi\, d\eta}{|\eta|^{1+2\varepsilon}}$$

$$\leq 2\left(\|(\widehat{\varphi}_n - \widehat{\varphi})\chi_{2K}\|_*^2 + \|(\widehat{\varphi}_n - \widehat{\varphi})\chi_{\widetilde{K}}\|_*^2 \right) \|{m_0}^{(k)}\|_{L^2(\mathbb{T})}^2 \left[\int_{|\eta|>1} \frac{d\eta}{|\eta|^{1+2\varepsilon}} \right]$$

$$\leq C \sup_{\xi\in\widetilde{K}} |(\widehat{\varphi}_n - \widehat{\varphi})(\xi)|^2 \to 0, \quad \text{as } n\to\infty.$$

On the other hand, when $|\eta| \leq 1$, we introduce the same function w as for B_5 to obtain

$$\|\chi_{2K}(\cdot)[(\widehat{\varphi}_n - \widehat{\varphi})(\cdot) - (\widehat{\varphi}_n - \widehat{\varphi})(\cdot+\eta)]\|_*^2 \leq$$

$$\|[w(\widehat{\varphi}_n - \widehat{\varphi})](\cdot) - [w(\widehat{\varphi}_n - \widehat{\varphi})](\cdot+\eta)\|_*^2.$$

If we assume that $0 < \varepsilon < \frac{1}{2}$, then Corollary 1.38 and Proposition 1.52 imply that

$$\|[w(\widehat{\varphi}_n - \widehat{\varphi})](\cdot) - [w(\widehat{\varphi}_n - \widehat{\varphi})](\cdot+\eta)\|_*^2 \leq$$

$$\leq C\,|\eta|\, \|w(\widehat{\varphi}_n - \widehat{\varphi})\|_{H^1(\mathbb{R})}^2 \leq C\,|\eta|\, \|\widehat{\varphi}_n - \widehat{\varphi}\|_{H^1_{loc},2\widetilde{K}}^2.$$

Thus,

$$\int_{|\eta|\leq 1} \int_{\mathbb{R}} |{m_0}^{(k)}(\xi)|^2\, |(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - (\widehat{\varphi}_n - \widehat{\varphi})(\xi+\eta)|^2 \,\chi_{2K}(\xi)\, \frac{d\xi\, d\eta}{|\eta|^{1+2\varepsilon}} \leq$$

$$\leq C\, \|{m_0}^{(k)}\|_{L^2(\mathbb{T})}^2 \left[\int_{|\eta|\leq 1} |\eta|\, \frac{d\eta}{|\eta|^{1+2\varepsilon}} \right] \|\widehat{\varphi}_n - \widehat{\varphi}\|_{H^1_{loc},2\widetilde{K}}^2 \to 0, \quad \text{as } n\to\infty,$$

where the convergence to 0 follows from (3.17), in case 2 above. This shows that $\omega_{\varepsilon,K}(B_6) \to 0$ and completes the proof of the theorem for the case $\alpha = k+\varepsilon$, when $0 < \varepsilon < \frac{1}{2}$. Suppose that $\frac{1}{2} \leq \varepsilon < 1$ and let $\gamma \in (2\varepsilon, 2)$. As before, by Corollary 1.38 and Proposition 1.52, we have

$$\|\,[w(\widehat{\varphi}_n - \widehat{\varphi})](\cdot) - [w(\widehat{\varphi}_n - \widehat{\varphi})](\cdot+\eta)\,\|_*^2 \leq |\eta|^\gamma\, \|\widehat{\varphi}_n - \widehat{\varphi}\|^2_{H_{loc}^{\frac{1+\gamma}{2}},2\widetilde{K}}.$$

Then, since $\frac{1+\gamma}{2} \in (1, \frac{3}{2})$, the case we just proved gives us

$$\int_{|\eta|\le 1}\int_{\mathbb{R}} |m_0{}^{(k)}(\xi)|^2 \, |(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - (\widehat{\varphi}_n - \widehat{\varphi})(\xi+\eta)|^2 \, \chi_{2K}(\xi) \, \frac{d\xi \, d\eta}{|\eta|^{1+2\varepsilon}} \le$$

$$\le C \, \|m_0{}^{(k)}\|^2_{L^2(\mathbb{T})} \left[\int_{|\eta|\le 1} \frac{d\eta}{|\eta|^{1-(\gamma-2\varepsilon)}} \right] \|\widehat{\varphi}_n - \widehat{\varphi}\|^2_{H^{\frac{1+\gamma}{2}}_{loc}, 2\widetilde{K}} \to 0, \quad \text{as } n \to \infty.$$

Hence, $\omega_{\varepsilon,K}(B_6)$ also goes to 0 in the case $\alpha = k+\varepsilon$, when $\frac{1}{2} \le \varepsilon < 1$. This concludes the proof of case 3 and establishes Theorem 3.9.

□

If we extend the domain of the mapping N defined in (2.43) from $\mathcal{E}_\alpha$ to $\mathcal{F}_\alpha$ then, as we pointed out in Remark 3.11, N is not necessarily continuous (when it takes values in $H^\alpha(\mathbb{R})$). The most we are able to say is the local convergence shown in Theorem 3.9. When we restrict N to $\mathcal{E}_\alpha$, Cohen's condition (2.15) will allow us to pass from local to global convergence (in H^α) because of the constraint that the MRA property (3.6) imposes on $\widehat{\varphi}_n$ and $\widehat{\varphi}$. The next lemma clarifies what we are saying.

Lemma 3.23 *Let $\alpha > \frac{1}{2}$. Suppose that $F_n \to m_0$ in $\mathcal{E}_\alpha$ and let $\widehat{\varphi}_n, \widehat{\varphi}$ be defined as in (3.10). Then,*

$$\|\widehat{\varphi}_n - \widehat{\varphi}\|^2_* = \sup_{\xi\in\mathbb{T}} \sum_{k\in\mathbb{Z}} |\widehat{\varphi}_n(\xi+2k\pi) - \widehat{\varphi}(\xi+2k\pi)|^2 \to 0, \quad \textit{as } n \to \infty. \tag{3.24}$$

Proof:

By (3.6) (and Corollary 1.18) we have that

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}_n(\xi+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+2k\pi)|^2 = 1, \quad \text{for every } \xi \in [-\pi,\pi], \; n \ge 1,$$

where all the series converge uniformly on $[-\pi,\pi]$. Let us fix a positive number $\varepsilon \in (0,1)$. Then, we can find an integer $k_0 \ge 1$ such that

$$\sum_{|k|\ge k_0} |\widehat{\varphi}(\xi+2k\pi)|^2 < \varepsilon, \quad \text{for all } \xi \in [-\pi,\pi]. \tag{3.25}$$

Consider the compact set $K = [-(2k_0+1)\pi, (2k_0+1)\pi]$. Then, by Lemma 3.5, we can find $n_0 \geq 1$ such that, for all $n \geq n_0$

$$|\widehat{\varphi}_n(\eta) - \widehat{\varphi}(\eta)| < \frac{\varepsilon}{2k_0+1}, \quad \text{for all } \eta \in K. \tag{3.26}$$

Therefore, when $\xi \in [-\pi, \pi]$ and $n \geq n_0$ we have

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}_n(\xi+2k\pi) - \widehat{\varphi}(\xi+2k\pi)|^2 \leq \sum_{|k|\leq k_0} |(\widehat{\varphi}_n - \widehat{\varphi})(\xi+2k\pi)|^2 +$$

$$+2\sum_{|k|\geq k_0+1} |\widehat{\varphi}(\xi+2k\pi)|^2 + 2\sum_{|k|\geq k_0+1} |\widehat{\varphi}_n(\xi+2k\pi)|^2 = I + II + III.$$

By (3.25) and (3.26), we easily see that

$$I + II < \sum_{|k|\leq k_0} \frac{\varepsilon^2}{(2k_0+1)^2} + 2\varepsilon \leq 3\varepsilon.$$

On the other hand,

$$\frac{1}{2} III = 1 - \sum_{|k|\leq k_0} |\widehat{\varphi}_n(\xi+2k\pi)|^2 =$$

$$= \sum_{|k|\geq k_0+1} |\widehat{\varphi}(\xi+2k\pi)|^2 + \Big(\sum_{|k|\leq k_0} |\widehat{\varphi}(\xi+2k\pi)|^2 - \sum_{|k|\leq k_0} |\widehat{\varphi}_n(\xi+2k\pi)|^2 \Big)$$

$$< \varepsilon + 2\sum_{|k|\leq k_0} |(\widehat{\varphi} - \widehat{\varphi}_n)(\xi+2k\pi)| < \varepsilon + 2\varepsilon = 3\varepsilon.$$

Putting these two estimations together, we obtain that $I + II + III < 9\varepsilon$ and, therefore, that

$$\sup_{\xi\in\mathbb{T}} \sum_{k\in\mathbb{Z}} |\widehat{\varphi}_n(\xi+2k\pi) - \widehat{\varphi}(\xi+2k\pi)|^2 \to 0, \quad \text{as } n \to \infty,$$

which completes the proof of the lemma.

□

We are now ready to state our main result.

Theorem 3.27 *Let $\alpha > \frac{1}{2}$. Suppose that $F_n, m_0 \in \mathcal{E}_\alpha$, $n = 1, 2, \ldots$, and that $F_n \to m_0$ in $\mathcal{E}_\alpha$. Then, $\widehat{\varphi}_n \to \widehat{\varphi}$ in $H^\alpha(\mathbb{R})$, where $\widehat{\varphi}_n, \widehat{\varphi}$ are defined as in (3.10).*

PROOF:

The idea of the proof is the same as for Theorem 3.9. Now, we have to replace the compact set K by the real line $\mathbb{R}$, and make use of Lemma 3.23 whenever we need to pass from local to global convergence (that is, from $\|(\widehat{\varphi}_n-\widehat{\varphi})\cdot\chi_K\|_* \to 0$, to $\|\widehat{\varphi}_n-\widehat{\varphi}\|_*\to 0$). For completeness, we inlude some of the estimations here, leaving the obvious ones to the reader. First of all note that

$$\lim_{n\to\infty}\|\widehat{\varphi}_n-\widehat{\varphi}\|^2_{L^2(\mathbb{R})} = \lim_{n\to\infty}\int_{\mathbb{T}}\sum_{k\in\mathbb{Z}}|(\widehat{\varphi}_n-\widehat{\varphi})(\xi+2k\pi)|^2\,d\xi = 0. \tag{3.28}$$

To show that $\|\widehat{\varphi}_n-\widehat{\varphi}\|_{H^\alpha(\mathbb{R})}\to 0$, as $n\to\infty$, we use, as in Theorem 3.9, the definition of norm for $H^\alpha(\mathbb{R})$ given in (1.4) (or (1.6)).

<u>*Case 1*</u>: $\frac{1}{2}<\alpha<1$.

We shall show that

$$\omega_\alpha(\widehat{\varphi}_n-\widehat{\varphi}) =$$
$$\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|(\widehat{\varphi}_n-\widehat{\varphi})(\xi)-(\widehat{\varphi}_n-\widehat{\varphi})(\xi+\eta)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}}\to 0,\quad \text{as } n\to\infty. \tag{3.29}$$

We proceed as when we showed (3.14), replacing K by $\mathbb{R}$. Using the decomposition

$$(\widehat{\varphi}_n-\widehat{\varphi})(\xi)-(\widehat{\varphi}_n-\widehat{\varphi})(\xi+\eta) = A+B+C,$$

given in (3.16), we obtain

$$\omega_\alpha(\widehat{\varphi}_n-\widehat{\varphi})\le\omega_\alpha(A)+\omega_\alpha(B)+\omega_\alpha(C).$$

Then, if we repeat the estimations right after (3.16) above, we obtain

$$\omega_\alpha(A)\le\sum_{\ell=1}^{\infty}\sum_{h=1}^{\ell-1}\|F_n-m_0\|_\infty\,2^{-\ell(\alpha-\frac{1}{2})}\,\|\widehat{\varphi}_n\|_*\left[\int_{\mathbb{R}}\int_{\mathbb{T}}|F_n(\xi)-F_n(\xi+\eta)|^2\,\frac{d\xi\,d\eta}{|\eta|^{1+2\alpha}}\right]^{\frac{1}{2}}\le$$
$$\le C\,\|F_n-m_0\|_{H^\alpha(\mathbb{T})}\to 0,\quad \text{as } n\to\infty.$$

One deals similarly with B. For C, we have

$$\omega_\alpha(C)\le\sum_{\ell=1}^{\infty}2^{-\ell(\alpha-\frac{1}{2})}\,\|\widehat{\varphi}_n-\widehat{\varphi}\|_*\left[\int_{|\eta|>\pi}4\,\frac{d\eta}{|\eta|^{1+2\alpha}}+\omega_\alpha(m_0)^2\right]^{\frac{1}{2}}$$

$$\leq C\,\|\widehat{\varphi}_n - \widehat{\varphi}\|_* \to 0, \quad \text{as } n \to \infty,$$

where the convergence to 0 follows from Lemma 3.23. This shows (3.29) and, together with (3.28), completes the proof of case 1.

<u>*Case 2*</u>: $\alpha = k \in \mathbb{Z}^+$.

Here, we need to show that:

$$\|D^{(k)}\widehat{\varphi}_n - D^{(k)}\widehat{\varphi}\|_{L^2(\mathbb{R})} \to 0, \quad \text{as } n \to \infty. \tag{3.30}$$

As when we proved (3.17), we use the decomposition

$$D^{(k)}\widehat{\varphi}_n(\xi) - D^{(k)}\widehat{\varphi}(\xi) = I + II + III + IV + V,$$

given in (3.18). Note that the same arguments right after (3.18) imply that

$$\|I\|_{L^2(\mathbb{R})}, \|III\|_{L^2(\mathbb{R})}, \|IV\|_{L^2(\mathbb{R})} \leq C\,\|F_n - m_0\|_{H^\alpha(\mathbb{T})} \to 0, \quad \text{as } n \to \infty.$$

For the other two cases we have:

$$\|II\|_{L^2(\mathbb{R})} \leq C\,\|\widehat{\varphi}_n - \widehat{\varphi}\|_{L^2(\mathbb{R})} \to 0, \quad \text{as } n \to \infty,$$

and

$$\|V\|_{L^2(\mathbb{R})} \leq C\,\|\widehat{\varphi}_n - \widehat{\varphi}\|_* \to 0, \quad \text{as } n \to \infty,$$

the first limit following from (3.28) and last one from Lemma 3.23. This shows (3.30) and completes the proof of the second case.

<u>*Case 3*</u>: $\alpha = k + \varepsilon, \quad k \in \mathbb{Z}^+,\ 0 < \varepsilon < 1.$

In this last case we need to show that

$$\omega_\varepsilon(D^{(k)}\widehat{\varphi}_n - D^{(k)}\widehat{\varphi}) =$$

$$\left[\int_{\mathbb{R}}\int_{\mathbb{R}} |D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi + \eta)|^2 \,\frac{d\xi\, d\eta}{|\eta|^{1+2\varepsilon}}\right]^{\frac{1}{2}} \to 0, \quad \text{as } n \to \infty. \tag{3.31}$$

By using the decomposition

$$D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi) - D^{(k)}(\widehat{\varphi}_n - \widehat{\varphi})(\xi + \eta) = [A_1 + \ldots + A_6] + [B_1 + \ldots + B_6],$$

given in (3.20), it is enough to estimate each of the terms $\omega_\varepsilon(A_j)$ and $\omega_\varepsilon(B_j)$, $j = 1, \ldots, 6$, separately. By following exactly the same arguments as right after (3.20), we can easily see that, when $j = 1, 2, 3, 5$, we have

$$\omega_\varepsilon(A_j) \leq C \, \|F_n - m_0\|_{H^\alpha(\mathbb{T})} \to 0, \quad \text{as } n \to \infty.$$

In the case A_4, we use Lemma 3.23 to obtain

$$\omega_\varepsilon(A_4) \leq C \, \|\widehat{\varphi}_n - \widehat{\varphi}\|_* \to 0, \quad \text{as } n \to \infty,$$

while for A_6 we have

$$\omega_\varepsilon(A_6) \leq C \, \omega_\varepsilon(\widehat{\varphi}_n - \widehat{\varphi}) \to 0, \quad \text{as } n \to \infty,$$

the convergence to 0 following now from (3.29) in case 1 above. To estimate the B_j's, one proceeds similarly. Note that B_1, B_2, B_3 and B_4 are treated by using (3.21), giving us

$$\omega_\varepsilon(B_j) \leq C \, \|F_n - m_0\|_{H^\alpha(\mathbb{T})} \to 0, \quad \text{as } n \to \infty, \; j = 1, 2, 3,$$

while

$$\omega_\varepsilon(B_4) \leq C \, \|\widehat{\varphi}_n - \widehat{\varphi}\|_* \to 0, \quad \text{as } n \to \infty.$$

To deal with B_5, the only difference with the local method of Theorem 3.9 is that the estimation in (3.22) should be replaced (by Corollary 1.38) by

$$\|\widehat{\varphi}_n(\cdot) - \widehat{\varphi}_n(\cdot + \eta)\|_*^2 \leq \begin{cases} C \, |\eta| \, \|\widehat{\varphi}_n\|_{H^1(\mathbb{R})}^2 , & \text{when } 0 < \varepsilon < \frac{1}{2} \\ C \, |\eta|^\gamma \, \|\widehat{\varphi}_n\|_{H^{\frac{1+\gamma}{2}}(\mathbb{R})}^2 , & \text{when } \frac{1}{2} \leq \varepsilon < 1, \end{cases}$$

for a suitable $\gamma \in (2\varepsilon, 2)$. In both cases, this implies

$$\omega_\varepsilon(B_5) \leq C \, \|F_n - m_0\|_{H^\alpha(\mathbb{T})} \to 0, \quad \text{as } n \to \infty.$$

Finally, for B_6, Corollary 1.38 tells us once more that

$$\|(\widehat{\varphi}_n - \widehat{\varphi})(\cdot) - (\widehat{\varphi}_n - \widehat{\varphi})(\cdot + \eta)\|_*^2 \leq \begin{cases} C\,|\eta|\,\|\widehat{\varphi}_n - \widehat{\varphi}\|^2_{H^1(\mathbb{R})}\,, & \text{when } 0 < \varepsilon < \frac{1}{2} \\ C\,|\eta|^\gamma\,\|\widehat{\varphi}_n - \widehat{\varphi}\|^2_{H^{\frac{1+\gamma}{2}}(\mathbb{R})}\,, & \text{when } \frac{1}{2} \leq \varepsilon < 1. \end{cases}$$

When $0 < \varepsilon < \frac{1}{2}$, this gives us

$$\omega_\varepsilon(B_6) \leq C\,\|\widehat{\varphi}_n - \widehat{\varphi}\|_{H^1(\mathbb{R})} \to 0, \quad \text{as } n \to \infty,$$

the convergence to 0 following from case 2 above. This would complete the proof of (3.31) when $0 < \varepsilon < \frac{1}{2}$. When $\frac{1}{2} \leq \varepsilon < 1$, we have

$$\omega_\varepsilon(B_6) \leq C\,\|\widehat{\varphi}_n - \widehat{\varphi}\|_{H^{\frac{1+\gamma}{2}}(\mathbb{R})} \to 0, \quad \text{as } n \to \infty,$$

where the convergence to 0 now follows from the fact that $1 < \frac{1+\gamma}{2} < \frac{3}{2}$ and the previous case. This completes the proof of (3.31) and, with it, establishes Theorem 3.27.

□

COROLLARY 3.32 *Let $\alpha > \frac{1}{2}$. Then $N\colon \mathcal{E}_\alpha \to \mathcal{S}_\alpha$, given by (2.43), is a homeomorphism of topological spaces with inverse $N^{-1} = M$, defined as in Lemma 2.7.*

The homeomorphism between $\mathcal{S}_\alpha$ and $\mathcal{E}_\alpha$ has some other important consequences, besides showing the interplay between scaling functions and low-pass filters. For instance, it allows us to work with the somewhat simpler (and better described) space $\mathcal{E}_\alpha$ of 2π-periodic functions whenever we want to find topological properties satisfied by $\mathcal{S}_\alpha$. In the next section we shall show that $\mathcal{E}_\alpha$ is a connected infinite dimensional manifold. Corollary 3.32 tells us that the same condition will automatically be satisfied by $\mathcal{S}_\alpha$. This will lead, in §5 below, to similar properties for the set of *α-localized wavelets.*

4 $\mathcal{E}_\alpha$ is a connected infinite dimensional manifold

In this section we study some topological properties of $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$. The limiting case when $\alpha = \infty$, in which $\mathcal{E}_\infty$ and $\mathcal{F}_\infty$ have the topology of the Fréchet space $C^\infty(\mathbb{T})$, was studied by A. Bonami, S. Durand and G. Weiss in [BDW]. They showed that both $\mathcal{E}_\infty$ and $\mathcal{F}_\infty$ are connected infinite dimensional manifolds (see Theorems 2.1 and 3.2 in [BDW]). We adapt their proofs to show similar properties for the spaces $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$, when $\alpha > \frac{1}{2}$. In most of what follows we will work with the latter instead of the former. This assumption is not too general since, as we proved in Theorem 3.8, $\mathcal{E}_\alpha$ is open in $\mathcal{F}_\alpha$. Our first theorem will show that $\mathcal{F}_\alpha$ is an *infinite dimensional submanifold* of the Hilbert space $H^\alpha(\mathbb{T})$. Formally, this comes from the fact that, for $m_0 \in \mathcal{F}_\alpha$, $|m_0|$ is completely determined by its values in $\frac{\mathbb{T}}{2} \equiv [\frac{-\pi}{2}, \frac{\pi}{2})$ (by (2.2)), while $\arg(m_0)$ can be any arbitrary real-valued function[{6}] in $H^\alpha(\mathbb{T})$. Roughly speaking, this gives us a local homeomorphism $m_0 \mapsto (|m_0|, \arg(m_0))$ between $\mathcal{F}_\alpha$ and $H^\alpha(\mathbb{T}/2) \times H^\alpha_{\mathbb{R}}(\mathbb{T})$, this product space being itself homeomorphic to $H^\alpha(\mathbb{T})$. The assumption $m_0(0) = 1$ in (2.2), for $m_0 \in \mathcal{F}_\alpha$, has also to be taken into consideration, and will tell us that $\mathcal{F}_\alpha$ actually lives in a "hyperplane" (and therefore a submanifold) of $H^\alpha(\mathbb{T})$. Let us now give a more precise meaning to what we are saying.

Suppose that $\alpha > \frac{1}{2}$ and consider, then, the following subset of $H^\alpha(\mathbb{T})$

$$\mathcal{H}_\alpha = \{\, f \in H^\alpha(\mathbb{T}) \mid f(0) = f(\pi) = 0 \,\}. \tag{4.1}$$

Note that $\mathcal{H}_\alpha$ is a closed subspace of the Hilbert space $H^\alpha(\mathbb{T})$ and, therefore, $\mathcal{H}_\alpha$ and $H^\alpha(\mathbb{T})$ are isomorphic. Geometrically, $\mathcal{H}_\alpha$ represents a hyperplane of codimension 2 of $H^\alpha(\mathbb{T})$.

Theorem 4.2 *Let $\alpha > \frac{1}{2}$. Then, $\mathcal{F}_\alpha$ is an infinite dimensional submanifold of $H^\alpha(\mathbb{T})$, in the sense that for every $m_0 \in \mathcal{F}_\alpha$, there exists a neighborhood of m_0,*

[{6}]When $z \in \mathbb{C}$, by $\arg z$ we denote a real number such that $z = |z| e^{i \arg z}$. Unless otherwise specified, we will consider only the main branch of the argument, that is, $\arg z \in [-\pi, \pi[$.

$U \subset \mathcal{F}_\alpha$, a neighborhood of $\mathbf{0}$, $V \subset \mathcal{H}_\alpha$, and a homeomorphism

$$\Phi : U \to V, \quad \textit{such that} \quad \Phi(m_0) = \mathbf{0}.$$

PROOF:

Let $m_0 \in \mathcal{F}_\alpha$ be fixed and define

$$m_1(\xi) = e^{i\xi}\, \overline{m_0(\xi + \pi)}, \quad \xi \in \mathbb{R}.$$

Then, $m_1 \in H^\alpha(\mathbb{T})$. Now, for every $\xi \in \mathbb{T}$ consider the unitary matrix

$$\mathcal{U}(\xi) = \begin{pmatrix} m_0(\xi) & m_0(\xi+\pi) \\ m_1(\xi) & m_1(\xi+\pi) \end{pmatrix}. \tag{4.3}$$

We use $\mathcal{U}$ to find a particular isomorphism between the Hilbert spaces [7] $H^\alpha(\mathbb{T})$ and $H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2)$.

LEMMA 4.4 *Let $\alpha > \frac{1}{2}$ and $m_0 \in \mathcal{F}_\alpha$ be fixed. Let $\mathcal{U}$ be the unitary matrix defined in (4.3) above. Then, the operator associated with the matrix:*

$$\begin{aligned} \mathcal{U} \quad &: H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2) \to H^\alpha(\mathbb{T}) \\ &(G, H) \mapsto \mathcal{U}(G, H) = G \cdot m_0 + H \cdot m_1 \end{aligned}$$

is an isomorphism of Hilbert spaces with inverse $\mathcal{U}^{-1}$ given by

$$F \in H^\alpha(\mathbb{T}) \mapsto \mathcal{U}^{-1}(F) = (G, H),$$

where

$$\left.\begin{aligned} G(\xi) &= F(\xi)\,\overline{m_0(\xi)} + F(\xi+\pi)\,\overline{m_0(\xi+\pi)}, \quad \xi \in \frac{\mathbb{T}}{2} \\ H(\xi) &= F(\xi)\,\overline{m_1(\xi)} + F(\xi+\pi)\,\overline{m_1(\xi+\pi)}, \quad \xi \in \frac{\mathbb{T}}{2} \end{aligned}\right\} \tag{4.5}$$

---

[7] Here, we say that $f \in H^\alpha(\mathbb{T}/2)$ if $f \in H^\alpha(\mathbb{T})$ and f is π-periodic; in this case, we define the norm $\|f\|_{H^\alpha(\mathbb{T}/2)} = \|f\|_{H^\alpha(\mathbb{T})}$. Equivalently, one can consider $H^\alpha(\mathbb{T}/2)$ as the Hilbert space of π-periodic functions that satisfy (1.20) (or (1.21) and (1.23)) when $\mathbb{T}$ is replaced by $\frac{\mathbb{T}}{2}$. In $H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2)$ we will consider the product norm $\|(f,g)\|_{H^\alpha(\mathbb{T}/2)\times H^\alpha(\mathbb{T}/2)} = \|f\|_{H^\alpha(\mathbb{T}/2)} + \|g\|_{H^\alpha(\mathbb{T}/2)}$, for all $(f,g) \in H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2)$.

PROOF:

The proof of this lemma is easy. It is clear that both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are linear and well-defined (by Lemma 1.30). Moreover, $\mathcal{U} \circ \mathcal{U}^{-1}(F) = F$ and $\mathcal{U}^{-1} \circ \mathcal{U}(G,H) = (G,H)$. We show that $\mathcal{U}$ and $\mathcal{U}^{-1}$ are bounded. Let $F \in H^\alpha(\mathbb{T})$ and $(G,H) \in H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2)$. Then, by using Lemma 1.30, we obtain

$$\begin{aligned}\|\mathcal{U}(G,H)\|_{H^\alpha(\mathbb{T})} &= \|Gm_0 + Hm_1\|_{H^\alpha(\mathbb{T})} \leq \|G\|_{H^\alpha(\mathbb{T})}\|m_0\|_{H^\alpha(\mathbb{T})} + \|H\|_{H^\alpha(\mathbb{T})}\|m_1\|_{H^\alpha(\mathbb{T})} \\ &\leq \left(\|m_0\|_{H^\alpha(\mathbb{T})} + \|m_1\|_{H^\alpha(\mathbb{T})}\right)\|(G,H)\|_{H^\alpha(\mathbb{T}/2)\times H^\alpha(\mathbb{T}/2)},\end{aligned}$$

and

$$\|\mathcal{U}^{-1}(F)\|_{H^\alpha(\mathbb{T}/2)\times H^\alpha(\mathbb{T}/2)} = \|F\,\overline{m_0} + F(\cdot+\pi)\overline{m_0(\cdot+\pi)}\|_{H^\alpha(\mathbb{T}/2)} +$$

$$\|F\,\overline{m_1} + F(\cdot+\pi)\overline{m_1(\cdot+\pi)}\|_{H^\alpha(\mathbb{T}/2)} \leq 2\left(\|m_0\|_{H^\alpha(\mathbb{T})} + \|m_1\|_{H^\alpha(\mathbb{T})}\right)\|F\|_{H^\alpha(\mathbb{T})}.$$

This shows that $\mathcal{U}$ and $\mathcal{U}^{-1}$ are bounded and completes the proof of the lemma.

$\square$

Before embarking on the proof of Theorem 4.2 we need to consider one more technical detail. Let $\alpha > \frac{1}{2}$ be fixed and consider the following notation:

$$\mathcal{I} \equiv \{f \in H^\alpha(\mathbb{T}/2) \mid f(0) = 0\} \quad \text{and} \quad \mathcal{I}_{\mathbb{R}} \equiv \{f \in \mathcal{I} \mid f \text{ is real-valued}\}.$$

Note that $\mathcal{I}$ and $\mathcal{I}_{\mathbb{R}}$ are both closed subspaces of the *real* Hilbert space $H^\alpha(\mathbb{T}/2)$ (say, with the *symmetric* inner product $<f,g> \equiv \frac{1}{2}\{(f,g)+(g,f)\}$). Thus, it follows from classical Hilbert space theory that there is an isomorphism between them (in fact, they are isometrically isomorphic to $\ell^2_{\mathbb{R}}$). For the sake of completeness, we present here the construction of one such isomorphism. Let

$$\begin{aligned} J_1 \;:\; & \mathcal{I}_{\mathbb{R}} \times \mathcal{I}_{\mathbb{R}} \to \mathcal{I} \\ & (F,G) \mapsto (F+iG). \end{aligned}$$

This is clearly an isomorphism of real Hilbert spaces.

On the other hand, we define $J_2 : \mathcal{I}_{\mathbb{R}} \to \mathcal{I}_{\mathbb{R}} \times \mathcal{I}_{\mathbb{R}}$ as follows; if $f(\theta) = \sum_{k\in\mathbb{Z}} c_k e^{ik(2\theta)}$ is a function in $\mathcal{I}_{\mathbb{R}}$ then, we let

$$(J_2(f))(\theta) \equiv (\, F(\theta)\, ,\, G(\theta)\,) = \left(\sum_{k\in\mathbb{Z}} a_k e^{ik(2\theta)}, \sum_{k\in\mathbb{Z}} b_k e^{ik(2\theta)} \right),$$

where

$$a_k = \begin{cases} c_{2k} + c_{2k-1}, & k \geq 1 \\ c_0, & k = 0 \\ c_{2k} + c_{2k+1}, & k \leq -1 \end{cases} \quad \text{and} \quad b_k = \begin{cases} c_{2k} - c_{2k-1}, & k \geq 1 \\ \displaystyle\sum_{\ell\in\mathbb{Z}} c_{2\ell+1} - \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\neq 0}} c_{2\ell}, & k = 0 \\ c_{2k} - c_{2k+1}, & k \leq -1. \end{cases}$$

Note that, if f is real valued and $f(0) = 0$, then, $c_{-k} = \overline{c_k}$ and $\sum_{k\in\mathbb{Z}} c_k = 0$. In particular, this implies that $a_{-k} = \overline{a_k}$ and $b_{-k} = \overline{b_k}$, $k \in \mathbb{Z}$, and $\sum_{k\in\mathbb{Z}} a_k = \sum_{k\in\mathbb{Z}} b_k = 0$. Thus, J_2 is a well-defined linear operator (between real Hilbert spaces). Moreover, its inverse $J_2^{-1}(F, G)$, for $F(\theta) = \sum_{k\in\mathbb{Z}} a_k e^{ik(2\theta)}$, $G(\theta) = \sum_{k\in\mathbb{Z}} b_k e^{ik(2\theta)} \in \mathcal{I}_{\mathbb{R}}$, is given by:

$$(J_2^{-1}(F, G))(\theta) = \sum_{k\in\mathbb{Z}} c_k e^{ik(2\theta)},$$

where

$$c_0 = a_0 \quad \text{and} \quad \begin{cases} c_{2k} = \dfrac{a_k + b_k}{2}, & & k \neq 0 \\ c_{2k-1} = \dfrac{a_k - b_k}{2}, & c_{-(2k-1)} = \dfrac{a_{-k} - b_{-k}}{2}, & k \geq 1. \end{cases}$$

We leave to the reader the verification that $J_2 \circ J_2^{-1}(F, G) = (F, G)$, $J_2^{-1} \circ J_2(f) = f$, and both J_2, J_2^{-1} are bounded. Then, $J \equiv J_1 \circ J_2 : \mathcal{I}_{\mathbb{R}} \to \mathcal{I}$ is one of the isomorphisms seeked.

We can now continue with the proof of Theorem 4.2. We are looking for a homeomorphism Φ between a neigborhood of $m_0 \in \mathcal{F}_\alpha$ and a neighborhood of $\mathbf{0} \in \mathcal{H}_\alpha$. In order to do so, we proceed via $H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2)$, as in the following diagram:

$$\begin{tikzcd} \mathcal{F}_\alpha \arrow[r, "\Phi"] \arrow[d, "\mathcal{U}^{-1}"'] & \mathcal{H}_\alpha \\ \mathcal{U}^{-1}(\mathcal{F}_\alpha) \arrow[r, "\tilde{\Phi}"] & \mathcal{U}^{-1}(\mathcal{H}_\alpha) \arrow[u, "\mathcal{U}"'] \end{tikzcd} \qquad (*)$$

Note that $\mathcal{U}^{-1}(m_0) = (\mathbf{1}, \mathbf{0})$ and $\mathcal{U}^{-1}(\mathbf{0}) = (\mathbf{0}, \mathbf{0})$. Then, by Lemma 4.4, it is enough to find a neighborhood $\tilde{U}$ of $(\mathbf{1}, \mathbf{0})$ in

$$\mathcal{U}^{-1}(\mathcal{F}_\alpha) = \{\, (G, H) \in H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2) \mid |G|^2 + |H|^2 = \mathbf{1} \text{ and } (G, H)(0) = (1, 0) \,\},$$

a neighborhood $\tilde{V}$ of $(\mathbf{0}, \mathbf{0})$ in

$$\mathcal{U}^{-1}(\mathcal{H}_\alpha) = \{\, (G, H) \in H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2) \mid (G, H)(0) = (0, 0) \,\},$$

and a homeomorphism

$$\tilde{\Phi} : \tilde{U} \to \tilde{V} \quad \text{such that} \quad \tilde{\Phi}(\mathbf{1}, \mathbf{0}) = (0, 0).$$

Indeed, if we do this, we can define $U = \mathcal{U}(\tilde{U}), V = \mathcal{U}(\tilde{V})$ and $\Phi = \mathcal{U} \circ \tilde{\Phi} \circ \mathcal{U}^{-1}$, and this would complete the proof of our theorem. Consider the following domain of the complex plane

$$\Omega = \left\{ z \in \mathbb{C} \mid \frac{\sqrt{3}}{2} < |z| < 2, \ |\arg z| < \frac{\pi}{6} \right\}.$$

(See Figure 4.2 below.)

We define

$$\tilde{U} = \{\, (G, H) \in H^\alpha(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2) \mid G(\mathbb{T}) \subset \Omega \text{ and } \|H\|_\infty < 1/2 \,\} \cap \mathcal{U}^{-1}(\mathcal{F}_\alpha).$$

By Sobolev's Imbedding Theorem 1.25, $\tilde{U}$ is an open neighborhood of $(\mathbf{1}, \mathbf{0})$ in $\mathcal{U}^{-1}(\mathcal{F}_\alpha)$. On the other hand, define

$$V^\sharp = \{\, (A, H) \in H^\alpha_{\mathbb{R}}(\mathbb{T}/2) \times H^\alpha(\mathbb{T}/2) \mid \|A\|_\infty < \pi/6, \ \|H\|_\infty < 1/2 \,\} \cap \mathcal{U}^{-1}(\mathcal{H}_\alpha),$$

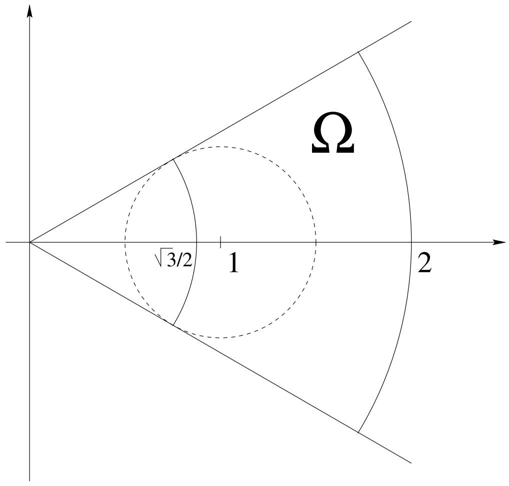

Figure 4.2: Sketch of the region Ω.

which is an open neighborhood of $(\mathbf{0},\mathbf{0})$ in $\mathcal{U}^{-1}(\mathcal{H}_\alpha)\cap[H^\alpha_{\mathbb{R}}(\mathbb{T}/2)\times H^\alpha(\mathbb{T}/2)]$. Then, we define the homeomorphism $\Phi^\sharp:\tilde{U}\to V^\sharp$ by

$$\Phi^\sharp(G,H)=(A,H),\quad \text{where}\quad A=\arg(G)=\frac{1}{i}\log\frac{G}{|G|},\quad \text{for } (G,H)\in\tilde{U}.$$

We claim that $\Phi^\sharp$ is well-defined and continuous. Indeed, by using the results in §1.3 about Banach algebras, we see that $G\mapsto G/|G|$ is continuous (into $H^\alpha(\mathbb{T}/2)$) when $\sigma(G)\subset\Omega$ (this is because $|G|=\sqrt{G\cdot\overline{G}}$ and the square root is well-defined since $\sigma(G\cdot\overline{G})=|G|^2(\mathbb{T}/2)\subset]\frac{3}{4},4[$, which is simply connected and does not contain 0). Moreover, since

$$\sigma(G/|G|)\subset\left\{\,|z|=1,\ |\arg z|<\frac{\pi}{6}\right\},$$

we can use the holomorphic functional calculus of Theorem 1.40 to conclude that $G\mapsto\log(G/|G|)$ is continuous in $H^\alpha(\mathbb{T}/2)$ and $(A,H)\in V^\sharp$. On the other hand, the candidate for inverse, $(\Phi^\sharp)^{-1}$, is given by

$$(A,H)\mapsto(e^{iA(\xi)}\sqrt{1-|H(\xi)|^2},H),\quad (A,H)\in V^\sharp.$$

Another use of the holomorphic functional calculus in $H^\alpha(\mathbb{T}/2)$ shows that $(\Phi^\sharp)^{-1}$ is continuous and $(\Phi^\sharp)^{-1}(A,H)\in\tilde{U}$. Moreover, since $\Phi^\sharp\circ(\Phi^\sharp)^{-1}(A,H)=(A,H)$, and $(\Phi^\sharp)^{-1}\circ\Phi^\sharp(G,H) = (G,H)$, for all $(A,H) \in V^\sharp$ and $(G,H) \in \tilde{U}$, we have that $\Phi^\sharp : \tilde{U} \to V^\sharp$ is a homeomorphism.

Now, the mapping

$$(A,H) \mapsto J^\sharp(A,H) \equiv (\, J(A), H\,)$$

gives an isomorphism between $V^\sharp = \{(A,H) \in \mathcal{I}_{\mathbb{R}} \times \mathcal{I} \mid \|A\|_\infty < \pi/6,\ \|H\|_\infty < 1/2\}$ and the open set $J^\sharp(V^\sharp) \equiv \tilde{V} \subset \mathcal{I} \times \mathcal{I} = \mathcal{U}^{-1}(\mathcal{H}_\alpha)$. Hence, if we define

$$\tilde{\Phi} = J^\sharp\circ\Phi^\sharp : \tilde{U} \to \tilde{V} \subset \mathcal{U}^{-1}(\mathcal{H}_\alpha)$$

we obtain a homeomorphism that completes the diagram $(*)$ above (with the mappings restricted to appropriate open sets). This establishes Theorem 4.2.

□

Remark 4.6 We have not considered here any "differential structure" in the *manifold* $\mathcal{F}_\alpha$. One can do so by noticing that for any pair (Φ_1, U_1), (Φ_2, U_2), as in the previous theorem, such that $U_1 \cap U_2 \neq \emptyset$, we have that

$$\Phi_1\circ\Phi_2^{-1} : \Phi_2(U_1 \cap U_2) \longrightarrow \Phi_1(U_1 \cap U_2)$$

is a C^∞-diffeomorphism between open sets in the *real* Hilbert space $\mathcal{H}_\alpha$. This actually follows from the construction of Φ above and from elementary properties of $H^\alpha(\mathbb{T})$ (of the type of what we said in Remark 1.41). In the terminology of infinite dimensional geometry (see, e.g., Chapter II of [LAN]), $\mathcal{F}_\alpha$ becomes a *Hilbert manifold* (or a *manifold modeled in the Hilbert space* $\mathcal{H}_\alpha \simeq H^\alpha(\mathbb{T})$). We restrict the investigation of this thesis to show the connectivity of $\mathcal{F}_\alpha$, but stronger properties arising from the differential structure might be true. We postpone this study to a future occasion.

Theorem 4.2 is also valid if we replace $\mathcal{F}_\alpha$ by $\mathcal{E}_\alpha$. This is just a consequence of the openess of this last set in the first one, which we proved in Theorem 3.8 above. From these considerations we deduce the following result:

COROLLARY 4.7 *Let $\alpha > \frac{1}{2}$. Then, $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ are locally connected topological spaces.*

PROOF: This follows from the fact that $\mathcal{H}_\alpha$ is connected and $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ are locally homeomorphic to $\mathcal{H}_\alpha$.

$\square$

In order to show that $\mathcal{F}_\alpha$ is "globally" connected, given a pair of functions (m_0, m_1) in $\mathcal{F}_\alpha$, we find a continuous path (in $\mathcal{F}_\alpha$) that joins one with the other. One candidate for such a path is $t \mapsto m_t \equiv \sqrt{(1-t)|m_0|^2 + t|m_1|^2}$ (note m_t satisfies (2.2)). Unfortunately, taking the square root does not preserve the smoothness of the functions m_0 and m_1 (although m_t still belongs to $L^2(\mathbb{T})$). It turns out that one can overcome this difficulty when m_0 and m_1 are trigonometric polynomials. Indeed, under these conditions, the Fejér-Riesz Lemma tells us that we may choose a trigonometric polynomial m_t so that $|m_t|^2 = (1-t)|m_0|^2 + t|m_1|^2$. The assumption that $m_0, m_1 \in \mathcal{T}$ is not too general, and can be justified from the local connectivity of $\mathcal{F}_\alpha$ (Corollary 4.7) and the following density result (see also Lemma 1.47):

THEOREM 4.8 *Let $\alpha > \frac{1}{2}$ and $F \in \mathcal{F}_\alpha$. Then, there exists a sequence of trigonometric polynomials $\{P_n\}_{n=1}^\infty$ such that $P_n(0) = 1$ and $P_n(\pi) = 0$, $n = 1, 2, \ldots$ and*

$$\frac{P_n}{\sqrt{|P_n|^2 + |P_n(\cdot + \pi)|^2}} \to F \quad \text{in } \mathcal{F}_\alpha, \quad \text{as } n \to \infty. \tag{4.9}$$

In particular, the set

$$\left\{ \frac{P}{\sqrt{|P|^2 + |P(\cdot + \pi)|^2}} \;\middle|\; P \in \mathcal{T},\ P(0) = 1,\ P(\pi) = 0,\ \ |P|^2 + |P(\cdot + \pi)|^2 \neq 0 \right\}$$

is a dense subset of $\mathcal{F}_\alpha$.

PROOF:

We showed in Lemma 1.47 that the set $\mathcal{T}$ of trigonometric polynomials is dense in $H^\alpha(\mathbb{T})$. Let $\{Q_n\}_{n=1}^\infty \subset \mathcal{T}$ be a sequence such that $Q_n \to F$ in $H^\alpha(\mathbb{T})$ (e.g., take the

n^{th}-symmetric partial sum of the Fourier series of F). Then, by Sobolev's Imbedding Theorem 1.25,

$$\lim_{n\to\infty} Q_n(0) = F(0) = 1 \quad \text{and} \quad \lim_{n\to\infty} Q_n(\pi) = F(\pi) = 0.$$

Therefore, for large n we have $|Q_n(0) - Q_n(\pi)| > \frac{1}{2}$. Define for each such n

$$P_n(\xi) = \frac{Q_n(\xi) - Q_n(\pi)}{Q_n(0) - Q_n(\pi)}, \quad \xi \in \mathbb{R}.$$

Then, $P_n \in \mathcal{T}$ and $P_n \to F$ in $H^\alpha(\mathbb{T})$. Moreover, $P_n(0) = 1$ and $P_n(\pi) = 0$. Once again, Sobolev's Theorem 1.25 tells us that $\sqrt{|P_n(\xi)|^2 + |P_n(\xi+\pi)|^2} \to 1$ uniformly in $\mathbb{T}$. Therefore, for n large enough, we must have that

$$|P_n(\xi)|^2 + |P_n(\xi+\pi)|^2 > \frac{1}{2}, \quad \text{for all } \xi \in \mathbb{T}.$$

From the properties in §1.3 about Banach algebras we see that

$$(|P_n(\cdot)|^2 + |P_n(\cdot+\pi)|^2)^{-1/2} \in H^\alpha(\mathbb{T})$$

and (4.9) holds. This completes the proof of the theorem.

□

REMARK 4.10 Note that if $F \in \mathcal{E}_\alpha$, the trigonometric polynomials $\{P_n\}$ in the previous theorem can be taken to satisfy Cohen's condition (2.15). This follows from the fact that $\mathcal{E}_\alpha$ is an open subset of $\mathcal{F}_\alpha$.

THEOREM 4.11 *Let $\alpha > \frac{1}{2}$. Then $\mathcal{F}_\alpha$ and $\mathcal{E}_\alpha$ are arcwise connected topological spaces.*

PROOF:

Let us choose, as a fixed element of $\mathcal{F}_\alpha$, the Haar filter: $m_H(\xi) = \frac{1+e^{i\xi}}{2}$, $\xi \in \mathbb{T}$. Let m_0 be an arbitrary element of $\mathcal{F}_\alpha$. We will construct a continuous path in $\mathcal{F}_\alpha$ that joins m_0 with m_H. Moreover, we will see that this path can be chosen within $\mathcal{E}_\alpha$ if $m_0 \in \mathcal{E}_\alpha$.

We borrow the following result from [BDW].

Proposition 4.12 : see Lemma 3.6 in [BDW]

Given a polynomial $Q(z)=\sum_{n=0}^{N} a_n z^n$ *such that* $Q(1)=1$, *there exists an integer* k, $0\le k\le N$, *and a continuous map*

$$[0,1] \longrightarrow C^\infty(\mathbb{C}) \tag{4.13}$$

$$t \longmapsto Q_t$$

such that

(i) $Q_0(z)=z^k$ *and* $Q_1=Q$.

(ii) Q_t *are all polynomials of degree* $\le N$, $t\in[0,1]$.

(iii) $|Q_t(z)|^2=(1-t)+t|Q(z)|^2$, *when* $|z|=1$, *for all* $0\le t\le 1$.

Remark 4.14 We are considering the usual Fréchet topology in the space $C^\infty(\mathbb{C})=C^\infty(\mathbb{R}^2)$ in (4.13) above. That is, if $f_n, f\in C^\infty(\mathbb{R}^2)$, we say that $f_n\to f$ in $C^\infty(\mathbb{R}^2)$ if, and only if, $D^{\mathbf{k}}f_n\to D^{\mathbf{k}}f$ uniformly in compact sets of $\mathbb{R}^2$ and for every multi-index $\mathbf{k}=(k_1,k_2)$ such that $k_i\ge 0$, $i=1,2$.

Remark 4.15 We will not prove Proposition 4.12 here, but a sketch of the proof is as follows. When $|z|=1$, consider the positive trigonometric polynomial $(1-t)+t|Q(z)|^2$ given in (iii) above. Then, an argument of the type used to establish the Fejér-Riesz Lemma (see, e.g., Lemma 3.16 in Chapter 2 of [HW]) allows us to construct a trigonometric polynomial Q_t such that $|Q_t(z)|^2=(1-t)+t|Q(z)|^2$, when $|z|=1$. The selection of the Q_t's is not unique in general, but with some care in the process of construction we can make the coefficients of Q_t depend continuously on t. The integer k in (i) appears when passing from $|z|=1$ to $z\in\mathbb{C}$, and is related to the multiplicity of the zeros of Q. For more details see [BDW].

Continuing with the proof of Theorem 4.11, let m_0 be a fixed element in $\mathcal{F}_\alpha$, and let P be a trigonometric polynomial (as in Theorem 4.8) such that m_0 and $P\left(|P|^2+|P(\cdot+\pi)|^2\right)^{-\frac{1}{2}}$ can be joined by a continuous path in $\mathcal{F}_\alpha$ (we can do this by

Corollary 4.7). We may assume that $|P(\xi)|^2+|P(\xi+\pi)|^2>\frac{1}{2}$, when $\xi\in\mathbb{T}$. Suppose that $P(\xi)=\sum_{n=-N}^{N}a_n e^{in\xi}$ has degree N. Then, since $P(\pi)=0$, we can write P as

$$P(\xi)=e^{-iN\xi}\left(\frac{1+e^{i\xi}}{2}\right)Q(e^{i\xi}),\quad \xi\in\mathbb{R},$$

where Q is a polynomial (in $\mathbb{C}$) of degree $\leq 2N-1$. Applying Proposition 4.12 to Q we obtain a continuous map $t\mapsto Q_t$, $t\in[0,1]$, as in (4.13), and an integer k, $0\leq k\leq 2N-1$, such that properties (i), (ii) and (iii) of the proposition are satisfied. Let us define

$$P_t(\xi)=e^{-iN\xi}\left(\frac{1+e^{i\xi}}{2}\right)Q_t(e^{i\xi}),\quad \xi\in\mathbb{R},\ t\in[0,1]. \tag{4.16}$$

Then, the map

$$\begin{aligned}[0,1]&\longrightarrow C^\infty(\mathbb{T})\\ t&\longmapsto P_t\end{aligned}$$

is continuous[8]. Moreover,

$$(i)\quad P_0(\xi)=e^{ip\xi}\,\frac{1+e^{i\xi}}{2}\quad \text{for some } p\in\mathbb{Z},\quad \text{and}\quad P_1=P.$$

(ii) the trigonometric polynomials P_t, $t\in[0,1]$, have all degree $\leq N$.

$$(iii)\quad |P_t(\xi)|^2=(1-t)\left|\frac{1+e^{i\xi}}{2}\right|^2+t|P(\xi)|^2,\quad \xi\in\mathbb{T},\quad t\in[0,1]. \tag{4.17}$$

In particular, note that (4.17) above implies that

$$|P_t(\xi)|^2+|P_t(\xi+\pi)|^2=(1-t)+t\left(|P(\xi)|^2+|P(\xi+\pi)|^2\right),\quad \xi\in\mathbb{T},\quad t\in[0,1]. \tag{4.18}$$

By our assumption on P, this last expression must always be greater than $\frac{1}{2}$. Therefore, we can define

$$\Phi_t(\xi)=\frac{P_t(\xi)}{\sqrt{|P_t(\xi)|^2+|P_t(\xi+\pi)|^2}},\quad \xi\in\mathbb{R},\quad t\in[0,1], \tag{4.19}$$

which is an element in $\mathcal{F}_\alpha$ for every $t\in[0,1]$. Now, since we have the continuous inclusion $C^\infty(\mathbb{T})\hookrightarrow H^\alpha(\mathbb{T})$, for all $\alpha>\frac{1}{2}$, we see that the map

[8] Here, the Fréchet space $C^\infty(\mathbb{T})$ is considered to have the topology of the uniform convergence in all the derivatives. That is, for $f_n, f\in C^\infty(\mathbb{T})$, $f_n\to f$ in $C^\infty(\mathbb{T})$ if and only if $D^{(k)}f_n\to D^{(k)}f$ uniformly in $\mathbb{T}$, for every integer $k\geq 0$.

$$[0,1] \longrightarrow \mathcal{F}_\alpha$$
$$t \longmapsto \Phi_t$$

is continuous and

$$(i) \quad \Phi_0(\xi) = e^{ip\xi}\,\frac{1+e^{i\xi}}{2} \qquad \text{for some } p \in \mathbb{Z},\ -N \le p \le N-1.$$
$$(ii) \quad \Phi_1(\xi) = P(\xi)\,(|P(\xi)|^2 + |P(\xi+\pi)|^2)^{-\frac{1}{2}}, \quad \xi \in \mathbb{R}.$$

Note that $\Phi_0 \ne m_H$ unless $p = 0$. To complete the proof of the theorem we need to join $m_H(\xi) = \frac{1+e^{i\xi}}{2}$ and $e^{i\xi}\,\frac{1+e^{i\xi}}{2}$ with a continuous arc in $\mathcal{E}_\alpha$. When $p = 1$ we saw how to do this in example 1 of §3 (just connect the "pairs" $(1,0)$ and $(0,0)$ continuously within the punctured circle $C \setminus (1,1)$). Iterating this process, we can connect, within $\mathcal{E}_\alpha$, the filters $\frac{1+e^{i\xi}}{2}$ and $e^{ip\xi}\,\frac{1+e^{i\xi}}{2}$, for any $p \in \mathbb{Z}$. This, together with the map Φ_t above and the local connectivity of Corollary 4.7, shows that $\mathcal{F}_\alpha$ is an arcwise connected topological space.

Suppose now that the original function m_0 belongs to $\mathcal{E}_\alpha$. Then, by Remark 4.10, we can take P as in Theorem 4.8 in such a way that $P \in \mathcal{E}_\alpha$. In particular, we can find a compact set K as in (2.15) such that $P(2^{-j}\xi) \ne 0,\ \xi \in K,\ j = 1, 2, \ldots$ Then, the trigonometric polynomials P_t defined in (4.16) must satisfy, by (4.17), the same condition; that is, $P_t(2^{-j}\xi) \ne 0,\ \xi \in K,\ j = 1, 2, \ldots, t \in (0,1]$. Therefore, the path $t \in [0,1] \mapsto \Phi_t$ defined in (4.19) lies within $\mathcal{E}_\alpha$ and, by the same argument as before, we conclude that $\mathcal{E}_\alpha$ is connected by arcs.

□

Remark 4.20 Note that, if m_0 is a trigonometric polynomial in $\mathcal{F}_\alpha$ of degree $\le N$, then, the path $t \mapsto \Phi_t$ constructed above consists of trigonometric polinomials in $\mathcal{E}_\alpha$ of degree $\le N$, for every $t \in [0,1)$. In particular, the set of trigonometric polynomials of degree $\le N$ in $\mathcal{F}_\alpha$ is connected. The same holds for the set of trigonometric polynomials of degree $\le N$ in $\mathcal{E}_\alpha$.

5 Connectivity in the set of α-localized wavelets

In this section we introduce our main results related to the theory of α-localized wavelets. The close link existing between these objects and the α-localized MRA's of §§ 2,3 and 4 (provided by a theorem of Lemarié) will move us along these lines to obtain some interesting properties, such as a complete decomposition of the set $\mathcal{W}_\alpha$ of α-localized wavelets into its connected components. It turns out that a wavelet belongs to one connected component or another depending on the homotopy degree of its "phase" $\nu(\xi)$, that is, the unimodular, 2π-periodic function necessary to reconstruct the wavelet from an MRA as in (3.9) of Chapter 1. In §5.3 we study the functional equation that determines when a "simple phase" can be chosen (such as $\nu(\xi) = \mathbf{1}$ or $e^{ik\xi}$), and we give a complete solution to it. We conclude the section by presenting two theorems on connectivity of more general sets of wavelets, due to different authors, in which no topological restriction such as the "homotopy degree" appears.

5.1 Definition and properties of α-localized wavelets

Definition 5.1 *Let $\alpha > \frac{1}{2}$. We say that a function $\psi \in L^2(\mathbb{R})$ belongs to $\mathcal{W}_\alpha$ when*

(i) ψ is an orthonormal wavelet (see Definition (2.1) in Chapter 1).

(ii) $\widehat{\psi} \in H^\alpha(\mathbb{R})$.

(iii) There exists an $\varepsilon > 0$ such that $\psi \in H^\varepsilon(\mathbb{R})$.

We consider $\mathcal{W}_\alpha$ as a topological space with the topology of $L^2((1+|x|^2)^\alpha\,dx)$; that is, $\psi_n \to \psi$ in $\mathcal{W}_\alpha$ if and only if $\widehat{\psi}_n \to \widehat{\psi}$ in $H^\alpha(\mathbb{R})$.

We say that $\psi \in L^2(\mathbb{R})$ belongs to $\mathcal{W}_\infty$ when $\psi \in \mathcal{W}_\alpha$ for all $\alpha > \frac{1}{2}$. In $\mathcal{W}_\infty$ we consider the topology generated by the seminorms $L^2((1+|x|^2)^N\,dx)$, $N = 1, 2, \ldots$, meaning that $\psi_n \to \psi$ in $\mathcal{W}_\infty$ if and only if $\widehat{\psi}_n \to \widehat{\psi}$ in $H^N(\mathbb{R})$ for all $N \geq 1$.

EXAMPLE 1

The *Haar wavelet*

$$\psi_H(x) = \chi_{[-1,0)}(2x-1) - \chi_{[-1,0)}(2x-2) = \begin{cases} 1\,, & \text{when } \ 0 \le x < \frac{1}{2} \\ -1\,, & \text{when } \ \frac{1}{2} \le x < 1. \end{cases}$$

It is not hard to see that ψ_H is an orthonormal wavelet. In fact, ψ_H arises from an MRA as in (3.10) of Chapter 1:

$$\widehat{\psi}_H(2\xi) = e^{-i\xi}\,\overline{m_H(\xi+\pi)}\,\widehat{\varphi}_H(\xi), \quad \xi \in \mathbb{R},$$

where $m_H(\xi) = \frac{1+e^{i\xi}}{2}$, $\xi \in \mathbb{T}$, is the Haar filter and $\varphi_H = \chi_{[-1,0]}$ is the Haar scaling function (see Example B in Chapter 2 in [HW]). It is clear that $\psi_H \in L^2((1+|x|^2)^\alpha\,dx)$, for all $\alpha > \frac{1}{2}$. To see that (iii) in Definition 5.1 holds note that

$$\widehat{\psi}_H(\xi) = 2i\,e^{-i\frac{\xi}{2}}\,\frac{1-\cos\frac{\xi}{2}}{\xi}, \quad \xi \in \mathbb{R},$$

and, therefore, $\widehat{\psi}_H \in L^2((1+|\xi|^2)^\varepsilon\,d\xi)$ when $0 < \varepsilon < \frac{1}{2}$. This shows that the Haar wavelet is in $\mathcal{W}_\infty$.

The Haar wavelet is just a particular instance of a more general case; namely, all wavelets arising from α-localized MRAs are also in $\mathcal{W}_\alpha$.

Theorem 5.2 *Let $\alpha > \frac{1}{2}$ and suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is an α-localized MRA, and let φ be any α-localized scaling function*[9] *with its corresponding low-pass filter m_0. Let ν be any unimodular 2π-periodic function in $H^\alpha(\mathbb{T})$. Then, ψ defined by*

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\,\overline{\nu(\xi)}\,\overline{m_0(\xi/2+\pi)}\,\widehat{\varphi}(\xi/2), \quad \xi \in \mathbb{R}, \tag{5.3}$$

is a wavelet in the class $\mathcal{W}_\alpha$.

The previous theorem tells us that the class $\mathcal{W}_\alpha$ is naturally associated with the class of α-localized MRAs. It turns out that the converse is also true:

[9] We remind the reader that an α-localized scaling function must satisfy (3.13) of Chapter 1 and (2.1) of §2 in this chapter.

THEOREM 5.4 *Let $\alpha > \frac{1}{2}$ and $\psi \in \mathcal{W}_\alpha$. Then, ψ is a wavelet arising from an α-localized MRA. More precisely, there exists an α-localized scaling function φ, with low-pass filter m_0, and a 2π-periodic unimodular function $\nu \in H^\alpha(\mathbb{T})$ such that ψ can be written in terms of φ, m_0 and ν as in (5.3).*

Moreover, suppose that $\tilde{\varphi}$ is another α-localized scaling function, with low-pass filter $\tilde{m}_0$, and $\tilde{\nu} \in H^\alpha(\mathbb{T})$ is unimodular, and suppose that one can also write ψ as:

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\, \overline{\tilde{\nu}(\xi)}\, \overline{\tilde{m}_0(\xi/2+\pi)}\, \hat{\tilde{\varphi}}(\xi/2), \quad \xi \in \mathbb{R}. \tag{5.5}$$

Then, there exists a unimodular $\mu \in H^\alpha(\mathbb{T})$ such that $\widehat{\varphi}(\xi) = \mu(\xi)\, \hat{\tilde{\varphi}}(\xi)$, $\xi \in \mathbb{R}$, and ν and $\tilde{\nu}$ are homotopically equivalent.

The two previous theorems are essentially due to P. G. Lemarié-Rieusset and their proofs are far away from being trivial. We give a sketch of them, but for more details we refer the reader to the original paper of Lemarié, [LEM], or to Chapter 3 of [KAH-LEM].

<u>Sketch of Proof of Theorem 5.2</u>

The first part of this theorem is not difficult. Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is an α-localized MRA with scaling function φ and low-pass filter m_0. Then, by Lemma 2.3 we have that $\widehat{\varphi} \in H^\alpha(\mathbb{R})$ and $m_0 \in H^\alpha(\mathbb{T})$. In this case, for any unimodular $\nu \in H^\alpha(\mathbb{T})$, the considerations in §3 of Chapter 1 and Lemma 1.35 tell us that ψ defined as in (5.3) is a wavelet with $\widehat{\psi} \in H^\alpha(\mathbb{R})$. In order to show that $\psi \in \mathcal{W}_\alpha$, one needs to verify that (iii) in Definition 5.1 holds. This last part follows from the following theorem of L. Hervé, whose proof we do not include here (see, e.g., Theorem 2 in Chapter 4 of [KAH-LEM], or the original work by Hervé in Chapter II of [HER]).

THEOREM 5.6 *Let $\alpha > \frac{1}{2}$ and $\widehat{\varphi} \in \mathcal{S}_\alpha$ (as in Definition 2.6). Then, there exists a real number $\varepsilon > 0$ such that $\varphi \in H^\varepsilon(\mathbb{R})$.*

To conclude the proof of Theorem 5.2 note that, by the theorem just stated, $\widehat{\varphi} \in L^2((1+|\xi|^2)^\varepsilon\, d\xi)$ and, since m_0 and ν are bounded, we must have that also

$\widehat{\psi} \in L^2((1+|\xi|^2)^\varepsilon\, d\xi)$. This shows that (iii) in Definition 5.1 holds and establishes Theorem 5.2.

□

Sketch of Proof of Theorem 5.4

Suppose that $\psi \in \mathcal{W}_\alpha$. To show that ψ arises from an α-localized MRA one follows the steps listed below:

1. Since $\widehat{\psi} \in H^\alpha(\mathbb{R})$ and $\psi \in H^\varepsilon(\mathbb{R})$, by using complex interpolation, it is possible to find two numbers $\alpha' > \frac{1}{2}$ and $\varepsilon' > 0$ such that $|\xi|^{\varepsilon'}\widehat{\psi}(\xi) \in H^{\alpha'}(\mathbb{R})$ (see Lemma 2 in Chapter 2 of [KAH-LEM]).

2. If this is the case, the *dimension function*, defined by the series:
$$D_\psi(\xi) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi+2k\pi))|^2, \quad \xi \in \mathbb{R}, \tag{5.7}$$
converges uniformly on compact sets of $\mathbb{R}\setminus 2\pi\mathbb{Z}$. Indeed, by Corollary 1.17 we know that $\sum_{k\in\mathbb{Z}} |\xi+2k\pi|^{2\varepsilon'}\,|\widehat{\psi}(\xi+2k\pi)|^2$ converges uniformly in $[-\pi,\pi]$. Then, for all $j \geq 1$ and $0 < \delta \leq |\xi| \leq \pi$,
$$\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi+2k\pi))|^2 \leq \frac{1}{(2^j\delta)^{2\varepsilon'}} \sum_{k\in\mathbb{Z}} |2^j(\xi+2k\pi)|^{2\varepsilon'}\,|\widehat{\psi}(2^j(\xi+2k\pi))|^2$$
$$\leq \frac{1}{(2^j\delta)^{2\varepsilon'}} \sup_{\eta\in\mathbb{T}} \sum_{k\in\mathbb{Z}} |\eta+2k\pi|^{2\varepsilon'}\,|\widehat{\psi}(\eta+2k\pi)|^2 \leq \frac{C}{(2^j\delta)^{2\varepsilon'}}\,\|\,|\eta|^{\varepsilon'}\widehat{\psi}(\eta)\|^2_{H^{\alpha'}(\mathbb{R})}.$$
Summing over $j \geq 1$ we obtain the uniform convergence in $[-\pi,\pi]\setminus(-\delta,\delta)$ of the series in (5.7).

3. Since the function D_ψ is integer-valued $a.e.$ and $\int_{-\pi}^{\pi} D_\psi(\xi)\,d\xi = 2\pi$ (see (3.8) in Chapter 7 and Lemma 4.16 in Chapter 3, respectively, of [HW]), we must have that $D_\psi(\xi) = 1$, $a.e.\,\xi \in \mathbb{R}$. Thus, Proposition 3.8, part (V), implies that ψ arises from an MRA.

4. Formally, step 3 establishes the theorem, except for the fact that the MRA might not be localized. The construction of a scaling function φ such that $\widehat{\varphi} \in H^{\alpha}(\mathbb{R})$ requires a more delicate analysis. For the interested reader, we present here the main ideas, but we strongly recommend a consultation of the original reference (Theorem 2 in Chapter 3 of [KAH-LEM]) for further details. Consider the orthogonal projection operator onto V_0, $P_0 \colon L^2(\mathbb{R}) \to V_0$, given by the kernel

$$p_0(x,y) = \sum_{j<0} q_j(x,y) = \delta(x-y) - \sum_{j\geq 0} q_j(x,y),$$

where each $q_j(x,y) = \sum_{k\in\mathbb{Z}} 2^j \psi(2^j x - k)\overline{\psi(2^j y - k)}$ is the kernel of the orthogonal projection $Q_j \colon L^2(\mathbb{R}) \to W_j$, for $j \in \mathbb{Z}$. The steps involved in the construction of φ are the following:

(i) One can show that P_0 commutes with translation by integers and that the kernel $p_0(x,y)$ satisfies some "nice" decay conditions away from the diagonal; more precisely, $p_0(x,y) \in L^1_{loc}(\mathbb{R}\times\mathbb{R})$ and

$$\int_{x\in[0,1]} \int_{y\in\mathbb{R}} |p_0(x,y)|^2 \, (1+|x-y|)^{2\alpha} \, dy \, dx < \infty,$$

this coming from $\psi \in H^{\varepsilon}(\mathbb{R}) \cap L^2((1+|x|^2)^{\alpha}\, dx)$.

(ii) For each $\xi \in \mathbb{T}$, one "periodizes" P_0 into $P_0^{\xi} \colon L^2([0,1]) \to V_0^{\xi}$, by considering the kernel $p_0^{\xi}(x,y) = \sum_{k\in\mathbb{Z}} p_0(x, y-k) e^{-i\xi(x-y+k)}$. One can show that P_0^{ξ} is an orthogonal projection and that V_0^{ξ} consists exactly of the periodization of all the "nice" functions in V_0. That is,

$$V_0^{\xi} = \{ f^{\xi}(x) = \sum_{k\in\mathbb{Z}} e^{-i\xi(x-k)} f(x-k) \mid f \in V_0 \cap L^2((1+|x|^2)^{\alpha}\, dx) \}.$$

(iii) It is not hard to see now that P_0^{ξ} is a compact operator and, therefore, that V_0^{ξ} is finite-dimensional. Moreover, $\dim V_0^{\xi} = 1$, for all $\xi \in \mathbb{T}$.

(iv) One carefully constructs a function $g \in V_0 \cap L^2((1+|x|^2)^\alpha\, dx)$ such that g^ξ is a "basis" for the 1-dimensional space V_0^ξ, for each $\xi \in \mathbb{T}$. Note that this happens if and only if

$$(g^\xi, g^\xi)_{L^2([0,1])} = \sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 \neq 0, \quad \xi \in \mathbb{T}.$$

Therefore, $\{g(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_0, and one can define φ by

$$\widehat{\varphi}(\xi) = \frac{\hat{g}(\xi)}{\sqrt{\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2}}, \quad \xi \in \mathbb{R},$$

which would satisfy the required conditions.

5. Once we have found our α-localized scaling function, the original wavelet ψ can now be recovered from φ and m_0 by formula (5.3), for some unimodular $\nu \in L^2(\mathbb{T})$. To show that $\nu \in H^\alpha(\mathbb{T})$ note that, if we write $\overline{\nu}$ in terms of its Fourier series, $\overline{\nu}(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}$, (5.3) implies that

$$c_k = \int_{\mathbb{R}} \psi(x)\, \overline{\psi^\sharp(x-k)}\, dx, \quad k \in \mathbb{Z},$$

where $\widehat{\psi^\sharp}(\xi) = e^{-i\frac{\xi}{2}} \overline{m_0(\frac{\xi}{2} + \pi)} \widehat{\varphi}(\frac{\xi}{2}) \in H^\alpha(\mathbb{R})$. Then, the same type of argument as the one following (2.5) gives us that $\sum_{k\in\mathbb{Z}} |c_k|^2 (1+|k|^2)^\alpha < \infty$. This completes the proof of the existence part of the theorem.

6. Finally, we turn to the "uniqueness" of the choice (ν, φ, m_0). Let $(\tilde{\nu}, \tilde{\varphi}, \tilde{m}_0)$ be as in (5.5). Then, by the remark after (3.1), there exists a unimodular function $\mu \in L^2(\mathbb{T})$ such that $\widehat{\varphi}(\xi) = \mu(\xi)\hat{\tilde{\varphi}}(\xi)$, $\xi \in \mathbb{R}$. The same argument as in 5 above gives us that $\mu \in H^\alpha(\mathbb{T})$. Moreover, we have

$$\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi) = \mu(2\xi)\overline{\mu(\xi)}\tilde{m}_0(\xi)\widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}.$$

Thus, the uniqueness of the coefficients in the orthogonal expansion of $\frac{1}{2}\varphi(\frac{\cdot}{2})$ in terms of $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ implies that

$$m_0(\xi) = \mu(2\xi)\overline{\mu(\xi)}\tilde{m}_0(\xi), \quad \xi \in \mathbb{R}.$$

On the other hand, (5.3) and (5.5) imply that

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\,\overline{\nu(\xi)}\,\overline{m_0(\xi/2+\pi)}\,\widehat{\varphi}(\xi/2) =$$

$$= e^{-i\frac{\xi}{2}}\,\overline{\tilde{\nu}(\xi)}\,\overline{\mu(\xi/2+\pi)}\,\mu(\xi)\,\overline{m_0(\xi/2+\pi)}\,\overline{\mu(\xi/2)}\,\widehat{\varphi}(\xi/2), \quad \xi \in \mathbb{R},$$

Again, the uniqueness of the coefficients in the orthogonal expansion of ψ in terms of $\{\psi^\sharp(\cdot - k)\}_{k\in\mathbb{Z}}$ (where $\psi^\sharp$ is defined as in step 5 above) gives us

$$\nu(\xi) = \tilde{\nu}(\xi)\,\mu(\frac{\xi}{2}+\pi)\,\overline{\mu(\xi)}\,\mu(\frac{\xi}{2}), \quad \xi \in \mathbb{T}. \tag{5.8}$$

Now, suppose that $\tilde{\nu}$ is homotopic to $e^{ik\xi}$ and that μ is homotopic to $e^{i\ell\xi}$, for two integers $k, \ell \in \mathbb{Z}$. That is, there exist two continuous paths

$$\begin{aligned} [0,1]\times[0,2\pi] &\to S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \\ (t,\xi) &\longmapsto H_t(\xi) \\ (t,\xi) &\longmapsto I_t(\xi) \end{aligned}$$

such that $H_0(\xi) = \tilde{\nu}(\xi)$, $H_1(\xi) = e^{ik\xi}$ and $I_0(\xi) = \mu(\xi)$, $I_1(\xi) = e^{i\ell\xi}$. Then, the path

$$(t,\xi) \mapsto J_t(\xi) = H_t(\xi)\,I_t(\xi/2+\pi)\,\overline{I_t(\xi)}\,I_t(\xi/2)$$

is a homotopy starting at $J_0(\xi) = \nu(\xi)$ and ending at $J_1(\xi) = (-1)^\ell\, e^{ik\xi}$. This implies that ν has homotopy degree k and, hence, that ν and $\tilde{\nu}$ are homotopically equivalent.

□

Remark 5.9 Theorems 5.2 and 5.4 also hold in the limiting case $\alpha = \infty$, after replacing the sets $\mathcal{W}_\alpha$ and $H^\alpha(\mathbb{T})$ by $\mathcal{W}_\infty$ and $C^\infty(\mathbb{T})$, respectively, and the object "α-localized scaling function" by "scaling function with polynomial decay", as defined in (2.14).

Definition 5.10 *We shall denote by $\mathcal{E}_\infty$ the set of low-pass filters corresponding to scaling functions with polynomial decay; that is, $\mathcal{E}_\infty = \cap_{\alpha>\frac{1}{2}} \mathcal{E}_\alpha = \mathcal{E}_\alpha \cap C^\infty(\mathbb{T})$, for any $\alpha > \frac{1}{2}$. In this context, $\mathcal{E}_\infty$ is endowed with the topology of the Fréchet space $C^\infty(\mathbb{T})$. For convenience, sometimes we will denote $C^\infty(\mathbb{T})$ by $H^\infty(\mathbb{T})$.*

Definition 5.11 *Let $\frac{1}{2} < \alpha \leq \infty$. We say that a 2π-periodic function ν belongs to the set $\mathcal{M}_\alpha$ if $\nu \in H^\alpha(\mathbb{T})$ and $|\nu| = 1$. Note that $\mathcal{M}_\alpha$ is a closed subset of $H^\alpha(\mathbb{T})$.*

With this new notation in mind one can restate Theorems 5.2 and 5.4 jointly as follows.

Corollary 5.12 *Let $\frac{1}{2} < \alpha \leq \infty$. Then,*

$$\begin{aligned} T\colon\ & \mathcal{M}_\alpha \times \mathcal{E}_\alpha \longrightarrow \mathcal{W}_\alpha \\ & (\nu, m_0) \longmapsto T(\nu, m_0) = \psi, \end{aligned}$$

where ψ is defined in terms of (ν, m_0) by

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\, \overline{\nu(\xi)}\, \overline{m_0(\xi/2+\pi)} \prod_{j=2}^{\infty} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R},$$

is a continuous map onto $\mathcal{W}_\alpha$.

5.2 Connectivity in $\mathcal{W}_\alpha$ and the homotopy degree of a wavelet

Note that part 6 in the proof of Theorem 5.4 shows that "(III)$\Rightarrow$(IV)" in Proposition 3.8 of Chapter 1 does not necessarily hold in the α-localized case, unless ν has homotopy degree 0. One can ask whether in this last case the "phase" $\nu = \mathbf{1}$ is attainable. The answer turns out to be "no", as the examples in §5.3 show. However, the importance of the homotopy degree of the phase has to be taken into consideration and motivates the following definition.

Definition 5.13 *We say that a wavelet $\psi \in \mathcal{W}_\alpha$, $\frac{1}{2} < \alpha \leq \infty$, has homotopy degree $k \in \mathbb{Z}$ if the function ν of Theorem 5.4 is homotopically equivalent to $e^{ik\xi}$. The set of all such wavelets will be denoted by $\mathcal{W}_\alpha^{(k)}$.*

Note that $\psi \in \mathcal{W}_\alpha$ has degree k if and only if $\psi(\cdot + k)$ has degree 0. In fact, the shifting map $\psi \mapsto \psi(\cdot + k)$ is a homeomorphism from $\mathcal{W}_\alpha^{(k)}$ onto $\mathcal{W}_\alpha^{(0)}$. In applications, the wavelets ψ and $\psi(\cdot + k)$, although distinct as functions, represent the same object, and do not require a separate study. The reason for this is that the wavelet coefficients of a given function $f \in L^2(\mathbb{R})$ coincide except from a shift by k:

$$< f, (\psi(\cdot + k))_{j,\ell} > = < f, \psi_{j,\ell-k} >, \quad j, \ell \in \mathbb{Z},$$

where $< f, g > = \int_{\mathbb{R}} f\overline{g}$ denotes the inner product in $L^2(\mathbb{R})$. We will see in (5.19) below how the degree k of a wavelet is associated to the "center of mass" of $|\psi|^2$. This relation will be decisive in the study of the connectivity of the space $\mathcal{W}_\alpha$. One can see this already from the following remarkable fact: within the same MRA, it is not possible to join a wavelet $\psi \in \mathcal{W}_\alpha$ with its shift $\psi(\cdot + k)$ by means of a continuous path in $L^2((1 + |x|^2)^\alpha \, dx)$, unless $k = 0$. Indeed, an informal argument for this assertion is as follows. Suppose there is a path of wavelets $t \mapsto \psi_t$, continuous in $L^2((1 + |x|^2)^\alpha \, dx)$, with $\psi_0 = \psi$ and $\psi_1 = \psi(\cdot + k)$, and such that, for each $t \in [0, 1]$, the wavelet ψ_t arises from the same MRA as ψ. Then, there must exist a 2π-periodic, unimodular function ν_t in $H^\alpha(\mathbb{T})$ such that

$$\widehat{\psi}_t(\xi) = \nu_t(\xi) \, \widehat{\psi}(\xi).$$

Now, $\widehat{\psi} \neq 0$ in a compact set $K \sim_{2\pi} \mathbb{T}$ (this follows from $\sum_{k \in \mathbb{Z}} |\widehat{\psi}(\xi + 2k\pi)|^2 = 1$, and the same arguments as in the proof of Cohen's condition, in Lemma 2.18). From here we can conclude that the map $t \mapsto \nu_t = \widehat{\psi}_t / \widehat{\psi}$ defines a continuous path of 2π-periodic, unimodular functions joining $\mathbf{1}$ with $e^{ik\xi}$ (in $H^\alpha(K) = H^\alpha(\mathbb{T})$). This is clearly a contradiction (unless $k = 0$).

In Theorem 5.34 below, we will prove (in detail) a more general statement; namely, that ψ and $\psi(\cdot + k)$ cannot be joined at all with a continuous path in $\mathcal{W}_\alpha$. These results contrast strongly with the fact that, if one only considers the $L^2(\mathbb{R})$-topology in the the set $\mathcal{W}_\alpha$, then, it is possible to connect these two wavelets with a continuous arc of wavelets, all of them belonging to the same MRA (see Remark 5.41 below). This kind of results on connectivity of sets of wavelets with the $L^2(\mathbb{R})$-topology have been obtained recently by the group of researchers known as "The WUTAM Consortium". It still remains open whether the set of all wavelets is connected in the $L^2(\mathbb{R})$-topology. We give an account of the existing partial results in §5.4.

THEOREM 5.14 *Let $\alpha > \frac{1}{2}$. Suppose that ψ is a wavelet in $\mathcal{W}_\alpha$ with homotopy degree $k \in \mathbb{Z}$. Then, there exists a continuous path*

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{W}_\alpha^{(k)} \\ t &\longmapsto \psi_t \end{aligned}$$

such that $\psi_0 = \psi$ and $\psi_1 = \psi_H(\cdot - k)$, where ψ_H is the Haar wavelet. In particular, the topological space $\mathcal{W}_\alpha^{(k)}$ is arcwise connected for each $k \in \mathbb{Z}$.

PROOF:

By Theorem 5.4 we know that there exists an α-localized scaling function φ with associated low-pass filter m_0, and a unimodular function $\nu \in H^\alpha(\mathbb{T})$ homotopically equivalent to $e^{ik\xi}$ such that ψ can written in terms of (φ, m_0, ν) as in (5.3). Now, Theorem 4.11 and Corollary 3.32 tell us that we can join $(\widehat{\varphi}, m_0)$ to $(\widehat{\varphi}_H, m_H)$ with a continuous path (in $\mathcal{S}_\alpha \times \mathcal{E}_\alpha$) of the form $t \mapsto (\widehat{\varphi}_t, m_t) = (\, N(m_t), m_t\,)$, $t \in [0,1]$. It suffices to show that we can find a continuous path $t \mapsto \nu_t$ in $H^\alpha(\mathbb{T})$ joining $\nu(\xi)$ to $e^{ik\xi}$. Indeed, if this is the case, the arc $t \mapsto \psi_t$, where ψ_t is given by

$$\widehat{\psi}_t(\xi) = e^{-i\frac{\xi}{2}}\, \overline{\nu_t(\xi)}\, \overline{m_t(\frac{\xi}{2} + \pi)}\, \widehat{\varphi}_t(\frac{\xi}{2}), \quad \xi \in \mathbb{R},$$

will do the job. The construction of ν_t is just a consequence of the homotopical equivalence between ν and $e^{ik\xi}$, as we show in the following lemma.

LEMMA 5.15 *Let $\alpha > \frac{1}{2}$ and suppose that $\nu\colon [0, 2\pi] \to S^1$ is a unimodular function in $H^\alpha(\mathbb{T})$ homotopically equivalent to $e^{ik\xi}$. Then, there exists a continuous map*

$$H : [0,1] \longrightarrow H^\alpha(\mathbb{T})$$

such that $H(0)=\nu$, $H(1)=e^{ik\xi}$, *and* $H(t)\in\mathcal{M}_\alpha$, $\forall t\in[0,1]$.

Proof: We may assume that $k=0$. If not, consider $\tilde{\nu}(\xi)=e^{-ik\xi}\nu(\xi)$ to find a map $\widetilde{H}$ as in the statement of the lemma. Then, $H(t)=e^{ik\xi}\widetilde{H}(t)$ will do for ν. Suppose, therefore, that ν is homotopic to the constant function $\mathbf{1}$. Without loss of generality, we may also assume that $\nu(0)=1$.

CLAIM

There exists $\theta\in H^\alpha(\mathbb{T})$*, real-valued, such that* $\nu(\xi)=e^{i\theta(\xi)}$, $\xi\in\mathbb{T}$.

If we show the claim, the lemma follows by taking $H(t)=e^{i(1-t)\theta(\xi)}$, $t\in[0,1]$.

Proof of CLAIM

To prove the claim we use the principle of analytic continuation{10}. Let us denote the disk centered at $z=1$, with radius $\frac{1}{3}$, by $\Delta_0=\mathbb{D}(1,\frac{1}{3})$, and let

$$g_0(z)\equiv\log z\in\mathcal{H}(\Delta_0)=\{f:\Delta_0\to\mathbb{C}\mid f\text{ is holomorphic in }\Delta_0\},$$

where we take the branch of the log defined in $\mathbb{C}\setminus\,]-\infty,0]\supset\Delta_0$. We may continue analytically (g_0,Δ_0) along the curve ν to find disks $\Delta_1,\ldots,\Delta_n$, a partition of $[0,2\pi]$: $0=s_0<s_1<\ldots<s_{n+1}=2\pi$ such that $\nu([s_i,s_{i+1}])\subset\Delta_i$, $i=0,1,\ldots,n$, and holomorphic functions $g_i\in\mathcal{H}(\Delta_i)$ such that

$$g_i(z)=g_{i+1}(z),\quad z\in\Delta_i\cap\Delta_{i+1}\neq\emptyset,\quad i=0,1,\ldots,n-1.$$

By the monodromy principle (see, e.g., Theorem 2 in Chapter 8 of [AHL]), and since ν is homotopic to $\mathbf{1}$, we must have that $g_n(z)=g_0(z)$, $z\in\Delta_n\cap\Delta_0$. Thus, we have found an analytic continuation of the log along ν. Define now

$$\eta(\xi)=g_i(\nu(\xi)),\quad \xi\in[s_i,s_{i+1}]=I_i,\ i=0,1,\ldots,n.$$

{10}For the definition of *analytic continuation* and related theory see Chapter 16 of [RUD2], or Chapter 8 of [AHL].

Since the range of $\nu_{|I_i}$ is a subset of Δ_i and g_i is holomorphic in that disk, the results about Banach algebras in §1.3 apply here and, after appropriate glueing of the extremes of the intervals I_i, one obtains that $\eta(\xi)$ belongs to $H^\alpha(\mathbb{T})$. Moreover, since the g_i's are continuations of the $\log z$, we have $e^{\eta(\xi)} = \nu(\xi)$. Then, $\theta(\xi) = \frac{1}{i}\eta(\xi)$ will do the job.

□ □

REMARK 5.16 Note that the proof of Theorem 5.14 also works for $\alpha = \infty$. This particular case was previously shown by A. Bonami, S. Durand and G. Weiss using similar techniques, although the concept of homotopy degree was not considered there (see Proposition 3.4 in [BDW]).

The matter of whether one can choose a "simple" phase $\nu(\xi)$ to write the wavelets in $\mathcal{W}_\alpha$ as in (5.3) is not yet very clear. If we do not require any localization condition, Proposition 3.8 in Chapter 1 shows that ν can be taken to be $\mathbf{1}$ (after an appropriate choice of the scaling function φ). However, in the α-localized case, ν has to keep the same degree of homotopy regardless of the scaling function that we take. A natural question would be if ν can be taken to be the "simplest" function with degree k, that is, $\nu(\xi) = e^{ik\xi}$. In the case $\alpha = \infty$ this can certainly be done, as we will show in the next Proposition. For $\frac{1}{2} < \alpha < \infty$, the answer is no. The role played by phases in the study of the connectivity of $\mathcal{W}_\alpha$ is extremely important, since they force the homotopical constraints that isolate each of the $\mathcal{W}_\alpha^{(k)}$ from one another. A more careful treatment of the possible choices of phases when $\frac{1}{2} < \alpha < \infty$ is postponed until §5.3, although the main ideas on how to choose them is given in Proposition 5.17.

PROPOSITION 5.17 : see Théorème 4 in [LEM]

Suppose that $\psi \in \mathcal{W}_\infty$, that is, ψ is a wavelet satisfying (2.14) and such that for some positive ε, $\psi \in H^\varepsilon(\mathbb{R})$. Then, there exists a scaling function[11] *φ with polynomial decay and with low-pass filter $m_0 \in C^\infty(\mathbb{T})$, and an integer $k \in \mathbb{Z}$ such that*

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\, e^{-ik\xi}\, \overline{m_0(\frac{\xi}{2}+\pi)}\, \widehat{\varphi}(\frac{\xi}{2}), \quad \xi \in \mathbb{R}. \tag{5.18}$$

The number k is uniquely determined by

$$k + \frac{1}{2} = \int_{\mathbb{R}} x|\psi(x)|^2 dx, \tag{5.19}$$

while the function $\widehat{\varphi}$ is unique up to a factor $(-1)^M e^{iM\xi}$, for some $M \in \mathbb{Z}$.

Proof: We already sketched in Theorem 5.4 how to find a scaling function $\tilde{\varphi}$ with polynomial decay and low-pass filter $\tilde{m}_0 \in C^\infty(\mathbb{T})$, and a unimodular function $\tilde{\nu} \in C^\infty(\mathbb{T})$ such that (5.5) holds (see [LEM] for more details). Suposse that $\tilde{\nu}$ has homotopy degree $k \in \mathbb{Z}$. We first prove the uniqueness part of the proposition. Suppose there is a scaling function φ, with polynomial decay and low-pass filter $m_0 \in C^\infty(\mathbb{T})$, such that (5.18) above holds. Then, $\widehat{\varphi}$ must be given by $\widehat{\varphi}(\xi) = \mu(\xi)\hat{\tilde{\varphi}}(\xi)$, for a unimodular $\mu \in C^\infty(\mathbb{T})$ and, therefore, its low-pass filter must satisfy

$$m_0(\xi) = \mu(2\xi)\,\overline{\mu(\xi)}\,\tilde{m}_0(\xi).$$

As in (5.8), this implies that $\tilde{\nu}$ can be written as:

$$\tilde{\nu}(\xi) = e^{ik\xi}\mu(\xi)\,\overline{\mu(\xi/2)\,\mu(\xi/2+\pi)}, \quad \xi \in \mathbb{T}. \tag{5.20}$$

Assume for the moment that μ has homotopy degree 0 (if $\deg(\mu) = M$, we can replace μ by $(-1)^M e^{-iM\xi}\mu$, which is also a solution to (5.20)). By studying equation (5.20) we will find an expression for μ in terms of $\tilde{\nu}$ that will lead to our uniqueness statement.

[11] In this proposition, the scaling function φ is not supposed to satisfy condition (2.1) (that is, $\widehat{\varphi}(0)$ might not be equal to 1). We can still define the filter m_0 uniquely in terms of φ, but the converse is not true; the infinite product formula (2.11) determines $\widehat{\varphi}$ up to a unimodular constant. That is, $\widehat{\varphi}(\xi) = \widehat{\varphi}(0)\prod_{j=1}^{\infty} m_0(2^{-j}\xi)$, $\xi \in \mathbb{R}$.

Now, since $\tilde{\nu}$ has homotopy degree k, we can write $\tilde{\nu}(\xi) = e^{i\theta(\xi)}$, where

$$\theta(\xi) = k\xi + \sum_{\ell\in\mathbb{Z}} \theta_\ell e^{-i\ell\xi}, \quad \xi \in \mathbb{T}, \tag{5.21}$$

the series on the right representing a real-valued $C^\infty(\mathbb{T})$ function (this decomposition actually follows from the proof of Lemma 5.15 we presented above). Similarly, we can write $\mu(\xi) = e^{i\gamma(\xi)}$, where $\gamma \in C^\infty(\mathbb{T})$ is real-valued and can be expressed as:

$$\gamma(\xi) = \sum_{\ell\in\mathbb{Z}} \gamma_\ell e^{-i\ell\xi}, \quad \xi \in \mathbb{T}. \tag{5.22}$$

Since the two series in (5.21) and (5.22) represent functions in $C^\infty(\mathbb{T})$, the Fourier coefficients must have polynomial decay, in the sense that, for any $N \geq 1$,

$$\sup_{\ell\in\mathbb{Z}} |\theta_\ell|\,|\ell|^N = C_N < \infty \quad \text{and} \quad \sup_{\ell\in\mathbb{Z}} |\gamma_\ell|\,|\ell|^N = C'_N < \infty. \tag{5.23}$$

Equation (5.20) can be written in terms of (5.21) and (5.22) as:

$$\theta(2\xi) + 2K\pi = \gamma(2\xi) - \gamma(\xi) - \gamma(\xi+\pi), \quad \xi \in \mathbb{T}, \tag{5.24}$$

for some constant $K \in \mathbb{Z}$. Expressing (5.24) in terms of Fourier coefficients, and since all the series involved converge absolutelly, we have

$$\theta_\ell = \gamma_\ell - 2\gamma_{2\ell}, \quad \ell \in \mathbb{Z}\setminus\{0\},$$

and, therefore,

$$\gamma_0 = -\theta_0 - 2K\pi \quad \text{and} \quad \gamma_\ell = \sum_{j=0}^{n} 2^j\theta_{2^j\ell} + 2^{n+1}\gamma_{2^{n+1}\ell} \to \sum_{j=0}^{\infty} 2^j\theta_{2^j\ell}, \tag{5.25}$$

the convergence of the limit on the right following from the decay of the γ_ℓ's in (5.23). This implies that if a solution (φ, m_0) for (5.18) exists, φ has to be given by $\widehat{\varphi}(\xi) = e^{i\gamma(\xi)}\hat{\tilde{\varphi}}(\xi)$ or, more generally, by $\widehat{\varphi}(\xi) = (-1)^M e^{iM\xi} e^{i\gamma(\xi)}\hat{\tilde{\varphi}}(\xi)$, where $M \in \mathbb{Z}$ and γ has its Fourier coefficients as in (5.25). To complete the proof of the uniqueness, if $(\varphi^\sharp, m_0^\sharp, k^\sharp)$ is another triad satisfying

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\, e^{-ik^\sharp\xi}\, \overline{m_0^\sharp(\frac{\xi}{2}+\pi)}\, \widehat{\varphi}^\sharp(\frac{\xi}{2}), \quad \xi \in \mathbb{R}, \tag{5.26}$$

by Theorem 5.4 we must have $k^\sharp = k$, while by the reasoning above there exists an integer $M^\sharp$ such that $\widehat{\varphi}^\sharp(\xi) = (-1)^{M^\sharp} e^{iM^\sharp\xi} e^{i\gamma(\xi)} \hat{\tilde{\varphi}}(\xi)$. Then,

$$\widehat{\varphi}^\sharp(\xi) = (-1)^{M^\sharp - M} e^{i(M^\sharp - M)\xi} \widehat{\varphi}(\xi) \quad \text{and} \quad m_0^\sharp(\xi) = e^{i(M^\sharp - M)\xi} m_0(\xi), \quad \xi \in \mathbb{R}.$$

This completes the proof of the uniqueness.

For the existence, if one defines γ by its Fourier series (as in (5.22)) with coefficients γ_ℓ given by (5.25) (we may take $K = 0$), then $\gamma \in C^\infty(\mathbb{T})$ and (5.24) holds. Here, the smoothness of γ follows from the decay of $\{\gamma_\ell\}_{\ell \in \mathbb{Z}}$:

$$|\gamma_\ell| \le \sum_{j=0}^{\infty} 2^j |\theta_{2^j\ell}| \le \sum_{j=0}^{\infty} 2^j \frac{C_N}{2^{jN}|\ell|^N} \le 4C_N \frac{1}{|\ell|^N}, \quad \forall N \ge 2.$$

Now, let $\mu(\xi) = e^{i\gamma(\xi)} \in C^\infty(\mathbb{T})$ and $\widehat{\varphi} = \mu\widehat{\tilde{\varphi}}$, and let m_0 be low-pass filter associated to φ. Then, (5.24) implies (5.20) and, therefore, the triad (φ, m_0, k) satisfies (5.18). This establishes the existence part of the proposition.

Finally, we show the validity of the formula (5.19) that uniquely determines the integer k in terms of ψ. Because of its own interest and simple proof, we state it separately as a lemma. For a different proof of this formula, in a more restricted case, see, e.g., Proposition 10.1 in [VILL].

Lemma 5.27 *Suppose ψ is a wavelet that can be written as in (5.18) above, where $k \in \mathbb{Z}$, φ is a scaling function with polynomial decay, and $m_0 \in C^\infty(\mathbb{T})$ is its low-pass filter. Then,*

$$k + \frac{1}{2} = \int_{\mathbb{R}} x|\psi(x)|^2 dx.$$

Proof:

Suppose that ψ can be written as in (5.18), then, using the Plancherel's theorem and the fact that $(x\psi(x))\hat{}(\xi) = i\widehat{\psi}'(\xi)$, we have

$$I_\psi \equiv \int_{\mathbb{R}} x|\psi(x)|^2\, dx = \frac{1}{2\pi} \int_{\mathbb{R}} (x\psi(x))\hat{}(\xi)\, \overline{\widehat{\psi}(\xi)}\, d\xi$$

$$= \frac{i}{2\pi} \int_{\mathbb{R}} \widehat{\psi}'(\xi)\, \overline{\widehat{\psi}(\xi)}\, d\xi.$$

Note that all the integrals above are absolutely convergent under the assumption that φ (and, hence, ψ) has polynomial decay. By taking derivatives in (5.18) we obtain

$$\widehat{\psi}'(\xi)\overline{\widehat{\psi}(\xi)} = -i(\frac{1}{2}+k)|\widehat{\psi}(\xi)|^2 + \frac{1}{2}|\widehat{\varphi}(\xi/2)|^2 m_0(\xi/2+\pi)\overline{m_0'(\xi/2+\pi)} +$$
$$+\frac{1}{2}|m_0(\xi/2+\pi)|^2\widehat{\varphi}'(\xi/2)\overline{\widehat{\varphi}(\xi/2)}$$

where again all the summands are integrable functions in $\mathbb{R}$. Thus,

$$I_\psi = (k+\frac{1}{2})\frac{1}{2\pi}\|\widehat{\psi}\|^2_{L^2(\mathbb{R})} + \frac{i}{2\pi}(A+B),$$

where, after a change of variables, A and B can be written as:

$$\begin{cases} A = \displaystyle\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 m_0(\xi+\pi)\overline{m_0'(\xi+\pi)}\,d\xi \\ B = \displaystyle\int_{\mathbb{R}} |m_0(\xi+\pi)|^2\widehat{\varphi}'(\xi)\overline{\widehat{\varphi}(\xi)}\,d\xi. \end{cases}$$

We will show that $A = \overline{B}$ and this will establish our result (because, by definition, I_ψ is a real number). Indeed, by a periodization argument, we can write:

$$\begin{aligned} A &= \sum_{\ell\in\mathbb{Z}}\int_{\mathbb{T}} |\widehat{\varphi}(\xi+2\ell\pi)|^2\, m_0(\xi+\pi)\,\overline{m_0'(\xi+\pi)}\,d\xi \\ &= \int_{\mathbb{T}} m_0(\xi+\pi)\,\overline{m_0'(\xi+\pi)}\,d\xi = \int_{\mathbb{T}} m_0(\xi)\,\overline{m_0'(\xi)}\,d\xi \\ &= \ldots = \int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2\, m_0(\xi)\,\overline{m_0'(\xi)}\,d\xi, \end{aligned}$$

where in the second and fourth equalities we have used $\sum_{\ell\in\mathbb{Z}}|\widehat{\varphi}(\xi+2\ell\pi)|^2 = 1$. On the other hand, using $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$, we have:

$$B = \int_{\mathbb{R}} |m_0(\xi+\pi)|^2\,\widehat{\varphi}'(\xi)\,\overline{\widehat{\varphi}(\xi)}\,d\xi =$$
$$= \int_{\mathbb{R}} \widehat{\varphi}'(\xi)\,\overline{\widehat{\varphi}(\xi)}\,d\xi - \int_{\mathbb{R}} |m_0(\xi)|^2\,\widehat{\varphi}'(\xi)\,\overline{\widehat{\varphi}(\xi)}\,d\xi.$$

Now, by differentiating in the scaling equation (3.4) of Chapter 1, and changing variables, we obtain

$$\int_{\mathbb{R}} \widehat{\varphi}'(\xi)\,\overline{\widehat{\varphi}(\xi)}\,d\xi = \int_{\mathbb{R}} m_0'(\xi)\,\overline{m_0(\xi)}\,|\widehat{\varphi}(\xi)|^2\,d\xi + \int_{\mathbb{R}} |m_0(\xi)|^2\,\widehat{\varphi}'(\xi)\,\overline{\widehat{\varphi}(\xi)}\,d\xi.$$

Thus, we can write B as:

$$B = \int_{\mathbb{R}} m_0'(\xi)\, \overline{m_0(\xi)}\, |\widehat{\varphi}(\xi)|^2\, d\xi = \overline{A}.$$

This shows that $I_\psi = k + \frac{1}{2}$, and establishes the lemma, and with it, the theorem.

$\square \quad \square$

As we pointed out in the statement of the proposition, the scaling function φ does not necessarily satisfy (2.1). It is true, however, that $|\widehat{\varphi}(0)| = 1$. By considering the unimodular constant $\widehat{\varphi}(0)$ apart, we can restate Proposition 5.17 as follows:

COROLLARY 5.28 *Let $\psi \in \mathcal{W}_\infty$. Then, there exists a function $m_0 \in \mathcal{E}_\infty$, an integer $k \in \mathbb{Z}$ and a unimodular constant $c \in \mathbb{C}$ such that*

$$\widehat{\psi}(\xi) = c\, e^{-i\frac{\xi}{2}}\, e^{-ik\xi}\, \overline{m_0(\xi/2 + \pi)} \prod_{j=2}^{\infty} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R}. \tag{5.29}$$

If $(m_0^\sharp, k^\sharp, c^\sharp) \in \mathcal{E}_\infty \times \mathbb{Z} \times S^1$ is another solution to (5.29), then $k^\sharp = k$ and there exists an integer $M \in \mathbb{Z}$ such that $c^\sharp = (-1)^M c$ and $m_0^\sharp(\xi) = e^{iM\xi} m_0(\xi)$, $\xi \in \mathbb{T}$.

PROOF: For the existence, use the solution in Proposition 5.17 with $c = \widehat{\varphi}(0)$. For the uniqueness, by Proposition 5.17 again, any other solution $(m_0^\sharp, k^\sharp, c^\sharp)$ to (5.29) must satisfy $m_0^\sharp(\xi) = e^{iM\xi} m_0(\xi)$ and $\widehat{\varphi}^\sharp(\xi) = c^\sharp \prod_{j=1}^{\infty} m_0^\sharp(2^{-j}\xi) = (-1)^M e^{iM\xi} \widehat{\varphi}(\xi) = (-1)^M e^{iM\xi} \widehat{\varphi}(0) \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$, for some integer $M \in \mathbb{Z}$ and all $\xi \in \mathbb{R}$. Solving for $c^\sharp$ we have $c^\sharp = (-1)^M \widehat{\varphi}(0) = (-1)^M c$.

$\square$

As a consequence of Corollary 5.28 we obtain the following characterization of the space $\mathcal{W}_\alpha$ in terms of $\mathcal{E}_\alpha$:

COROLLARY 5.30 *Let $k \in \mathbb{Z}$ and let*

$$\begin{aligned} T_k\colon \quad S^1 \times \mathcal{E}_\infty &\longrightarrow \mathcal{W}_\infty^{(k)} \\ (c, m_0) &\longmapsto T_k(c, m_0) = \psi \end{aligned}$$

where $\widehat{\psi}$ *is given by (5.29). Then,* T_k *is continuous and onto and every element* $\psi \in \mathcal{W}_\infty^{(k)}$ *has a discrete fiber* $T_k^{-1}\psi$*.*

In particular, if we let the equivalence relation "$\sim$" in $S^1 \times \mathcal{E}_\infty$*:*

$$(c, m_0) \sim (c^\sharp, m_0^\sharp) \qquad \Longleftrightarrow \qquad \exists M \in \mathbb{Z} \mid c = (-1)^M c^\sharp,\ m_0(\xi) = e^{iM\xi} m_0^\sharp(\xi)$$

and we introduce the quotient topology in $(S^1 \times \mathcal{E}_\infty)/\sim$*, then:*

$$\begin{aligned} \overline{T}_k\colon \quad (S^1 \times \mathcal{E}_\infty)/\sim &\longrightarrow \mathcal{W}_\infty^{(k)} \\ [(c, m_0)] &\longmapsto T_k(c, m_0) = \psi \end{aligned}$$

is a continuous bijection of topological spaces.

Remark 5.31 Note that, although $\overline{T}_k$ is a continuous bijection from $(S^1 \times \mathcal{E}_\infty)/\sim$ onto $\mathcal{W}_\infty^{(k)}$, it might not be a homeomorphism. In order to obtain a complete topological identification of $\mathcal{W}_\infty^{(k)}$ it seems that one must consider as well the topology of the spaces $H^\varepsilon(\mathbb{R})$ in the time domain of ψ, as we assumed this property to hold in Definition 5.1 *(iii)*. However, since it goes out of the scope of this work to deal with finer topologies, we shall take up this matter somewhere else.

As an example, we illustrate how the equivalence relation of the previous corollary deforms the figure given at the beginning of §3, of the subset of filters in $\mathcal{E}_\alpha$ which are trigonometric polynomials of degree ≤ 3, to convert it from a punctured circle into something resembling a ribbon[12]. Indeed, after identifying the three points $(0,0), (1,0), (0,1)$, corresponding to the Haar filter (and noting that all the other points belong to different equivalence classes), we obtain:

[12] We wish to thank Aline Bonami for pointing this out to us.

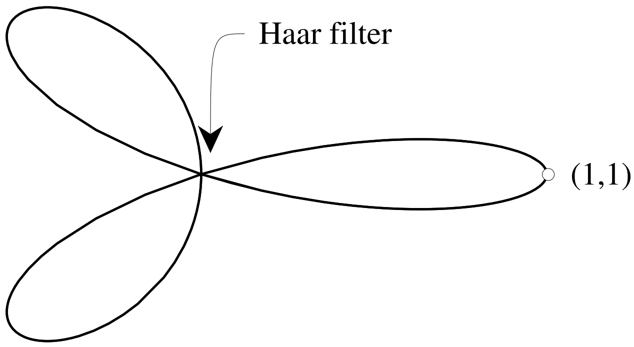

Figure 5.3: Figure 3.1 after identification with quotient map in Corollary 5.30.

We proceed now with a closer study of the of phases $\mathcal{M}_\alpha$. From here, we will extend formula (5.19) to the case $\frac{1}{2} < \alpha < \infty$, and show that the sets $\mathcal{W}_\alpha^{(k)}$, $k \in \mathbb{Z}$, are disjoint connected components in $\mathcal{W}_\alpha$.

DEFINITION 5.32 *Let $\frac{1}{2} < \alpha \leq \infty$ and $k \in \mathbb{Z}$. We say that $\nu \in \mathcal{M}_\alpha^{(k)}$ if $\nu \in \mathcal{M}_\alpha$ and ν is homotopically equivalent to $e^{ik\xi}$.*

Note that, for each $k \in \mathbb{Z}$, $\mathcal{M}_\alpha^{(k)}$ is a closed and open subset of $\mathcal{M}_\alpha$. This is a consequence of the following general result:

LEMMA 5.33 *If $\{h_n\}_{n=0}^\infty \subset C(\mathbb{T})$, $|h_n| = 1$, for $n \geq 0$, and $h_n \to h_0$ uniformly in $\mathbb{T}$, then there exists a positive integer n_0 such that, for all $n \geq n_0$,* $\deg(h_n) = \deg(h_0)$.

PROOF: The proof is easy. Suppose first that $h_0 = \mathbf{1}$. Then, if we take n_0 large enough so that $\|h_n - \mathbf{1}\|_\infty < \frac{1}{2}$, $n \geq n_0$, we must have $\deg(h_n) = \deg(\mathbf{1}) = 0$.

For a general h_0, if $h_n \to h_0$, then $h_n\overline{h_0} \to \mathbf{1}$. Thus, there exists a positive integer n_0 such that $\deg(h_n\overline{h_0}) = 0$, for $n \geq n_0$. Now for a fixed $n \geq n_0$, let $t \in [0,1] \mapsto H_t$ be a homotopy map such that $H_0 = h_n\overline{h_0}$ and $H_1 = \mathbf{1}$. Then, the map $t \mapsto h_0H_t$ is a homotopy starting at h_n and ending at h_0, which implies that $\deg(h_n) = \deg(h_0)$.

$\square$

We collect these and other properties of $\mathcal{M}_\alpha^{(k)}$ in the following theorem.

THEOREM 5.34 *Let* $\frac{1}{2} < \alpha \leq \infty$ *and let* $\nu(\xi) = \sum_{\ell\in\mathbb{Z}} c_\ell e^{i\ell\xi} \in \mathcal{M}_\alpha$. *Then:*

$$(i) \quad \sum_{\ell\in\mathbb{Z}} \ell\,|c_\ell|^2 \in \mathbb{Z}. \tag{5.35}$$

$$(ii) \quad \text{For } k\in\mathbb{Z}, \quad \nu \in \mathcal{M}_\alpha^{(k)} \quad \text{if and only if} \quad k = \sum_{\ell\in\mathbb{Z}} \ell\,|c_\ell|^2.$$

Moreover, for every $k \in \mathbb{Z}$, $\mathcal{M}_\alpha^{(k)}$ *is a closed and open subset of* $\mathcal{M}_\alpha$ *and* $\mathcal{M}_\alpha$ *can be decomposed in connected components by* $\mathcal{M}_\alpha = \cup\{\mathcal{M}_\alpha^{(k)} \mid k \in \mathbb{Z}\}$.

PROOF:

Suppose first that $\nu \in C^\infty(\mathbb{T})$. Then, by Parseval's equality

$$\frac{1}{2\pi i}\int_\nu \frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\nu'(t)}{\nu(t)}\,dt = \frac{1}{2\pi i}\int_0^{2\pi} \nu'(t)\overline{\nu(t)}\,dt$$

$$= \frac{1}{i}\sum_{\ell\in\mathbb{Z}} i\ell c_\ell \overline{c_\ell} = \sum_{\ell\in\mathbb{Z}} \ell|c_\ell|^2.$$

But the integral on the left represents the *winding number*[13] of the curve ν, which is always an integer. Moreover, ν is homotopically equivalent to $e^{ik\xi}$ if and only if the winding number is k. This shows (i) and (ii) for the case $\alpha = \infty$.

Suppose now that $\frac{1}{2} < \alpha < \infty$. Then, the series in (5.35) is absolutelly convergent (by (1.20)). If we consider the symmetric partial sums of the Fourier series of ν, $s_N(\xi) = \sum_{|\ell|\leq N} c_\ell e^{i\xi\ell}$, for $N \geq 1$, we have that $s_N \in C^\infty(\mathbb{T})$ and, by Lemma 1.47, that $|s_N| > \frac{1}{2}$ for N large enough. Now, the Banach algebra convergence properties of §1.3 and Lemma 1.47 again, imply that

$$\nu_N \equiv \frac{s_N}{|s_N|} \to \nu \quad in \quad H^\alpha(\mathbb{T}).$$

Thus,

$$|\sum_{\ell\in\mathbb{Z}} \ell\,(\,|c_\ell(\nu_N)\,|^2 - |c_\ell(\nu)\,|^2\,)\,| \;=\; |\sum_{\ell\in\mathbb{Z}} \ell\,(\,|c_\ell(\nu_N)\,| + |c_\ell(\nu)\,|\,)\,(\,|c_\ell(\nu_N)\,| - |c_\ell(\nu)\,|\,)\,|$$

[13] For a definition of winding number and properties see, e.g., Chapter 10 of [RUD2].

$$\leq \Big(\sum_{\ell\in\mathbb{Z}} |\ell|\,(\,|c_\ell(\nu_N)\,| + |c_\ell(\nu)\,|\,)^2\Big)^{\frac{1}{2}} \Big(\sum_{\ell\in\mathbb{Z}} |\ell|\,|\,c_\ell(\nu_N) - c_\ell(\nu)\,|^2\Big)^{\frac{1}{2}} \leq C\,\|\nu_N - \nu\|_\alpha.$$

But this last expression goes to 0 as N approaches infinity, which implies that

$$\lim_{N\to\infty}\sum_{\ell\in\mathbb{Z}} \ell\,|c_\ell(\nu_N)\,|^2 = \sum_{\ell\in\mathbb{Z}} \ell\,|c_\ell(\nu)\,|^2 < \infty. \tag{5.36}$$

Since every term on the left hand side of the above equality is an integer (because $\nu_N \in C^\infty(\mathbb{T})$), there must exist a $k \in \mathbb{Z}$ and an $N_0 \geq 1$ such that for every $N \geq N_0$, $\sum_{\ell\in\mathbb{Z}} \ell\,|c_\ell(\nu_N)\,|^2 = k$ and $\nu_N \in \mathcal{M}_\infty^{(k)} \subset \mathcal{M}_\alpha^{(k)}$. Now, by using that $\mathcal{M}_\alpha^{(k)}$ is closed we conclude that $\sum_{\ell\in\mathbb{Z}} \ell\,|c_\ell(\nu)\,|^2 = k$ and $\nu \in \mathcal{M}_\alpha^{(k)}$. For the converse of (ii) note that, since $\mathcal{M}_\alpha^{(k)}$ is open, if we knew that $\nu \in \mathcal{M}_\alpha^{(k)}$, for some $k \in \mathbb{Z}$, then for N large enough also $\nu_N \in \mathcal{M}_\alpha^{(k)}$. This and the considerations right after Definition 5.32 complete the proof of the theorem.

□

REMARK 5.37 Note that the set $\mathcal{M}_\alpha$, $\frac{1}{2} < \alpha \leq \infty$, is not connected even if we endow it with the (weaker) relative topology of $C(\mathbb{T})$. In this case we still have the homotopical constraint that keeps us from joining the functions $\mathbf{1}$ and $e^{i\xi}$. However, if we consider $\mathcal{M}_\alpha$ with the (even weaker) topology of $L^2(\mathbb{T})$ we do obtain an arcwise connected space. It is not hard to construct a continuous path within $(\mathcal{M}_\infty, \|\cdot\|_{L^2(\mathbb{T})})$ starting at $e^{i\xi}$ and ending at $\mathbf{1}$. Indeed, let $b \in C^\infty(\mathbb{T})$ be such that $0 \leq b \leq 1$, $b|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 0$ and $b|_{\{|\xi|\geq 1\}} \equiv 1$. Then, for $t \in (0,1]$, define the function

$$\beta_t(\xi) = \begin{cases} \xi\, b(\xi/t), & \xi \in [0,\pi] \\ \xi\, b((2\pi-\xi)/t), & \xi \in [\pi, 2\pi] \end{cases}$$

and extend it 2π-periodically to $\mathbb{R}$, that is, $\beta_t(\xi) = \beta_t(\xi - 2k\pi)$, if $\xi \in [2k\pi, 2(k+1)\pi]$ (see Figure 5.4 below).

Note that $\beta_t \in C^\infty(\mathbb{T})$, $0 \leq \beta_t \leq 2\pi$ and if $t_0 \in (0,1]$, then $\beta_t \to \beta_{t_0}$ in $C^\infty(\mathbb{T})$, as $t \to t_0$. Let us define $\nu_t(\xi) = e^{i\beta_t(\xi)}$. Then, clearly, $\nu_t \in \mathcal{M}_\infty$ and $\nu_t \to \nu_{t_0}$ in $C^\infty(\mathbb{T})$ (and hence in $L^2(\mathbb{T})$) whenever $t_0 \in (0,1]$ and $t \to t_0$. We show that if $t \to 0^+$, then

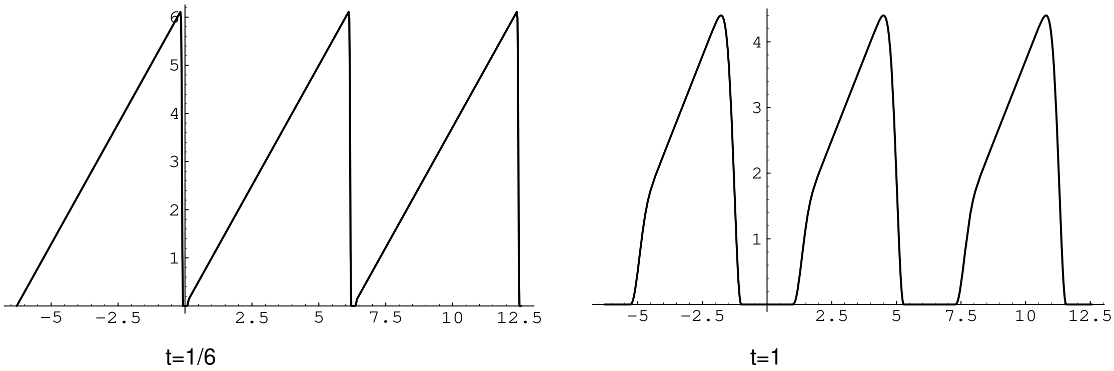

Figure 5.4: The function β_t for $t = 1/6, 1$.

$\int_0^{2\pi} |\nu_t(\xi) - e^{i\xi}|^2 \, d\xi \to 0$. Indeed,

$$\int_0^{2\pi} |\nu_t(\xi) - e^{i\xi}|^2 \, d\xi = \int_0^{2\pi} |e^{i(\beta_t(\xi)-\xi)} - 1|^2 \, d\xi =$$

$$= \int_0^t |e^{i\xi(b(\frac{\xi}{t})-1)} - 1|^2 \, d\xi + \int_{2\pi-t}^{2\pi} |e^{i\xi(b(\frac{2\pi-\xi}{t})-1)} - 1|^2 \, d\xi \leq 8t \to 0, \quad \text{as } t \to 0^+.$$

Hence, $t \mapsto \nu_t$ is a continuous path from $[0,1]$ into $(\mathcal{M}_\infty, \|\cdot\|_{L^2(\mathbb{T})})$ and connects $e^{i\xi}$ with $\nu_1(\xi) = e^{i\beta_t(\xi)} \in \mathcal{M}_\infty^{(0)}$. By using Lemma 5.15 we can extend this path (continuously) to end at $\mathbf{1}$ and, from here, the connectivity of $(\mathcal{M}_\infty, \|\cdot\|_{L^2(\mathbb{T})})$ follows easily.

We turn now to the continuous analogue of Theorem 5.34 in terms of the space $\mathcal{W}_\alpha$, which clarifies to a large extent the role played by the "homotopy degree" k of Definition 5.13, while it gives a decomposition of $\mathcal{W}_\alpha$ in connected components.

Theorem 5.38 *Let $\frac{1}{2} < \alpha \leq \infty$ and $\psi \in \mathcal{W}_\alpha$, then:*

(i) $\quad \int_{\mathbb{R}} x|\psi(x)|^2 \, dx \in \frac{1}{2} + \mathbb{Z}.$

(ii) $\quad$ *For $k \in \mathbb{Z}$, $\quad \psi \in \mathcal{W}_\alpha^{(k)}$ $\quad$ if and only if* $\quad \int_{\mathbb{R}} x|\psi(x)|^2 \, dx = k + \frac{1}{2}.$

Moreover, for every $k \in \mathbb{Z}$, $\mathcal{W}_\alpha^{(k)}$ is a closed and open subset of $\mathcal{W}_\alpha$ and $\mathcal{W}_\alpha$ can be decomposed in connected components by $\mathcal{W}_\alpha = \cup\{\mathcal{W}_\alpha^{(k)} \mid k \in \mathbb{Z}\}$.

PROOF:

The proof depends in the following simple lemma, continuous version of (5.36).

LEMMA 5.39 *Suppose that* $\widehat{\psi}_n \in H^{\frac{1}{2}}(\mathbb{R})$, *for* $n = 0, 1, 2, \ldots$ *and that* $\widehat{\psi}_n \to \widehat{\psi}_0$ *in* $H^{\frac{1}{2}}(\mathbb{R})$. *Then,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} x|\psi_n(x)|^2\,dx = \int_{\mathbb{R}} x|\psi_0(x)|^2\,dx. \tag{5.40}$$

Proof of Lemma 5.39

The proof is exactly the same as in the discrete case. Note that all the integrals involved are absolutely convergent. Then, we have:

$$\left| \int_{\mathbb{R}} x\,(\,|\psi_n(x)|^2 - |\psi_0(x)|^2\,)\,dx \right| =$$

$$= \left| \int_{\mathbb{R}} x\,(\,|\psi_n(x)| + |\psi_0(x)|\,)\,(\,|\psi_n(x)| - |\psi_0(x)|\,)dx \right| \leq$$

$$\leq \left(\int_{\mathbb{R}} |x|\,(\,|\psi_n(x)| + |\psi_0(x)|\,)^2\,dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |x|\,|\psi_n(x) - \psi_0(x)|^2\,dx \right)^{\frac{1}{2}}$$

$$\leq C\,\|\widehat{\psi}_n - \widehat{\psi}_0\|_{H^{\frac{1}{2}}(\mathbb{R})} \to 0, \quad \text{as} \quad n \to \infty,$$

and this proves (5.40).

□

To show Theorem 5.38, it is enough to establish (i) and (ii) in the statement of the theorem, since the property that $\mathcal{W}_\alpha^{(k)}$ is open and closed for every $k \in \mathbb{Z}$ will follow then from this and Lemma 5.39, while the connectivity of $\mathcal{W}_\alpha^{(k)}$ was shown in Theorem 5.14 above. For the case $\alpha = \infty$, the validity of (i) and (ii) was proven in Proposition 5.17. Let us consider then the case $\frac{1}{2} < \alpha < \infty$. By Theorem 5.4, we can write ψ as:

$$\widehat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\,\overline{\nu(\xi)}\,\overline{m_0(\xi/2 + \pi)}\,\widehat{\varphi}(\xi/2),$$

where $\nu \in \mathcal{M}_\alpha$, $m_0 \in \mathcal{E}_\alpha$ and $\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi) \in \mathcal{S}_\alpha$. Suppose that ν is homotopically equivalent to $e^{ik\xi}$, that is, $\nu \in \mathcal{M}_\alpha^{(k)}$. Then, in the same way as in the proof of Theorem 5.34, we can find a sequence $\{\nu_N\}_{N=1}^{\infty} \subset \mathcal{M}_\infty^{(k)}$ such that $\nu_N \to \nu$

in $H^\alpha(\mathbb{T})$. By Remark 4.10, we can find another sequence $\{m_N\}_{N=1}^\infty \subset \mathcal{E}_\infty$ such that $m_N \to m_0$ in $H^\alpha(\mathbb{T})$. Then, by Theorem 5.2 and Corollary 5.12, the functions ψ_N defined by

$$\widehat{\psi}_N(\xi) = e^{-i\frac{\xi}{2}}\,\overline{\nu_N(\xi)}\,\overline{m_N(\xi/2+\pi)}\prod_{j=2}^\infty m_N(2^{-j}\xi), \quad \xi \in \mathbb{R},\ N \geq 1,$$

are wavelets in $\mathcal{W}_\infty^{(k)}$ and $\widehat{\psi}_N \to \widehat{\psi}$ in $H^\alpha(\mathbb{R})$. Another application of Lemma 5.39 gives us that

$$\int_{\mathbb{R}} x|\psi(x)|^2\,dx = \lim_{N\to\infty}\int_{\mathbb{R}} x|\psi_N(x)|^2\,dx = k + \frac{1}{2},$$

and completes the proof of (i) and the "only if" part of (ii). The remaining implication follows from these two and the fact that the family $\{\mathcal{W}_\alpha^{(k)}\}_{k\in\mathbb{Z}}$ forms a partition of $\mathcal{W}_\alpha$.

$\square$

Remark 5.41 Theorem 5.38 asserts that, in particular, there is no way to join continuously a wavelet ψ with its shift $\psi(\cdot + k)$ within the topological space $\mathcal{W}_\alpha$, while by the considerations in Remark 5.37, such a path would exist (and can be taken to consist of wavelets from the same MRA as ψ) if we endow $\mathcal{W}_\alpha$ with the weaker topology of $L^2(\mathbb{R})$. This answers the question we rose right before Theorem 5.14 above.

5.3 The problem of the phase in $\mathcal{W}_\alpha$

In this subsection we go back to the problem posed right after Remark 5.16 of finding a "simple phase" for wavelets in $\mathcal{W}_\alpha$, when $\frac{1}{2} < \alpha < \infty$. Unfortunately, difficulties appear very soon, since the counterpart of Proposition 5.17 is not true in the α-localized case. In general, for a given $\psi \in \mathcal{W}_\alpha$, $\frac{1}{2} < \alpha < \infty$, it is not always possible to express $\widehat{\psi}$ in terms of $(\tilde{\nu}, \tilde{m}_0, \tilde{\varphi})$ (as in (5.5)) with a simple choice of the phase $\tilde{\nu}$ (such as $e^{ik\xi}$) if we still want to keep $\widehat{\tilde{\varphi}} \in H^\alpha(\mathbb{R})$. We will present an example below to illustrate what we are saying. As we saw in the proof of Propostion 5.17, the existence of a nice phase for ψ is very closely related to the functional equation

$$\tilde{\nu}(2\xi) = \mu(2\xi)\,\overline{\mu(\xi)\mu(\xi+\pi)}, \quad \xi \in \mathbb{T}, \tag{5.42}$$

where $\tilde{\nu} \in \mathcal{M}_{\alpha}^{(0)}$ is given, and μ is the unknown, to be in the space $\mathcal{M}_{\alpha'}^{(0)}$, for the largest possible α'. Indeed, if we could write ψ as in (5.18), for a scaling function φ, with associated low-pass filter m_0 and such that $\widehat{\varphi} \in H^{\alpha'}(\mathbb{R})$, then, for some unimodular $\mu \in H^{\alpha'}(\mathbb{T})$, we must have $\widehat{\varphi}(\xi) = \mu(\xi)\hat{\tilde{\varphi}}(\xi)$ and the same argument as in the proof of Proposition 5.17 would imply that (5.42) must hold[14]. Now, if we write $\tilde{\nu}(\xi) = e^{i\theta(\xi)}$ and $\mu(\xi) = e^{i\gamma(\xi)}$, where θ and γ are defined in terms of their Fourier series by

$$\theta(\xi) = \sum_{\ell\in\mathbb{Z}} \theta_\ell e^{-i\ell\xi} \quad \text{and} \quad \gamma(\xi) = \sum_{\ell\in\mathbb{Z}} \gamma_\ell e^{-i\ell\xi}, \quad \xi \in \mathbb{T}, \tag{5.43}$$

then, (5.42) implies that the difference equation

$$\begin{cases} \theta_\ell = \gamma_\ell - 2\gamma_{2\ell}, & \ell \neq 0 \\ \theta_0 = -\gamma_0 + 2K\pi, & \end{cases} \tag{5.44}$$

holds, for some fixed constant $K \in \mathbb{Z}$, and conversely. Thus, all wavelets ψ that can be represented in terms of a simple phase as in (5.18), are associated with a sequence $\{\gamma_\ell\}$, solution to the difference equation (5.44), and conversely. In our example, we consider a particular choice of $\theta \in H^\alpha(\mathbb{T})$ for which there is no possible solution γ to (5.44) that belongs to $H^\alpha(\mathbb{T})$.

EXAMPLE 2

Suppose that $\alpha \geq 1$ and let us fix a number $\frac{1}{2} < \beta \leq 1$. We define the following lacunary series $\{\theta_\ell\}_{\ell\in\mathbb{Z}} \in H^\alpha(\mathbb{T})$:

$$\theta_\ell = \begin{cases} \frac{1}{2^{\alpha 2^j} j^\beta}, & \text{if } \ell = 2^{2^j},\ j \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

[14] For simplicity, and without loss of generality, we will assume that the homotopy degrees of $\tilde{\nu}$ and μ are both 0. The general case follows immediately from here after multiplication by $e^{ik\xi}$ when necessary.

Note that $\sum_{\ell\in\mathbb{Z}} |\theta_\ell|^2\,|\ell|^{2\alpha} = \sum_{j\geq 1}\frac{1}{j^{2\beta}} < \infty$ if $\beta > \frac{1}{2}$. On the other hand, if we expect $\{\gamma_\ell\}_{\ell\in\mathbb{Z}}$ to be a function in $H^\alpha(\mathbb{T})$ we must have that

$$\lim_{n\to\infty}(2^{n+1}|k|)^\alpha|\gamma_{2^{n+1}k}| = 0, \quad \text{for all } \ k\in\mathbb{Z}.$$

In addition, if γ is a solution to (5.44), then the equalities in (5.25) must hold. In the case $\alpha = 1$ this is already a contradiction since we would have $\gamma_1 = \sum_{j=0}^\infty 2^j\theta_{2^j} = \sum_{r=1}^\infty \frac{1}{r^\beta} = \infty$, when $\beta \leq 1$. For the cases $\alpha > 1$, we have

$$\sum_{\ell\in\mathbb{Z}} |\gamma_\ell|^2\,|\ell|^{2\alpha} = \sum_{m=1}^\infty 2^{2m\alpha}|\gamma_{2^m}|^2 = \sum_{m=1}^\infty 2^{2m\alpha}\,|\sum_{j=0}^\infty 2^j\theta_{2^{j+m}}|^2$$

$$= \sum_{m=1}^\infty 2^{2(\alpha-1)m}\,|\sum_{j=m}^\infty 2^j\theta_{2^j}|^2 \geq \sum_{m=2}^\infty 2^{2(\alpha-1)m}\left|\sum_{r=[\log_2 m]+1}^\infty 2^{2^r}\frac{1}{2^{\alpha 2^r}r^\beta}\right|^2$$

$$\geq C\sum_{m=2}^\infty 2^{2(\alpha-1)m}\left|\frac{2^m}{2^{\alpha m}(\log_2 m)^\beta}\right|^2 = C\sum_{m=2}^\infty\frac{1}{(\log_2 m)^{2\beta}} = \infty,$$

which is also a contradiction.

However, by modifying slightly the argument given in the proof of Proposition 5.17 we can still find solutions to the difference equation (5.44) with a minimal loss of smoothness.

THEOREM 5.45 *Let $1 < \alpha < \infty$ and $\{\theta_\ell\}_{\ell\in\mathbb{Z}} \in H^\alpha(\mathbb{T})$. Then, there exists a unique solution $\{\gamma_\ell\}_{\ell\in\mathbb{Z}}$ to the difference equation (5.44) in $H^1(\mathbb{T})$. Moreover, in this case, $\{\gamma_\ell\} \in H^{\alpha'}(\mathbb{T})$ for all $\alpha' < \alpha$.*

PROOF:

Following the same ideas as in the proof of Proposition 5.17, let us define $\{\gamma_\ell\}$ by:

$$\gamma_0 = -\theta_0 + 2K\pi \quad \text{and} \quad \gamma_\ell = \sum_{j=0}^\infty 2^j\theta_{2^j\ell}, \ \ \ell\in\mathbb{Z}\setminus\{0\}.$$

We claim that $\sum_{\ell\in\mathbb{Z}}(1+|\ell|^2)^{\alpha'}\,|\gamma_\ell|^2 < \infty$, for all $\alpha' < \alpha$. Indeed,

$$\sum_{\ell\neq 0}|\gamma_\ell|^2\,|\ell|^{2\alpha'} \leq \sum_{\ell\neq 0}(\sum_{j=0}^\infty 2^j|\theta_{2^j\ell}|)^2\,|\ell|^{2\alpha'} \leq \sum_{\ell\neq 0}(\sum_{j=0}^\infty\frac{1}{2^{2j(\alpha-1)}})(\sum_{j=0}^\infty 2^{2j\alpha}|\theta_{2^j\ell}|^2)\,|\ell|^{2\alpha'}$$

$$= \frac{2^{2(\alpha-1)}}{2^{2(\alpha-1)}-1}\sum_{p=0}^{\infty}\sum_{m\in 2\mathbb{Z}+1}\sum_{j=0}^{\infty}(2^{j+p}|m|)^{2\alpha}\,|\theta_{2^{j+p}m}|^2\,|2^p m|^{2(\alpha'-\alpha)}$$

$$\leq \frac{4^{\alpha-1}}{4^{\alpha-1}-1}\sum_{p=0}^{\infty}\big(\sum_{s\in\mathbb{Z}}|s|^{2\alpha}|\theta_s|^2\big)\frac{1}{2^{2(\alpha-\alpha')p}} \leq C(\alpha,\alpha')\sum_{s\in\mathbb{Z}}|s|^{2\alpha}|\theta_s|^2.$$

Thus, $\{\gamma_\ell\} \in \cap_{\alpha'<\alpha}H^{\alpha'}(\mathbb{T})$ and this shows the existence part of the theorem. For the uniqueness note that if we assume $\{\tilde{\gamma}_\ell\} \in H^1(\mathbb{T})$ then an estimation of the type $\gamma_\ell = o(|\ell|)$, as $|\ell| \to \infty$, must hold, and iterating the difference equation we obtain, for $\ell \neq 0$,

$$\tilde{\gamma}_\ell = \sum_{j=0}^{n} 2^j\theta_{2^j\ell} + 2^{n+1}\tilde{\gamma}_{2^{n+1}\ell} \to \sum_{j=0}^{\infty} 2^j\theta_{2^j\ell} = \gamma_\ell.$$

This completes the proof of the theorem.

□

Remark 5.46 We point out that the choice of $\alpha' = 1$ for the "uniqueness" in Theorem 5.45 is optimal. Indeed, the lacunary series

$$\gamma_\ell = \begin{cases} 2^{-j}, & \text{if } \ell = 2^j,\ j \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

satisfies $\sum_{\ell\in\mathbb{Z}}|\gamma_\ell|^2(1+|\ell|^2)^{\alpha'} < \infty$, for all $\alpha' < 1$, and $\gamma_\ell = 2\gamma_{2\ell}, \ell \neq 0$, $\gamma_0 = 0$, so is a non-trivial solution to the difference equation (5.44) with $\theta_\ell \equiv 0$ (and $K = 0$). In fact, multiplying $\{\gamma_\ell\}$ by any constant we obtain infinitely many other solutions.

With respect to the existence, we have left aside in the previous theorem the cases when $\frac{1}{2} < \alpha \leq 1$. It is still possible to find a solution $\{\gamma_\ell\} \in \cap_{\alpha'<\alpha}H^{\alpha'}(\mathbb{T})$ to the difference equation (5.44), when $\{\theta_\ell\}$ is a given element in $H^\alpha(\mathbb{T})$, in spite of the fact that the series $\sum_{j=0}^{\infty} 2^j\theta_{2^j}$ might be divergent (as happens, for instance, with the sequence $\{\theta_\ell\}_{\ell\in\mathbb{Z}}$ of Example 2). To construct such a solution we iterate the difference equation (5.44) "backwards", assuming *a priori* that $\gamma_{2\ell+1} = 0,\ \ell \in \mathbb{Z}$. Then, we can define the coefficients γ_ℓ by:

$$\gamma_\ell = \begin{cases} -\theta_0, & \text{if } \ell = 0 \\ 0, & \text{if } \ell \in 2\mathbb{Z}+1 \\ -\sum_{j=1}^{n} 2^{-j}\theta_{2^{n-j}p}, & \text{if } \ell = 2^n p,\ n \geq 1,\ p \in 2\mathbb{Z}+1. \end{cases} \tag{5.47}$$

An easy computation shows that $\gamma_\ell = \theta_\ell + 2\gamma_{2\ell}$, $\ell \in \mathbb{Z}$. We claim, further, that $\{\gamma_\ell\}_{\ell\in\mathbb{Z}} \in \cap_{\alpha'<\alpha} H^{\alpha'}(\mathbb{T})$. Indeed, let $\alpha' < \alpha \leq 1$; then

$$\begin{aligned}
\sum_{\ell\neq 0} |\ell|^{2\alpha'} |\gamma_\ell|^2 &= \sum_{n=1}^{\infty} \sum_{p\in 2\mathbb{Z}+1} |2^n p|^{2\alpha'} \, |\sum_{j=1}^{n} 2^{-j} \theta_{2^{n-j}p}|^2 \\
&\leq \sum_{n=1}^{\infty} \sum_{p\in 2\mathbb{Z}+1} |2^n p|^{2\alpha'} \left(\sum_{j=1}^{n} 2^{2j(\alpha-1)}\right) \left(\sum_{j=1}^{n} 2^{-2j\alpha} |\theta_{2^{n-j}p}|^2\right) \\
&\leq \sum_{n=1}^{\infty} \sum_{p\in 2\mathbb{Z}+1} \sum_{j=1}^{n} 2^{2n\alpha'} |p|^{2\alpha'} \, n \, 2^{-2j\alpha} \, |\theta_{2^{n-j}p}|^2 \\
&= \sum_{n=1}^{\infty} \sum_{p\in 2\mathbb{Z}+1} \sum_{j=1}^{n} |2^{n-j}p|^{2\alpha} \, |\theta_{2^{n-j}p}|^2 \, \frac{1}{|p|^{2(\alpha-\alpha')}} \, \frac{n}{2^{2n(\alpha-\alpha')}} \\
&\leq \sum_{n=1}^{\infty} \sum_{p\in 2\mathbb{Z}+1} \sum_{s=0}^{n-1} |2^s p|^{2\alpha} \, |\theta_{2^s p}|^2 \, \frac{n}{2^{2n(\alpha-\alpha')}} \\
&= \sum_{s=0}^{\infty} \sum_{p\in 2\mathbb{Z}+1} |2^s p|^{2\alpha} \, |\theta_{2^s p}|^2 \sum_{n=s+1}^{\infty} \frac{n}{2^{2n(\alpha-\alpha')}} \\
&\leq C_{\alpha,\alpha'} \sum_{r\in\mathbb{Z}} |r|^{2\alpha} \, |\theta_r|^2 < \infty.
\end{aligned}$$

The non-uniqueness of solutions to the difference equation (5.44) is again clearly seen in this construction from the freedom we have to choose the odd coefficients of γ.

The statement we just made about existence is also sharp, in the sense that we cannot improve $\cap_{\alpha'<\alpha} H^{\alpha'}(\mathbb{T})$ with the space $H^\alpha(\mathbb{T})$ for a general solution $\{\gamma_\ell\}$ to (5.44). Indeed, when $\{\theta_\ell\}_{\ell\in\mathbb{Z}}$ is taken as in Example 2 (now for $\frac{1}{2} < \alpha < 1$), the same solution $\{\gamma_\ell\}_{\ell\in\mathbb{Z}}$ to (5.44) we have just found cannot be in $H^\alpha(\mathbb{T})$ since

$$\begin{aligned}
\sum_{\ell\neq 0} |\ell|^{2\alpha} \, |\gamma_\ell|^2 &= \sum_{n=1}^{\infty} 2^{2n\alpha} \, |\sum_{j=1}^{n} 2^{-j} \theta_{2^{n-j}}|^2 = \sum_{n=3}^{\infty} 2^{2n(\alpha-1)} \left| \sum_{r=1}^{[\log_2(n-1)]} 2^{2^r} \frac{1}{2^{\alpha 2^r} r^\beta} \right|^2 \\
&\geq C \sum_{n=3}^{\infty} 2^{2n(\alpha-1)} \left| 2^{(1-\alpha)n} \frac{1}{(\log_2 n)^\beta} \right|^2 = C \sum_{n=3}^{\infty} \frac{1}{(\log_2 n)^{2\beta}} = \infty.
\end{aligned}$$

Furthermore, no other solution $\{\gamma_\ell\}_{\ell\in\mathbb{Z}}$ to the difference equation (5.44) will belong to $H^\alpha(\mathbb{T})$, for if it did, and since by iteration we always have that $\gamma_{2^n} = \frac{1}{2^n}\gamma_1 - \sum_{j=1}^{n} \frac{1}{2^j} \theta_{2^{n-j}}$, $n \geq 1$, then

$$\infty > \sum_{n=3}^{\infty} 2^{2n\alpha} \, |\gamma_{2^n}|^2 = \sum_{n=3}^{\infty} 2^{2n(\alpha-1)} \, |\gamma_1 - \sum_{j=1}^{n} 2^{n-j} \theta_{2^{n-j}}|^2 \geq$$

$$\sum_{n=3}^{\infty} 2^{2n(\alpha-1)} \left\{ |\gamma_1|^2 + |\sum_{j=1}^{n} 2^{n-j}\theta_{2^{n-j}}|^2 - 2\,|\gamma_1|\,|\sum_{j=1}^{n} 2^{n-j}\theta_{2^{n-j}}| \right\} \geq$$
$$|\gamma_1|^2 \sum_{n=3}^{\infty} 2^{2n(\alpha-1)} + C \sum_{n=3}^{\infty} \frac{1}{(\log_2 n)^{2\beta}} - 2|\gamma_1|\,\widetilde{C} \sum_{n=3}^{\infty} \frac{2^{n(\alpha-1)}}{(\log_2 n)^{\beta}} = +\infty,$$

which is a contradiction.

This, together with what was said in Proposition 5.17, gives a complete description of the solutions to the functional equation (5.42), that can be compiled in the following theorem:

Theorem 5.48 *Let $\frac{1}{2} < \alpha \leq \infty$ and $\tilde{\nu} \in \mathcal{M}_{\alpha}^{(0)}$ and write $\tilde{\nu}(\xi) = e^{i\theta(\xi)}$, for some real-valued $\theta(\xi) = \sum_{\ell\in\mathbb{Z}} \theta_\ell e^{-i\ell\xi} \in H^{\alpha}(\mathbb{T})$. Consider the functional equation (5.42). Then,*

(i) If $1 < \alpha \leq \infty$, (5.42) has a unique solution $\mu(\xi) = e^{i\gamma(\xi)} \in \mathcal{M}_1^{(0)}$ given by $\gamma(\xi) = \sum_{\ell\in\mathbb{Z}} \gamma_\ell e^{-i\ell\xi}$, where $\gamma_0 = -\theta_0$ and $\gamma_\ell = \sum_{j=0}^{\infty} 2^j \theta_{2^j\ell}$, $\ell \neq 0$. Moreover, in this case $\mu \in \cap_{\alpha'<\alpha} H^{\alpha'}(\mathbb{T})$. If $\alpha \neq \infty$, there are examples for which the solution $\mu \notin H^{\alpha}(\mathbb{T})$.

(ii) If $\frac{1}{2} < \alpha \leq 1$, (5.42) has infinitely many solutions $\mu \in \cap_{\alpha'<\alpha}\mathcal{M}_{\alpha'}^{(0)}$. There are examples for which none of the solutions belongs to $H^{\alpha}(\mathbb{T})$.

We can rewrite this result in terms of α-localized wavelets and its associated low-pass filters as we did in Corollary 5.28. Its proof is a repetition of the one given for Proposition 5.17 and is left to the reader.

Corollary 5.49 *Let $\frac{1}{2} < \alpha \leq \infty$, $k \in \mathbb{Z}$ and $\psi \in \mathcal{W}_{\alpha}^{(k)}$. Then, there exists a low-pass filter $m_0 \in \cap_{\alpha'<\alpha}\mathcal{E}_{\alpha'}$ such that (5.29) holds for some unimodular constant $c \in \mathbb{C}$. Furthermore, if (m_0, c) and $(m_0^{\sharp}, c^{\sharp})$ are two solutions to (5.29) such that $m_0, m_0^{\sharp} \in H^1(\mathbb{T})$, then, there exists an integer $M \in \mathbb{Z}$ such that $c^{\sharp} = (-1)^M c$ and $m_0^{\sharp}(\xi) = e^{iM\xi} m_0(\xi)$.*

5.4 Some results in connectivity for other sets of wavelets

We have already pointed out that when $\mathcal{W}_\alpha$ is regarded as a topological subspace of $L^2(\mathbb{R})$, then it is arcwise-connected (see Remark 5.41). In fact, as long as we use the topology of $L^2(\mathbb{R})$, much more general sets of wavelets, where no localization or smoothness are assumed, can be shown to be connected. This matter has been treated by different authors in the series of papers: [DAI-LIA], [HAN-LU], [ION-LAR-PEA], [SPEE], ... The article [WUTAM] compiles most of the results in the others and gives appropriate credit to each author, stating some of the open questions that still remain in the subject. It is not known, for instance, whether the set $\mathcal{W}$ of all wavelets is connected in the topology of $L^2(\mathbb{R})$. For completeness, we include here the two main results known in this direction:

Theorem 5.50 : [DAI-LIA], [HAN-LU], [WUTAM].

Let $\mathcal{W}_{MRA}$ denote the set of all MRA wavelets with the topology of $L^2(\mathbb{R})$. Then, for any pair $\psi_0, \psi_1 \in \mathcal{W}_{MRA}$ there exists a continuous path

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{W}_{MRA} \\ t &\longmapsto \psi_t \end{aligned}$$

starting at ψ_0 and ending at ψ_1. Moreover, each wavelet in the path can be taken so that $\widehat{\psi}_t \in C(\mathbb{R})$ and $\widehat{\psi}_t \equiv 0$ in a neighborhood of $\xi = 0$, provided $t \in]0,1[$.

Theorem 5.51 : [SPEE], [ION-LAR-PEA].

Let $\mathcal{W}_{MSF}$ denote the set of orthonormal wavelets ψ such that $\widehat{\psi} = \chi_K$, for some measurable set $K \subset \mathbb{R}$, endowed with the topology of $L^2(\mathbb{R})$. Then, $\mathcal{W}_{MSF}$ is arcwise connected; that is, for any pair $\psi_0, \psi_1 \in \mathcal{W}_{MSF}$, such that $\widehat{\psi}_j = \chi_{K_j}$, $j = 0, 1$, there exists a family of measurable sets $K_t \subset \mathbb{R}$, $0 < t < 1$, such that $\psi_t = (\chi_{K_t})\check{}$ is a wavelet, for every $t \in [0,1]$, and the map

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{W}_{MSF} \\ t &\longmapsto \psi_t = (\chi_{K_t})\check{} \end{aligned}$$

is continuous into $L^2(\mathbb{R})$.

Remark 5.52 This version of Theorem 5.51 was proved by D. Speegle in [SPEE]. In [ION-LAR-PEA] it is proved a similar theorem, but with an additional, non-trivial, assumption on the sets K_0, K_1. This assumption, satisfied by most MSF-wavelets (but not by all of them), allows the authors of this paper to construct their family of sets K_t in such a way that $K_t \subset K_0 \cup K_1$, for all $t \in [0,1]$.

Appendix A

Examples and counter-examples

Equations (2.8) and (2.9) in Chapter 1 can be used, among other things, to construct a variety of examples and counterexamples in wavelet theory. In this appendix we present some of these, trying to answer questions related to the behavior of the Fourier transform of a wavelet at the point $\xi = 0$.

1 A band-limited wavelet with Fourier transform discontinuous at 0

Let us point out to the reader that, from equation (2.8) (i) in Chapter 1, it follows that if ψ is a wavelet whose Fourier transform is continuous at 0, then it must have $\widehat{\psi}(0) = 0$ (this is a version of what we said in §1 that ψ must have "mean" zero: $\widehat{\psi}(0) = \int_{\mathbb{R}} \psi = 0$). In fact, something stronger is true if we assume, in addition, that ψ is *band-limited*, that is, that supp $\widehat{\psi} \subset [-M, M]$ for some $M > 0$. In this case, it can be shown (see Theorem 2.7 in [BSW], or the same theorem in chapter 3 of [HW]) that $\widehat{\psi}$ must vanish in a neighborhood of 0. One can ask whether it is really necessary to assume that $\widehat{\psi}$ is continuous at 0 in order to obtain this result. A partial answer was given in [HWW] for wavelets with supp $\widehat{\psi} \subset [-8\pi/3, 8\pi/3]$ (or, more generally, with supp $\widehat{\psi} \subset [-4\pi + 4\alpha/3, 8\alpha/3]$, for some $0 < \alpha \leq \pi$). Indeed, when this is the case, it can be shown that $\widehat{\psi}$ is continuous at the origin and, moreover, vanishes on $[-2\pi/3, 2\pi/3]$ (respectively, on $[-2\pi + 4\alpha/3, 2\alpha/3]$). It was not known to the authors of [HWW] whether this property could be true for larger domains. We will start by proving the following:

Proposition 1.1 *There is a wavelet ψ whose Fourier transform has support contained in $\subset [-4\pi, \pi]$ but does not vanish in any neighborhood of* 0.

Remark 1.2 At the time we discovered it, this example seemed to be the only one in the literature with these characteristics. However, some time afterwards, we found an example of W. Madych of a scaling function with similar properties (see 3.1.2 in [MAD]). After carrying out some computations one obtains from it an MRA wavelet ψ with supp $\widehat{\psi} \subset [-3\pi/2, 4\pi]$ and $\widehat{\psi}$ discontinuous at 0 on the left.

Proof:

The wavelet we construct to prove Proposition 1.1 will be an MSF wavelet. That is, $\widehat{\psi} = \chi_K$, for an appropriate set $K \subset \mathbb{R}$. To construct K, we find a partition of $I = [-2\pi, -\pi) \cup (\pi, 2\pi]$ satisfying (2.6) of Chapter 1.

The partition of $(\pi, 2\pi]$ is obtained as follows: divide $(\pi, 2\pi]$ in two halves and keep the right-hand one; do the same with the left-hand part. Continue this process a countable number of times. The intervals obtained are:

$$I_n = \left(\frac{(2^n+1)\pi}{2^n}, \frac{(2^{n-1}+1)\pi}{2^{n-1}}\right], \quad n = 1, 2, 3, \ldots$$

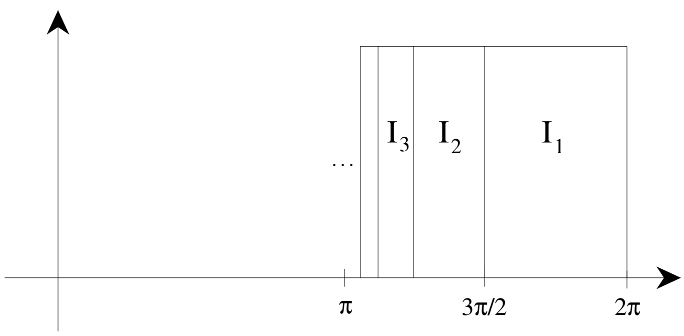

FIGURE 1

A partition of $(-2\pi, -\pi]$ is obtained by translating by -2π a partition of $(0, \pi]$. The points $\frac{(2^n+1)\pi}{2^{2n+1}}$, $n = 1, 2, \ldots$, satisfy:

$$\frac{\pi}{2^{n+1}} < \frac{(2^n+1)\pi}{2^{2n+1}} < \frac{\pi}{2^n}.$$

Let

$$\begin{aligned} \widetilde{G}_n &= \left(\frac{(2^n+1)\pi}{2^{2n+1}}, \frac{\pi}{2^n}\right], \quad n = 1, 2, 3, \ldots \quad \text{and} \\ \widetilde{H}_n &= \left(\frac{\pi}{2^{n+1}}, \frac{(2^n+1)\pi}{2^{2n+1}}\right], \quad n = 0, 1, 2, \ldots \end{aligned}$$

(See Figures 2 and 3 below.)

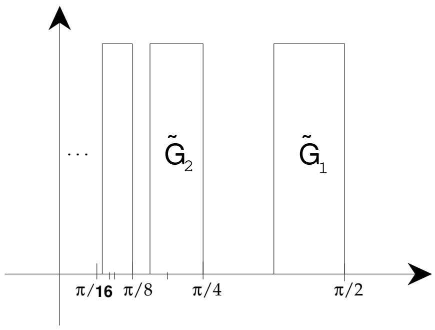

FIGURE 2

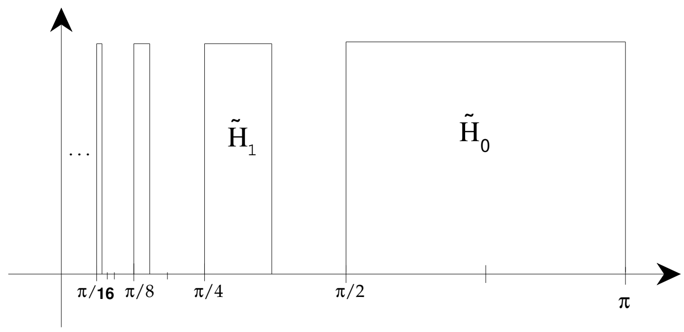

FIGURE 3

Then, $\{\widetilde{G}_n\}_{n=1}^{\infty} \cup \{\widetilde{H}_n\}_{n=0}^{\infty}$ forms a partition of $(0, \pi]$ and, hence,

$$\begin{aligned} G_n &= \widetilde{G}_n - 2\pi, \quad n = 1, 2, 3, \ldots \\ H_n &= \widetilde{H}_n - 2\pi, \quad n = 0, 1, 2, \ldots \end{aligned}$$

forms a partition of $(-2\pi, -\pi]$. Now, we appropriately dilate these intervals to form a partition of our set K as in (2.6) of Chapter 1:

$$K = \{\cup_{n=1}^{\infty} 2^{-n} I_n\} \cup \{\cup_{n=1}^{\infty} G_n\} \cup \{\cup_{n=0}^{\infty} 2H_n\}.$$

(See Figure 4 below.)

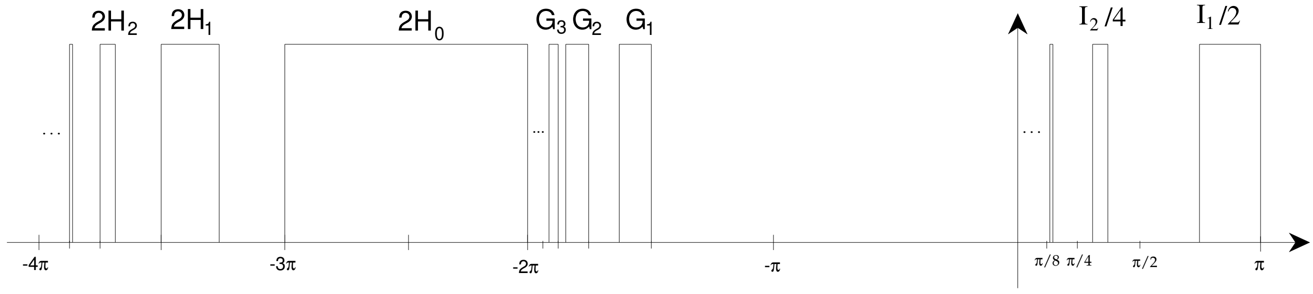

FIGURE 4

Finally, we show that this new family of intervals can be translated back to form another partition of I. Indeed, the set

$$2H_0 + 4\pi = 2[H_0 + 2\pi] = 2\widetilde{H}_0 = (\pi, 2\pi]$$

is a partition of $(\pi, 2\pi]$. Appropriate translations of the remaining intervals form a partition of $(-2\pi, -\pi]$:

$$\{2H_n + 2\pi\}_{n=1}^{\infty}, \quad \{2^{-n} I_n - 2\pi\}_{n=1}^{\infty}, \quad \{G_n\}_{n=1}^{\infty}.$$

This can be seen by showing that

$$\left.\begin{array}{rcll} 2H_n + 4\pi = 2\widetilde{H}_n & = & \left(\dfrac{\pi}{2^n}, \dfrac{(2^n+1)\pi}{2^{2n}}\right], & n = 1, 2, \ldots \\ 2^{-n} I_n & = & \left(\dfrac{(2^n+1)\pi}{2^{2n}}, \dfrac{(2^{n-1}+1)\pi}{2^{2n-1}}\right], & n = 1, 2, \ldots \\ \text{and} \qquad G_{n-1} + 2\pi = \widetilde{G}_{n-1} & = & \left(\dfrac{(2^{n-1}+1)\pi}{2^{2n-1}}, \dfrac{\pi}{2^{n-1}}\right], & n = 2, 3, \ldots \end{array}\right\} \tag{1.3}$$

form a partition of $(0, \pi]$. Note that the right-end point of $2H_n + 4\pi$ coincides with the left-end point of $2^{-n} I_n$, and the right-end point of $2^{-n} I_n$ coincides with the left-end point of $G_{n-1} + 2\pi$.

Thus, $\psi = (\chi_K)\check{}$, is a band-limited wavelet, with supp $\widehat{\psi} = \overline{K} \subset [-4\pi, -\pi] \cup [0, \pi]$, and such that $|\widehat{\psi}|$ is discontinuous at 0 from the right.

$\square$

Remark 1.4 This method of constructing MSF wavelets via the characterization in (2.6) of Chapter 1 leads tipically to non-MRA wavelets (see, for instance, [FAN-WAN], [DAI-LAR-SPE] or [SOA-WEI]). It is a remarkable fact that the wavelet ψ constructed above *does* arise from an MRA, although none of the MRA conditions have been used in the process. As a consequence, one obtains as well the existence of band-limited scaling functions whose Fourier transform is discontinuous at the origin.

To show that ψ comes from an MRA it is enough to check that

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi + 2k\pi))|^2 = 1, \quad a.e.\, \xi \in (-\pi, \pi] \tag{1.5}$$

(see Proposition 3.8 in Chapter 1 of this thesis). Now, if $\xi \in (-\pi, 0)$, the sum in (1.5) reduces to

$$D_\psi(\xi) = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2 = 1,$$

the last equality following from (2.5) of Chapter 1 (see also Figure 4). On the other hand, if $\xi \in (0, \pi]$ and $j \geq 1$ is fixed, the only k's that contribute to the series in (1.5) are $k = 0, -1$:

$$\begin{aligned} D_\psi(\xi) &= \sum_{j=1}^{\infty} \{\, |\widehat{\psi}(2^j\xi)|^2 + |\widehat{\psi}(2^j(\xi - 2\pi))|^2 \,\} \\ &= \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2 + |\widehat{\psi}(2(\xi - 2\pi))|^2 = A + B. \end{aligned}$$

Now,

$$\begin{aligned} B = 1 \quad &\Longleftrightarrow \quad 2(\xi - 2\pi) \in \cup_{n=0}^{\infty} 2H_n \iff \xi \in \cup_{n=0}^{\infty}\{H_n + 2\pi\} = \cup_{n=0}^{\infty} \widetilde{H}_n \\ &\Longleftrightarrow \quad \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2 = 0 \iff A = 0, \end{aligned}$$

the previous to the last equivalence because, by (1.3), $2\widetilde{H}_n \subset \widetilde{H}_{n-1} \setminus 2^{-n} I_n$ (see also Figures 2, 3 and 4).

2 More wavelets with Fourier transform discontinuous at 0

The construction in the previous example might seem a little mysterious[1], however, more intuitive constructions can be performed by using the method of the MRA's. We present below one such example, solution to a different but closely related problem.

We already mentioned at the beginning of §1 of the appendix that if $|\widehat{\psi}|$ is continuous at 0, there must be an interval around it in which $\widehat{\psi}$ vanishes identically. In fact, for wavelets with supp $\widehat{\psi} \subset [-4\pi + 4\alpha/3, 8\alpha/3]$, $0 < \alpha \leq \pi$, this interval $(= [-2\pi + 4\alpha/3, 2\alpha/3])$ has length, at least, $4\pi/3$ (see [HWW]). At first, this suggested to the authors of [HWW] that a lower bound for the length of the "gap" in which $\widehat{\psi}$ vanishes identically might exist. However, they proved later that there are band-limited wavelets for which the length of the "gap" can be made arbitrarily small (see Remark 1 in §3.2 of [HW]). Wavelets satisfying this property can be easily constructed from their low-pass filters. A graphic idea is as follows. Let a, b be two positive fixed numbers such that $a^2 + b^2 = 1$ and define m_0 2π-periodically as in Figure 5:

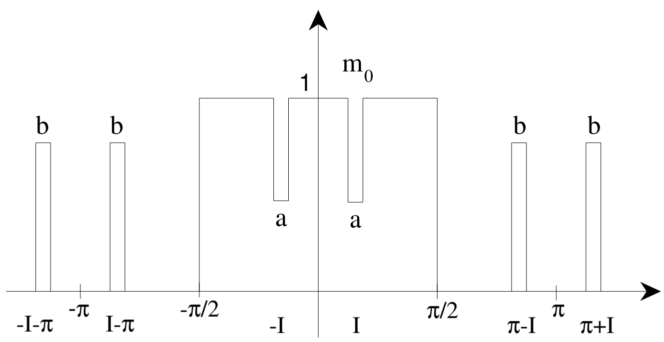

FIGURE 5

Here, the interval I can be chosen as close to 0 as we wish (with the only requirement

[1]A careful reader will realize that there is a pattern in the way of choosing the intervals $\widetilde{H}_n$ and $\widetilde{G}_n$ that explains why $(2^n + 1)\pi/2^{2n+1}$ is the most suited extreme point between them.

that $(I/2) \cap I = \emptyset$); one easily shows, then, that m_0 is a low-pass filter and that ψ given by

$$\widehat{\psi}(\xi) = e^{i\frac{\xi}{2}} \overline{m_0(\xi/2+\pi)} \prod_{j=2}^{\infty} m_0(2^{-j}\xi), \tag{2.6}$$

is a band-limited wavelet for which $\widehat{\psi}$ is an even function not vanishing on $2I$ (for more details, see [HW]).

A natural question would be if one could somehow modify the construction of the filters above to obtain a wavelet with "gap" of length exactly 0. The answer is yes, but under certain restrictions. By taking a filter m_0 as in Figure 6 below, one obtains a wavelet ψ for which $\widehat{\psi}$ is an even function not vanishing identically in any symmetric interval around 0. Thus, the "gap" disappears! However, this construction has the incovenience that ψ is no longer band-limited. At this point we do not know whether there are wavelets ψ with Fourier transform even and discontinuous at 0, but still having compact support.

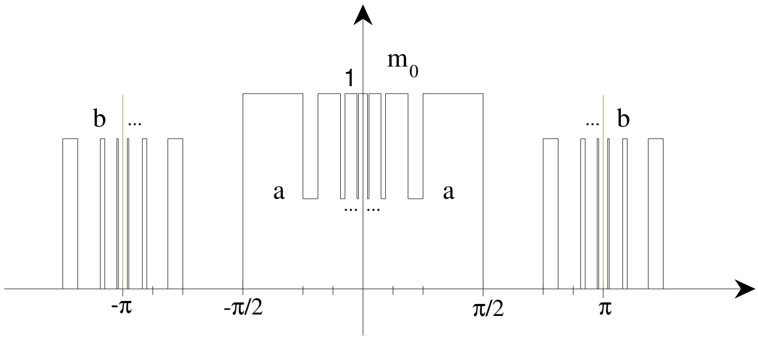

FIGURE 6

We say a few more words about the example in Figure 6 before going into details. By taking $a = 0$, we obtain a non-band-limited MSF wavelet that arises from an MRA. This example seemed also to be new at its time[2], when the only non-band-limited

[2] We learned later, in a wavelet conference held in North Carolina, that X. Dai and R. Liang found similar examples while studying connectivity properties of wavelets.

MSF wavelets known did not arise from an MRA (see [FAN-WAN] or [DAI-LAR-SPE]). For completeness, and its own interest, we present here a little more detailed discussion of this example.

Divide the interval $(\pi/4, \pi/2]$ in halves and call the right hand subinterval obtained J_1. Divide, then, the left one in halves and call the right hand subinterval obtained J_2. Continue this process a countable number of times to obtain:

$$J_n = \left(\frac{(2^n+1)\pi}{2^{n+2}}, \frac{(2^{n-1}+1)\pi}{2^{n+1}} \right], \quad n = 1, 2, 3, \ldots$$

(See Figure 1 above for a "dilated" picture of this partition.)

Then, define the sets

$$I_n = 2^{-n} J_n, \quad n = 1, 2, 3, \ldots \quad \text{and} \quad I = \cup_{n=1}^{\infty} I_n, \tag{2.7}$$

and the function

$$m_0(\xi) = \chi_{(0,\pi/2]\setminus I}(\xi) + \sum_{n=1}^{\infty} \{\, a\chi_{I_n}(\xi) \; + \; b\chi_{\pi - I_n}(\xi)\,\}, \quad \xi \in (0, \pi],$$

where $a, b > 0$ are such that $a^2 + b^2 = 1$. Extend m_0 evenly to $[-\pi, 0)$ and 2π-periodically to $\mathbb{R}$. Then, by construction we have that:

(i) $\quad |m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1, \quad a.e.\, \xi \in \mathbb{R}.$

(ii) For every $\xi \neq 0$, $\exists\, n_0 = n_0(\xi) \in \mathbb{Z} \mid \forall\, n > n_0 \Longrightarrow m_0(2^{-n}\xi) = 1.$

Moreover, if we define $\widehat{\varphi}(\xi) = \prod_{n=1}^{\infty} m_0(2^{-n}\xi)$, $\xi \neq 0$, then, $\widehat{\varphi} \in L^2(\mathbb{R})$ and

(iii) $\lim_{n\to\infty} \widehat{\varphi}(2^{-n}\xi) = 1, \quad \xi \neq 0.$

(iv) For $\quad a.e.\, \xi \in K = (-\pi, \pi], \quad$ have $\quad \widehat{\varphi}(\xi) \geq a.$

It follows from this and a standard argument{3} that φ satisfies the hypotheses of Proposition 3.12 in Chapter 1 and, therefore, is a scaling function of an MRA (see

{3}The "standard argument" we refer to is the following. Define $\hat{f}_n = \chi_{2^n K} \prod_{j=1}^{n} m_0(2^{-j}\cdot)$. Then, $\{f_n(\cdot + k)\}_{k\in\mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{R})$ for each $n \geq 1$. Since $0 \leq f_n \leq \frac{1}{a}\widehat{\varphi}$, the Dominated Convergence Theorem implies that $\{\varphi(\cdot + k)\}$ must be an orthonormal system as well and, hence, (3.6) of Chapter 1 holds. For more details, see Theorem 4.8 in chapter 7 of [HW].

Chapter 7 of [HW]). In particular, if a function ψ is defined as in (2.6), then, it will be a wavelet for which, clearly, $\widehat{\psi}$ is even and $\widehat{\psi}(\xi) \geq ab,\ \xi \in 2I$. It is easy to see that $\widehat{\varphi}$ (and, therefore, $\widehat{\psi}$) does not have compact support (try, for instance, $\xi \in 2^n(\pi - I_{n-1})$, for n large). Hence, we have constructed a non-band-limited wavelet with even Fourier transform discontinuous at 0.

We claimed above that by letting $a = 0$ in this example, one obtains a non-band-limited MSF wavelet arising from an MRA. This is true but the proof requires some modifications. Note that, when $a = 0$, condition (iv) above is vacuous and we cannot use the "standard argument" to show that φ is a scaling function. Thus, condition (i) in Proposition 3.12 needs to be verified directly this time. However, in this case, m_0 and $\widehat{\varphi}$ are characteristic functions of measurable sets in $\mathbb{R}$, say E and S, respectively, and this condition is equivalent to:

$$\sum_{k\in\mathbb{Z}} \chi_S(\xi + 2k\pi) = 1, \quad a.e.\, \xi \in (-\pi, \pi]. \tag{2.8}$$

In fact, it is enough to show the inequality "$\geq$" in (2.8), because "$\leq$" always holds when $S = \cap_{n=1}^{\infty} 2^{-n} E$ and $E = E + 2\pi,\ E \cap (E + \pi) = \emptyset$ (see Lemma 2.21 in [HWW2]).

For this example we take $I_2, I_3, \ldots$ defined as in (2.7) above, but assuming this time that I_1 is empty (in the course of the proof we will see why this assumption is required). The same definition of m_0 and $\widehat{\varphi}$ applies as well. Consider the following sets:

$$K_p^+ = 2^p(\cup_{j=p}^{\infty} I_{j-1}), \quad \text{for } p \geq 1 \quad \text{and} \quad K^+ = \cup_{p=1}^{\infty} K_p^+.$$

(For consistency with the notation we assume also $I_0 = \emptyset$.) Note that, by construction, $K^+ \subset (0, \pi]$ and, for $\xi \in (0, \pi]$, $\widehat{\varphi}(\xi) = 0$ if and only if $\xi \in K^+$.

Now, we translate each of the K_p^+'s to obtain new sets where $\widehat{\varphi} \equiv 1$. Let

$$S_p^+ = K_p^+ - 2^p\pi, \quad p \geq 1.$$

We claim that $\widehat{\varphi}(\xi) = 1$, $a.e.\, \xi \in S_p^+$. Equivalently, we need to show that $m_0(2^{-j}\xi) = 1$, for all $j \geq 1$, $a.e.\, \xi \in S_p^+$. We have three cases:

1. Suppose $1 \leq j \leq p-1$ (this condition is empty if $p=1$). Then,

$$m_0(2^{-j}\xi) = m_0(2^{-j}\xi + 2^{p-j}\xi) = m_0(2^{-j}(\xi + 2^p\pi)) = 1,$$

   because $\xi + 2^p\pi \in K_p^+ = 2^p \cup_{n=p}^{\infty} I_{n-1}$ and $j \neq p$.

2. If $j = p$, then $2^{-p}\xi \in 2^{-p}K_p^+ - \pi \subset I - \pi$, so that $m_0(2^{-j}\xi) = 1$ $(= b)$.

3. If $j = p + k,\ k \geq 1$. Then,

$$2^{-j}\xi \in 2^{-k}(2^{-p}K_p^+ - \pi) \subset 2^{-k}(I - \pi) \subset 2^{-k}\left(-\pi, -\frac{3\pi}{4}\right],$$

   but this is a dilation of the interval "I_1" defined in (2.7) that we removed this time from the set I. Hence, $m_0(2^{-j}\xi) = 1$.

This establishes our claim. Repeating this process symmetrically with $K^- = -K^+$, we obtain that

$$S = (-\pi, \pi] \setminus (K^+ \cap K^-) \cup (\cup_{p=1}^{\infty} S_p^+) \cup (\cup_{p=1}^{\infty} S_p^-)$$

and (2.8) holds.

3 One last example

Our last example in this section is of a different nature, but still related to MSF wavelets. We already mentioned that the equations:

$$\tau(\xi) \equiv \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + 2k\pi)|^2 = 1, \qquad a.e.\ \xi \in (-\pi, \pi] \tag{3.9}$$

$$\delta(\xi) \equiv \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1, \qquad a.e.\ \xi \in (-\pi, -\frac{\pi}{2}] \cup (\frac{\pi}{2}, \pi] \tag{3.10}$$

are necessary and sufficient conditions for a function ψ such that $|\widehat{\psi}| = \chi_K$ to be a wavelet. In particular, MSF wavelets are characterized in terms of their "amplitude", $|\widehat{\psi}|$, and admit arbitrary "phases", meaning that $(e^{i\alpha}|\widehat{\psi}|)^{\check{}}$ is a wavelet for any (measurable) real-valued function α on $\mathbb{R}$.

This is not the case when ψ is a non-MSF wavelet. In fact, if a wavelet has $\widehat{\psi} \geq 0$ then it must be an MSF wavelet. Indeed, if $\widehat{\psi} \geq 0$ and, for a fixed $\xi \in \mathbb{R}$, $\widehat{\psi}(\xi) \neq 0$, the orthogonality relations in (5.6) of Chapter 2 imply that $\widehat{\psi}(2^j \xi) = 0$, $\forall j \geq 1$. Then, if $\widehat{\psi}(\xi) < 1$, there must be a $j_0 < 0$ such that $\widehat{\psi}(2^{j_0}\xi) \neq 0$ (by (3.10)) but, again, the orthogonality relations (5.6) would imply that $\widehat{\psi}(2^{j_0+k}\xi) = 0$, $\forall k \geq 1$, which is contradiction. Thus, $\widehat{\psi}(\xi) = 1$ and $\widehat{\psi}$ is the characteristic function of a set.

Therefore, the two equations above do not imply, in general, that $\psi = (|\widehat{\psi}|)^\vee$ is a wavelet, unless an appropriate phase is chosen. Something stronger is true: these two equations do *not* imply, in general, that such a phase exists. That is, there are functions $\widehat{\psi}$ satisfying (3.9) and (3.10) for which no real-valued function $\alpha = \alpha(\xi)$ makes $(e^{i\alpha}|\widehat{\psi}|)^\vee$ into a wavelet. The following simple example illustrates what we are saying. Let

$$\widehat{\psi}(\xi) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } \xi \in (-\pi, -\frac{\pi}{4}] \cup (\pi, \frac{7\pi}{4}] \cup (2\pi, 3\pi] \cup (6\pi, 7\pi] \\ 1, & \text{if } \xi \in (\frac{7\pi}{4}, 2\pi] \\ 0, & \text{elsewhere} \end{cases}$$

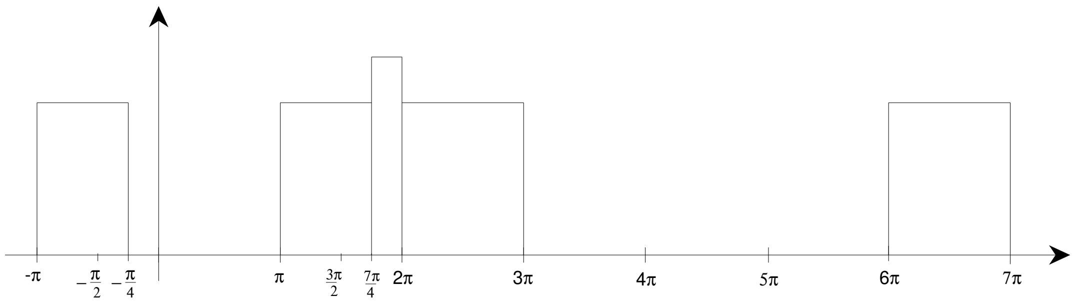

FIGURE 7

It is easy to see (by just looking at the graph) that $\widehat{\psi}$ satisfies (3.9) and (3.10). However, $(e^{i\alpha}|\widehat{\psi}|)^\vee$ can never be a wavelet because

$$t_{-1}(\xi) = \frac{1}{2} e^{i(\alpha(\xi) - \alpha(\xi - 2\pi))} \neq 0, \quad \text{if } \xi \in \left(\pi, \frac{3\pi}{2}\right],$$

and this would contradict Theorem 2.7 in Chapter 1.

The question of what phases are attainable for a given wavelet is still not very well understood, especially in the case of non-MRA wavelets. A more detailed discussion, and other properties of phases can be found in [BSW], [WUTAM] or §5 in Chapter 3 of this thesis.

Appendix B

Bibliography

1. [ADAM] Adams, R. "Sobolev Spaces". Academic Press (1975).

2. [AHL] Ahlfors, L. V. "Complex Analysis". Third Edition. McGraw-Hill (1979).

3. [AUS] Auscher, P. "Solutions to two problems on wavelets". The Journal of Geometric Analysis (1995), Vol 5, number 2 (pp.181-237).

4. [BDW] Bonami, A. , Durand, S. and Weiss, G. "Wavelets obtained by continuous deformations of the Haar wavelet". Revista Matemática Iberoamericana Vol. 12, Nº 1 (1996), pp. 1-18.

5. [BSW] Bonami, A. , Soria, F. & Weiss, G. "Band-Limited Wavelets". The Journal of Geometric Analysis Vol. 3, Number 6, 1993, pp. 543-578.

6. [CAL] Calogero, A. "Characterization of wavelets on general lattices". Preprint, 1997. (To appear in The Journal of Fourier Analysis and Applications.)

7. [CHU-SHI] Chui, C. K. , Shi, X. "Bessel Sequences and Affine Frames". Applied and Computational Harmonic Analysis (1993), Vol 1, number 1 (pp. 22-49).

8. [COH] Cohen, A. "Ondelettes et traitement numérique du signal". Recherches en Mathématiques Appliquées 25. Masson, Paris (1992).

9. [DAI-LAR-SPE] Dai, X. , Larson, D. R. , Speegle, D. M. "Wavelet Sets in $\mathbb{R}^n$". The Journal of Fourier Analysis and Applications, Vol 3, Number 4, 1997, pp. 451-456.

10. [DAI-LIA] Dai, X. , Liang, R. "Path Connectivity and Wavelets". Preprint (1996).

11. [DAUB] Daubechies, I. "Ten Lectures on Wavelets". SIAM CBMS 61 (1992).

12. [DAV] David, G. "Wavelets and Singular Integrals on Curves and Surfaces". Lecture Notes in Mathematics 1465. Springer-Verlag, 1991.

13. [FGWW] Frazier, M., Garrigós, G., Wang, K. and Weiss, G. "A Characterization of Functions that Generate Wavelet and Related Expansion [*sic*]". The Journal of Fourier Analysis and Applications, Vol. 3, Special Issue 1997, pp. 883-906.

14. [FJ] Frazier, M. , Jawerth, B. "The φ-Transform and Applications to Distribution Spaces". Springer Lecture Notes in Mathematics 1302 (1985) (pp.223-246).

15. [FJW] Frazier, M. , Jawerth, B. , Weiss, G. L. "Littlewood-Paley Theory and the Study of Function Spaces". AMS Regional Conference Series in Mathematics Vol. 79 (1991).

16. [FOL] Folland, G. B. "Real Analysis. Modern Techniques and their applications". John Wiley & Sons, 1984.

17. [GRIP] Gripenberg, G. "A necessary and sufficient condition for the existence of a father wavelet". Studia Mathematica 114 (3) 1995 (pp.207-226).

18. [HAN-LAR] Han, Deguang and Larson, D.R. "Frames, Bases and Group Representations". Preprint, 1997.

19. [HAN-LU] Han, D. , Lu, S. "Smoothing Approximations and Pathwise Connectivity of MRA Wavelets". Preprint (1996).

20. [HER] Hervé, L. "Methodes d'operateurs quasi-compacts en analyse multiresolution, applications a la construction de bases d'ondelettes et a l'interpolation". Thèse, Laboratoire de Probabilités, Université de Rennes-I, 1992.

21. [HUB] Hubbard, B. B. "The World According to Wavelets. The story of a mathematical technique in the making." AK Peters, 1996.

22. [HW] Hernández, E. , Weiss, G. L. "A First Course on Wavelets". CRC Press (1996).

23. [HWW] Hernández, E. , Wang, X. & Weiss, G. "Smoothing Minimally Supported Frequency Wavelets. Part I". The Journal of Fourier Analysis and Applications Vol. 2, Number 4, 1996, pp. 329-340.

24. [HWW2] Hernández, E. , Wang, X. & Weiss, G. "Smoothing Minimally Supported Frequency Wavelets. Part II". The Journal of Fourier Analysis and Applications Vol. 3, Number 1, 1997, pp. 23-41.

25. [ION-LAR-PEA] Ionascu, E. J. , Larson, D. R. , Pearcy, C. M. "On Wavelet Sets". Preprint (1996).

26. [KAH-LEM] Kahane, J. P. and Lemarié-Rieusset, P. G. "Fourier Series and Wavelets". Gordon & Breach Publishers, 1995.

27. [KATZ] Katznelson, Y. "An introduction to Harmonic Analysis". John Wiley & Sons, Inc. 1968.

28. [LAN] Lang, Serge. "Differential and Riemannian Manifolds". Third Edition, Springer-Verlag 160 (1995).

29. [LEM] Lemarié-Rieusset, P. G. "Projecteurs invariants, matrices de dilatation, ondelettes et analyses multi-résolution". Revista Matemática Iberoamericana Vol. 10, Nº 2 (1994), pp. 283-347.

30. [MAD] Madych, W.R. "Some Elementary Properties of Multiresolution Analyses of $L^2(\mathbb{R}^n)$". Wavelets- A Tutorial in Theory and Applications. C.K. Chui (ed.), pp. 259-294, (1992).

31. [MEY] Meyer, Y. "Wavelets and Operators". Cambridge University Press (1992).

32. [PSK] Papadakis, M. , Stavropoulos, T. and Kalouptsidis, N. "An Equivalence Relation between Multiresolution Analysis [*sic*] of $L^2(\mathbb{R})$". Applied and Computational Harmonic Analysis 3 (1996), pp. 366-371.

33. [PSW] Papadakis, M. , Šikić, H. and Weiss, G. L. "Characterizations and structure of functions associated with wavelets". Preprint, 1997.

34. [RON-SHEN] Ron, Amos and Shen, Zuowei "Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator". Journal of Functional Analysis 148, pp. 408-447 (1997).

35. [RON-SHEN2] Ron, Amos and Shen, Zuowei "Affine systems in $L_2(\mathbb{R}^d)$ II: Dual Systems". The Journal of Fourier Analysis and Applications Vol. 3, Number 5, 1997, pp. 617-638.

36. [RUD1] Rudin, W. "Functional Analysis". McGraw-Hill (1973).

37. [RUD2] Rudin, W. "Real and Complex Analysis". Second Edition. McGraw-Hill(1974).

38. [SIN] Singer, I. "Bases in Banach Spaces I". Springer-Verlag 1970.

39. [SOA-WEI] Soardi, P. , Weiland, D. "MSF Wavelets in Several Dimensions". Preprint (1996).

40. [SPEE] Speegle, D. M. "S-elementary wavelets and the into $C(K)$ extension property". Ph.D. Thesis. Texas A & M University, 1997.

41. [STE] Stein, E. M. "Singular Integrals and Differentiability Properties of Functions". Princeton University Press (1970).

42. [STRI] Strichartz, R. S. "Multipliers in fractional Sobolev spaces". Journal of Mathematics and Mechanics 16 (1967), pp. 1031-1060.

43. [TAIB] Taibleson, M. H. "On the theory of Lipschitz spaces and distributions on Euclidean n-space, I", Journal of Mathematics and Mechanics 13 (1964), pp. 407-480.

44. [VILL] Villemoes, L. F. "Energy moments in time and frequency for two-scale difference equation solutions and wavelets". SIAM, Journal on Mathematical Analysis, Vol 23, No 6, 1992, pp. 1519-1543.

45. [WAN] Wang, Xihua "The Study of Wavelets from the Properties of their Fourier Transforms". Ph. D. Thesis, Washington University 1995.

46. [WOJ] Wojtaszczyk, P. "A Mathematical Introduction to Wavelets". London Mathematical Society, Student Texts 37, 1997.

47. [WUTAM] The Wutam Consortium. "Basic Properties of Wavelets". Preprint, 1997. To appear in the Journal of Fourier Analysis and Applications.