LECTURE NOTES ON BERGMAN PROJECTORS IN TUBE DOMAINS OVER CONES: AN ANALYTIC AND GEOMETRIC VIEWPOINT

DAVID BÉKOLLÉ, ALINE BONAMI, GUSTAVO GARRIGÓS, CYRILLE NANA, MARCO M. PELOSO, AND FULVIO RICCI

Abstract. In these notes we present in printed form the content of a series of lectures given by five of the authors at the International Workshop in Classical Analysis held in Yaoundé in December 2001. Our purpose is to introduce the problem of L^p-boundedness of weighted Bergman projectors on tube domains over symmetric cones, and show some of the latest progress obtained in this subject. We begin with a complete description of the situation on the upper half-plane. Next, we introduce the geometric machinery necessary to study the problem in higher dimensions. This includes the riemannian structure of symmetric cones, the induced Whitney decomposition and the introduction of a wider class of spaces with mixed $L^{p,q}$-norms. Our main result is the boundedness of the weighted Bergman projector on the weighted mixed norm spaces $L^{p,q}_\nu$, for an appropriate range of indices ν, p, q. Finally, we conclude by discussing various applications, further results, and open questions.

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Foreword

In December 2001 the "*International Workshop in Classical Analysis, Partial Differential Equations and Applications*" was held in Yaoundé, Cameroon.

Here we present an outgrowth of the notes of a series of lectures that five of us[1] delivered on that occasion. These notes were carefully taken by Cyrille Nana[2], who also wrote the first coherent draft.

We provide in these lecture notes an introduction to the analysis of weighted Bergman spaces on tubes over symmetric cones and, at the same time, a self-contained presentation of the joint results we have obtained during the past few years ([5],[2], [3], [8], [4]).

During the academic year 2001/02, the first named author gave a graduate course based on the same notes and he is indebted to his students for many corrections and improvements.

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[1]D. Békollé, A. Bonami, G. Garrigós, M. Peloso and F. Ricci.

[2]A Ph. D. student of D. Békollé.

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1. Bergman spaces in the upper half-plane of the 1-D complex space

We begin our presentation with the simplest case: the upper half-plane $\mathbf{H}$ of the one-dimensional complex space $\mathbf{C}$. This is the prototype of a tube domain over a cone, the cone being the positive real half-line $(0, \infty)$. This approach gives us the opportunity to introduce the subject in a more familiar context, and to describe and prove many of our results using elementary techniques. Such presentation will also help understanding the problem in higher dimensions, where many new limitations appear and subtle difficulties must be overcome.

The results we present here in one-dimension are all well-known, and can be found scattered in the literature (see e.g., [23], [13] or [26] for basic properties of Hardy and Bergman spaces). In our presentation, we will try to be as self-contained as possible, emphasizing the proofs which are keener to be generalized to higher dimensions (specially arguments involving group invariance). This approach will end up with three apparently different problems which will be considered later in higher dimension, and which will turn out to be equivalent, providing a common point where geometry, real and complex analysis merge. We solve them completely in the one dimensional case.

1.1. Definitions and basic properties.

Let $\mathcal{H}(\mathbf{H})$ be the space of holomorphic functions on $\mathbf{H}$, where this domain denotes the upper half-plane in $\mathbf{C}$,

$$\mathbf{H} = \mathbf{R} + i(0, \infty) = \{x + iy \in \mathbf{C} : y > 0\}.$$

We first define the Bergman spaces.

Definition 1.1. Given $p \in [1, \infty)$, the *(unweighted) Bergman space* $A^p = A^p(\mathbf{H})$ is defined by

$$\begin{aligned} A^p &= \mathcal{H}(\mathbf{H}) \cap L^p(\mathbf{H}, dxdy) \\ &= \left\{ F \in \mathcal{H}(\mathbf{H}) : \|F\|_{A^p}^p = \int_0^\infty \int_{\mathbf{R}} |F(x+iy)|^p dxdy < \infty \right\}. \end{aligned}$$

Given $\nu > 0$ and $p \in [1, \infty)$, the *weighted Bergman space* $A_\nu^p = A_\nu^p(\mathbf{H})$ is defined by

$$A_\nu^p = \left\{ F \in \mathcal{H}(\mathbf{H}) : \|F\|_{A_\nu^p}^p = \int_0^\infty \int_{\mathbf{R}} |F(x+iy)|^p dx\, y^\nu \tfrac{dy}{y} < \infty \right\}.$$

We shall denote by L_ν^p the Lebesgue space associated with the measure $y^{\nu-1} dxdy$. Observe that $A_\nu^p = A^p$ for $\nu = 1$.

Below, we give examples of functions in these spaces. We leave the verification as an exercise to the reader.

Example 1.2. Let $\alpha > 0$ be fixed.

(i) The holomorphic function F_α defined on $\mathbf{H}$ by $F_\alpha(z) = \frac{1}{(z+i)^\alpha}$ belongs to A^p_ν if and only if $\nu > 0$ and $p > \frac{\nu+1}{\alpha}$;

(ii) The holomorphic function G_α defined on $\mathbf{H}$ by $G_\alpha(z) = \frac{e^{iz}}{z^\alpha}$ belongs to A^p_ν if and only if $\nu > 0$ and $\frac{1}{\alpha} < p < \frac{\nu+1}{\alpha}$.

The next proposition gives basic inequalities for functions in A^p_ν.

Proposition 1.3. *Let $p \in [1, \infty)$ and $\nu > 0$.*

(i) *There exists a constant $C = C(p, \nu) > 0$ such that for all $x + iy \in \mathbf{H}$ and for all $F \in A^p_\nu$, the following inequality holds:*

$$|F(x+iy)| \leq C y^{-\frac{\nu+1}{p}} \|F\|_{A^p_\nu}.$$

(ii) *There exists a constant $C = C(p, \nu) > 0$ such that for all $y \in (0, \infty)$ and for all $F \in A^p_\nu$, the following inequality holds:*

$$\|F(\cdot + iy)\|_p \leq C y^{-\frac{\nu}{p}} \|F\|_{A^p_\nu}.$$

(iii) *For all $F \in A^p_\nu$, and for all $y > 0$, the following holds:*

$$\lim_{|x| \to \infty} F(x+iy) = 0. \tag{1.4}$$

Proof: Before starting the proof, let us remark that A^p_ν is invariant by translations and dilations. More precisely, we fix $x \in \mathbf{R}$ and $y > 0$, and consider the translate of $F \in A^p_\nu$ under x, given by $F_1(u+iv) = F(x+u+iv)$. Then F_1 is also in A^p_ν with same norm as F. Analogously, if we define the dilate of F by $F_2(u+iv) = F(y(u+iv))$, then F_2 is in A^p_ν, with norm

$$\|F_2\|_{A^p_\nu} = y^{-\frac{\nu+1}{p}} \|F\|_{A^p_\nu}.$$

Let us now prove (i). From the invariance properties above, it suffices to consider the case $x = 0$ and $y = 1$ (which can be applied afterwards to $(F_1)_2$).

Let $D(z_0, r)$ denote the disc of center z_0 and radius r. The mean value property, Hölder's inequality and the fact that $v^{\nu-1}$ is bounded below on the interval $[\frac{1}{2}, \frac{3}{2}]$, imply that

$$\begin{aligned}
|F(i)| &= C \Big| \iint_{D(i,\frac{1}{2})} F(u+iv)\,du\,dv \Big| \\
&\leq C_p \Big(\iint_{D(i,\frac{1}{2})} |F(u+iv)|^p\,du\,dv \Big)^{\frac{1}{p}} \\
&\leq C_{p,\nu} \left[\int_{\frac{1}{2}}^{\frac{3}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |F(u+iv)|^p\,du\, v^{\nu-1}\,dv \right]^{\frac{1}{p}} \leq C_{p,\nu} \|F\|_{A^p_\nu}.
\end{aligned}$$

(ii) Again, by invariance it suffices to prove the case $y = 1$. Proceeding as above we obtain

$$|F(x+i)|^p \leq C_p \int_{|v-1|\leq\frac{1}{2}} \int_{|u-x|\leq\frac{1}{2}} |F(u+iv)|^p du dv. \tag{1.5}$$

Then, integration with respect to x gives:

$$\begin{aligned}\|F(\cdot+i)\|_p^p &\leq C_p \int_{|v-1|\leq\frac{1}{2}} \left[\int_{\mathbf{R}} \int_{|u-x|\leq\frac{1}{2}} dx\, |F(u+iv)|^p du\right] dv \\ &\leq C_{p,\nu} \int_{|v-1|\leq\frac{1}{2}} (\int_{\mathbf{R}} |F(u+iv)|^p du) v^{\nu-1} dv \leq C_{p,\nu} \|F\|^p_{A^p_\nu}.\end{aligned}$$

(iii) Once more, we may assume that $y = 1$. Rewriting (1.5) we see that:

$$|F(x+i)|^p \leq C \int_{|v-1|\leq\frac{1}{2}} \int_{\mathbf{R}} \chi_{[x-\frac{1}{2},x+\frac{1}{2}]}(u) |F(u+iv)|^p du\, v^{\nu-1} dv.$$

Then, from the dominated convergence theorem it follows that

$$\lim_{|x|\to\infty} |F(x+i)|^p = 0.$$

□

Exercise 1.6. Modify the proof of part (iii) above to show that the limit in (1.4) holds uniformly in y over compact sets of $(0, \infty)$. Combine this fact with (i) to show that, when $\nu > 0$, $y_0 > 0$, then for all $F \in A^p_\nu$,

$$\lim_{\substack{z\to\infty \\ \Im z > y_0}} F(z) = \lim_{\substack{|x|+y\to\infty \\ y>y_0}} F(x+iy) = 0.$$

Corollary 1.7. *Let $p \in [1, \infty)$ and $\nu > 0$. Then for every compact set K of $\mathbf{C}$ contained in $\mathbf{H}$, there exists a constant $C_K = C_K(p, \nu) > 0$ such that for every $F \in A^p_\nu$, the following estimate holds:*

$$\sup_{z\in K} |F(z)| \leq C_K \|F\|_{A^p_\nu}.$$

Proof: This follows immediately from assertion (i) of Proposition 1.3. □

Corollary 1.8. *For all $p \in [1, \infty)$ and $\nu > 0$, the Bergman space A^p_ν is a Banach space.*

Proof: The function $F \mapsto \|F\|_{A^p_\nu}$ defines a norm on A^p_ν, because of the equality $\|\cdot\|_{A^p_\nu} = \|\cdot\|_{L^p_\nu}$ and A^p_ν is complete in this norm. Indeed, let $\{F_q\}$ be a Cauchy sequence in A^p_ν. By Corollary 1.7, for every compact set K of $\mathbf{C}$ contained in $\mathbf{H}$, we get:

$$\sup_K |F_q - F_r| \leq C_K \|F_q - F_r\|_{A^p_\nu};$$

it then follows that the sequence $\{F_q\}$ converges uniformly on every compact set K of $\mathbf{C}$ contained in $\mathbf{H}$. By Weierstrass Theorem, its limit F is a holomorphic function on $\mathbf{H}$. On the other hand, since the space L^p_ν is complete, the Cauchy sequence $\{F_q\}$ converges in L^p_ν to a function $G \in L^p_\nu$. Therefore, we can extract a subsequence F_{q_k} that converges a.e. to G. This implies that $G = F$ almost everywhere on $\mathbf{H}$. Furthermore, $\{F_q\} \to F$ in A^p_ν. □

We point out that we could as well have defined the spaces A^p_ν for non positive ν, and prove the same propositions. This is of no interest, since an easy consequence of Proposition 1.3 in this case is the fact that the weighted Bergman spaces reduce to $\{0\}$ when $\nu \leq 0$. Before giving the proof of this fact, we need to recall basic properties of the Hardy classes, which may be seen, in some way, as the limit classes when ν tends to 0.

1.2. Hardy spaces on the upper half-plane.

Proofs and details of the results surveyed here can be found, e.g., in [10, Ch. 11] and [13, Ch.II].

Definition 1.9. For $p \in [1, \infty)$, the *Hardy space* $H^p = H^p(\mathbf{H})$ is the space of holomorphic functions on $\mathbf{H}$ which satisfy the estimate

$$\|F\|_{H^p} := \sup_{y>0} \left\{ \int_{-\infty}^{\infty} |F(x+iy)|^p dx \right\}^{\frac{1}{p}} < \infty.$$

It is clear that $\|\cdot\|_{H^p}$ is a norm on H^p. The next lemma follows from the mean value property and Hölder's inequality (proceeding as in Proposition 1.3 above).

Lemma 1.10. *For every $z = x + iy \in \mathbf{H}$ and for every $F \in H^p$*

$$|F(z)| \leq \left(\frac{4}{\pi y}\right)^{\frac{1}{p}} \|F\|_{H^p}.$$

Moreover, for every compact set K of $\mathbf{H}$ we have

$$\sup_{z\in K} |F(z)| \leq \left(\frac{4}{\pi dist(K, \partial\mathbf{H})}\right)^{\frac{1}{p}} \|F\|_{H^p}.$$

Corollary 1.11. *For all $p \in [1, \infty)$, H^p is a Banach space.*

Again, we use the same kind of proof as for Bergman spaces.

The next result is an easy consequence of the residue theorem, and gives the Cauchy integral representation for functions satisfying an H^p-integrability condition.

Proposition 1.12. *Let $F \in H^p, 1 \le p < \infty$. Then for all $z = x + iy \in \mathbf{H}$ and $\epsilon \in (0, y)$, we have*

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t + i\epsilon)}{t + i\epsilon - z} dt.$$

Definition 1.13. The kernel $C(x + iy) = C_y(x) = \frac{1}{2\pi} \frac{i}{x+iy}$ is called the *Cauchy kernel of* $\mathbf{H}$.

In the sequel, for every function $\phi : \mathbf{H} \to \mathbf{C}$ and for every $y > 0$, we denote by ϕ_y the function defined on $\mathbf{R}$ by $\phi_y(x) := \phi(x + iy)$. Then the Cauchy integral formula may be written as follows: for all $y > 0$ and $\epsilon \in (0, y)$

$$F_y = F_\epsilon * C_{y-\epsilon}.$$

The next theorem is also well known, and gives the existence of boundary values for functions in H^p. The proof for $p > 1$ is simple and only makes use of harmonicity. We state below the full result comprising also the case $p = 1$, for which we refer, e.g., to [10, pp. 190–195] or [18, pp. 317–323] (the latter, in the case of the disk).

Theorem 1.14. *Let $F \in H^p$, $1 \le p < \infty$. Then,*

(i) *The function $y \mapsto \|F_y\|_p$ is non-increasing and continuous for $y \in (0, \infty)$;*

(ii) *$\|F_y\|_p$ tends to $\|F\|_{H^p}$ as y tends to zero;*

(iii) *There exists a function $F_0 \in L^p(\mathbf{R})$ such that F_y converges to F_0 in the L^p norm as y tends to zero; also $F_y = F_0 * C_y = F_0 * P_y$ for every $y > 0$. Moreover, F_y tends to F_0 in L^p when y tends to 0.*

Here $P_y(x)$ denotes the *Poisson kernel in* $\mathbf{H}$. It is given by

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2},$$

has integral 1, and defines an approximate identity (while the Cauchy kernel does not).

Let us now give applications of the last theorem for weighted Bergman spaces. We first remark that Assertion (ii) of Proposition 1.3 can be read in the following way. For $F \in A^p_\nu$ and $\varepsilon > 0$, the function $F(\cdot + i\varepsilon)$ is in H^p, with norm bounded by $C\varepsilon^{-\frac{\nu}{p}}$. Moreover, we have the following proposition.

Proposition 1.15. *Let $F \in A^p_\nu$, $1 \le p < \infty$. Then,*

(i) *The function $y \mapsto \|F_y\|_p$ is non-increasing and continuous for $y \in (0, \infty)$;*

(ii) *$F(\cdot + i\varepsilon)$ is in A^p_ν for positive ε, and tends to F in A^p_ν as ϵ tends to zero.*

Proof: The proof of (i) is a direct consequence of the fact that $F(\cdot + i\varepsilon)$ is in H^p. It is clear from (i) that $F(\cdot + i\varepsilon)$ is in A^p_ν for positive ε. It remains to prove that $\int_0^\infty \|F_y - F_{y+\varepsilon}\|_p^p y^{\nu-1} dy$ tends to 0. This is an easy consequence of the Dominated Convergence Theorem. □

REMARK 1.16. Let us now prove that, if $\nu \leq 0$ then $A^p_\nu = \{0\}$ for every $p \in [1, \infty)$. Indeed, it follows from Proposition 1.3 adapted to this case that, for every $F \in A^p_\nu$, the function $G(z) := \frac{F(z)}{(z+i)^m}$ belongs to the Hardy space H^p for m large enough. Hence, the function $g : (0, \infty) \to [0, \infty)$ defined by $g(y) = \int_{\mathbf{R}} |G(x + iy)|^p dx$ is non-increasing. Moreover

$$\begin{aligned} \|F\|^p_{A^p_\nu} &\geq \|G\|^p_{A^p_\nu} = \int_0^\infty g(y) y^{\nu-1} dy \\ &\geq \int_0^{y_0} g(y) y^{\nu-1} dy \geq g(y_0) \times \infty. \end{aligned}$$

So, $g(y) = 0$ for every $y > 0$. This implies that G (and also F) is identically zero on $\mathbf{H}$.

We now prove a density result which will be used often below.

PROPOSITION 1.17. *Let $\nu > 0$ and $1 \leq p < \infty$. Then, for all $\mu > 0$ and $1 \leq q < \infty$ the set $A^q_\mu \cap A^p_\nu$ is dense in A^p_ν.*

PROOF: Let $m \geq 1$ be large enough so that so that

$$G_m(z) = \frac{1}{(-iz + 1)^m} \in A^q_\mu \tag{1.18}$$

(see Example 1.2 above). Given $F \in A^p_\nu$ and $\varepsilon > 0$ we consider

$$F^{(\varepsilon)}(z) = G_m(\varepsilon z) F(z + i\varepsilon), \quad z \in \mathbf{H},$$

which belongs to $A^q_\mu \cap A^p_\nu$ since both factors are bounded (the second one, by Proposition 1.3). Further, the pointwise limit of $F^{(\varepsilon)}(z)$ equals $F(z)$ when $\varepsilon \to 0$. We have already seen that $F(\cdot + i\varepsilon)$ tends to F in A^p_ν. It remains to see that the same is valid for $G_m(\varepsilon \cdot)F$. Again, it follows from the Dominated Convergence Theorem.

□

1.3. A Paley-Wiener Theorem.

Let us first recall the version of the Paley-Wiener Theorem which is adapted to Hardy spaces.

PROPOSITION 1.19. (i) *For every $g \in L^2(0, \infty)$, the following integral is absolutely convergent,*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{iz\xi} g(\xi) d\xi \qquad (z \in \mathbf{H}), \tag{1.20}$$

and defines a function $F \in H^2$ which satisfies

$$\|F\|^2_{H^2} = \int_0^\infty |g(\xi)|^2 d\xi. \tag{1.21}$$

(ii) *The converse holds, i.e., for every* $F \in H^2$, *there exists* $g \in L^2(0,\infty)$ *such that* (1.20) *and* (1.21) *hold.*

Proof: Let us prove (i). The integral on the right hand side of (1.20) is absolutely convergent, and defines a holomorphic function. Moreover, it follows from the inverse Fourier formula that the Fourier transform of F_y is given by

$$\widehat{F_y}(\xi) = \sqrt{2\pi}g(y)e^{-y\xi}.$$

By Plancherel formula, $\|F_y\|_2^2 = \int_0^\infty e^{-2y\xi}|g(\xi)|^2 d\xi$, and Formula 1.21 follows at once. Conversely, using Theorem 1.14 and the fact that the Fourier transform of the Poisson kernel is equal to $e^{-y|\xi|}$, if $\sqrt{2\pi}g$ is the Fourier transform of F_0, we get that

$$F(x+iy) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ix\xi}e^{-y|\xi|}g(\xi)d\xi \qquad (z \in \mathbf{H}).$$

It remains to show that g is supported in $(0,\infty)$. But, if we cut the integral into two parts, the integrals over $(-\infty,0)$ and over $(0,\infty)$, the first one gives an anti-holomorphic function, while the second one gives a holomorphic function. Since F is holomorphic, it means that the first one is 0. By Fourier uniqueness, this implies that g vanishes on $(-\infty,0)$, and allows to conclude.

Let us now consider the weighted Bergman spaces.

Theorem 1.22. (Paley-Wiener) (i) *For every* $g \in L^2((0,\infty),\xi^{-\nu}d\xi)$ *the following integral is absolutely convergent,*

$$F(z) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{iz\xi}g(\xi)d\xi \qquad (z \in \mathbf{H}), \tag{1.23}$$

and defines a function $F \in A^2_\nu$ *which satisfies*

$$\|F\|^2_{A^2_\nu} = \frac{\Gamma(\nu)}{2^\nu}\int_0^\infty |g(\xi)|^2\frac{d\xi}{\xi^\nu}. \tag{1.24}$$

(ii) *The converse holds, i.e., for every* $F \in A^2_\nu$, *there exists* $g \in L^2((0,\infty),\xi^{-\nu}d\xi)$ *such that* (1.23) *and* (1.24) *hold.*

Proof: (1) Again, the integral on the right hand side of (1.23) is absolutely convergent, since by Schwarz's inequality

$$\begin{aligned}\int_0^\infty |e^{iz\xi}g(\xi)|d\xi &= \int_0^\infty (e^{-y\xi}\xi^{\frac{\nu}{2}})(\xi^{-\frac{\nu}{2}}|g(\xi)|)d\xi \\ &\le \left(\int_0^\infty \xi^\nu e^{-2y\xi}d\xi\right)^{\frac12}\left(\int_0^\infty |g(\xi)|^2\frac{d\xi}{\xi^\nu}\right)^{\frac12} < \infty,\end{aligned}$$

(recall that $\nu > 0$). This implies that the right hand side of (1.23) defines a function F which is holomorphic in $\mathbf{H}$.

To prove (1.24), we use the Plancherel formula. By (1.23), we have that

$$\int_{\mathbf{R}} |F(x+iy)|^2 dx = \int_0^\infty e^{-2y\xi} |g(\xi)|^2 d\xi$$

and therefore

$$\begin{aligned}\int_0^\infty \left(\int_{\mathbf{R}} |F(x+iy)|^2 dx\right) y^\nu \frac{dy}{y} &= \int_0^\infty |g(\xi)|^2 \left(\int_0^\infty e^{-2y\xi} y^\nu \frac{dy}{y}\right) d\xi \\ &= \Gamma(\nu) \int_0^\infty |g(\xi)|^2 \frac{d\xi}{(2\xi)^\nu}.\end{aligned}$$

To prove (ii), we use Paley-Wiener Theorem for Hardy classes. For every $\varepsilon > 0$, there exists g_ε which is in $L^2(0,\infty)$ such that

$$F(z+i\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{iz\xi} g_\varepsilon(\xi) d\xi.$$

The uniqueness of the Fourier transform implies that

$$e^{-\varepsilon'\xi} g_\varepsilon(\xi) = e^{-\varepsilon\xi} g_{\varepsilon'}(\xi).$$

We take $g(\xi) = e^{\varepsilon\xi} g_\varepsilon(\xi)$ to conclude that F is given by the required formula. Again, by the Plancherel formula and Fubini's Theorem for positive functions as above,

$$\|F\|^2_{A^2_\nu} = \int_0^\infty |g(\xi)|^2 \left(\int_0^\infty e^{-2y\xi} y^{\nu-1} dy\right) d\xi = \Gamma(\nu) \int_0^\infty |g(\xi)|^2 \frac{d\xi}{(2\xi)^\nu}.$$

This last integral is finite, which we wanted to prove. □

Exercise 1.25. Let $\nu > 0$ and $1 \le p < 2$. Show that for all $F \in A^p_\nu$ there exists $g \in L^{p'}((0,\infty), \xi^{-\nu\frac{p'}{p}}\, d\xi)$ such that (1.23) holds and

$$\|g\|_{L^{p'}((0,\infty),\xi^{-\nu\frac{p'}{p}}\, d\xi)} \le C\, \|F\|_{A^p_\nu}.$$

(Hint: use Hausdorff-Young's inequality.)

1.4. Bergman kernels and Bergman projectors.

Definition 1.26. Let H denote a Hilbert space consisting of complex functions on an open set E. We call *reproducing kernel for H*, a complex function $K : E \times E \to \mathbf{C}$ such that, if we put $K_w(z) = K(z,w)$, then the following two properties hold:

(1) for every $w \in E$, the function K_w belongs to H;
(2) for all $f \in H$ and $w \in E$, we have

$$f(w) = \langle f, K_w \rangle.$$

It is worth noticing that these two properties imply that such a kernel K satisfies the identity $K(z,w) = \overline{K(w,z)}$, for all $z, w \in E$.

PROPOSITION 1.27. *For every $\nu > 0$, the Bergman space A^2_ν in $\mathbf{H}$ possesses a reproducing kernel.*

PROOF: By Corollary 1.7 used for the compact set $\{w\}$, we know that $F \mapsto F(w)$ is a continuous linear functional on the Hilbert space A^2_ν. We combine this with the Riesz representation theorem for such functionals. □

DEFINITION 1.28. The reproducing kernel for $A^2(\mathbf{H})$ is called the *Bergman kernel of* $\mathbf{H}$ and is denoted by $B(z, w)$. More generally, for $\nu > 0$ the reproducing kernel for A^2_ν is called the *weighted Bergman kernel of* $\mathbf{H}$ and it is denoted by $B_\nu(z, w)$.

We will see that the weighted Bergman kernel can be explicitly computed.

In what follows, the notation $\operatorname{Log} z$ and $z^\alpha = e^{\alpha \operatorname{Log} z}$, $\Re e\, z > 0$, $\alpha \in \mathbf{C}$, corresponds to the determination of the logarithm which is real in the positive real axis.

THEOREM 1.29. *If $\nu > 0$, then the weighted Bergman kernel is given by the formula*

$$B_\nu(z, w) = \frac{2^{\nu-1}\nu}{\pi} \left(\frac{z - \overline{w}}{i} \right)^{-\nu-1}.$$

PROOF: By the Paley-Wiener theorem, every function $F \in A^2_\nu$ can be written as

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{iz\xi} g(\xi) d\xi, \tag{1.30}$$

for some $g \in L^2_\nu(0, +\infty)$. Since $B_\nu(\cdot, w) \in A^2_\nu$, there exists $g_w \in L^2_\nu$ such that

$$B_\nu(z, w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{iz\xi} g_w(\xi) d\xi.$$

Now, since the kernel $B_\nu(\cdot, w)$ is reproducing for A^2_ν, polarizing the isometry in the Paley-Wiener theorem gives

$$F(w) = \langle F, K_w \rangle = \Gamma(\nu) \int_0^\infty g(\xi)\overline{g_w(\xi)} \frac{d\xi}{(2\xi)^\nu}.$$

The identification with (1.30) gives that

$$g_w(\xi) = \frac{1}{\sqrt{2\pi}\,\Gamma(\nu)} \frac{e^{-i\overline{w}\xi}}{(2\xi)^\nu}.$$

Hence,

$$\begin{aligned} B_\nu(z,w) &= \frac{1}{2\pi\Gamma(\nu)}\int_0^\infty e^{i(z-\overline{w})\xi}(2\xi)^\nu d\xi \\ &= \frac{2^\nu}{2\pi\Gamma(\nu)}\frac{1}{(-i(z-\overline{w}))^{\nu+1}}\int_0^\infty e^{-\xi}\xi^\nu d\xi \\ &= \frac{2^\nu}{2\pi\Gamma(\nu)}\frac{\Gamma(\nu+1)}{\left(\frac{z-\overline{w}}{i}\right)^{\nu+1}} = \frac{2^{\nu-1}\nu}{\pi}\left(\frac{z-\overline{w}}{i}\right)^{-\nu-1}. \end{aligned}$$

□

Definition 1.31. The orthogonal projector from the Hilbert space $L^2 = L^2(\mathbf{H})$ onto its closed subspace A^2 is called the *Bergman projector of* $\mathbf{H}$ and it is denoted by P. More generally, for every $\nu > 0$, the orthogonal projector from the Hilbert space L^2_ν onto its closed subspace A^2_ν is called the *weighted Bergman projector of* $\mathbf{H}$ and it is denoted P_ν.

Proposition 1.32. *For every $f \in L^2_\nu$ and $z \in \mathbf{H}$ we have that*

$$P_\nu f(z) = \int_{\mathbf{H}} B_\nu(z, u+iv) f(u+iv) du\, v^{\nu-1} dv. \tag{1.33}$$

Proof: By the reproducing property of $B_\nu(z,w)$ and the self-adjointness of P_ν in $L^2_\nu(\mathbf{H})$ we have:

$$\begin{aligned} P_\nu f(z) &= \langle P_\nu f\,,\, B_\nu(\cdot,z)\rangle_{L^2_\nu} = \langle f\,,\, P_\nu B_\nu(\cdot,z)\rangle_{L^2_\nu} \\ &= \langle f\,,\, B_\nu(\cdot,z)\rangle_{L^2_\nu} = \int_{\mathbf{H}} B_\nu(z,u+iv) f(u+iv) du\, v^{\nu-1} dv. \end{aligned}$$

□

1.5. Problem 1: The boundedness of the Bergman projector.

We have just found an explicit formula for the orthogonal projector P_ν from L^2_ν onto the subspace A^2_ν. It is natural to ask whether this operator extends in some meaningful way to L^p_ν for $p \neq 2$, and in that case whether the reproduction property of $B_\nu(z,w)$ (i.e., $P_\nu F = F$) holds in A^p_ν spaces. The first observation in this direction is that, for fixed z, the function $B_\nu(\cdot,z)$ belongs to L^q_ν if and only if $q > 1$ (cf. Example 1.2). Therefore, the right hand side of (1.33) is always well-defined whenever $f \in L^p_\nu$, $1 \le p < \infty$, and moreover, it coincides with f when this last function belongs to $A^2_\nu \cap A^p_\nu$. We already mentioned the density of this last set in A^p_ν (Proposition 1.17), so the reproduction property in A^p_ν will hold whenever P_ν defines a bounded operator on L^p_ν. The next theorem gives a complete answer to these questions, that is, it characterizes when P_ν is a bounded projector from L^p_ν onto A^p_ν.

Theorem 1.34. *Let $1 \leq p < \infty$. Then the Bergman projector P_ν is a bounded operator in L^p_ν if and only if $p > 1$. In this case, the operator P^+_ν with positive kernel $|B_\nu(z,w)|$ is also bounded in L^p_ν.*

Proof: We first prove the necessary condition for $p > 1$. We test P_ν on a specific function which is in all L^p_ν, namely $f(w) = \chi_B(w)v^{-\nu+1}$; where $w = u + iv$, and B is the ball of radius $1/2$ centered at i. Then the mean value property applied to the harmonic function $B_\nu(z,\cdot)$ gives us immediately that

$$P_\nu f(z) = cB_\nu(z,i)$$

for some constant c. This function is in L^p_ν if and only if $p > 1$, which proves the necessary condition.

To finish the proof of the theorem, it is clearly sufficient to prove that P^+_ν is bounded in L^p_ν. The main tool for the boundedness of operators with positive kernels is Schur's lemma, that now we state.

Lemma 1.35. (**Schur's Lemma**) *Let (X,μ) be a measure space and $K(x,y)$ a positive kernel on $X \times X$. Let T be the operator defined by*

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y).$$

For $1 < p < \infty$, let p' be the conjugate exponent. Suppose that there exist a positive function φ and a constant C such that

$$\begin{aligned} \int_X K(x,y)\varphi(y)^{p'}d\mu(y) &\leq C\varphi(x)^{p'} \\ \int_X K(x,y)\varphi(x)^{p}d\mu(x) &\leq C\varphi(y)^{p}. \end{aligned}$$

Then the operator T is well defined on $L^p(X,\mu)$, and it is and bounded on $L^p(X,\mu)$.

Proof: To prove Schur's Lemma, it is sufficient to consider positive functions f. An appeal to Hölder's inequality and the use of the first inequality gives that

$$\begin{aligned} Tf(x)^p &= \left(\int_X K(x,y)f(y)\varphi(y)^{-1}\varphi(y)d\mu(y)\right)^p \\ &\leq C^{p/p'}\varphi(x)^p \int_X K(x,y)f(y)^p\varphi(y)^{-p}d\mu(y). \end{aligned}$$

Integrating in x and using the second inequality we obtain the result. □

Let us go back to the proof of Theorem 1.34. We will do it in two steps. Again, we write

$$P_\nu f(x+iy) = c\int_0^\infty \left(\int_{-\infty}^{+\infty} (x-u+i(y+v))^{-\nu-1} f(u+iv)du\right) v^{\nu-1}dv,$$

and notice that the operator inside the parentheses is a convolution operator whose norm, when acting on $L^p(\mathbf{R})$, is bounded by the $L^1(\mathbf{R})$ norm of the function $(\cdot+i(y+v))^{-\nu-1}$. This quantity is easily computed, for y and v fixed, and it is equal to $c(y+v)^{-\nu}$. Thus, using Minkowski inequality for integrals, we get

$$\|P_\nu f(\cdot+iy)\|_p \leq c\int_0^\infty (y+v)^{-\nu}\|f(\cdot+iv)\|_p v^{\nu-1}dv.$$

Since the function $v\mapsto \|f(\cdot+iv)\|_p$ belongs to $L^p\big((0,\infty), v^{\nu-1}dv\big)$, it remains to prove that the operator with kernel $(y+v)^{-\nu}$ is bounded on $L^p((0,\infty), v^{\nu-1}dv)$. We use Schur's Lemma with the function $\varphi(v) = v^{-\alpha}$. It is sufficient to choose $\alpha>0$ such that $\nu-\alpha p'>0$, as well as $\nu-\alpha p>0$, and to use the homogeneity of the kernel. $\square$

Exercise 1.36. Use the ideas above and the Hardy-Littlewood maximal function to show that P^+ is bounded from $L^1_\nu(\mathbf{H})$ into $L^{1,\infty}_\nu$, the latter denoting the *weak-L^1 space* associated with the measure $y^{\nu-1}dx\,dy$.

Remark 1.37. It is possible to give a shorter proof of Theorem 1.34, using directly Schur's Lemma for P^+_ν. The advantage of the proof presented here is that it can be easily adapted to have boundedness of the operator in mixed norm spaces which will be introduced below.

1.6. Problem 2: Hardy-type inequalities in A^p_ν.

The Cauchy formula allows to estimate F' in terms of F: writing $F'(x+iy)$ as an integral along the circle of radius $y/4$ centered at $x+iy$, one gets that

$$|F'(x+iy)| \leq \frac{4}{y}\sup_{|w-x-iy|<y/4}|F(w)|.$$

As before, this quantity can be bounded in terms of the integral of F inside the ball of radius $y/2$ centered at $x+iy$:

$$y^p|F'(x+iy)|^p \leq \frac{C}{y^2}\int_{y/2<v<2y}\Big(\int_{|x-u|<y}|F(u+iv)|^p du\Big)dv. \tag{1.38}$$

Integrating on $\mathbf{H}$, we obtain the inequality

$$\int_{\mathbf{H}} y^p|F'(x+iy)|^p y^{\nu-1}dxdy \leq C\int_{\mathbf{H}}|F(u+iv)|^p v^{\nu-1}dudv. \tag{1.39}$$

Indeed, just change the order of integration in the right hand side of (1.38), and use that

$$\int_{y/2<v<2y} \left(\int_{|x-u|<y} dx\right) y^{\nu-1} y^{-2} dy = cv^{\nu-1},$$

for some positive constant c.

The converse inequality of (1.39) is much more interesting, and can be seen as a regularity property for the PDE $F' = G$, when G is a holomorphic data with a certain integrability condition. Such type of property is commonly known as a *Hardy-type inequality.* Clearly, there cannot be a version of it for $p = \infty$ because of constant functions. In the next proposition we show that, for all $1 \leq p < \infty$, there is a Hardy-type inequality in the Bergman spaces A^p_ν.

PROPOSITION 1.40. *For all $1 \leq p < \infty$, $\nu > 0$, the derivation operator maps continuously A^p_ν into $A^p_{\nu+p}$. Conversely, when $1 \leq p < \infty$, there exists a constant C_p such that the following Hardy-type inequality holds:*

$$\int_{\mathbf{H}} |F(u+iv)|^p v^{\nu-1} dudv \leq C_p \int_{\mathbf{H}} y^p |F'(x+iy)|^p y^{\nu-1} dxdy. \tag{1.41}$$

PROOF: To prove (1.41), we shall give an explicit formula for F in terms of its derivative. In fact, since the function F is holomorphic, we can replace F' by the partial derivative in y. Since F vanishes at ∞ (Exercise 1.6), we can write $-F(x+iy)$ as an integral of its derivative from y to $+\infty$, and get that

$$|F(x+iy)| \leq \int_y^{+\infty} |F'(x+iv)| dv.$$

As before, we use Minkowski integral inequality to see that the L^p norm in the x variable of an integral in v is bounded by the integral of the L^p norm in x. Doing this, we are reduced to a problem on $L^p((0,\infty), v^{\nu-1}dv)$. The estimate (1.41) now follows easily from the following result.

LEMMA 1.42. (cf. e.g. [20], p. 272) *For all $1 \leq p < \infty$, there exists a constant C such that, for all positive functions g on $(0, +\infty)$,*

$$\int_0^{+\infty} \left(\int_y^{+\infty} g(v)dv\right)^p y^{\nu-1} dy \leq C \int_0^{+\infty} y^p g(y)^p y^{\nu-1} dy. \tag{1.43}$$

PROOF: This is very classical, and may be found for instance in (cf. e.g. [20], p. 272). We give its prrof for completeness. We are again considering an operator with positive kernel. Moreover, this one is equal to $\frac{1}{v}\chi_{v>y}(v)$, which is clearly bounded by $c(y+v)^{-\nu} v^{\nu-1}$ that we have already considered. This gives the proof for $p > 1$. It is a simple consequence of Fubini's Theorem when $p = 1$.

□

1.7. Problem 3: Boundary values of functions in A^p_ν.

This paragraph is more difficult since it requires a good understanding of distributions. It may be left aside at first reading.

We have seen that all functions in the Bergman space A^2_ν can be obtained as a Fourier-Laplace transform of some function g in a weighted L^2 space (Theorem 1.22). In a sense, the distribution $f = \hat{g}$ can be seen as a "boundary limit" of $F \in A^2_\nu$, since at least formally,

$$F(x+iy) \to \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ix\xi} g(\xi)\, d\xi, \quad \text{when } y \to 0.$$

We want to give a precise meaning to this limit, ask whether such a boundary limit exists for other values of p, and whether an elegant characterization similar to Theorem 1.22 holds in that case. This question is more delicate now, and the answer will make use of the Littlewood-Paley decomposition for distributions with spectrum in $[0,\infty)$. We also point out that the language of distributions is necessary when we look at boundary limits in $\mathbf{H}$ (rather than on the Fourier transform side). Indeed, the elementary example $F(z) = \frac{e^{iz}}{z}$ belongs to A^2_ν for all $\nu > 1$ (see Example 1.2), but we cannot give a reasonable meaning to its *pointwise limit* $\frac{e^{ix}}{x}$ since it is not a distribution.

In order to present the Littlewood-Paley construction, we start with an elementary lemma on the existence of $\mathcal{C}^\infty$ functions with compact support (cf. [21]).

Lemma 1.44. *There exists a non-negative function ϕ on $\mathbf{R}$, which is of class $\mathcal{C}^\infty$ with compact support in $(1/2, 2)$, and satisfying the following identity*

$$\phi(\xi) + \phi(\xi/2) = 1 \qquad \textit{for } 1 \le \xi \le 2.$$

As a consequence,

$$\sum_{j\in\mathbf{Z}} \phi(2^{-j}\xi) = 1 \qquad \text{for } \xi > 0.$$

We define ψ as the inverse Fourier transform of ϕ, and $\psi_j(\cdot) = 2^j\psi(2^j\cdot)$. It follows from the identity above that

$$\sum_j f * \psi_j = f \tag{1.45}$$

when f is a tempered distribution whose Fourier transform is supported in $(0,+\infty)$. The candidate for space of boundary limits can now be defined as follows.

Definition 1.46. Let $\nu \in \mathbf{R}$ and $1 \le p < \infty$. The *(homogeneous) Besov space* B^p_ν is the space of classes of tempered distributions on $\mathbf{R}$, modulo polynomials, having Fourier transform with support in $[0, \infty)$ and such that

$$\|f\|^p_{B^p_\nu} = \sum_{j \in \mathbf{Z}} 2^{-\nu j} \|f * \psi_j\|^p_p < \infty. \tag{1.47}$$

Besov spaces arise naturally in the theory of partial differential equations when proving theorems on existence, uniqueness and regularity of solutions. For complex analysis we shall content ourselves to consider tempered distributions whose Fourier transform is supported in $[0, \infty)$, while in PDE one needs a more general space, without this restriction. We also remark that the "Besov norm" given by (1.47) vanishes if and only if the Fourier transform of f is supported in $\{0\}$, that is, if and only if f is a polynomial. So we get a norm on the quotient space that we consider.

We can now state the main theorem of this subsection.

Theorem 1.48. *Let $1 < p < \infty$ and $\nu > 0$. For all $f \in B^p_\nu$, the following series of Fourier-Laplace transforms*

$$F(z) = \sum_{j \in \mathbf{Z}} \tfrac{1}{\sqrt{2\pi}} \int_0^\infty e^{iz\xi} \widehat{f * \psi_j}(\xi) d\xi \qquad (z \in \mathbf{H}), \tag{1.49}$$

converges absolutely, and defines a holomorphic function which belongs to the space A^p_ν. Moreover, all functions F in A^p_ν can be written in this form for a unique (equivalence class) $f \in B^p_\nu$, and there exists a constant C, independent of f, such that

$$C^{-1} \|F\|_{A^p_\nu} \le \|f\|_{B^p_\nu} \le C \|F\|_{A^p_\nu}. \tag{1.50}$$

Proof: We shall show the absolute convergence of the series in (1.49) by a duality argument. For this, we use the observation that B^p_ν and $B^{p'}_{-\nu p'/p}$ are dual spaces with the duality pairing given by

$$\langle f, g \rangle = \lim_{J \to \infty} \sum_{|j|,|k| \le J} \int_{\mathbf{R}} f * \psi_j(x) \overline{g * \psi_k(x)} dx.$$

Using Plancherel's theorem, we can make this duality look closer to the expression in (1.49), by replacing the integral above by an integral involving $(\widehat{f * \psi_j})(\widehat{g * \psi_k})$. Indeed, it is easily seen that such factors vanish unless $|j - k| \le 1$. As a consequence, we see that the duality pairing is also equal to

$$\lim_{J \to \infty} \sum_{|j| \le J} \int_{\mathbf{R}} f * \psi_j(x) \overline{g(x)} dx = \lim_{J \to \infty} \sum_{|k| \le J} \int_{\mathbf{R}} f(x) \overline{g * \psi_k(x)} dx.$$

Going back to the statement of the theorem, we first observe that, at least formally, the function F in (1.49) is actually given by the equality

$$F(z) = \langle f, g_z \rangle, \quad z \in \mathbf{H},$$

where g_z is the distribution whose Fourier transform is given by

$$\widehat{g_z}(\xi) = \chi_{\xi>0}(\xi)e^{iz\xi},$$

To show that the expression in (1.49) is well defined, it is enough to see that $g_z \in B^{p'}_{-\nu p'/p}$. As a first step, we compute in the next lemma the L^r norm of $g_z * \psi_j$.

LEMMA 1.51. *For $1 \le r \le \infty$, there exists a constant C_r such that*

$$\|g_z * \psi_j\|_r \le C_r 2^{j/r'} e^{-2^j y/4} \qquad j \in \mathbf{Z}.$$

PROOF: By a change of variable, we may also assume $j = 0$, since

$$\|g_z * \psi_j\|_r = 2^{j/r'} \|g_{2^j z} * \psi\|_r.$$

Moreover, if $z = x + iy$, $g_z * \psi_j = g_{iy} * \psi_j(x + \cdot)$ so that we may assume $x = 0$. For $r = \infty$, it is sufficient to prove the same estimate for the L^1 norm of the Fourier transform, which is equal to $e^{-y\cdot}\phi$. This last one is a direct consequence of the assumption on the support of ϕ. To prove the lemma for other values of r, we remark that the same kind of estimates hold for the L^1 norms of all derivatives $\frac{d}{d\xi}$ of $e^{-y\,\xi}\phi(\xi)$, and in particular for its Laplacian (i.e., the second derivative), which is the Fourier transform of $|t|^2(g_z * \psi)(t)$, up to a constant. In particular, we get the estimate

$$|g_z * \psi(t)| \le C\frac{e^{-y/4}}{1 + |t|^2}.$$

The conclusion of the lemma now follows immediately. □

Using the lemma we obtain an estimate for the norm of g_z in $B^{p'}_{-\nu p'/p}$ by a constant times

$$\left(\sum_j 2^{j(\nu+1)\frac{p'}{p}} e^{-2^j yp'/4}\right)^{\frac{1}{p'}}.$$

Each summand is equivalent to an integral over the interval $(2^j, 2^{j+1}]$, so that the norm of g_z is bounded by

$$C\left(\int_0^\infty t^{(\nu+1)\frac{p'}{p}} e^{-yp't/8}\frac{dt}{t}\right)^{\frac{p}{p'}} = C'y^{-(\nu+1)},$$

which is finite. Thus, we have shown that $F(z)$ is well defined, and from here we deduce that it is holomorphic by a routine argument.

To prove that $F \in A^p_\nu$ is a little more tricky. Instead of giving a bound for $F(z)$, as above, we first estimate its norm in the x variable, keeping y fixed.

We write $F_y(x) = F(x + iy)$. Then, by Minkowski's inequality

$$\|F_y\|_p \le \sum_j \|F_y * \psi_j\|_p.$$

Since F_y is given through its Fourier transform, it is easy to compute the Fourier transform of $F_y * \psi_j$ and to see that it is, up to a constant,

$$e^{-y\cdot}\widehat{\psi_j}\hat{f} = (e^{-y\cdot}\widehat{\psi_{j-1}} + e^{-y\cdot}\widehat{\psi_j} + e^{-y\cdot}\widehat{\psi_{j+1}})\widehat{\psi_j}\hat{f} \tag{1.52}$$

because of the support condition on φ. Hence $F_y * \psi_j$ is the convolution of $f * \psi_j$ with a sum of three terms, for which we have already computed the L^1 norm (see Lemma 1.51). Therefore,

$$\|F_y\|_p \le C \sum_j e^{-2^j y/4} \|f * \psi_j\|_p,$$

and we are lead to prove that

$$\int_0^\infty \Big(\sum_j e^{-2^j y/4} \|f * \psi_j\|_p \Big)^p y^{\nu-1} dy \le C \sum_j 2^{-\nu j} \|f * \psi_j\|_p^p.$$

Equivalently, we have to prove that there exists a positive constant C_p such that, for every positive sequence (a_j), we have the inequality

$$\int_0^\infty \Big(\sum_j e^{-2^j\, y} a_j \Big)^p y^{\nu-1} dy \le C \sum_j 2^{-\nu j} a_j^p.$$

This can be thought as a Schur-type lemma: we shall multiply and divide inside the series by $2^{j\alpha}$, for some small positive α. From Hölder's inequality we deduce that

$$\Big(\sum_j e^{-2^j\, y} a_j \Big)^p \le C_p\, y^{-\alpha p} \sum_j e^{-2^j y} 2^{-\alpha j p} a_j^p,$$

using the elementary fact that

$$\sum_j e^{-2^j y} 2^{\alpha j p'} \le C_p y^{-\alpha p'}$$

(as one can check by replacing this sum by an integral). A last integration gives the required estimate provided we chose $0 < \alpha < \frac{\nu}{p}$.

We have proved the left hand side inequality of (1.50). Let us now prove the right hand side, which is much more elementary. We want to estimate $\|f * \psi_j\|_p$. Let us choose $y \in (2^{-j}, 2^{-j+1})$, so that if ξ is in the support of $\widehat{\psi_j}$, the product $y\xi$ is between $1/2$ and 4. We write, as in (1.52)

$$\widehat{f * \psi_j} = e^{y\cdot}\,\widehat{\psi_j}\,\widehat{F_y},$$

and, as before, compute the L^1 norm of the function whose Fourier transform is $e^{y\cdot}\widehat{\psi_j}$. It is easy to see that this is bounded by a uniform constant when

$y \sim 2^{-j}$. Thus,

$$\begin{aligned}\sum_j 2^{-\nu j}\|f * \psi_j\|_p^p &\le C\sum_j 2^{-\nu j}\int_{2^{-j}}^{2^{-j+1}} \|F_y\|_p^p \,\frac{dy}{2^{-j}} \\ &\le C\sum_j \int_{2^{-j}}^{2^{-j+1}} \|F_y\|_p^p \, y^{\nu-1} dy = \|F\|_{A^p_\nu}^p.\end{aligned}$$

To conclude the proof of the theorem, it remains to show that every function $F \in A^p_\nu$ may be written as the Laplace transform of the Fourier transform of some distribution $f \in B^p_\nu$. Now, the Paley-Wiener theorem and the above estimate ensure that this is the case when F is in the dense subset $A^2_\nu \cap A^p_\nu$. Then standard arguments of functional analysis give the result for all $F \in A^p_\nu$. □

1.8. Some remarks on Hardy spaces. One may ask what happens for the three problems under consideration when the weighted Bergmans spaces are replaced by the Hardy spaces.

Let us start with the third one. The characterization of those functions which arise as boundary values of H^p functions is now much simpler than for the Bergman case.

Theorem 1.53. *Let $1 \le p < \infty$. Then, the mapping*

$$\begin{array}{ccl} H^p(\mathbf{H}) & \longrightarrow & L^p(\mathbf{R}) \\ F & \longmapsto & F_0 \end{array}$$

is an isometric isomorphism from $H^p(\mathbf{H})$ onto the subspace of $L^p(\mathbf{R})$ defined as $E^p \equiv \{f \in L^p(\mathbf{R}) : supp\, \hat{f} \subset [0,\infty)\}$.

Proof: By Theorem 1.14, the correspondence above is an isometry. We have already showed the support condition on the Fourier transform for $p = 2$. For general $p \ne 2$ one proceeds by density of $H^2 \cap H^p$ in H^p (similar to Proposition 1.17 above).

We have also shown surjectivity when $p = 2$ (in Proposition 1.19). For general $p \ne 2$, since the mapping is an isometry it suffices to show that the range is dense. For $p < 2$, if $f \in L^p$ with $\operatorname{supp} \hat{f} \subset [0,\infty)$, and if $\{\phi_\varepsilon\}$ is a smooth approximation of the identity, then $\lim_{\varepsilon\to 0}\|f - f * \phi_\varepsilon\|_p = 0$, while by Young's inequality $f * \phi_\varepsilon \in E^2 \cap L^p$. When $p > 2$ and $f \in E^p$, one considers $f^\varepsilon(x) = G_m(\varepsilon x)(f * \phi_\varepsilon)(x)$, where G_m is defined as in (1.18) with $m = m(p)$ large enough so that $G_m(x) \in L^{\frac{2p}{p-2}}(\mathbf{R}) \cap L^\infty$. In particular, by Hölder's inequality $f^\varepsilon \in L^2 \cap L^p$. Also, the Fourier transform is supported in the sum of the spectra of each of the factors, which is contained in $[0,\infty)$. We have shown $f^\varepsilon \in E^2 \cap L^p$, while $\lim_{\varepsilon\to 0}\|f - f^\varepsilon\|_p = 0$ follows easily by the Dominated Convergence Theorem.

□

The subspace E^p of $L^p(\mathbf{R})$ is sometimes denoted[1] by $H^p(\mathbf{R})$. In particular, for $p = 2$, the Hardy space $H^2(\mathbf{H})$ is a Hilbert space which can be identified to the closed subspace $H^2(\mathbf{R})$ of $L^2(\mathbf{R})$. This leads to the following expression for the orthogonal projector.

Proposition 1.54. *The orthogonal projector S from $L^2(\mathbf{R})$ to $H^2(\mathbf{R})$ is given by the following three properties:*

(i) *Sf is the inverse Fourier transform of $\hat{f}\chi_{[0,\infty)}$;*

(ii) *$Sf = \lim_{y\to 0} f * C_y$;*

(iii) *$Sf(x) = \frac{1}{2}f(x) + \frac{i}{2\pi}Tf(x)$, where Tf is the* Hilbert transform *of f, i.e. the convolution of f with the principal value of $\frac{1}{x}$.*

Proof: The proofs of properties (i) and (ii) are easily deduced from the previous results. The equality between expressions (2) and (3) follows from a well known limiting argument, which can be found, e.g., in [23, p. 218]). □

We can now pose the question of the L^p-boundedness of the orthogonal projector in H^2, which leads to one of the first examples of a singular integral operator in Harmonic Analysis: *the Hilbert transform*. The answer to this question requires more sophisticated techniques than Schur's lemma, which are developed in any classical book on Complex or Harmonic Analysis (cf. e.g. [10, p. 54] and [23, p. 186]).

Theorem 1.55. (M. Riesz). *For all $1 < p < \infty$, the projector S extends to a bounded projector from $L^p(\mathbf{R})$ onto $H^p(\mathbf{R})$.*

We will not consider Problem 2 in the context of Hardy spaces, since it does not really makes sense.

To conclude this section we point out that, at least heuristically, the Hardy spaces H^p can be seen as a "limit" of the Bergman spaces A^p_ν, as $\nu \to 0^+$. When $p = 2$ this is quite obvious by the Paley-Wiener integral in (1.23): $A^2_\nu(\mathbf{H})$ is isometrically identified with $L^2((0,\infty), \frac{\Gamma(\nu)}{(2\xi)^\nu}\,d\xi)$, while, for $H^2(\mathbf{H})$, it is with $L^2((0,\infty), d\xi)$. Thus, for good enough functions $F \in H^2 \cap A^2_{\nu_0}$, we have

$$\lim_{\nu\to 0^+} \nu\|F\|^2_{A^2_\nu} = \|F\|^2_{H^2}.$$

This property remains true for general $p \geq 1$, based on the fact that, in the sense of distributions:

$$\nu y^{\nu-1}\chi_{(0,\infty)}(y)\,dy \to \delta_{\{0\}}, \quad \text{as } \nu \to 0^+.$$

[1]However, it shouldn't be confused with the *real Hardy space*, defined, e.g., in [21, Ch. III].

The interested reader can try to state (and prove!) a correct theorem with this principle. As we shall see, this principle is no longer true in several complex variables: the limiting space of the weighted Bergman family... is not the Hardy space! (see §6.2 below).

2. Geometry of symmetric cones

This section is devoted to the theory of symmetric cones. These objects provide a natural substitute to the half-line in higher dimensions, leading also to many non-trivial (yet interesting) questions in the analysis of the associated Bergman projectors. To be able to handle these problems in future sections we first need to exploit the rich geometry of symmetric cones, and establish the right analytic setting where complex theory can be carried out. We do not intend to give here a detailed account of statements and proofs which can be found in many texts (such as [11]), but we shall instead focus in describing the main properties in three model cases: the cone of positive real numbers, the Lorentz cone, and the cone of positive definite symmetric matrices. The goal is to present to the non specialist a general overview of the group theory involved in this problem, without having to face with the deeper results and more specialized notation appearing in most geometry books.

2.1. Convex cones.

Let V be an Euclidean vector space of finite dimension n, endowed with an inner product $(\cdot|\cdot)$. A subset Ω of V is said to be a *cone* if, for every $x \in \Omega$ and $\lambda > 0$, we have $\lambda x \in \Omega$. Clearly, a subset Ω of V is a convex cone if and only if $x, y \in \Omega$ and $\lambda, \mu > 0$ imply that $\lambda x + \mu y \in \Omega$.

Before giving examples, we give the next definition.

Definition 2.1. *Let $\Omega \subset V$ an open convex cone. The* open dual cone *of Ω is defined by*

$$\Omega^\star = \{y \in V : (y|x) > 0, \ \ \forall x \in \overline{\Omega} \setminus \{0\}\}. \tag{2.2}$$

We say that Ω is self-dual *whenever $\Omega = \Omega^\star$.*

Example 2.3.

1. *The octant*: $\Omega = (0, \infty)^n$ in $V = \mathbf{R}^n$;
2. *The Lorentz cone in $V = \mathbf{R}^n$* (or forward light-cone), when $n \geq 3$:

   $$\Lambda_n = \{y \in \mathbf{R}^n : \Delta(y) > 0 \text{ and } y_1 > 0\},$$

   where the quadratic function $\Delta(y) = y_1^2 - y_2^2 - ... - y_n^2$ is called the *Lorentz form*.

(3) *The cone of positive definite symmetric matrices* in $V = \mathrm{Sym}(r, \mathbf{R})$, the space of all $r \times r$ real symmetric matrices. Here the dimension is $n = \frac{r(r+1)}{2}$, $r \geq 1$. The natural inner product on the vector space $\mathrm{Sym}(r, \mathbf{R})$ is given by

$$(X|Y) = \mathrm{Tr}\ (XY) = \sum_{i=1}^{r} x_{ii} y_{ii} + 2 \sum_{1 \leq i < j \leq r} x_{ij} y_{ij}$$

whenever $X = (x_{ij})_{1 \leq i,j \leq r}$ and $Y = (y_{ij})_{1 \leq i,j \leq r}$ are $r \times r$ real symmetric matrices. We denote by $\mathrm{Sym}_+(r, \mathbf{R})$ the cone, consisting of all *positive definite* symmetric matrices (i.e., matrices with positive eigenvalues).

It is easily seen that all three families of examples are self-dual cones (see [11, pp. 7–10]). When we set $V = \mathbf{R}^n$, it means that we endow it with the canonical inner product. Even if all finite dimensional Euclidean spaces are isometric to some $\mathbf{R}^n$, it is more convenient to denote by V the ambiant space.

There exist examples of cones in $\mathbf{R}^n$ $(n \geq 4)$ which are not self-dual for any inner product in $\mathbf{R}^n$: see [11, Ex. 1.10], even if we restrict to homogeneous cones (see Definition 2.8 below). In this paper we shall only be interested in self-dual cones. For further properties of general cones the reader can consult [11, §I.1].

Before going on, let us remark that two of our examples coincide in dimension 3.

Lemma 2.4. The identification $\Lambda_3 \equiv \mathrm{Sym}_+(2, \mathbf{R})$. *Consider the mapping*

$$\Phi : \mathbf{R}^3 \longrightarrow Sym(2, \mathbf{R})$$

given by

$$y = (y_1, y_2, y_3) \longmapsto \Phi(y) = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 + y_2 & y_3 \\ y_3 & y_1 - y_2 \end{pmatrix},$$

which is an isometry from the Euclidean space $\mathbf{R}^3$ onto the inner product space $Sym(2, \mathbf{R})$. Then $y \in \Lambda_3$ if and only if $\Phi(y) \in Sym_+(2, \mathbf{R})$.

Proof: Observe that, if $Y = \Phi(y)$, then

$$2\Delta(y) = \det Y \quad \text{and} \quad \sqrt{2} y_1 = \mathrm{tr}\ Y,$$

□

The next lemma gives a characterization of $\overline{\Omega}$ for a self-dual cone Ω.

Lemma 2.5. *Let Ω be an open convex cone which is self-dual in $(V, (\cdot|\cdot))$. Then,*

$$\overline{\Omega} = \{y \in \mathbf{R}^n : (y|x) \geq 0, \ \forall x \in \Omega\}. \tag{2.6}$$

In particular, Ω is the interior of its closure, i.e., $\Omega = \overset{\circ}{(\overline{\Omega})}$.

Proof: The inclusion "$\subset$" is immediate from (2.2) and the self-duality of Ω. The converse is also easy: if y belongs to the right-hand side of (2.6), and we choose any $\varepsilon > 0$ and $\mathbf{e} \in \Omega$ fixed, then we have

$$(y + \varepsilon\mathbf{e}|x) = (y|x) + \varepsilon(\mathbf{e}|x) > 0, \quad \forall\, x \in \overline{\Omega} - \{0\}.$$

Thus, by self-duality $y + \varepsilon\mathbf{e} \in \Omega$, and letting $\varepsilon \to 0$, we get $y \in \overline{\Omega}$.

For the last assertion, observe that $\Omega \subset \overset{\circ}{\overline{\Omega}}$ is always true since Ω is an open set. For the converse, just notice that the interior of the right hand side of (2.6) is contained in the right hand side of (2.2), which by self-duality equals Ω. □

2.2. The automorphism group and homogeneous cones.

Let Ω be a fixed open convex cone in V, and let $\mathrm{GL}(V)$ denote the group of all linear invertible transformations of $\mathbf{R}^n$. We define the *automorphism group* $G(\Omega)$ *of the cone* by

$$G(\Omega) = \{g \in \mathrm{GL}(V) : g\Omega = \Omega\}.$$

The group $G(\Omega)$ is a closed subgroup of $\mathrm{GL}(\mathbf{R}^n)$, and in particular, a Lie group. This is a straightforward consequence of the following lemma. The reader can prove it as an exercise when the cone is self-dual, using the fact $\Omega = (\overline{\Omega})^{\circ}$, shown in Lemma 2.5 (this fact is also true for all open convex cones).

Lemma 2.7. *An element $g \in GL(V)$ belongs to $G(\Omega)$ if and only if $g\overline{\Omega} = \overline{\Omega}$.*

We denote by G the *connected component of the identity* in $G(\Omega)$. It is easy to verify that G is a closed subgroup of $G(\Omega)$. Indeed, G is closed in $G(\Omega)$ as a connected component of $G(\Omega)$. Also, G is a group because GG and G^{-1} are the ranges of the connected sets $G \times G$ and G under the respective continuous maps $(g, h) \mapsto g \circ h$ and $g \mapsto g^{-1}$; so GG and G^{-1} are connected subsets of $G(\Omega)$ which both contain the identity and therefore $GG \subset G$ and $G^{-1} \subset G$.

In this paper we are interested in a special class of cones which behave well enough under the action of $G(\Omega)$.

Definition 2.8. An open convex cone Ω is said to be *homogeneous* if the group $G(\Omega)$ acts transitively on Ω, i.e., for all $x, y \in \Omega$, there exists $g \in G(\Omega)$ such that $y = gx$. An open convex cone Ω is said to be *symmetric* if it is homogeneous and self-dual.

A simple exercise is the following:

Exercise 2.9. *Let Ω be a symmetric cone in $(V, (\cdot|\cdot))$. Then $G(\Omega)^* = G(\Omega)$ and $G^* = G$, where "$*$" denotes the adjoint under the inner product $(\cdot|\cdot)$.*

Example 2.10.

(1) The cone $\Omega = (0,\infty)$ is symmetric in $\mathbf{R}$. Indeed, the automorphism group is $G(\Omega) = G = \mathbf{R}_+$, and we can identify Ω with $G \cdot 1$. The situation is similar for the *octant* $\Omega = (0,\infty)^n$, for which the identity component $G = \{\text{Diag } (a_1, \dots, a_n) \mid a_j > 0\} \equiv \mathbf{R}_+^n$. We leave as an exercise determining the larger group $G(\Omega)$ (careful!, there are $n!$ identity components).

(2) The Lorentz cone $\Omega = \Lambda_n$ is symmetric in $\mathbf{R}^n$. To show this, we consider the *Lorentz group*

$$O(1, n-1) = \{g \in \mathrm{GL}(n, \mathbf{R}) \mid \Delta(gx) = \Delta(x),\ \forall\, x \in \mathbf{R}^n\},$$

and its subgroup $O_+(1, n-1) = \{g \in O(1, n-1) \mid g_{11} > 0\}$. In this case, we will also describe completely the group $G(\Omega)$. We shall show that

$$G(\Omega) = \mathbf{R}_+\, O_+(1, n-1) \quad \text{and} \quad \Omega = G(\Omega) \cdot \mathbf{e}, \tag{2.11}$$

where $\mathbf{e} = (1, 0, \dots, 0)$. For the first equality, the inclusion "$\subset$" is clear by definition of the Lorentz group (and condition $g_{11} > 0$). Let us now prove the second equality. Using *hyperbolic coordinates*, an arbitrary point $y \in \Omega$ can be written as

$$y = (r\mathrm{ch}\ t, r\mathrm{sh}\ t\ \omega), \quad r > 0,\ t \geq 0,\ \omega \in \mathbf{R}^{n-1}\ :\ |\omega| = 1. \tag{2.12}$$

This is the same as saying $y = r\,\kappa\,\alpha(t) \cdot \mathbf{e}$, where

$$\alpha(t) = \begin{pmatrix} \mathrm{ch}\ t & \mathrm{sh}\ t & 0 \\ \mathrm{sh}\ t & \mathrm{ch}\ t & 0 \\ 0 & 0 & I \end{pmatrix} \quad \text{and} \quad \kappa = \begin{pmatrix} 1 & 0 \\ 0 & \kappa_0 \end{pmatrix}, \tag{2.13}$$

for some $\kappa_0 \in SO(n-1)$ (to prove the existence of κ_0, we have used the fact that every point of the unit sphere in $\mathbf{R}^{n-1}$ is obtained as the image through a rotation of the vector $(1, 0, \cdots, 0)$). This implies that y is obtained from $\mathbf{e}$ using the action of the linear transformation $r\,\kappa\,\alpha(t)$ which is clearly in $\mathbf{R}_+\, O_+(1, n-1)$. At this point, we have already proved that Ω is homogeneous.

Let us go on with the description of $G(\Omega)$. It remains to show that every element $g \in G(\Omega)$ belongs to $\mathbf{R}_+\, O_+(1, n-1)$. Since we already know that this last subgroup acts transitively on Ω, it is sufficient to consider an element g which fixes $\mathbf{e}$. Then, g fixes the whole x_1-axis and also the cone boundary, and therefore it must take the form of a two block matrix as in the right hand side of (2.13), for some κ_0. Moreover, since g preserves the cone, restricting to the plane $\{x_1 = 1\}$, we see that κ_0 must leave invariant the unit sphere in $\mathbf{R}^{n-1}$, and thus $\kappa_0 \in O(n-1)$. This finishes the proof.

We point out that our arguments show actually more: if we define the subgroups

$$A = \{r\alpha(t) \mid r > 0,\ t \in \mathbf{R}\} \quad \text{and} \quad K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix} \mid \kappa \in O(n-1) \right\},$$

then we have found the *Cartan decomposition of* $G(\Omega)$, i.e., $G(\Omega) = KAK$, where K is compact and A abelian. Also, in (2.12) we have given a "polar decomposition" for every $y \in \Omega$, which allows us to identify Ω with the set $SO(n-1) \times A_+$ (with $A_+ = \{r\alpha(t) \in A \mid t \geq 0\}$). We point out that this last set is not a group, so this identification will not say much about the "geometry" of Ω (compare with §2.3 below). Finally, we leave as an exercise to the reader the verification of

$$G = \mathbf{R}_+\, SO_+(1, n-1) = SO(n-1)\, A\, SO(n-1),$$

where the "S" in front of a subgroup indicates that the linear transformations have all determinant 1.

(3) The cone $\Omega = \mathrm{Sym}_+(r, \mathbf{R})$ is symmetric in $\mathrm{Sym}(r, \mathbf{R})$. Indeed, just consider $\mathrm{GL}(r, \mathbf{R})$ as a subgroup of $G(\Omega)$ via the *adjoint action*:

$$(g \in \mathrm{GL}(r, \mathbf{R}), Y \in \mathrm{Sym}(r, \mathbf{R})) \longmapsto g \cdot Y = gYg^* \in \mathrm{Sym}(r, \mathbf{R}). \tag{2.14}$$

Now, every positive-definite symmetric matrix $Y \in \Omega$ can be written as $Y = X^2 = X \cdot I$, for another such $X \in \Omega$ (e.g., by diagonalizing Y). Thus, we have shown $\Omega = \mathrm{GL}(r, \mathbf{R}) \cdot I$ and the cone is homogeneous.

One can prove more: *the automorphism group* $G(\Omega)$ *coincides with* $GL(r, \mathbf{R})$*, via the adjoint action in (2.14).* This is shown, e.g. in [11, Ch. VI], using an analysis of their Lie algebras.

(4) A simple example of a cone Ω which is self-dual but not homogeneous is the (regular) *pentagonal cone in* $\mathbf{R}^3$. Indeed, a linear transformation preserving this cone must send each of the 5 boundary lines into another one of these lines, and similarly for the 5 boundary faces. Therefore, if we consider the triangular cones formed by the convex hull of three consecutive boundary lines, we see that each of these must be sent into another such triangular cone. Thus, there is a smaller pentagonal cone inside Ω which is left invariant by any linear transformation in $G(\Omega)$, implying that Ω is not homogeneous.

2.3. Group structure of symmetric cones.

After having seen in some detail the examples above, we are ready to state the main theorem about symmetric cones. We recall that, if G is a subgroup of $\mathrm{GL}(n, \mathbf{R})$ and $\mathbf{e} \in \mathbf{R}^n$, then $G_{\mathbf{e}} = \{g \in G \mid g\mathbf{e} = \mathbf{e}\}$ is called the *stabilizer subgroup of* $\mathbf{e}$ *in* G. Also $O(n)$ denotes the *orthogonal group in* $\mathbf{R}^n$, i.e. the group of all $n \times n$ real matrices such that $k^* = k^{-1}$, where k^* is the adjoint of k under the Euclidean scalar product on $\mathbf{R}^n$. Finally, a subgroup H of $\mathrm{GL}(n, \mathbf{R})$ is said to *act simply transitively on a set* Ω if for all $x, y \in \Omega$, there exists a unique $h \in H$ such that $y = hx$. We write as well $O(V)$, $\mathrm{GL}(V)$, etc, when $\mathbf{R}^n$ is replaced by the euclidean space V.

Theorem 2.15. *Let Ω be a symmetric cone in V. Then,*

1. *The identity component G of $G(\Omega)$ acts transitively on Ω;*
2. *There exists a point $\mathbf{e} \in \Omega$ such that*
$$G(\Omega)_{\mathbf{e}} = G(\Omega) \cap O(V) \quad \textit{and} \quad G_{\mathbf{e}} = G \cap O(V).$$
3. *There exists a subgroup H of G which acts simply transitively on Ω; i.e., for all $y \in \Omega$ we can find $h \in H$ such that $y = h\mathbf{e}$. Moreover, $G = H\,K$, the latter denoting the compact group $K = G_{\mathbf{e}}$.*

This result is well-known and can be found in most geometry books which deal with symmetric spaces. For the first two points we can refer, e.g., to Propositions I.1.9 and I.4.3 of [11]. The second assertion is due to E. B. Vinberg, being also valid in the more general setting of homogeneous cones [25]. A complete proof for symmetric irreducible cones can be found in [11, Th. VI.3.6].

Rather than trying to describe the proof (which makes use of deeper results on Lie algebras), we shall verify the thesis of the theorem in our main example $\Omega = \mathrm{Sym}_+(2, \mathbf{R})$. For this we use the fact that $G(\Omega) = \mathrm{GL}(2, \mathbf{R})$, as described in (3) of the previous subsection (in fact, such equality is also a consequence of (2)). We point out that that the ideas in this proof are completely general, and can be extended (with a little more complicated notation) to the cone $\mathrm{Sym}_+(r, \mathbf{R})$. Understanding this example will also help the reader who wants to see the proof given in [11, Th. VI.3.6] for general symmetric cones. To follow this general proof, one should have at his disposal the language of *Jordan algebras*.

Proof of Theorem 2.15 for $\Omega = \mathbf{Sym}_+(2, \mathbf{R})$:

The first two statements in Theorem 2.15 are immediate (with $\mathbf{e} = I$, the identity matrix), so we shall focus only in the third assertion. Observe that the group $K = SO(2)$. Now, take a positive-definite symmetric 2×2 matrix

$$Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix} \in \mathrm{Sym}(2, \mathbf{R}).$$

Then $y_1 > 0$ and $y_1 y_2 - y_3^2 > 0$. Next consider the *Gauss factorization of* Y:

$$Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y_3/y_1 & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 - \frac{y_3^2}{y_1} \end{pmatrix} \begin{pmatrix} 1 & y_3/y_1 \\ 0 & 1 \end{pmatrix}. \tag{2.16}$$

To understand this decomposition, one should recall the Gauss reduction of the quadratic form with matrix Y:

$$Q(\xi) = y_1 \xi_1^2 + 2y_3 \xi_1 \xi_2 + y_2 \xi_2^2 = y_1\,(\xi_1 + \xi_2 \tfrac{y_3}{y_1})^2 + (y_2 - \tfrac{y_3^2}{y_1})\,(\xi_2)^2,$$

which gives us the factorization $Q(\xi) = \xi^* Y \xi = (P\xi)^* D (P\xi)$, for the diagonal matrix $D = \text{Diag}\,(y_1, y_2 - \frac{y_3^2}{y_1})$ and the change of basis $P = \begin{pmatrix} 1 & \frac{y_3}{y_1} \\ 0 & 1 \end{pmatrix}$. Moreover, since D is a positive matrix, we can rewrite (2.16) as:

$$Y = (P^* \sqrt{D})(P^* \sqrt{D})^* = (P^* \sqrt{D}) \cdot I. \tag{2.17}$$

In particular, if we define

$$N = \left\{ \left(\begin{array}{cc} 1 & 0 \\ v & 1 \end{array} \right) \mid v \in \mathbf{R} \right\} \quad \text{and} \quad A = \left\{ \left(\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right) \mid \lambda_j > 0 \right\},$$

we have shown that every $Y \in \Omega$ can be written uniquely as $Y = (na) \cdot I$, for some $n \in N$ and $a \in A$. Therefore, *the (semidirect) product $H = NA$ acts simply transitively on $Sym_+(2, \mathbf{R})$*. Moreover, just by multiplying we observe that H is precisely the set of *lower triangular* 2×2-*matrices*, and thus a group as stated in the theorem. Finally, we notice that this reasoning gives us as well the *Iwasawa decomposition of $G = NAK$, with K compact, A abelian and N nilpotent.* □

Now that we have described the main groups acting on the cone $\text{Sym}_+(2, \mathbf{R})$, let us use the identification of the latter with the Lorentz cone Λ_3 to have a graphical image of the corresponding orbits.

EXAMPLE 2.18. *Group action in the light-cone Λ_3 of $\mathbf{R}^3$.* Using the mapping Φ in (4) of §2.1, we have the following correspondence for the action of the groups N, A, K in Λ_3:

$$\text{Nilpotent group } N: \quad \left(\begin{array}{cc} 1 & 0 \\ v & 1 \end{array} \right) \cdot Y \quad \longmapsto \quad \left(\begin{array}{ccc} 1 + \frac{v^2}{2} & \frac{v^2}{2} & -v \\ -\frac{v^2}{2} & 1 - \frac{v^2}{2} & v \\ -v & v & 1 \end{array} \right) y$$

$$\text{Abelian group } A: \quad \left(\begin{array}{cc} re^t & 0 \\ 0 & re^{-t} \end{array} \right) \cdot Y \quad \longmapsto \quad r^2 \left(\begin{array}{ccc} \text{ch } 2t & \text{sh } 2t & 0 \\ \text{sh } 2t & \text{ch } 2t & 0 \\ 0 & 0 & 1 \end{array} \right) y$$

$$\text{Compact group } K: \quad \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right) \cdot Y \quad \longmapsto \quad \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{array} \right) y.$$

The orbits of N are parabolas lying in the planes $\{x_1 + x_2 = c\}$, $c > 0$, which cut transversally the cone. The orbits of K are circles lying in the planes $\{x_1 = c\}$, $c > 0$, which cut horizontally the cone. Finally, the orbits of A are straight half-lines through the origin (for usual dilations) and hyperbolas (for dilations of type $\alpha(t)$) contained in the half-plane $\{x_3 = 0, x_1 > 0\}$, which cut vertically the cone.

From this example $\Lambda_3 = \mathrm{Sym}_+(2, \mathbf{R})$, we have the intuition of the analogous description in higher dimension for these two families of examples. We give it now, and leave the proof as an exercise.

EXAMPLE 2.19. *Group action in $Sym_+(r, \mathbf{R})$.*
In this case, $\mathbf{e}$ is the identity matrix. We describe the three subgroups N, A, K, which act via the adjoint action, and identify with subgroups of $\mathrm{GL}(r, \mathbf{R})$. Then K identifies with $SO(r, \mathbf{R})$, A identifies with diagonal matrices with positive elements on the diagonal, N identifies with lower triangular matrices whose diagonal entries are 1.

EXAMPLE 2.20. *Group action in Λ_n.*
In this case, $\mathbf{e}$ is as before the vector $(1, 0, \cdots, 0)$. We have already described the subgroup K, which identifies with $SO(n-1)$. We have also described A, which identifies with $\mathbf{R}^+ \times \mathbf{R}^+$. For $a = (a_1, a_2)$, its action $a \cdot y$ is given by $r^2\alpha(2t)y$, with $a_1 = re^t$ and $a_2 = re^{-t}$. It remains to describe the action of N, which identifies with $\mathbf{R}^{n-2}$. If $h \in N$ is given by the vector column vector v, then, the action of h is given by

$$h \cdot y = \begin{pmatrix} 1 + \frac{|v|^2}{2} & \frac{|v|^2}{2} & -v^* \\ -\frac{|v|^2}{2} & 1 - \frac{|v|^2}{2} & v^* \\ -v & v & 1 \end{pmatrix} y.$$

REMARK 2.21. The identification between a symmetric cone and the group H is actually *topological*. I.e., the correspondence $h \in H \mapsto h \cdot \mathbf{e} \in \Omega$ is a homeomorphism, when H is endowed with Lie subgroup topology of $\mathrm{GL}(n, \mathbf{R})$. To verify this in the case $\Omega = \mathrm{Sym}_+(2, \mathbf{R})$, just observe that the inverse mapping is given by:

$$Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix} \in \mathrm{Sym}_+(2, \mathbf{R}) \longmapsto \begin{pmatrix} \sqrt{y_1} & 0 \\ \frac{y_3}{\sqrt{y_1}} & \sqrt{y_2 - \frac{y_3^2}{y_1}} \end{pmatrix} \in H,$$

and hence it is continuous. This fact will be used in the next subsection, where we exploit the structure of H as a riemannian manifold.

For general symmetric cones, one has also the Iwasawa decomposition $G = NAK$, with $H = NA$. We then define the *rank of a symmetric cone* Ω as the dimension of the subgroup A in the decomposition. Equivalently, the rank of Ω is the largest positive integer for which there is a linear invertible change of coordinates in V that transforms Ω into another cone contained in $(0, \infty)^r \times \mathbf{R}^{n-r}$. In our examples above one has

$$\text{rank } (0, \infty)^n = n, \quad \text{rank } \Lambda_n = 2, \quad \text{and} \quad \text{rank } \mathrm{Sym}_+(r, \mathbf{R}) = r.$$

For a different, but equivalent definition of the rank in terms of Jordan algebras see [11, p. 28].

We will also say that a symmetric cone Ω is *irreducible* whenever it is not linearly equivalent to the product of at least two lower-dimensional symmetric cones. E.g., $(0,\infty)^n$ is clearly reducible, while $\mathrm{Sym}_+(r,\mathbf{R})$ and Λ_n $(n \geq 3)$ are irreducible. Observe that the cone Λ_2 is equivalent to $(0,\infty)^2$, and hence is also reducible. From now on, we will restrict to irreducible symmetric cones. Most results possess a generalization to reducible ones.

REMARK 2.22. Using the theory of Jordan algebras it is possible to classify *all irreducible symmetric cones of rank* r. Roughly speaking, these are $(0,\infty)$ for $r=1$, Λ_n $(n \geq 3)$ for $r=2$, and the cones of positive-definite matrices $\mathrm{Sym}_+(r,\mathbf{R})$, $\mathrm{Her}_+(r,\mathbf{C})$, $\mathrm{Her}_+(r,\mathbf{H})$, $\mathrm{Her}_+(3,\mathbf{O})$, when $r \geq 3$. Here $\mathbf{H}$ denotes the non-commutative field of *quaternions*, and $\mathbf{O}$ the non-associative of *octonions*, being this last cone only symmetric when $r=3$. Her stands for Hermitian matrices. In view of this classification, it is clear that we are not so far from the general case by just restricting to the examples presented in §2.1. The reader wishing to learn more on the classification of symmetric cones is referred to Chapter V of [11].

2.4. Riemannian structure and dyadic decomposition.

Having identified in Theorem 2.15 a symmetric cone with a subgroup H of G, it is possible to endow a *riemannian metric in* Ω as follows: given a point $p \in \Omega$, consider the bilinear form

$$\mathcal{G}_p : V \times V \longrightarrow \mathbf{R}$$

defined as

$$\mathcal{G}_p(\xi,\eta) = (h^{-1}\xi | h^{-1}\eta), \quad \text{whenever } p = h\mathbf{e},\ h \in H.$$

It is clear that for each $p \in \Omega$, $\mathcal{G}_p(\cdot,\cdot)$ is an inner product on V, and therefore, $\mathcal{G}$ defines a non-degenerate smooth metric in Ω. Moreover, this metric is G-*invariant*, that is, for all $g \in G$, $p \in \Omega$,

$$\mathcal{G}_{gp}(g\xi, g\eta) = \mathcal{G}_p(\xi,\eta), \quad \xi,\eta \in V. \tag{2.23}$$

This is obvious by definition when $g \in H$. In general, if $p = h\mathbf{e}$, since $G = HK$, there exists $k \in K$ such that $ghk \in H$, and then, since $K = G_{\mathbf{e}}$, we can write $gp = gh\mathbf{e} = ghk\mathbf{e}$. Applying the definition of the metric we obtain

$$\mathcal{G}_{gp}(g\xi, g\eta) = ((ghk)^{-1}g\xi, (ghk)^{-1}g\eta) = (k^{-1}h^{-1}\xi, k^{-1}h^{-1}\eta) = (h^{-1}\xi|h^{-1}\eta)$$

(by the orthogonality of the group $K = O(n) \cap G$), establishing our claim.

Associated with the riemannian metric $\mathcal{G}$ there is a *distance function* $d\colon \Omega \times \Omega \to \mathbf{R}_+$ defined as usual: for $p, q \in \Omega$

$$d(p,q) = \inf_{\gamma} \left\{ \int_0^1 \sqrt{\mathcal{G}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}\, dt \right\}, \tag{2.24}$$

where the infimum is taken over the smooth curves $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$. The following proposition is an easy exercise using the G-invariance of $\mathcal{G}$.

Proposition 2.25. *The Riemannian distance d is invariant under the action of the group G, i.e. $d(gp, gq) = d(p, q)$, for all $g \in G$, $p, q \in \Omega$.*

In order to understand how this distance looks like in a general symmetric cone, we first consider the trivial case of the 1-dimensional situation.

Example 2.26. For $n = 1$, we consider the symmetric cone $\Omega = (0, \infty)$. Recall that, in this elementary case, $G(\Omega) = G = H = \mathbf{R}_+$. We identify the cone $\Omega = (0, \infty)$ with the multiplicative group $H = \mathbf{R}_+$. Then, for every $p \in \Omega$, the riemannian metric takes the form:

$$\mathcal{G}_p(\xi, \eta) = p^{-1}\xi . p^{-1}\eta = \frac{\xi \cdot \eta}{p^2}, \quad \xi, \eta \in \mathbf{R}.$$

The corresponding distance on Ω is therefore given by

$$d(p, q) = \inf_\gamma \left\{ \int_0^1 \frac{|\dot\gamma(t)|}{\gamma(t)}\, dt \right\},$$

where the infimum is taken over all smooth curves $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Assume that $p < q$. We claim that $d(p, q) = \log\left(\frac{q}{p}\right)$. Indeed, on the one hand, for every γ,

$$\int_0^1 \frac{|\dot\gamma(t)|}{\gamma(t)}\, dt \geq \int_0^1 \frac{\dot\gamma(t)}{\gamma(t)}\, dt = \log\gamma(1) - \log\gamma(0) = \log(q/p).$$

Conversely, for the segment $\gamma(t) = (1-t)p + tq$, we have

$$\int_0^1 \frac{|\dot\gamma(t)|}{\gamma(t)}\, dt = \int_0^1 \frac{q-p}{t(q-p)+p}\, dt = \log(q/p).$$

Notice the trivial invariance of the distance under the action of the group $G = \mathbf{R}_+$: $d(gp, gq) = d(p, q)$ for every $g > 0$. As a consequence, a natural covering of the cone with invariant balls is the *dyadic decomposition of* $(0, \infty)$:

$$(2^{j-1}, 2^{j+1}) = B_{\log 2}(2^j) = \{p \in \Omega : d(p, 2^j) < \log 2\}, \quad j \in \mathbf{Z}.$$

Let us recall that this dyadic decomposition has played an important role in the analysis of Besov spaces related to the upper half-plane.

Our example above suggests an analogue to the dyadic decomposition for general symmetric cones. This will be defined in terms of *G-invariant balls*, using the riemannian distance d in (2.24). That is, given $y \in \Omega$ and $\delta > 0$ we denote $B_\delta(y) = \{\xi \in \Omega \mid d(\xi, y) < \delta\}$. We recall that the topology generated by these balls is equivalent to the original topology of H, and thus to the relative topology of Ω as a subset of V (by Remark 2.21). Then we have the following result.

Theorem 2.27. *Let Ω be a symmetric cone. Then, there exists a sequence $\{\xi_k\}_{k=1}^{\infty}$ of points of Ω such that the following three properties hold:*

(i) *The balls $B_1(\xi_k)$ are pairwise disjoint;*
(ii) *The balls $B_3(\xi_k)$ form a covering of Ω;*
(iii) *There is an integer $N = N(\Omega)$ such that every $y \in \Omega$ belongs to at most N balls $B_3(\xi_k)$ ("finite overlapping property").*

Proof: Let $\{B_j\}_{j=1}^{\infty}$ be any countable covering of Ω with open balls of d-radius 1 (it exists since the topology of Ω is locally compact). By induction, we can select a subsequence $\{B_{j_k}\}_{k=1}^{\infty}$ so that, for each k, the ball B_{j_k} is disjoint with B_{j_1}, ..., $B_{j_{k-1}}$. Then, the sequence $\{\xi_k\}_{k=1}^{\infty}$ of centers of such balls satisfies properties (i) and (ii). Indeed, the first one is immediate, while for the second, take any point $y \in \Omega$ and a ball B_j containing the point. Then, by construction, B_j must intersect some ball B_{j_k}, from which it follows that $y \in B_3(\xi_k)$.

To show (iii), let μ denote a left Haar measure in H, and $\tilde{\mu}$ the induced *H-invariant measure in* Ω. That is, $\tilde{\mu}$ is defined by:

$$\tilde{\mu}(E) = \mu\left(\{h \in H \mid h\mathbf{e} \in E\}\right), \quad E \subset \Omega,$$

and satisfies $\tilde{\mu}(h \cdot E) = \tilde{\mu}(E)$, for all $h \in H$. Then, if $y \in \Omega$ and we denote by $J_y = \{k \in \mathbf{Z}_+ \mid y \in B_3(\xi_k)\}$, we shall show that $\mathrm{Card}(J_y)$ is bounded by a constant depending only on Ω. Indeed, since $y \in \cap_{k\in J_y} B_3(\xi_k)$, we have $\cup_{k\in J_y} B_1(\xi_k) \subset B_4(y)$. Since these balls are disjoint and the measure $\tilde{\mu}$ is H-invariant we have

$$\begin{aligned} \tilde{\mu}(B_4(\mathbf{e})) = \tilde{\mu}(B_4(y)) &\geq \mu(\cup_{k\in J_y} B_1(\xi_k)) \\ &= \sum_{k\in J_y} \tilde{\mu}(B_1(\xi_k)) = \mathrm{Card}\ (J_y)\, \tilde{\mu}(B_1(\mathbf{e})). \end{aligned}$$

Thus,

$$\mathrm{Card}\ (J_y) \leq \tilde{\mu}(B_4(\mathbf{e}))/\tilde{\mu}(B_1(\mathbf{e})) = N(\Omega), \quad \forall\, y \in \Omega,$$

which is a finite constant since the Haar is finite and does not vanish over open bounded sets. $\square$

Remark 2.28. Sequences $\{\xi_k\}$ satisfying the properties of Theorem 2.27 are called 1-*lattices of* Ω. They will play the same role as the dyadic grid in $(0, \infty)$, and in particular, we shall use them in the analysis of functions with spectrum in Ω, e.g., to define Besov norms or to discretize multipliers. This will be a crucial point where geometry and analysis merge for the solution of our problem. Of course, there is nothing special about the radius 1, and we could have as well considered δ-*lattices in* Ω, for all $\delta > 0$. These have the additional remarkable property that, for each $\delta_0 > 0$, the number N in the Finite Overlapping Property is independent of δ as long as $0 < \delta \leq \delta_0$ (see 2.44 below) .

2.5. Analysis of symmetric cones.

In this last section we give an account of the most important functions defined on a cone, in the sense that they preserve a fair amount of its geometric properties. We do it for our two families of irreducible symmetric cones, but it can be done in general, see the remark below. When Ω is the positive real line, which is of rank 1, the analysis of the cone makes an intensive use of the function $\Delta(y) = y$ which is clearly related to the automorphism group. We want to find its equivalent in higher rank.

(1) *Determinants and principal minors.*

Consider first the vector space $V = \mathrm{Sym}(r, \mathbf{R})$. Then, we define the k^{th}-*principal minor of* $Y \in V$ as the determinant

$$\Delta_k(Y) = \begin{vmatrix} y_{11} & \cdots & y_{1k} \\ \vdots & \ddots & \\ y_{1k} & \cdots & y_{kk} \end{vmatrix}, \quad k = 1, 2, \ldots, r.$$

Of course, $\Delta(Y) := \Delta_r(Y)$ is the usual determinant of Y as a linear operator in $\mathbf{R}^r$, and hence, independent of the basis. The lower principal minors, however, will depend on the choice of the orthonormal basis. Observe that,

$$\begin{aligned} &(2.29) & \Delta_k(Y) &= \lambda_1 \cdots \lambda_k, \quad \text{when } Y = \mathrm{Diag}\ (\lambda_1, \ldots, \lambda_r), \\ &(2.30) & \Delta_k(a \cdot Y) &= \lambda_1^2 \cdots \lambda_k^2\, \Delta_k(Y), \ \text{when } a = \mathrm{Diag}\ (\lambda_1, \ldots, \lambda_r) \in A, \\ &(2.31) & \Delta_k(n \cdot Y) &= \Delta_k(Y), \quad \text{when } n \in N, \end{aligned}$$

where in the last case we recall that N is the subgroup of $\mathrm{GL}(r, \mathbf{R}) \equiv G(\Omega)$ consisting of lower triangular matrices with 1's in the main diagonal. From these two properties and the fact $\Omega = (NA) \cdot I$ it can be shown easily that

$$\Omega = \mathrm{Sym}_+(r, \mathbf{R}) = \{Y \in V \mid \Delta_k(Y) > 0,\ \forall\, k = 1, \ldots, r\}.$$

Finally, we have the following *homogeneity property*:

$$\Delta(g \cdot Y) = (\mathrm{Det}\ g)^{\frac{r}{n}}\, \Delta(Y), \quad \text{when } g \in G(\Omega), \tag{2.32}$$

where Det g denotes the determinant of g *as a linear transformation*[1] *in* V which preserves the subset Ω (recall the original definition of $G(\Omega)$).

(2) *Determinant and principal minor for the forward light cones.*

The previous cone had rank r, and we have defined r functions. Since the forward light cone Λ_n has rank 2, we define two functions, which are still called the principal minors, by

$$\Delta_1(y) = y_1 + y_2 \quad \text{and} \quad \Delta(y) = \Delta_2(y) = y_1^2 - (y_2^2 + \ldots + y_n^2), \quad y \in \mathbf{R}^n.$$

[1]Observe that, when we identify $G(\Omega) \equiv \mathrm{GL}(r, \mathbf{R})$, via the adjoint action $Y \mapsto gYg^*$, then Det $g = (\det g)^{\frac{2n}{r}}$, where the latter is the usual determinant as a matrix in $\mathrm{GL}(r, \mathbf{R})$. From this equality, (2.32) follows easily.

As an elementary exercise, the reader can verify the equivalent of the above properties for these two functions, that is

$$\Delta_1(a\cdot y) = a_1^2\Delta_1(y), \text{ when } a=(a_1,a_2)\in A, \tag{2.33}$$

$$\Delta_2(a\cdot y) = a_1^2a_2^2\,\Delta_2(y), \text{ when } a=(a_1,a_2)\in A, \tag{2.34}$$

$$\Delta_k(n\cdot y) = \Delta_k(y), \quad \text{when } n\in N, \tag{2.35}$$

$$\Delta_2(gy) = (\mathrm{Det}\ g)^{\frac{r}{n}}\,\Delta_2(y), \quad \text{when } g\in G(\Omega). \tag{2.36}$$

Now A and N are the two subgroups of the group G related to the cone Λ_n (see the example 2.20 above).

Remark 2.37. The two cases above are two particular cases of a general situation. That is, for a general symmetric cone Ω of rank r, we can define r determinant functions Δ_k, which coincide with the previous ones in these two families of examples, and which have the invariance properties given by Equations 2.30, 2.31, **??** in terms of the groups N, A, K which appear in the Iwasawa decomposition of G. This is done by using the *theory of Jordan algebras* (see [11, p. 114]). For the purposes of this paper, we will use below general notations. This may be understood as an unified notation for the two families of examples in a first reading. But it may also be used to understand the general situation in a deeper study of symmetric cones.

(2) *Generalized powers.*

A *generalized power in a symmetric cone* Ω of rank r is defined by:

$$\Delta^{\mathbf{s}}(y) = \Delta_1(y)^{s_1-s_2}\,\Delta_2(y)^{s_2-s_3}\cdots\Delta_r(y)^{s_r}, \quad \mathbf{s}=(s_1,\dots,s_r)\in\mathbf{C}^r,$$

where Δ_k are the principal minors from the previous paragraph, and $y\in\Omega$ [11, p. 122]. For the case of the light-cone $\Omega=\Lambda_n$ we have

$$\Delta^{s_1,s_2}(y) = \Delta_1(y)^{s_1-s_2}\,\Delta(y)^{s_2} = (y_1+y_2)^{s_1-s_2}\,(y_1^2-(y_2^2+\ldots+y_n^2))^{s_2}.$$

These functions will play an important role when looking at the Bergman projectors, since they are the natural test functions for Schur's Lemma. From a more geometrical point of view, their importance is justified by the fact that they constitute precisely the *set of characters of the group* H; i.e., any continuous multiplicative function on H is necessarily of the form:

$$h\in H\mapsto\Delta^{\mathbf{s}}(h\mathbf{e}), \quad \text{for some } \mathbf{s}\in\mathbf{C}^r$$

(see, e.g., [14, Lemma 2.4]).

For the analysis in subsequent sections, the main property of generalized powers states that these remain essentially constant within each invariant ball.

Theorem 2.38. *Let Ω be a symmetric cone. Then, there exists a constant $C=C(\Omega)\geq 1$ such that, for all $k=1,\dots,r$,*

$$\frac{1}{C}\leq\frac{\Delta_k(y)}{\Delta_k(y_0)}\leq C, \quad \textit{whenever } y\in B_1(y_0). \tag{2.39}$$

Proof: Suppose first that $y_0 = \mathbf{e}$. Since the invariant ball $B_1(\mathbf{e})$ is relatively compact in Ω and the functions

$$y \in \Omega \longmapsto \frac{\Delta_k(y)}{\Delta_k(\mathbf{e})}, \quad k = 1, \dots, r,$$

are continuous and positive, there must exist a constant $C \geq 1$ such that

$$\frac{1}{C} \leq \frac{\Delta_k(y)}{\Delta_k(\mathbf{e})} \leq C, \quad \text{whenever } y \in B_1(\mathbf{e}),$$

establishing (2.39) when $y_0 = \mathbf{e}$.

In the general case, write $y_0 = h \cdot \mathbf{e}$. Then, the H-invariance of the distance d gives

$$y \in B_1(h \cdot \mathbf{e}) = h \cdot B_1(\mathbf{e}) \iff h^{-1} \cdot y \in B_1(\mathbf{e}). \tag{2.40}$$

Moreover, we claim that the following homogeneity property is true:

$$\frac{\Delta_k(y)}{\Delta_k(h \cdot \mathbf{e})} = \frac{\Delta_k(h^{-1} \cdot y)}{\Delta_k(\mathbf{e})}, \quad \forall\, h \in H, \tag{2.41}$$

which combined with (2.40) and the previous case will gives us (2.39). Now, property (2.41) for the determinant (i.e., $k = r$) is obvious from (2.32). In the case of principal minors we shall only prove it for $\Omega = \mathrm{Sym}_+(r, \mathbf{R})$. Use the identity $H = NA$ to write $h = na$, where $n \in N$ is a lower triangular $r \times r$-matrix, and $a = \mathrm{Diag}\,(\lambda_1, \dots, \lambda_r)$ is diagonal with positive eigenvalues. Now, $h^{-1} = a^{-1}n^{-1}$, with $n^{-1} \in N$ and $a^{-1} = \mathrm{Diag}\,(\lambda_1^{-1}, \dots, \lambda_r^{-1})$. Thus, using properties (2.29) to (2.32) we conclude with:

$$\Delta_k(h^{-1} \cdot y) = \lambda_1^{-2} \cdots \lambda_k^{-2}\, \Delta_k(n^{-1} \cdot y) = \frac{\Delta_k(y)}{\lambda_1^2 \cdots \lambda_k^2} = \frac{\Delta_k(y)}{\Delta_k(na \cdot \mathbf{e})}.$$

□

(3) *The invariant measure in a cone.*

We used in the proof of Theorem 2.27 the existence of an H-invariant measure in Ω, that is, a measure $\tilde{\mu}$ satisfying $\tilde{\mu}(h \cdot E) = \tilde{\mu}(E)$ for all $h \in H$ and $E \subset \Omega$. We obtained this measure from a left-Haar measure on the group H. In this section we use the determinant function to obtain an explicit expression for a G-invariant measure in Ω.

Proposition 2.42. *Let Ω be a symmetric cone. Consider the measure in Ω:*

$$\mu(E) = \int_E \frac{dy}{\Delta(y)^{\frac{n}{r}}}, \quad E \subset \Omega.$$

Then μ is G-invariant, i.e., $\mu(g \cdot E) = \mu(E)$ for all $g \in G$.

Proof: This measure is locally finite (over Ω) since $\Delta(y)$ is bounded above and below on compact sets of Ω. For the G-invariance, just perform a change of variables and use property (2.32). □

Corollary 2.43. *Let Ω be a symmetric cone and $B_1(y_0)$ an invariant ball centered at $y_0 \in \Omega$. Then,*

$$|B_1(y_0)| \sim \Delta(y_0)^{\frac{n}{r}},$$

where $|\cdot|$ denotes the Lebesgue measure and the constants in "$\sim$" depend only on Ω.

Proof: Write $y_0 = h \cdot \mathbf{e}$, for some $h \in H$. Then, using Theorem 2.38 and the G-invariance of μ we get:

$$\begin{aligned} |B_1(y_0)| &= \int_{B_1(y_0)} dy \sim \Delta(y_0)^{\frac{n}{r}} \int_{B_1(y_0)} \frac{dy}{\Delta(y)^{\frac{n}{r}}} \\ &= \Delta(y_0)^{\frac{n}{r}}\, \mu(B_1(y_0)) = \Delta(y_0)^{\frac{n}{r}}\, \mu(B_1(\mathbf{e})) = c\, \Delta(y_0)^{\frac{n}{r}}. \end{aligned}$$

□

Finally, it is well-known that, in a riemannian manifold, Euclidean balls and riemannian balls centered at a fixed point are comparable when the radii are small enough. That is, there are constants $0 < c_1 < c_2 < \infty$, depending only on Ω, such that

$$\{x \in \mathbf{R}^n \mid |x - \mathbf{e}| < c_1\delta\} \subset B_\delta(\mathbf{e}) \subset \{x \in \mathbf{R}^n \mid |x - \mathbf{e}| < c_2\delta\},$$

for all $\delta \in (0, 1]$ (see, e.g., [19, §9.22]). This, and the properties of the invariant measure, imply that

$$\mu(B_\delta(y_0)) = \mu(B_\delta(\mathbf{e})) \sim \Delta(\mathbf{e})^{-\frac{n}{r}} \int_{B_\delta(\mathbf{e})} dy \sim \delta^n.$$

Thus, a repetition of the proof of Theorem 2.27 yields the following:

Corollary 2.44. *Let $0 < \delta \le 1$ be fixed and $\{\xi_j\}$ be a δ-lattice in Ω. Then, there exists a constant N* depending only on Ω, *such that every point in Ω belongs to al most N balls of the family $\{B_{3\delta}(\xi_j)\}$.*

(4) *Trace and inner product.*

Let Ω be a symmetric cone in V with inner product $(\cdot|\cdot)$. Let $\mathbf{e}$ be the "identity point" defined in Theorem 2.15. We define the *trace* of a vector $y \in V$ (associated with $\{\Omega, \mathbf{e}, (\cdot|\cdot)\}$) by:

$$\mathrm{tr}\ (y) = (y|\mathbf{e}).$$

Observe that this apparently obvious definition extends the usual one in the case of symmetric matrices. For the light-cone, tr $(y) = y_1$.

The main result for this function states that, as it happens with the principal minors, the trace remains essentially constant within invariant balls. In fact, we have a stronger result:

THEOREM 2.45. *Let Ω be a symmetric cone. Then, there exists a constant $C = C(\Omega) \geq 1$ such that, for all $\xi \in \Omega$,*

$$\frac{1}{C} \leq \frac{(y|\xi)}{(y_0|\xi)} \leq C, \quad \textit{whenever } y \in B_1(y_0). \tag{2.46}$$

PROOF: Assume first $y_0 = \mathbf{e}$. The proof then is easy, since the inner product $(y|\xi)$ is a positive and continuous function when $y, \xi \in \Omega$ (by self-duality). Thus, restricted to the compact set $\overline{B_1(\mathbf{e})} \times \{\xi \in \Omega \mid |\xi| = 1\}$ this function of two variables is between two positive constants C_1 and C_2. Replacing ξ by $\xi/|\xi|$ in numerator and denominator of (2.46), we establish the theorem for $y_0 = \mathbf{e}$.

For the general case, just write $y_0 = h \cdot \mathbf{e}$, for $h \in H$, and notice that

$$\frac{(y|\xi)}{(y_0|\xi)} = \frac{(h^{-1} \cdot y|h^* \cdot \xi)}{(\mathbf{e}|h^* \cdot \xi)}.$$

Then, one concludes easily using the first case and (2.40) (we are also using that $\eta = h^* \cdot \xi \in \Omega$; see Exercise 2.9). □

REMARK 2.47. We point out that, although the above definition is convenient for us, in the theory of Jordan algebras the trace typically appears with a different definition, independent of the inner product of the underlying vector space (see [11, p. 29]). In fact, it is precisely from such definition of trace how one chooses a "distinguished" inner product in a Jordan algebra (see [11, pp. 37,51]). A clever reader will find out the reason for using the "natural" inner product Tr (XY) in the space $\mathrm{Sym}(r, \mathbf{R})$ (see §2.1).

(5) *The inverse transformation of the cone.*

Let Ω be a symmetric cone and H the subgroup of $G(\Omega)$ for which $\Omega = H \cdot \mathbf{e}$ (according to Theorem 2.15). Then, for every point of the cone $y = h \cdot \mathbf{e}$, $h \in H$, we define its *inverse* by:

$$y^{-1} = (h^*)^{-1} \cdot \mathbf{e}. \tag{2.48}$$

Observe that y^{-1} belongs also to the cone (since $G^* = G$), and moreover $(y^{-1})^{-1} = y$. To verify the latter, take any $k \in K = G_{\mathbf{e}} = G \cap O(n)$ such that $\tilde{h} = (h^*)^{-1} k \in H$ (recall that $G = HK$). Then, using that $k^* = k^{-1}$ we conclude:

$$(y^{-1})^{-1} = (\tilde{h} \cdot \mathbf{e})^{-1} = (\tilde{h}^*)^{-1} \cdot \mathbf{e} = (hk) \cdot \mathbf{e} = h \cdot \mathbf{e} = y.$$

Observe also the following property of the determinant:

$$\Delta(y^{-1}) = \Delta(y)^{-1}, \quad y \in \Omega, \tag{2.49}$$

which follows easily from (2.48) and (2.32).

Again, this notion of inverse extends the usual one for positive-definite symmetric matrices. Indeed, for one such matrix $Y = h \cdot I = hIh^*$, we have

$$Y^{-1} = (h^*)^{-1} I h^{-1} = (h^*)^{-1} \cdot I.$$

In the case of the light-cone in $\mathbf{R}^3$, a direct computation gives:

$$y^{-1} = \tfrac{1}{\Delta(y)} \, (y_1, -y_2, -y_3), \quad y \in \Lambda_3.$$

In the higher dimensional Lorentz cone Λ_n, the same formula holds replacing the vector by $(y_1, -y_2, \ldots, -y_n)$.

The main property of the inverse transformation, for the purposes of these notes, is stated in the following theorem.

Theorem 2.50. *Let Ω be a symmetric cone, and $(H, \mathbf{e})$ be as in Theorem 2.15. Then, the transformation*

$$y \in \Omega \longmapsto \mathcal{I}(y) = y^{-1} \in \Omega,$$

is an involute isometry in Ω. I.e., $\mathcal{I}^2(y) = y$ and $d(y^{-1}, y_0^{-1}) = d(y, y_0)$, for all $y, y_0 \in \Omega$.

Proof: The involution was already shown above. We will prove the isometry only for the cone $\Omega = \mathrm{Sym}_+(r, \mathbf{R})$ of positive-definite symmetric matrices. In this case we have $\mathcal{I}(Y) = Y^{-1}$, with the usual inverse. By definition of the distance, being an isometry is equivalent to

$$\mathcal{G}_{\mathcal{I}(Y)} \left(D_Y \mathcal{I}\,[\xi], D_Y \mathcal{I}\,[\eta] \right) = \mathcal{G}_Y \left(\xi, \eta \right), \quad \forall\, \xi, \eta \in \mathrm{Sym}(r, \mathbf{R}). \tag{2.51}$$

Now, it is a classical exercise in algebra to compute the differential of $\mathcal{I}$, which equals

$$D_Y \mathcal{I}\,[\xi] = -Y^{-1} \xi Y^{-1}, \quad \xi \in \mathrm{Sym}(r, \mathbf{R}).$$

Thus, if $Y = h \cdot I = hh^*$, then $Y^{-1} = (h^*)^{-1} \cdot I = ((h^*)^{-1} k) \cdot I$, for some $k \in G_I = G \cap O(\mathrm{Sym}(r, \mathbf{R}))$ such that the matrix in parenthesis belongs to H. Then, by definition of the metric $\mathcal{G}$:

$$\begin{aligned}
\mathcal{G}_{\mathcal{I}(Y)} \left(D_Y \mathcal{I}\,[\xi], D_Y \mathcal{I}\,[\eta] \right) &= \mathcal{G}_{(h^*)^{-1} k \cdot I} \left(Y^{-1} \xi Y^{-1}, Y^{-1} \eta Y^{-1} \right) \\
&= \langle\, (k^{-1} h^*) \cdot (Y^{-1} \xi Y^{-1}) \,,\, (k^{-1} h^*) \cdot (Y^{-1} \eta Y^{-1}) \,\rangle \\
&= \langle\, h^* \cdot ((h^*)^{-1} h^{-1} \xi (h^*)^{-1} h^{-1}) \,,\, h^* \cdot ((h^*)^{-1} h^{-1} \eta (h^*)^{-1} h^{-1}) \,\rangle \\
&= \langle\, h^{-1} \xi (h^{-1})^* \,,\, h^{-1} \eta (h^{-1})^* \,\rangle \;=\; \mathcal{G}_{h \cdot I}(\xi, \eta).
\end{aligned}$$

□

A simple consequence of the previous, which we shall use often below, is the identity: $B_1(y_0^{-1}) = (B_1(y_0))^{-1}$. In particular, we have the following:

Corollary 2.52. *Let Ω be a symmetric cone and $\{\xi_j\}_{j=1}^\infty$ a 1-lattice as in* §2.4. *Then the sequence $\{\xi_j^{-1}\}_{j=1}^\infty$ is also a 1-lattice, and moreover it holds*

$$\frac{1}{C} \leq (y|\xi) \leq C, \quad \forall y \in B_1(\xi_j^{-1}),\ \xi \in B_1(\xi_j),$$

for a constant $C = C(\Omega) > 0$. The sequence $\{\xi_j^{-1}\}$ is called the dual lattice *of $\{\xi_j\}$.*

Proof: Simple exercise using the definition of 1-lattice and Theorems 2.45 and 2.50. □

2.6. Two remarks on Jordan algebras and symmetric spaces.

Remark 2.53. We point out that the inverse transformation is very deeply related with the *algebraic structure* of symmetric cones, and their underlying vector spaces. The definition we gave above (using the group H) is good enough for our purposes, but it is not the usual way to introduce it in the literature. For example, in the vector space of symmetric matrices $V = \mathrm{Sym}(r, \mathbf{R})$ there is a *product composition law* for which Y^{-1} is an algebraic inverse element. This is not the usual matrix product (which does not preserve V) but the so-called *symmetric product*:

$$X \circ Y = \frac{XY + YX}{2}, \quad X, Y \in \mathrm{Sym}(r, \mathbf{R}).$$

Observe this product is commutative, but not associative! Also, the identity matrix $I = I_r$ is a neutral element for "$\circ$", and any invertible matrix X in the usual sense is also invertible for "$\circ$", being $X{-}1$ an inverse element. The inverse element for $\circ$, however, may not be unique[2] in V, but it will be unique in the subspace of polynomials $\mathcal{P}[X] = \mathrm{span}\ \{I, X, X^2, \ldots\}$ (see [11, p. 30]). This is therefore the right definition of $\circ$-inverse, which whenever it exists, coincides with the usual X^{-1}. Finally, it is important to notice that from the inverse operation one can recover the cone $\Omega = \mathrm{Sym}_+(r, \mathbf{R})$, as the identity component of the set of invertible (or $\circ$-invertible in the above sense) elements in V.

These properties are not restricted to symmetric matrices, and in general there is a deep theorem (due to Vinberg) stating that *the underlying vector space of a symmetric cone can be endowed with a commutative (but not associative) product for which Ω is the identity component of the set of invertible elements in V* (see Theorem III.3.1 in [11]). The product law obtained in this theorem satisfies the axioms of a *Jordan product*, and with it the vector space V becomes a *Euclidean Jordan algebra*. It is from this important theorem how

[2]Consider the simple example, $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & a \\ a & -1 \end{pmatrix}$, for which $X \circ X = X \circ Y = I$ (see, [11, p. 30]).

one can classify all irreducible symmetric cones (see Remark 2.22), and the reason why Jordan algebras enter into play to understand this theory. The reader wishing to learn more on this topic is encouraged to read the first eight chapters of the text [11].

Remark 2.54. There is yet another approach to the inverse transformation arising from riemannian geometry, and which avoids completely the use of Jordan algebras. Roughly speaking, the approach is the following: associated with a self-dual open convex cone Ω there is a positive function

$$\phi(x) = \int_\Omega e^{-(x|y)}\, dy, \quad x \in \Omega,$$

called *the characteristic function of* Ω. It can be shown that the function $\log \phi$ is strictly convex (i.e., the second derivative $D^2 \log \phi$ is positive definite), and thus it defines a riemannian metric in Ω by:

$$\widetilde{\mathcal{G}}_x(\xi, \eta) = D_\xi D_\eta \log \phi(x), \quad \xi, \eta \in V.$$

Associated with ϕ, one can also define an involution in Ω by:

$$x^\star = -\nabla \log \phi(x), \quad x \in \Omega,$$

with the property that $x \to x^\star$ has unique fixed point (say $\mathbf{e} \in \Omega$) and $(x|x^*) = n$. With these definitions Ω becomes a *riemannian symmetric space*, and the involution $x \mapsto x^\star$ an isometry for $\widetilde{\mathcal{G}}$. One can show that this approach is equivalent to the previous one, and in fact, $y^\star = \frac{n}{r}\, y^{-1}$ and $\widetilde{\mathcal{G}} = \frac{n}{r}\, \mathcal{G}$ (see [11, pp. 15,58]). To learn more on this approach the reader can consult Chapter I of [11].

3. Weighted Bergman spaces on tube domains over symmetric cones

In this section we extend to several complex variables the results proved in the first four paragraphs of Section 1 for the upper half-plane. Thus, we establish an appropriate analytic setting where Bergman spaces and Bergman projectors can be studied, introducing the right concepts with which a Paley-Wiener Theorem can be proved. The structure of symmetric cones is exploited in two ways: first because they constitute domains of positivity (i.e., have a partial order), providing us with the right properties for Hardy-type norms; second because of the group action and the homogeneity of determinants, which lead easily to an explicit expression for the Bergman kernel. Many of these results are known in the literature, and we refer to Chapter III of [23] for results dealing with Hardy spaces, and to Chapters IX and $XIII$ of [11] for the L^2 theory of Bergman spaces. We also give a detailed account of (weighted) mixed norm Bergman spaces, which appear in some papers of the authors (see, e.g., [5]).

3.1. Weighted Bergman spaces and weighted Bergman kernels.

In the sequel, we shall assume that $\Omega \subset V$, with V of dimension $n \geq 3$, is an *irreducible symmetric cone*, that is, a symmetric cone which is not linearly equivalent to the product of at least two lower-dimensional symmetric cones. We denote by r the rank of the cone Ω.

The following property defines a crucial element in the analysis of symmetric cones: *the Gamma function of* Ω. We refer to Chapter VII of [11] for more properties on it.

Proposition 3.1. ([11], Corollary VII.1.3)

(1) *For* $\lambda \in \mathbf{R}$*, the integral*

$$\Gamma_\Omega(\lambda) = \int_\Omega e^{-(x|\mathbf{e})} \Delta(x)^{\lambda - \frac{n}{r}} dx$$

converges if and only if $\lambda > \frac{n}{r} - 1$*. In this case, if* $d = \frac{2(\frac{n}{r}-1)}{r-1}$*, we have*

$$\Gamma_\Omega(\lambda) = \pi^{\frac{n}{r}-1}\Gamma(\lambda)\Gamma(\lambda - \tfrac{d}{2})...\Gamma(\lambda - (r-1)\tfrac{d}{2}),$$

where $\Gamma(x)$ *denotes the usual Euler gamma function.*

(2) *For* y *in* Ω *and* $\lambda > \frac{n}{r} - 1$ *we have*

$$\int_\Omega e^{-(x|y)} \Delta(x)^{\lambda - \frac{n}{r}} dx = \Gamma_\Omega(\lambda)\Delta(y)^{-\lambda}.$$

We refer to [11] for the proof of (1). Let us remark that (2) can be obtained from (1) by a change of variables which maps $\mathbf{e}$ to y.

Even if we shall not use it right now, we write the generalization of this proposition to generalized powers of the Determinant function.

Proposition 3.2. *For* $y \in \Omega$ *and* $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbf{C}^r$ *with* $\Re e\, s_j > (j-1)\frac{n/r-1}{r-1}$, $j = 1, \ldots, r$*, then*

$$\int_\Omega e^{-(\xi|y)}\, \Delta^{\mathbf{s}}(\xi)\, \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} = \Gamma_\Omega(\mathbf{s})\, \Delta^{\mathbf{s}}(y^{-1}).$$

Moreover, this integral converges absolutely if and only if the condition on $\mathbf{s}$ *is satisfied.*

We refer to [11] for the explicit value of the constant, $\Gamma_\Omega(\mathbf{s})$. Let us remark that the condition on $\mathbf{s}$ is the condition for local integrability of $\Delta^{\mathbf{s}}$ relatively to the invariant measure. y^{-1} is the inverse of y, which has been defined in the previous section.

In the sequel, we shall call $T_\Omega = V + i\Omega$ the *tube domain with base* Ω *in the complexified vector space* $V + iV$.

Remark 3.3. From Proposition 3.1 (2), for $\lambda > \frac{n}{r} - 1$ fixed and $z = x + iy \in T_\Omega$, it follows that the integral

$$z \longmapsto \frac{1}{\Gamma_\Omega(\lambda)} \int_\Omega e^{-(\xi | \frac{z}{i})} \Delta(\xi)^{\lambda - \frac{n}{r}} d\xi$$

is absolutely convergent and defines a holomorphic function in the tube domain T_Ω. This holomorphic function is an extension of the function $\Delta(y)^{-\lambda}$ defined on Ω and so we shall denote it by $\Delta^{-\lambda}\left(\frac{z}{i}\right)$.

Corollary 3.4. *Let $\lambda > \frac{n}{r} - 1$ be fixed. Then,*

$$\begin{array}{lll} (i) & \Delta(y + y') \geq \Delta(y), & \forall\, y, y' \in \Omega \\ (ii) & |\Delta^{-\lambda}((x+iy)/i)| \leq \Delta(y)^{-\lambda}, & \forall\, x \in \mathbf{R}^n,\ y \in \Omega. \end{array}$$

Proof: Immediate from the second part of the previous proposition. □

Let ν be a real number and $1 \leq p \leq \infty$. We shall denote by L^p_ν the weighted Lebesgue space $L^p(T_\Omega, \Delta^{\nu - \frac{n}{r}}(y) dx dy)$. We define the *Bergman space* $A^p = A^p(T_\Omega)$ as the subspace of $L^p = L^p_{\frac{n}{r}}$ consisting of holomorphic functions. We define the *weighted Bergman space* A^p_ν as the subspace of L^p_ν consisting of holomorphic functions. We write the norm as $\|\cdot\|_{A^p_\nu} = \|\cdot\|_{L^p_\nu}$.

We first state two basic properties of weighted Bergman spaces on tube domains over symmetric cones. The following result is the extension of Proposition 1.3 to several complex variables.

Proposition 3.5. *Let $p \in [1, \infty)$ and $\nu \in \mathbf{R}$. Then, the following properties hold.*

(i) *There exists a constant $C = C(p, \nu) > 0$ such that for all $x + iy \in T_\Omega$ and for all $F \in A^p_\nu$,*

$$|F(x+iy)| \leq C \Delta^{-\frac{\nu + \frac{n}{r}}{p}}(y) \|F\|_{A^p_\nu}.$$

(ii) *There exists a constant $C = C(p, \nu) > 0$ such that for all $y \in \Omega$ and for all $F \in A^p_\nu$, we have*

$$\|F(\cdot + iy)\|_p \leq C \Delta^{-\frac{\nu}{p}}(y) \|F\|_{A^p_\nu}.$$

Proof: The weighted Bergman space A^p_ν is invariant through translations and automorphisms of the cone Ω. Then it suffices to prove that for all $F \in A^p_\nu$,

$$F(i\mathbf{e}) \leq C \|F\|_{A^p_\nu}$$

and

$$\|F(\cdot + i\mathbf{e})\|_p \leq C \|F\|_{A^p_\nu}.$$

These follow using the mean value property in the same way as in the proof of the analogous results in one variable (Proposition 1.3). □

We are linked to use Hardy spaces, as in the one-dimensional case. Let us give their definitions and first properties.

DEFINITION 3.6. For $p \in [1,\infty)$, the Hardy space $H^p = H^p(T_\Omega)$ is the space of holomorphic functions on T_Ω which satisfy the estimate

$$\|F\|_{H^p} = \sup_{y\in\Omega}\left\{\int_{\mathbf{R}^n} |F(x+iy)|^p dx\right\}^{\frac{1}{p}}.$$

We have the analogue of the main theorem in the one-dimensional case.

THEOREM 3.7. (1) *Given $F \in H^p$, the function*

$$y \in \Omega \mapsto \|F(\cdot + iy)\|_p$$

in non-increasing in the sense of the partial ordering on Ω defined in (3.9). Moreover, for every $t \in \Omega$,

$$\lim_{y\to 0, y\in\Omega}\int_{\mathbf{R}^n} |F(x+i(y+t)) - F(x+it)|^p dx = 0.$$

(2) *Given $F \in A^{p,q}_\nu$, then for every $t \in \Omega$, the function $F_t(z) = F(z+it)$ is in the Hardy space H^s for every $s \geq p$.*

PROOF: For the proof (1), see [23, Th. III.5.6]. Assertion (2) is a consequence of Proposition 3.22 (2). □

Let us give a first application. We prove, as in the upper-half plane, that the space is reduced to 0 when the weight is not locally integrable, that is $\nu \leq \frac{n}{r} - 1$. The proof is due to to Daniele Debertol.

PROPOSITION 3.8. *Let $1 \leq p < \infty$. Then, for all $\nu \leq \frac{n}{r} - 1$ we have $A^p_\nu = \{0\}$.*

PROOF: Assume first that $0 \leq \nu \leq \frac{n}{r} - 1$. Then by part (ii) of Proposition 3.5 and part (i) of Corollary 3.4, for every $F \in A^p_\nu$, the function $G(z) = F(z + i\mathbf{e})$ belongs to the Hardy space H^p on tube domain T_Ω (for the definition and basic properties of Hardy spaces, see §3.3 below). Therefore, by Theorem 3.7, the function

$$y \in \Omega \longmapsto g(y) = \int_V |G(x+iy)|^p dx$$

is non-increasing with respect to the partial ordering $\prec$ of the cone; that is

$$x \prec y \quad \text{iff} \quad y - x \in \Omega. \tag{3.9}$$

Then, $\|F(\cdot + iy)\|_p \geq \|F(\cdot + i(y+\mathbf{e}))\|_p$, and therefore

$$\begin{aligned}\|F\|^p_{A^p_\nu} \geq \|G\|^p_{A^p_\nu} &= \int_\Omega g(y)\Delta^{\nu-\frac{n}{r}}(y)dy \\ &\geq \int_{\substack{y\prec\mathbf{e}\\ y\in\Omega}} g(y)\Delta^{\nu-\frac{n}{r}}(y)dy \;\geq\; g(\mathbf{e})\int_{y\prec \underline{e}} \Delta^{\nu-\frac{n}{r}}(y)dy.\end{aligned}$$

Now, by Theorem VII.1.7 of [11], the latter integral is infinite when $\nu \leq \frac{n}{r} - 1$. Since $\|F\|^p_{A^p_\nu} < \infty$, we conclude that $g(\mathbf{e}) = 0$ and as a consequence, $g(y) = 0$

for every $y \in \Omega$ such that $\mathbf{e} \prec y$. This implies that G (and also F) is identically zero on T_Ω.

Assume next that $\nu < 0$. The result still follows because the function $H(z) = F(z)\Delta^{\frac{\nu}{p}}\left(\frac{z+i\mathbf{e}}{i}\right)$ belongs to A^p_0. □

As a consequence, *in the sequel we shall always assume that* $\nu > \frac{n}{r} - 1$. It follows from Proposition 3.5 (i) that for every $z \in T_\Omega$, the point evaluation linear functional $F \mapsto F(z)$ is continuous on A^p_ν. We can prove in the same way as in the 1-dimensional case that for $\nu > \frac{n}{r} - 1$ and $1 \le p < \infty$, A^p_ν is a Banach space (this is also valid for Hardy spaces). In particular, equipped with the inner product

$$\langle F, G\rangle = \int_\Omega F(z)\overline{G(z)}\Delta^{\nu-\frac{n}{r}}(y)dxdy, \quad z = x+iy,$$

A^2_ν is a Hilbert space. So, by the Riesz representation theorem, for every $z \in T_\Omega$, there exists a unique function $k_z \in A^2_\nu$ such that

$$F(z) = \langle F, k_z\rangle.$$

The kernel $B_\nu(z,\zeta) = \overline{k_z(\zeta)}$ is called *Bergman kernel* of T_Ω when $\nu = \frac{n}{r}$ and *weighted Bergman kernel* of T_Ω for all $\nu > \frac{n}{r} - 1$. Moreover, the orthogonal projector P_ν from the Lebesgue Hilbert space L^2_ν onto its closed subspace A^2_ν is called the *Bergman projector* of T_Ω when $\nu = \frac{n}{r}$, and the *weighted Bergman projector* of T_Ω for all other values of ν. It can be shown, in the same way as in Proposition 1.32, that P_ν is given by

$$P_\nu f(z) = \int_{T_\Omega} B_\nu(z, s+it)f(s+it)\Delta^{\nu-\frac{n}{r}}(t)dsdt \qquad (f \in L^2_\nu). \tag{3.10}$$

We shall adopt the notation $L^2_{(\mu)}(\Omega) = L^2\left(\Omega, \Delta(\xi)^{-\mu}d\xi\right)$.

Theorem 3.11. (Paley-Wiener) *Let $\nu > \frac{n}{r} - 1$. Given $g \in L^2_{(\nu)}(\Omega)$, the formula*

$$F(z) = \int_\Omega e^{i(z|\xi)}g(\xi)d\xi, \quad z \in T_\Omega, \tag{3.12}$$

defines an element of A^2_ν; moreover,

$$\|F\|^2_{A^2_\nu} = C_\nu \|g\|^2_{L^2_\nu(\Omega)}, \tag{3.13}$$

with $C_\nu = (2\pi)^n\,\Gamma_\Omega(\nu)2^{-r\nu}$.

Conversely, given $F \in A^2_\nu$, (3.12) and (3.13) hold for some $g \in L^2_{(\nu)}(\Omega)$.

Proof: The proof of this theorem is very similar to its counterpart in dimension 1 (Theorem 1.22). We first prove the direct part. The integral on the right-hand side of (3.12) is absolutely convergent, because by Schwarz's inequality, for $z = x + iy$,

$$\begin{aligned}\int_\Omega |e^{i(z|\xi)} g(\xi)| d\xi &= \int_\Gamma \left(e^{-(y|\xi)}\Delta^{\frac{\nu}{2}}(\xi)\right)\left(|g(\xi)|\Delta^{-\frac{\nu}{2}}(\xi)\right) d\xi \\ &\leq \left(\int_\Omega e^{-2(y|\xi)}\Delta^\nu(\xi)d\xi\right)^{\frac12}\left(\int_\Omega |g(\xi)|^2\Delta^{-\nu}(\xi)d\xi\right)^{\frac12} \\ &= \left(\Gamma_\Omega(\nu+\tfrac{n}{r})\,\Delta^{-(\nu+\frac{n}{r})}(2y)\right)^{\frac12}\|g\|_{L^2_\nu(\Omega)} < \infty.\end{aligned}$$

The latter equality follows by Proposition 3.1 since $\nu > -1$.

To prove (3.13) we use the Plancherel formula to obtain the equality

$$\int_{\mathbf{R}^n} |F(x+iy)|^2 dx = (2\pi)^n \int_\Omega e^{-2(y|\xi)}|g(\xi)|^2 d\xi.$$

Moreover, by Fubini's theorem and Proposition 3.1, we get

$$\begin{aligned}\|F\|^2_{A^2_\nu} &= \int_\Omega\left(\int_{\mathbf{R}^n}|F(x+iy)|^2dx\right)\Delta^{\nu-\frac{n}{r}}(y)dy \\ &= (2\pi)^n\int_\Omega\left(\int_\Omega e^{-2(y|\xi)}|g(\xi)|^2d\xi\right)\Delta^{\nu-\frac{n}{r}}(y)dy \\ &= (2\pi)^n\int_\Omega|g(\xi)|^2\left(\int_\Omega e^{-2(y|\xi)}\Delta^{\nu-\frac{n}{r}}(y)dy\right)d\xi \\ &= C_\nu\int_\Omega|g(\xi)|^2\Delta^{-\nu}(\xi)d\xi = C_\nu\|g\|^2_{L^2_\nu(\Omega)}.\end{aligned}$$

We prove the converse part as in the one-dimensional case, using the corresponding Paley-Wiener for the Hardy space $H^2(T_\Omega)$. Basically, once we know that functions in $H^2(T_\Omega)$ may be written as in (**??**), with g a square integrable function which is supported in Ω, the proof is exactly the same. This is an easy consequence of the following result, which is well known.

Lemma 3.14. (Theorem III.2.3 of [23].) *Let B be an open connected subset of $\mathbf{R}^n$ and let T_B denote the tube domain over B. Then for every function F in the Hardy space $H^2(T_B)$, there exists a measurable $g : \mathbf{R} \to \mathbf{C}$ satisfying the estimate*

$$\sup_{y\in B}\int_{\mathbf{R}^n} e^{-2(y|\xi)}|g(\xi)|^2d\xi < \infty,$$

such that for every $z \in T_B$

$$F(z) = \int_{\mathbf{R}^n} e^{iz\cdot\xi}g(\xi)d\xi.$$

Indeed, when B is Ω, the integrability condition forces g to vanish outside Ω, letting y tend to infinity, with $(y|\xi) < 0$. □

From Theorem 3.11, we obtain an explicit expression for the weighted Bergman kernel B_ν in the tube T_Ω.

THEOREM 3.15. *The weighted Bergman kernel B_ν of T_Ω is given by*

$$B_\nu(w,z) = d_\nu\, \Delta\left(\frac{w-\bar z}{i}\right)^{-\nu-\frac{n}{r}}$$

with $d_\nu = C_\nu^{-1}\Gamma(\nu+\frac{n}{r})$.

Here, $\Delta\left(\frac{w-\bar z}{i}\right)^{-\nu-\frac{n}{r}}$ is the determination of the power defined in Remark 3.3.

PROOF: Since $z \in T_\Omega$ the functions F and $B_\nu(.,z)$ are in A^2_ν. By the Paley-Wiener theorem (Theorem 3.11), there exist two functions $g, g_z \in L^2_\nu(\Omega)$ such that for $w \in T_\Omega$

$$F(w) = \int_\Omega e^{i(w|\xi)} g(\xi) d\xi \tag{3.16}$$

and

$$B_\nu(w,z) = \int_\Omega e^{i(w|\xi)} g_z(\xi) d\xi.$$

The polarization of the isometry (3.13) gives

$$F(z) = \langle F, B_\nu(\cdot,z)\rangle_{A^2_\nu} = C_\nu \langle g, g_z\rangle_{L^2_\nu(\Omega)} \;=\; C_\nu \int_\Omega g(\xi)\overline{g_z(\xi)}\Delta^{-\nu}(\xi)d\xi. \tag{3.17}$$

Comparing (3.16) and (3.17) implies

$$g_z(\xi) = C_\nu^{-1} e^{-i\bar z\cdot\xi}\Delta^\nu(\xi).$$

Therefore by Remark 3.3,

$$B_\nu(w,z) = C_\nu^{-1}\int_\Omega e^{i(w|\xi)} e^{-i(\bar z|\xi)}\Delta^\nu(\xi)d\xi = d_\nu\,\Delta\left(\frac{w-\bar z}{i}\right)^{-\nu-\frac{n}{r}}. \tag{3.18}$$

□

3.2. Mixed norm weighted Bergman spaces. Our main interest is the study of the three problems that we have completely solved in the one-dimensional case. We will see later that the solution is simpler in different spaces, which are part of a larger family of Bergman spaces. This is why we enlarge our class of spaces, by introducing mixed norms. For $1 \le p, q \le \infty$, let

$$L^{p,q}_\nu = L^q_\nu\left(\Omega, \Delta(y)^{\nu-\frac{n}{r}}dy; L^p(\mathbf{R}^n, dx)\right)$$

be the space of functions $F(x+iy)$ on T_Ω such that

$$\|F\|_{L^{p,q}_\nu} = \left(\int_\Omega \|F(\cdot+iy)\|_p^q \Delta(y)^{\nu-\frac{n}{r}}dy\right)^{\frac{1}{q}} < \infty$$

(with the obvious modification if $p = \infty$). We call $A^{p,q}_\nu$ the closed subspace of $L^{p,q}_\nu$ consisting of holomorphic functions. These spaces will be called *mixed norm weighted Bergman spaces.* For $p = q$, we have $L^{p,p}_\nu = L^p_\nu$ and $A^{p,p}_\nu = A^p_\nu$.

Before proceeding further, we give some examples of functions in $A^{p,q}_\nu$. Given $\beta \in \mathbf{R}$, we denote by $\Delta^\beta(\frac{x+iy}{i})$ the holomorphic determination of the $\beta-$ power which reduces to the function $\Delta^\beta(y)$ when $x = 0$. To illustrate our examples we need the following lemma, which also defines beta functions on the cone.

Lemma 3.19. ([11]) *For α, β real, the integral*

$$I_{\alpha,\beta}(t) = \int_\Omega \Delta^\alpha(y+t)\Delta^\beta(y)dy$$

is convergent if and only if $\beta > -1$ and $\alpha + \beta < -\frac{2n}{r} + 1$. In this case, $I_{\alpha,\beta}(t) = C_{\alpha,\beta}\Delta^{\alpha+\beta+\frac{n}{r}}(t)$.

Proof: We shall show that this is an easy consequence of Proposition 3.1. The condition $\beta > -1$ is necessary for the local integrability. Since $\Delta(y+t)$ is bounded below for fixet t, it is sufficient to restrict to the case when $\alpha < -\frac{n}{r}+1$. Then, we can write

$$\Delta(y+t) = c\int_\Omega e^{-(y+t|\xi)}\Delta(\xi)^{-\alpha-\frac{n}{r}}d\xi.$$

Using Fubini Theorem, and integrating first in y, we have to consider the integral

$$\int_\Omega e^{-(t|\xi)}\Delta(\xi)^{-\alpha-\beta-\frac{2n}{r}}d\xi.$$

The necessary and sufficient condition on $\alpha + \beta$ is given in Proposition 3.1, which allows to conclude easily.

Lemma 3.20. *Let $\alpha \in \mathbf{R}$. Then,*

(1) *the integral*

$$J_\alpha(y) = \int_{\mathbf{R}^n}\left|\Delta^{-\alpha}\left(\frac{x+iy}{i}\right)\right|dx \tag{3.21}$$

converges if and only if $\alpha > \frac{2n}{r} - 1$. In this case, $J_\alpha(y) = C_\alpha\Delta^{-\alpha+\frac{n}{r}}(y)$, where $C_\alpha = \left[2^{-\alpha r+n}\Gamma_\Omega(\alpha - \frac{n}{r})\right]\left[\Gamma_\Omega(\frac{\alpha}{2})\right]^{-2}$.

(2) *The function $F(z) = \Delta^{-\alpha}\left(\frac{z+it}{i}\right)$, with $t \in \Omega$, belongs to $A^{p,q}_\nu$ if and only if $\alpha > max\left(\frac{\frac{2n}{r}-1}{p}, \frac{n}{rp} + \frac{\nu+\frac{n}{r}-1}{q}\right)$. In this case,*

$$\|F\|_{A^{p,q}_\nu} = C_{\alpha,p,q}\Delta^{-\alpha q+\frac{nq}{rp}+\nu}(t).$$

Proof: (1) Interpret the integral in (3.21) as the L^2 norm of $\Delta^{-\frac{\alpha}{2}}\left(\frac{\cdot+iy}{i}\right)$. By Proposition 3.1, Remark 3.3 and the Plancherel formula, the integral $J_\alpha(y)$ is finite if and only if $\alpha > \frac{2n}{r} - 1$.

(2) The conclusion follows from part (1) and Lemma 3.19. □

We record the following extension of Proposition 3.5 to mixed norm Bergman spaces. The proof is the same. The reader will observe that we use the invariance of the spaces under the action of translations in x, and the action of the group G.

Proposition 3.22. *Let $p, q \in [1, \infty)$ and $\nu > \frac{n}{r} - 1$.*

(1) There exists a positive constant $C = C(p, q, \nu)$ such that for all $x + iy \in T_\Omega$ and for all $F \in A^{p,q}_\nu$,

$$|F(x+iy)| \leq C\Delta^{-\frac{n}{rp}-\frac{\nu}{q}}(y)\|F\|_{A^{p,q}_\nu}.$$

(2) Let $F \in A^{p,q}_\nu$. For $y \in \Omega$, the function $F(\cdot + iy)$ belongs to $L^s(\mathbf{R}^n)$ for all $s \geq q$. Moreover, there exists a positive constant $C = C(p, q, s, \nu)$ such that for all $y \in \Omega$,

$$\|F(\cdot + iy)\|_s \leq C\Delta^{-\frac{\nu}{q}-\frac{n}{r}(\frac{1}{p}-\frac{1}{s})}(y)\|F\|_{A^{p,q}_\nu}.$$

We pass now to the density theorem. For the particular case of the Lorentz cone, this was proved in [5], Corollary 4.5.

Theorem 3.23. *For all $p, q, \rho, \sigma \in [1, \infty)$ and $\mu, \nu > \frac{n}{r} - 1$, the subspace $A^{p,q}_\nu \cap A^{\rho,\sigma}_\mu$ is dense in the space $A^{p,q}_\nu$.*

Proof: Let $F \in A^{p,q}_\nu$. Given $\alpha \geq 0$ and $\epsilon > 0$, let

$$F_{\epsilon,\alpha}(z) = F(z + i\epsilon\mathbf{e})\Delta^{-\alpha}\left(\frac{\epsilon z + i\mathbf{e}}{i}\right).$$

We claim that:

(1) $F_{\epsilon,\alpha} \in A^{p,q}_\nu$ with $\|F_{\epsilon,\alpha}\|_{A^{p,q}_\nu} \leq \|F\|_{A^{p,q}_\nu}$;
(2) $\lim_{\epsilon\to 0} \|F - F_{\epsilon,\alpha}\|_{A^{p,q}_\nu} = 0$;
(3) for α large enough, $F_{\epsilon,\alpha} \in A^{\rho,\sigma}_\nu$.

For claim (1), by Remark 3.3, observe that if $z = x + iy$,

$$|\Delta^{-\alpha}\left(\frac{\epsilon z + i\mathbf{e}}{i}\right)| \leq \Delta^{-\alpha}(\epsilon y + \mathbf{e}) \leq \Delta^{-\alpha}(\mathbf{e}) = 1. \tag{3.24}$$

The desired conclusion then follows because $\|F(\cdot + i\epsilon\mathbf{e})\|_{A^{p,q}_\nu} \leq \|F\|_{A^{p,q}_\nu}$.

For claim (2), using (3.24) and Theorem 3.7, we get

$$\|F(\cdot + iy) - F_{\epsilon,\alpha}(\cdot + iy)\|_p \leq 2\|F(\cdot + iy)\|_p.$$

On the other hand,

$$\begin{aligned}&\|F(\cdot + iy) - F_{\epsilon,\alpha}(\cdot + iy)\|_p\\ &\leq \|F(\cdot + iy) - F(\cdot + i(y + i\epsilon\mathbf{e})\|_p + \|F(\cdot + iy)\left(1 - \Delta^{-\alpha}\left(-i\epsilon(\cdot + iy) + \mathbf{e}\right)\right)\|_p.\end{aligned}$$

The first norm on the right-hand side tends to zero by assertions 1 and 2 of Theorem 3.7 and so does the second one by dominated convergence. Now,

$$\|F - F_{\epsilon,\alpha}\|_{A^{p,q}_\nu} = \int_\Omega \|F(\cdot + iy) - F_{\epsilon,\alpha}(\cdot + iy)\|_p^q \Delta^{\nu - \frac{n}{r}}(y)dy$$

which also tends to zero by dominated convergence.

Finally, to prove claim (3), first assume that $\rho \geq p$. Observe that if $\epsilon < 1$, then $\Delta(\epsilon y + \mathbf{e}) \geq \epsilon^r \Delta(y + \mathbf{e})$ and similarly for $\Delta(y + \epsilon\mathbf{e})$. By (3.24) and Proposition 3.22, there exists a positive number τ and a positive constant $C_{\epsilon,\alpha,\tau}$ such that

$$\begin{aligned}\|F_{\epsilon,\alpha}(\cdot + iy)\|_\rho &\leq \Delta^{-\alpha}(\epsilon y + \mathbf{e})\|F(\cdot + i(y + \epsilon\mathbf{e}))\|_\rho \\ &\leq C_{\epsilon,\rho}\Delta^{-(\alpha+\tau)}(y + \mathbf{e})\|F\|_{A^{p,q}_\nu}.\end{aligned}$$

Then

$$\|F_{\epsilon,\alpha}\|_{A^{\rho,\sigma}_\mu} \leq C_{\epsilon,\rho,\tau}\|F\|_{A^{p,q}_\nu}\left(\int_\Omega \Delta^{-(\alpha+\tau)\sigma}(y + \mathbf{e})\Delta^{\mu - \frac{n}{r}}(y)dy\right)^{\frac{1}{\sigma}}.$$

By Lemma 4.8 below, we can take α large enough so that the previous integral converges.

Next, if $\rho < p$, we use Hölder's inequality to obtain that

$$\|F_{\epsilon,\alpha}(\cdot + iy)\|_\rho \leq \|F(\cdot + i(y + \epsilon\mathbf{e})\|_p \left\|\Delta^{-\alpha}\left(-i\epsilon(\cdot + iy) + \mathbf{e}\right)\right\|_{\frac{p\rho}{p-\rho}}.$$

By Proposition 3.22,

$$\|F(\cdot + i(y + \epsilon\mathbf{e}))\|_p \leq C_\epsilon \Delta^{\frac{-\nu}{q}}(y + \mathbf{e})\|F\|_{A^{p,q}_\nu}$$

and by Lemma 3.20, if α is chosen large enough,

$$\left\|\Delta^{-\alpha}\left(-i\epsilon(\cdot + iy) + \mathbf{e}\right)\right\|_{\frac{p\rho}{p-\rho}} \leq C_\epsilon \Delta^{-\alpha + \frac{n(p-\rho)}{rp\rho}}(y + \mathbf{e}).$$

Therefore,

$$\|F_{\epsilon,\alpha}\|_{A^{\rho,\sigma}_\mu} \leq C_\alpha\left(\int_\Omega \Delta^{\left(-\alpha + \frac{n(p-\rho)}{rp\rho} - \frac{\nu}{q}\right)\sigma}(y + \mathbf{e})\Delta^{\mu - \frac{n}{r}}(y)dy\right)^{\frac{1}{\sigma}}$$

which again converges if α is large enough, by Lemma 4.8. $\square$

We intend now to show that the mixed norm Bergman spaces are simpler in the case when $p = 2$. The L^2 norm in the x variable can then be computed using Plancherel formula, and the geometric tools of the last section can be used.

First, recall that for $p = q = 2$, by the Paley-Wiener theorem (Theorem 3.11), $F \in A^{2,2}_\nu$ if and only if $F = \mathcal{L}g$, with $g \in L^2_\nu(\Omega)$. Here the Laplace transform $\mathcal{L}g$ of g is defined by

$$\mathcal{L}g(z) = \int_\Omega g(\xi) e^{iz\cdot\xi} d\xi.$$

Moreover,

$$\|F\|^2_{A^{2,2}_\nu} = C_\nu \int_\Omega |g(\xi)|^2 \Delta^{-\nu}(\xi) d\xi.$$

Using the dyadic decomposition of the cone Ω (Theorem 2.27), if we write $B_j = B_3(\xi_j)$, we have

$$\begin{aligned}\|F\|^2_{A^{2,2}_\nu} &= C_\nu \int_{\cup_j B_j} |g(\xi)|^2 \Delta^{-\nu}(\xi) d\xi \\ &\leq C_\nu \sum_j \int_{B_j} |g(\xi)|^2 \Delta^{-\nu}(\xi) d\xi \;\leq\; C'_\nu \sum_j \Delta^{-\nu}(\xi_j) \int_{B_j} |g(\xi)|^2 d\xi,\end{aligned}$$

where the latter inequality follows by Theorem 2.38.

Conversely,

$$\begin{aligned}\sum_j \Delta^{-\nu}(\xi_j) \int_{B_j} |g(\xi)|^2 d\xi &\leq c_\nu \sum_j \int_{B_j} |g(\xi)|^2 \Delta^{-\nu}(\xi) d\xi \\ &= c_\nu \int_\Omega |g(\xi)|^2 \Delta^{-\nu}(\xi) \sum_j \chi_{B_j}(\xi) d\xi.\end{aligned}$$

By the finite overlapping property of the balls B_j, there exists a positive integer N such that for every $\xi \in \Omega$, $\sum\limits_j \chi_{B_j}(\xi) \leq N$. Then

$$\sum_j \Delta^{-\nu}(\xi_j) \int_{B_j} |g(\xi)|^2 d\xi \leq c_\nu N \int_\Omega |g(\xi)|^2 \Delta^{-\nu}(\xi) d\xi = c'_\nu N \|F\|^2_{A^{2,2}_\nu}.$$

We have thus established the following result:

Proposition 3.25. *There exists a constant $C = C(\nu) > 1$ such that for every $F \in A^{2,2}_\nu$, if $F = \mathcal{L}g$ with $g \in L^2_\nu$ we have*

$$\frac{1}{C} \sum_j \Delta^{-\nu}(\xi_j) \int_{B_j} |g(\xi)|^2 d\xi \leq \|F\|^2_{A^{2,2}_\nu} \leq C \sum_j \Delta^{-\nu}(\xi_j) \int_{B_j} |g(\xi)|^2 d\xi.$$

We intend to extend this proposition to the mixed norm weighted Bergman spaces $A^{2,q}_\nu$, when it is possible. The first inequality will be proved hereafter, while the second one is postponed to the next subsection. We will see that this is related with the third problem of the first section (boundary values of Bergman spaces), and that the boundedness of the Bergman projection is involved in the values of q for which it is valid.

We denote by b^q_ν the space of all measurable functions g on Ω such that

$$\|g\|_{b^q_\nu} = \left(\sum_j \Delta^{-\nu}(\xi_j)\left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} < \infty. \tag{3.26}$$

Let

$$\tilde{q}_\nu = \begin{cases} \frac{\nu+\frac{n}{r}-1}{\frac{n}{2r}-1} & \text{if } n > 2r \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 3.27. *Assume $q < \tilde{q}_\nu$. Then, there exists $C > 0$ such that for every $g \in b^q_\nu$ we have*

$$\int_\Omega |g(\xi)| e^{-(y|\xi)} d\xi \leq C\|g\|_{b^q_\nu} \Delta(y)^{-(\frac{\nu}{q}+\frac{n}{2r})}.$$

In particular, g is a locally integrable function in Ω.

Proof: By Lemma 2.38 and Schwarz's inequality, we get

$$\begin{aligned}\int_\Omega |g(\xi)| e^{-(y|\xi)} d\xi &\leq \sum_j \int_{B_j} |g(\xi)| e^{-(y|\xi)} d\xi \\ &\leq \sum_j e^{-\gamma(y|\xi_j)} \int_{B_j} |g(\xi)| d\xi \\ &\leq \sum_j e^{-\gamma(y|\xi_j)} \left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{B_j} d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

Recall that $\Delta^{-\frac{n}{r}}(\xi)d\xi$ is a G-invariant measure on Ω (Proposition 2.42). Also, by Corollary 2.43 we have

$$|B_j| \sim \Delta^{\frac{n}{r}}(\xi_j). \tag{3.28}$$

Now, the bound (3.28) implies that

$$\begin{aligned}\int_\Omega |g(\xi)| e^{-(y|\xi)} d\xi &\leq C \sum_j e^{-\gamma(y|\xi_j)} \left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{1}{2}} \Delta^{\frac{n}{2r}}(\xi_j) \\ &= C \sum_j \left(\Delta^{-\frac{\nu}{q}}(\xi_j)\left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{1}{2}}\right)\left(e^{-\gamma(y|\xi_j)} \Delta^{\frac{n}{2r}+\frac{\nu}{q}}(\xi_j)\right) \\ &\leq \|g\|_{b^q_\nu} \left(\sum_j e^{-\gamma q'(y|\xi_j)} \Delta^{q'(\frac{n}{2r}+\frac{\nu}{q})}(\xi_j)\right)^{\frac{1}{q'}};\end{aligned}$$

where the last step follows by Hölder's inequality. Again, (3.28) implies

$$\sum_j e^{-\gamma q'(y|\xi_j)} \Delta^{q'(\frac{n}{2r}+\frac{\nu}{q})}(\xi_j) \leq C \sum_j e^{-\gamma q'(y|\xi_j)} \Delta^{q'(\frac{n}{2r}+\frac{\nu}{q})-\frac{n}{r}}(\xi_j) \int_{B_j} d\xi.$$

Therefore, by Theorem 2.38, the finite overlapping property and Proposition 3.1, we obtain

$$\begin{aligned}\sum_j e^{-\gamma q'(y|\xi_j)}\Delta^{q'(\frac{n}{2r}+\frac{\nu}{q})}(\xi_j) &\leq C\sum_j \int_{B_j} e^{-\gamma_1 q'(y|\xi)}\Delta^{q'(\frac{n}{2r}+\frac{\nu}{q})-\frac{n}{r}}(\xi)d\xi \\ &\leq CN\int_\Omega e^{-\gamma_1 q'(y|\xi)}\Delta^{q'(\frac{n}{2r}+\frac{\nu}{q})-\frac{n}{r}}(\xi)d\xi \\ &= C\,N\,\Gamma_\Omega\left(q'(\tfrac{n}{2r}+\tfrac{\nu}{q})\right)\,\Delta^{-q'(\frac{n}{2r}+\frac{\nu}{q})}(\gamma_1 q' y) < \infty\end{aligned}$$

since $q'(\frac{n}{2r}+\frac{\nu}{q}) > \frac{n}{r}-1$. The conclusion now follows. □

THEOREM 3.29. *Let $q < \tilde{q}_\nu$. Given $F \in A_\nu^{2,q}$, there is a unique function $g \in b_\nu^q$ such that $F = \mathcal{L}g$ and*

$$\|g\|_{b_\nu^q} \leq C\|F\|_{A_\nu^{2,q}}.$$

PROOF: By density, take $F \in A_\nu^{2,q} \cap A_\nu^{2,2}$. By the Paley-Wiener theorem (Theorem 3.11), there exists $g \in L^2_{(\nu)}(\Omega)$ such that

$$F(z) = \mathcal{L}g(z) = \int_\Omega g(\xi)e^{i(z|\xi)}d\xi \qquad (z \in T_\Omega).$$

Recall that for $y \in B_j$, $y^{-1} \in B_j^{-1} = B_3(\xi_j^{-1})$ since the mapping $x \mapsto x^{-1}$ is an isometry (see Theorem 2.50). Moreover, by Theorem 2.52, there exists a constant A such that for all j, $\xi \in B_j$ and $y \in B_j^{-1}$, we have $\frac{1}{A} \leq (\xi|y) \leq A$. Thus, for all $y \in B_j^{-1}$, if $C = e^{2A}$,

$$\begin{aligned}\int_{B_j}|g(\xi)|^2 d\xi &\leq C\int_{B_j}|g(\xi)|^2 e^{-2(y|\xi)}d\xi \\ &\leq C\int_\Omega |g(\xi)|^2 e^{-2(y|\xi)}d\xi \;=\; C'\int_{\mathbf{R}^n}|F(x+iy)|^2 dx,\end{aligned}$$

by the Plancherel formula. Therefore,

$$|B_j^{-1}|\left(\int_{B_j}|g(\xi)|^2 d\xi\right)^{\frac{q}{2}} \leq C_q\int_{B_j^{-1}}\left(\int_{\mathbf{R}^n}|F(x+iy)|^2 dx\right)^{\frac{q}{2}}dy.$$

Furthermore, if we write $F_y = F(\cdot + iy)$ and $y_j = \xi_j^{-1}$, since there is a constant C such that for every j, $|B_j^{-1}| \geq C\Delta^{\frac{n}{r}}(y_j)$ (see (3.28)), we get

$$\left(\int_{B_j}|g(\xi)|^2 d\xi\right)^{\frac{q}{2}} \leq C\Delta^{-\frac{n}{r}}(y_j)\int_{B_j^{-1}}\|F_y\|_2^q dy.$$

Moreover, by (2.49) we have $\Delta(\xi_j) = \Delta(y_j)^{-1}$, and thus

$$\begin{aligned}\Delta^{-\nu}(\xi_j)\left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} &\le C\Delta^{\nu-\frac{n}{r}}(y_j)\int_{B_j^{-1}} \|F_y\|_2^q dy \\ &\le C'\int_{B_j^{-1}} \Delta^{\nu-\frac{n}{r}}(y)\|F_y\|_2^q dy.\end{aligned}$$

Therefore, since the balls B_j^{-1} also form a dyadic decomposition of Ω (see Corollary 2.44),

$$\begin{aligned}\sum_j \Delta^{-\nu}(\xi_j)\left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} &\le C'\sum_j \int_{B_j^{-1}} \Delta^{\nu-\frac{n}{r}}(y)\|F_y\|_2^q dy \\ &\le C''\int_{\Omega} \Delta^{\nu-\frac{n}{r}}(y)\|F_y\|_2^q dy\end{aligned}$$

by the finite overlapping property for the balls B_j^{-1}. This finishes the proof. □

4. Mapping properties of the weighted Bergman projectors

4.1. Statement of the main problem.

Recall that, for $p \in [1,\infty]$ and $\nu \in \mathbf{R}$ we denote by

$$L^p_\nu = L^p\left(T_\Omega,\, \Delta^{\nu-\frac{n}{r}}(y)dxdy\right)$$

the weighted Lebesgue spaces and by A^p_ν, $\nu > \frac{n}{r} - 1$, the weighted Bergman spaces. We consider the weighted Bergman projector P_ν defined in (3.10) as

$$P_\nu f(z) = \int_\Omega \left(\int_{\mathbf{R}^n} B_\nu(z, u+iv) f(u+iv)du\right)\Delta^{\nu-\frac{n}{r}}(v)dv \qquad (f \in L^2_\nu),$$

where B_ν denotes the weighted Bergman kernel whose expression was given in Theorem 3.15.

Our main goal is to determine the values of $p \in [1,\infty]$ for which P_ν extends to a bounded operator on L^p_ν, in which case it is a bounded projector from L^p_ν to A^p_ν. We observe that P_ν is a self-adjoint operator and hence P_ν is bounded on L^p_ν if and only if it is bounded on $L^{p'}_\nu$, where p' is the conjugate exponent of p. We denote P^+_ν the positive integral operator defined for $f \in L^2_\nu$) by

$$P^+_\nu f(z) = \int_\Omega \left(\int_{\mathbf{R}^n} |B_\nu(z, u+iv)| f(u+iv)du\right)\Delta^{\nu-\frac{n}{r}}(v)dv. \tag{4.1}$$

We set

$$q_\nu = 1 + \frac{\nu}{\frac{n}{r} - 1}, \qquad p_\nu = q_\nu + 1, \quad \text{and} \quad \tilde{p}_\nu = \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1}. \tag{4.2}$$

Observe that $2 < q_\nu < p_\nu < \tilde{p}_\nu$. Finally, notice that if P_ν^+ is bounded on L^p_ν, then P_ν extends to a bounded operator from L^p_ν to A^p_ν. The converse is also true in the case $n = 1$ (see Section 1). This is no more the case for $n \geq 3$ as the following theorem shows.

THEOREM 4.3. *The following properties hold:*

1. *The operator P_ν^+ is bounded on L^p_ν if and only if $q'_\nu < p < q_\nu$;*
2. *If P_ν extends to a bounded operator from L^p_ν to A^p_ν, then $\tilde{p}'_\nu < p < \tilde{p}_\nu$;*
3. *The operator P_ν extends to a bounded operator from L^p_ν to A^p_ν if $p'_\nu < p < p_\nu$.*

Let us make some comments on this theorem. When considering simultaneously assertions (1) and (3), we see that there are values of p for which the Bergman projector P_ν extends to a bounded operator from L^p_ν to A^p_ν while the associated positive integral operator P_ν^+ is not bounded on L^p_ν. This is a new phenomenon compared to all cases for which the Bergman projector is known to satisfy L^p estimates. The proof of assertion (1) uses basically the same methods as in the upper half-plane, that is Schur's lemma, which gives L^p continuity properties for integral operators with positive kernels. Hence, in order to get the larger range of values of p given in assertion (3), we must exploit the oscillations of the Bergman kernel. While trying to do this, we are lead to use the Fourier transform in the x variables and consequently to focus on L^2 norms in these variables. This is the reason why we enlarged our class of spaces, by introducing mixed norms.

We recall that for $p, q \in [1, \infty]$ we set

$$L^{p,q}_\nu = L^q\left(\Omega, \Delta(y)^{\nu - \frac{n}{r}} dy; L^p(\mathbf{R}^n, dx)\right), \qquad A^{p,q}_\nu = L^{p,q}_\nu \cap \mathcal{H}(T_\Omega).$$

Assertion (1) will be proved in subsection 4.3. In fact, we shall prove a more general result giving necessary and sufficient conditions on p, q for the $L^{p,q}_\nu$ boundedness of P_ν^+. In subsection 4.4, we prove $L^{2,q}_\nu$ estimates for P_ν. Finally, we shall prove assertion (3) in subsection 4.5 using interpolation methods. More precisely, assertion (3) will be obtained as a particular case of a result giving a sufficient condition on p, q under which P_ν extends to a bounded operator from $L^{p,q}_\nu$ to $A^{p,q}_\nu$. We also prove a necessary condition on p, q so that P_ν extends to a bounded operator from $L^{p,q}_\nu$ to $A^{p,q}_\nu$, fact that includes assertion (2) as a particular case.

4.2. Positive integral operators on the cone.

We consider the following positive integral operator T defined on the cone Ω by

$$Tg(y) = \int_\Omega \Delta^{-\nu}(y+v)g(v)\Delta^{\nu-\frac{n}{r}}(v)dv. \tag{4.4}$$

In the next subsection we shall see that T is closely related to the operator P_ν^+. Recall that $q_\nu = 1 + \frac{\nu}{\frac{n}{r}-1}$. We shall need the following theorem.

Theorem 4.5. *The operator T is bounded on $L^q\left(\Omega, \Delta^{\nu-\frac{n}{r}}(v)dv\right)$ if and only if $q_\nu' < q < q_\nu$.*

Proof: (*Sufficiency*) We will use Schur's lemma (Lemma 1.35) as in the one-dimensional case. For $K(y,v) = \Delta^{-\nu}(y+v)$, it suffices to find a positive function ϕ on Ω such that the following two properties are satisfied:

$$\int_\Omega K(y,v)\phi(v)^{q'}\Delta^{\nu-\frac{n}{2}}(v)dv \leq C\phi(y)^{q'} \tag{4.6}$$

and

$$\int_\Omega K(y,v)\phi(y)^{q}\Delta^{\nu-\frac{n}{2}}(y)dy \leq C\phi(v)^{q}. \tag{4.7}$$

As Schur's test functions we take $\phi = \Delta^{\mathbf{s}}$, for appropriate $\mathbf{s}$. We rely on the following lemma, which may be found in [5], Lemma 3.3, for the light cone.

Lemma 4.8. *For $v \in \Omega$, the integral*

$$I_{\mathbf{s}}(t) = \int_\Omega \Delta(y+t)^\beta \Delta^{\mathbf{s}}(y)\Delta^{-\frac{n}{r}}(y)dy$$

is convergent if and only if the following conditions hold:

$$s_j > (j-1)\frac{n/r-1}{r-1}, \quad s_j + \beta < -(r-j)\frac{n/r-1}{r-1} \quad \text{for } j = 1, \ldots, r.$$

In this case,

$$I_{\mathbf{s}}(t) = C\Delta^{\mathbf{s}}(t)\Delta^\beta(t).$$

Proof: The scheme of the proof is the same as for Lemma 3.19. The conditions on $\mathbf{s}$ allow to restrict on values of β for which $\Delta(y+t)^\beta$ can be written as a Laplace transform, using Proposition 3.1. We then use Proposition 3.2 to reduce to the integral

$$\int_\Omega e^{-(t|\xi)}\Delta_{\mathbf{s}}(\xi^{-1})\Delta^{-\beta-\frac{n}{r}}(\xi)\, d\xi.$$

To go on with the proof, one needs to write $\Delta^{\mathbf{s}}(\xi^{-1})$ in terms of ξ. We refer to [11] for the general case, and go on for the light cone, where all formulas are explicit. We get

$$\Delta^{\mathbf{s}}(\xi^{-1}) = (\Delta_1^*)^{s_1-s_2}(\xi)\Delta_2^{-s_1}(\xi),$$

where we note

$$\Delta_1^*(\xi) = \xi_1 + \xi_2. \tag{4.9}$$

To get the result, we change of variables so that ξ_2 is replaced by $-\xi_2$, use Proposition 3.2 again (the second range of conditions on $\mathbf{s}$ comes from it), and use (4.9) with t in place of ξ. This finishes the proof in this particular case. □

Let us go on with the proof of the sufficiency for the forward light cone. An application of Lemma 4.8 in this particular case gives that (4.6) holds when we take $\phi = \Delta_1^{\beta_1}\Delta^{\beta_2}$ whenever β_1, β_2 satisfy

$$\frac{1}{q'}\left(\frac{n}{2} - \nu - 1\right) < \beta_1 < 0, \quad -\frac{\nu}{q'} < \beta_1 + \beta_2 < \frac{1}{q'}\left(-\frac{n}{2} + 1\right)$$

and estimate (4.7) holds when

$$\frac{1}{q}\left(\frac{n}{2} - \nu - 1\right) < \beta_1 < 0, \quad -\frac{\nu}{q} < \beta_1 + \beta_2 < \frac{1}{q}\left(-\frac{n}{2} + 1\right).$$

Thus, both of β_1 and $\beta_1 + \beta_2$ must lie in the intersection of two intervals. Assume $q \geq q'$, i.e. $q \geq 2$. Then β_1 must lie in $(\frac{1}{q'}\left(\frac{n}{2} - \nu - 1\right), 0)$ which is a non-empty interval. For β_2, we must have

$$\beta_1 + \beta_2 \in \left(-\frac{\nu}{q'}, \frac{1}{q'}\left(-\frac{n}{2} + 1\right)\right) \cap \left(-\frac{\nu}{q}, \frac{1}{q}\left(-\frac{n}{2} + 1\right)\right).$$

Since $q \geq 2$, then $-\frac{\nu}{q} > -\frac{\nu}{q'}$ and therefore, the previous intersection is non-empty if $-\frac{\nu}{q} < \frac{1}{q'}\left(-\frac{n}{2} + 1\right)$, i.e. if $q < 1 + \frac{\nu}{\frac{\nu}{r} - 1} = q_\nu$. The case $q < q'$ can be treated accordingly; it gives the dual condition $q'_\nu < q < 2$.

The general proof, for an arbitrary symmetric cone, follows the same lines, using Lemma 4.8.

(*Necessity*) We prove the necessity part of the theorem in the case of an arbitrary symmetric cone. If we take the characteristic function of the invariant ball $B_1(\mathbf{e})$ as a test function g, from Theorem 2.38 we know that $\Delta(v)$ and $\Delta(y+v)$ are almost constant on the support of $g(v)$ as functions of the variable v. So if T_g is bounded on $L^q\left(\Omega, \Delta^{\nu - \frac{n}{r}}(v)dv\right)$, the function $\Delta^{-\nu}(y + \mathbf{e})$ is in $L^q\left(\Omega, \Delta^{\nu - \frac{n}{r}}(y)dy\right)$. Using Lemma 3.19, we get the necessary condition $q > q'_\nu$. The dual condition $q < q_\nu$ follows from the self-adjointness of T. □

4.3. Estimates for the positive integral operator P_ν^+.

Recall that $q_\nu = 1 + \frac{\nu}{\frac{n}{r} - 1}$. We shall prove the following extension of Theorem 4.3 (1).

Theorem 4.10. *Let $p, q \in [1, \infty]$. The operator P_ν^+ defined by (4.1) is bounded on $L_\nu^{p,q}$ if and only if*

$$q'_\nu < q < q_\nu.$$

Proof: For a function $g : T_\Omega \to \mathbf{C}$, we write $g_y(x) = g(x+iy)$. It is sufficient to consider non-negative functions f. Then,

$$\begin{aligned} P_\nu^+ f(x+iy) &= \left(P_\nu^+ f\right)_y(x) \\ &= d_\nu \int_\Omega \left(\int_{\mathbf{R}^n} |\Delta_{y+v}^{-(\nu+\frac{n}{r})}(x-u)| f_v(u) du \right) \Delta^{\nu-\frac{n}{r}}(v) dv \\ &= d_\nu \int_\Omega \left(|\Delta_{y+v}^{-(\nu+\frac{n}{r})}| * f_v \right)(x) \Delta^{\nu-\frac{n}{r}}(v) dv. \end{aligned}$$

By the Minkowski inequality and the Young inequality, we obtain that

$$\begin{aligned} \| \left(P_\nu^+ f\right)_y \|_{L^p(\mathbf{R}^n)} &= \left(\int_{\mathbf{R}^n} \left(\left(P_\nu^+ f\right)_y(x) \right)^p dx \right)^{\frac{1}{p}} \\ &= d_\nu \left(\int_{\mathbf{R}^n} \left(\int_\Omega \left(|\Delta_{y+v}^{-(\nu+\frac{n}{r})}| * f_v \right)(x) \Delta^{\nu-\frac{n}{r}}(v) dv \right)^p dx \right)^{\frac{1}{p}} \\ &\leq d_\nu \int_\Omega \left(\int_{\mathbf{R}^n} \left(\left(|\Delta_{y+v}^{-(\nu+\frac{n}{r})}| * f_v \right)(x) \right)^p dx \right)^{\frac{1}{p}} \Delta^{\nu-\frac{n}{r}}(v) dv \\ &= d_\nu \int_\Omega \| |\Delta_{y+v}^{-(\nu+\frac{n}{r})}| * f_v \|_p \Delta^{\nu-\frac{n}{r}}(v) dv \\ &\leq d_\nu \int_\Omega \| \Delta_{y+v}^{-(\nu+\frac{n}{r})} \|_1 \| f_v \|_p \Delta^{\nu-\frac{n}{r}}(v) dv. \end{aligned}$$

The L^1 norm of $\Delta_{y+v}^{-(\nu+\frac{n}{r})}$ is given by assertion 1 in Lemma 3.20. This implies that

$$\begin{aligned} \| \left(P_\nu^+ f\right)_y \|_{L^p(\mathbf{R}^n)} &\leq C_\nu \int_\Omega \Delta^{-\nu}(y+v) \| f_v \|_p \Delta^{\nu-\frac{n}{r}}(v) dv \\ &= T\left(\| f_v \|_p \right)(y), \end{aligned}$$

where T is the positive integral operator defined in 4.4 on the cone Ω. Recall that by Theorem 4.5, this operator T is bounded on $L^q\big(\Omega, \Delta^{\nu-\frac{n}{r}}(v)dv\big)$ if $q_\nu' < p < q_\nu$. Therefore,

$$\begin{aligned} \| P_\nu^+ f \|_{L_\nu^{p,q}} &= \left(\int_\Omega \| \left(P_\nu^+ f\right)_y \|_{L^p(\mathbf{R}^n)}^q \Delta^{\nu-\frac{n}{r}}(y) dy \right)^{\frac{1}{q}} \\ &\leq C_\nu \left(\int_\Omega \big(T(\| f_v \|)(y) \big)^q \Delta^{\nu-\frac{n}{r}}(y) dy \right)^{\frac{1}{q}} \\ &\leq C_\nu \| T \| \cdot \big\| \| f_v \|_p \big\|_{L^q\left(\Omega,\ \Delta^{\nu-\frac{n}{r}}(v)dv\right)} \\ &= C_\nu \| T \| \cdot \| f \|_{L_\nu^{p,q}} \end{aligned}$$

if $q_\nu' < p < q_\nu$. This finishes the proof of the sufficiency part.

(Necessity) We need to show that P_ν^+ is unbounded on $L_\nu^{p,q}$ when $q \geq q_\nu$. To do this, we will show that, if P_ν^+ is bounded on $L_\nu^{p,q}$, then T is bounded on $L^q\left(\Omega, \Delta^{\nu-\frac{n}{r}}(v)dv\right)$. This fact will follow from the next lemma and a homogeneity (dilation) argument. We adapt the proof from [8], where it is given for $p = q$.

Lemma 4.11. *There are positive constants γ and c such that, for all $z = x + iy \in T_\Omega$ and $v \in \Omega$ with $|v| < \gamma$ and $|y| < \gamma$,*

$$\int_{|u|\leq 1} |B_\nu(z, u + iv)|du \geq c\Delta(y + v)^{-\nu}.$$

Proof: It is sufficient to prove the inequality

$$\int_{B_1(y)} |\Delta(x + iy)|^{-a}dx \geq c\Delta(y)^{-a+\frac{n}{r}} \ .$$

Indeed, one can also show that the Euclidean norm is almost constant on invariant balls of radius 1 (this is proved in detail in [4]: $|t|/|y|$ is bounded by an universal constant on the ball $B_1(y)$. As a consequence, the invariant ball $B_1(y)$ is contained in the Euclidean ball $\{|x| < 1\}$ if $|y| < \gamma$, for some γ. Now, we can use the fact that Δ is almost constant on the invariant ball, which allows to write that the left hand side is equivalent to

$$\Delta(y)^{\frac{n}{r}} \int_{B_1(y)} |\Delta(x + iy)|^{-a} \frac{dx}{\Delta(x)^{\frac{n}{r}}} \ .$$

Using the action of G and the formula of change of variable for Δ, we see that this last quantity is equal to $\Delta(y)^{-a+\frac{n}{r}}$, multiplied by the same integral when computed for $y = \mathbf{e}$. This last factor is clearly a positive constant.

□

To get the announced implication, using Lemma 4.11, we test P_ν^+ on specific $L_\nu^{p,q}$ functions, namely $g(z) = \chi_{|x|<2}(x)k(y)$, $z = x + iy$, with $k \in L^q\left(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy\right)$ supported in $\Omega \cap \{|y| < \gamma\}$. For x such that $|x| < 1$, and $y \in \Omega$ such that $|y| < \gamma$, one has the inequality

$$P_\nu^+ f(x + iy) \geq c \int_\Omega \Delta(y + v)^{-\nu} g(v) Q(v)^{\nu-\frac{n}{r}} dv.$$

By assumption, there exists a constant C independent of g, such that

$$\int_{y\in\Omega, |y|<\gamma} \left(\int_\Omega \Delta(y + v)^{-\nu} g(v)\Delta(v)^{\nu-\frac{n}{r}} dv\right)^q \Delta(y)^\nu dy \\ \leq C \int_\Omega g(v)^q \, \Delta(v)^{\nu-\frac{n}{r}} dv.$$

By homogeneity of the kernel, we can replace the constant γ by any positive constant N: for every positive function g on Ω, we have the inequality

$$\int_{y\in\Omega,|y|<N}\left(\int_\Omega \Delta(y+v)^{-\nu}g(v)\Delta(v)^{\nu-\frac{n}{r}}dv\right)^q \Delta(y)^{\nu-\frac{n}{r}}dy \\ \leq C\int_{v\in\Omega,|v|<N} g(v)^q\,\Delta(v)^{\nu-\frac{n}{r}}dv.$$

Using the density of compactly supported functions, we get the same inequality without any bound on integrals. This means that the operator T is bounded, and gives the restriction on q.

4.4. The boundedness of P_ν on $L^{2,q}_\nu$.

We recall that $q_\nu = 1+\frac{\nu}{\frac{n}{r}-1}$ and set $Q_\nu = 2q_\nu$.

We will first show how to relate the spaces b^q_ν and $A^{2,q}_\nu$.

THEOREM 4.12. *Assume $1\leq q<Q_\nu$. Given $g\in b^q_\nu$, then $\mathcal{L}g\in A^{2,q}_\nu$ and*

$$\|\mathcal{L}g\|_{A^{2,q}_\nu}\leq C\|g\|_{b^q_\nu}.$$

PROOF: Write $F=\mathcal{L}g$. For every $y\in\Omega$, $F_y(x)$ is the inverse Fourier transform of the function $\xi\mapsto g(\xi)e^{-(y|\xi)}$. By the Plancherel theorem,

$$\begin{aligned}\|F\|^q_{A^{2,q}_\nu} &= \int_\Omega \|F_y\|_2^q\Delta^{\nu-\frac{n}{r}}(y)dy\\ &= \int_\Omega\left(\int_\Omega |g(\xi)|^2e^{-2(y|\xi)}d\xi\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy.\end{aligned}\tag{4.13}$$

By Theorem 2.38 and (4.13), we deduce that

$$\|F\|^q_{A^{2,q}_\nu}\leq\int_\Omega\left(\sum_j e^{-2\gamma(y|\xi_j)}\int_{B_j}|g(\xi)|^2d\xi\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy.\tag{4.14}$$

First assume that $1\leq q<2$. We recall that for $\delta\in(0,1)$,

$$\left(\sum_j a_j\right)^\delta\leq\sum_j a_j^\delta.$$

Since $\frac{q}{2} < 1$, it follows from (4.14) and Proposition 3.1 that

$$\begin{aligned}\|F\|^q_{A^{2,q}_\nu} &\leq C \int_\Omega \sum_j e^{-q\gamma(y|\xi_j)} \Big(\int_{B_j} |g(\xi)|^2 d\xi \Big)^{\frac{q}{2}} \Delta^{\nu - \frac{n}{r}}(y) dy \\ &\leq \sum_j \Big(\int_{B_j} |g(\xi)|^2 d\xi \Big)^{\frac{q}{2}} \int_\Omega e^{-q\gamma(y|\xi_j)} \Delta^{\nu-\frac{n}{r}}(y) dy \\ &= C\Gamma_\Omega(\nu) \sum_j \Big(\int_{B_j} |g(\xi)|^2 d\xi \Big)^{\frac{q}{2}} \Delta^{-\nu}(q\gamma\xi_j) \\ &= C_{\nu,q,\gamma} \sum_j \Big(\int_{B_j} |g(\xi)|^2 d\xi \Big)^{\frac{q}{2}} \Delta^{-\nu}(\xi_j) \; = \; C_{\nu,q,\gamma} \|g\|^q_{b^q_\nu}.\end{aligned}$$

Assume next that $2 < q < Q_\nu$. At this point, our intention is to use Hölder's inequalitywith the introduction of some factor related to some generalized power of the Delta function. Again, to simplify the computations, we restrict ourselves to the particular case in which Ω is the Lorentz cone Λ_n, so that $r = 2$ and $\Delta(y) = y_1^2 - y_2^2 - ... - y_n^2$ and $\Delta_1(y) = y_1 + y_2$. To simplify the notation, we call $\rho = q/2$. We also take a real multi-index $s = (s_1, s_2)$ to be selected later. An application of Hölder's inequality gives

$$\begin{aligned}\sum_j e^{-2\gamma(y|\xi_j)} \int_{B_j} |g(\xi)|^2 d\xi \leq &\Big(\sum_j e^{-2\gamma(y|\xi_j)} \big(\int_{B_j} |g(\xi)|^2 d\xi \big)^\rho \Delta_1^{-s_1\rho}(\xi_j) \Delta^{-s_2\rho}(\xi_j) \Big)^{\frac{1}{\rho}} \\ &\times \Big(\sum_j e^{-2\gamma(y|\xi_j)} \Delta_1^{s_1\rho'}(\xi_j) \Delta^{s_2\rho'}(\xi_j) \Big)^{\frac{1}{\rho'}}.\end{aligned}$$

From (4.14) it follows that

$$\begin{aligned}\|F\|^q_{A^{2,q}_\nu} \leq C \int_{\Lambda_n} &\Big(\sum_j e^{-2\gamma(y|\xi_j)} \big(\int_{B_j} |g(\xi)|^2 d\xi \big)^\rho \Delta_1^{-s_1\rho}(\xi_j) \Delta^{-s_2\rho}(\xi_j) \Big) \\ &\times \Big(\sum_j e^{-2\gamma(y|\xi_j)} \Delta_1^{s_1\rho'}(\xi_j) \Delta^{s_2\rho'}(\xi_j) \Big)^{\frac{\rho}{\rho'}} \Delta^{\nu - \frac{n}{2}}(y).\end{aligned}$$

Notice that by Theorem 2.38, (3.28) and the finite overlapping property of the balls B_j, the sum in the second parenthesis on the right-hand side is bounded by

$$I = C \int_{\Lambda_n} e^{-2\gamma(y|\xi)} \Delta_1^{s_1\rho'}(\xi) \Delta^{s_2\rho'}(\xi) \frac{d\xi}{\Delta^{\frac{n}{2}}(\xi)}.$$

We use Proposition 3.2, as well as (4.9) to obtain that

$$I = C\Delta^{-(s_1+s_2)\rho'}(2\gamma y) \Delta_1^{\star s_1\rho'}(2\gamma y)$$

if $s_2\rho' > \frac{n}{2} - 1$ and $(s_1 + s_2)\rho' > 0$. In this case, Proposition 3.2, if $-(s_1 + s_2)\rho + \nu > \frac{n}{2} - 1$ and $-s_2\rho + \nu > 0$ we have

$$\begin{aligned}\|F\|^q_{A^{2,q}_\nu} &\leq C \sum_j \left(\int_{B_j} |g(\xi)|^2 d\xi\right)^\rho \Delta_1^{-s_1\rho}(\xi_j)\Delta^{-s_2\rho}(\xi_j) \\ &\qquad \times \left(\int_{\Lambda_n} e^{-2\gamma(y|\xi_j)} \Delta_1^{\star s_1\rho}(y)\Delta^{-(s_1+s_2)\rho+\nu-\frac{n}{2}}(y)dy\right) \\ &= C\sum_j \Delta^{s_2\rho-\nu}(2\gamma\xi_j)\Delta_1^{s_1\rho}(2\gamma\xi_j)\Delta^{-s_2\rho}(\xi_j)\Delta_1^{-s_1\rho}(\xi_j)\left(\int_{B_j} |g(\xi)|^2 d\xi\right)^\rho \\ &= C\sum_j \Delta^{-\nu}(\xi_j)\left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} \; = \; C\|g\|^q_{b^q_\nu};\end{aligned}$$

where the constant C depends on the involved parameters.

Therefore the conclusion follows if we choose s_1 and s_2 such that the following conditions are satisfied:

$$s_2\rho' > \frac{n}{2} - 1, \quad (s_1 + s_2)\rho' > 0;$$

and

$$s_2\rho < \nu, \quad (s_1 + s_2)\rho < \nu - \frac{n}{2} + 1.$$

The parameter s_2 can be suitably chosen since $\frac{n}{2} - 1 < \nu$. For s_1, $s_1 + s_2$ must lie in $\left(0, \frac{\nu - \frac{n}{2}+1}{\rho}\right)$ which is a non-empty interval. □

The statement of Theorem 4.12 is false for $q \geq Q_\nu$ as the next theorem shows.

Theorem 4.15. *For $q \geq Q_\nu$, there is a function $g \in b^q_\nu$ such that $\mathcal{L}g$ does not belong to $L^{p,q}_\nu$.*

Proof: We give the proof for the particular case of the cone $\mathrm{Sym}_+(2, \mathbf{R})$ of 2×2 real positive-definite symmetric matrices. For $\xi = \begin{pmatrix}\xi_1 & \xi_3 \\ \xi_3 & \xi_2\end{pmatrix} \in \mathrm{Sym}_+(2, \mathbf{R})$, we recall that $\Delta(\xi) = \mathrm{Det}\,\xi = \xi_1\xi_2 - \xi_3^2$. Take

$$g(\xi) = e^{-\xi_1-\xi_2}\Delta^{-\frac{1}{2}}(\xi)\left(1 + |\log \frac{\Delta(\xi)}{\xi_1}|\right)^{-\frac{1}{2}}.$$

Then if I denotes the 2×2 identity matrix,

$$\|(\mathcal{L}g)_I\|_2^2 = \int_\Omega e^{-4\xi_1-4\xi_2}\Delta^{-1}(\xi)\left(1 + |\log \frac{\Delta(\xi)}{\xi_1}|\right)^{-1} d\xi \tag{4.16}$$

and

$$\begin{aligned}
\|g\|_{b^q_\nu}^q &= \sum_j \Delta^{-\nu}(\xi_j)\left(\int_{B_j}|g(\xi)|^2 d\xi\right)^{\frac{q}{2}}\\
&= \sum_j \Delta^{-\nu}(\xi_j)\left(\int_{B_j} e^{-2\xi_1}\Delta^{-1}(\xi)\left(1+|\log\frac{\Delta(\xi)}{\xi_1}|\right)^{-1} d\xi\right)^{\frac{q}{2}}\\
&\leq C\sum_j\left(\int_{B_j}\Delta^{-\frac{2\nu}{q}-1}(\xi)e^{-2\xi_1}\Delta^{-1}(\xi)\left(1+|\log\frac{\Delta(\xi)}{\xi_1}|\right)^{-1} d\xi\right)^{\frac{q}{2}}\\
&\leq C'\sum_j\int_{B_j}\Delta^{-\nu-\frac{n}{2}+(\frac{n}{2}-1)\frac{q}{2}}(\xi)e^{-q\xi_1}\left(1+|\log\frac{\Delta(\xi)}{\xi_1}|\right)^{-\frac{q}{2}} d\xi
\end{aligned}$$

by Theorem 2.38, the Hölder's inequality and (3.28). Hence, by the finite overlapping property,

$$\|g\|_{b^q_\nu}^q \leq C\int_{\mathrm{Sym}_+(2,\mathbf{R})} e^{-q\xi_1}\Delta^{(\frac{n}{2}-1)\frac{q}{2}-\nu+\frac{n}{2}}(\xi)\big(1+|\log\frac{\Delta(\xi)}{\xi_2}|\big)^{-\frac{q}{2}}d\xi. \tag{4.17}$$

It now suffices to show that the right-hand side of (4.17) is infinite while the right-hand side of (4.16) is finite. This is given by the next lemma.

Lemma 4.18. *For α and β real, the integral*

$$I_{\alpha,\beta} = \int_{Sym_+(2,\mathbf{R})} e^{-\xi_1-\xi_2}\Delta^\alpha(\xi)\big(1+|\log\frac{\Delta(\xi)}{\xi_1}|\big)^\beta d\xi$$

is finite if and only if one of the following two conditions is satisfied:

(1) $\alpha > -1$;
(2) $\alpha = -1$ *and* $\beta < -1$.

Proof: We use the Gauss coordinates of $\xi \in \mathrm{Sym}_+(2,\mathbf{R})$ defined in §2.3 by

$$\xi_1 = \lambda^2, \quad \xi_2 - \frac{\xi_3^2}{\xi_1} = \mu^2, \quad \frac{\xi_3}{\xi_1} = v.$$

Then $\Delta(\xi) = \lambda^2\mu^2$ and

$$\begin{aligned}
I_{\alpha,\beta} &= 4\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbf{R}} e^{-\mu^2-(v^2+1)\lambda^2}\lambda^{2\alpha}\mu^{2\alpha}\left(1+2|\log\mu|\right)^\beta\lambda^3\mu d\lambda d\mu dv\\
&= J_{\alpha,\beta}K_\alpha,
\end{aligned}$$

where

$$J_{\alpha,\beta} = 4\int_0^\infty e^{-\mu^2}\mu^{2\alpha+1}\left(1+2|\log\mu|\right)^\beta d\mu$$

and

$$K_\alpha = \int_0^\infty \left(\int_{-\infty}^\infty e^{-v^2\lambda^2} dv \right) \lambda^{2\alpha+3} e^{-\lambda^2} d\lambda = C \int_0^\infty e^{-\lambda^2} \lambda^{2\alpha+2} d\lambda.$$

Next observe that $K_\alpha < \infty$ if and only if $\alpha > -\frac{3}{2}$ while $J_{\alpha,\beta} < \infty$ if and only if either $\alpha > -1$ or both $\alpha = -1$ and $\beta < -1$. □

We will now show how this last theorem is related to the boundedness of the weighted Bergman projection. We consider the following commutative diagram:

$$\begin{tikzcd} L^2_\nu \arrow[r, "P_\nu"] \arrow[d, "P_\nu"'] & A^2_\nu \\ A^2_\nu \arrow[r, "\mathcal{L}^{-1}"] & L^2_{(\nu)}(\Omega) \arrow[u, "\mathcal{L}"'] \end{tikzcd}$$

Notice that $\mathcal{L} : L^2_{(\nu)}(\Omega) \to A^2_\nu$ is invertible by Paley-Wiener theorem (Theorem 3.11). Given $\phi \in L^2_\nu$ and $F = \mathcal{L}g \in A^2_\nu$, since $P_\nu F = F$, the self-adjointness of P_ν implies

$$\langle P_\nu \phi, F \rangle_{A^2_\nu} = \langle \phi, F \rangle_{L^2_\nu} = \langle \phi, \mathcal{L}g \rangle_{L^2_\nu}.$$

Now, by the Plancherel formula, if $\mathcal{F}^{-1}$ denotes the inverse Fourier transform,

$$\begin{aligned} \langle \phi, \mathcal{L}g \rangle_{L^2_\nu} &= \int_{T_\Omega} \phi(x+iy) \left(\overline{\int_\Omega g(\xi) e^{i(x+iy)\cdot\xi} d\xi} \right) \Delta^{\nu-\frac{n}{r}}(y) dxdy \\ &= \int_\Omega \left(\int_{\mathbf{R}^n} \phi_y(x) \overline{\mathcal{F}^{-1}(g(\xi)e^{-(y|\xi)})(x)} dx \right) \Delta^{\nu-\frac{n}{r}}(y) dy \\ &= \int_\Omega \left(\int_\Omega \hat{\phi}_y(\xi) \overline{g(\xi)} e^{-(y|\xi)} d\xi \right) \Delta^{\nu-\frac{n}{r}}(y) dy \\ &= \int_\Omega \left(\Delta^\nu(\xi) \int_\Omega \hat{\phi}_y(\xi) e^{-(y|\xi)} \Delta^{\nu-\frac{n}{r}}(y) dy \right) \overline{g(\xi)} \Delta^{-\nu}(\xi) d\xi, \end{aligned} \tag{4.19}$$

where equality (4.19) follows by Fubini's theorem. Therefore, for $g \in L^2_{(\nu)}(\Omega)$, equality (4.19) and the polarization of isometry (3.13) in the Paley-Wiener theorem imply that

$$\begin{aligned} \langle \phi, \mathcal{L}g \rangle_{L^2_\nu} &= \langle P_\nu \phi, F \rangle_{A^2_\nu} \\ &= C_\nu \langle \mathcal{L}^{-1}(P_\nu \phi, g), g \rangle_{L^2_{(\nu)}(\Omega)} = \langle T\phi, g \rangle_{L^2_{(\nu)}(\Omega)}. \end{aligned} \tag{4.20}$$

Comparing (4.19) and (4.20) then gives

$$T\phi(\xi) = \Delta^\nu(\xi) \int_\Omega \hat{\phi}_y(\xi) e^{-(y|\xi)} \Delta^{\nu-\frac{n}{r}}(y) dy.$$

We shall need the following lemma.

Lemma 4.21. *If $q > 2$, then for all $\phi \in L^{2,q}_\nu$, $T\phi \in b^q_\nu$ and $\|T\phi\|_{b^q_\nu} \le C \|\phi\|_{L^{2,q}_\nu}$.*

Proof: By Schwarz's inequality and Proposition 3.1,

$$\begin{aligned}|T\phi(\xi)| &\le \Delta^{\nu}(\xi)\left(\int_{\Omega}|\hat{\phi}_y(\xi)|^2 e^{-(y|\xi)}\Delta^{\nu-\frac{n}{r}}(y)dy\right)^{\frac{1}{2}}\left(\int_{\Omega}e^{-(y|\xi)}\Delta^{\nu-\frac{n}{r}}(y)dy\right)^{\frac{1}{2}}\\ &= C_{\nu}\Delta^{\frac{\nu}{2}}(\xi)\left(\int_{\Omega}|\hat{\phi}_y(\xi)|^2 e^{-(y|\xi)}\Delta^{\nu-\frac{n}{r}}(y)dy\right)^{\frac{1}{2}}.\end{aligned}$$

Furthermore, by Hölder's inequality and Theorem 2.38,

$$\begin{aligned}\left(\int_{B_j}|T\phi(\xi)|^2 d\xi\right)^{\frac{q}{2}} &\le C_{\nu}\left(\int_{B_j}\Delta^{\nu}(\xi)\left(\int_{\Omega}|\hat{\phi}_y(\xi)|^2 e^{-(y|\xi)}\Delta^{\nu-\frac{n}{r}}(y)dy\right)d\xi\right)^{\frac{q}{2}}\\ &\le C'_{\nu}\Delta^{\frac{q\nu}{2}}(\xi_j)\left(\int_{\Omega}e^{-\gamma(y|\xi_j)}\left(\int_{B_j}|\hat{\phi}_y(\xi)|^2 d\xi\right)\Delta^{\nu-\frac{n}{r}}(y)dy\right)^{\frac{q}{2}}\\ &\le C'_{\nu}\Delta^{\frac{q\nu}{2}}(\xi_j)\left(\int_{\Omega}\left(\int_{B_j}|\hat{\phi}_y(\xi)|^2 d\xi\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy\right)\\ &\qquad\times\left(\int_{\Omega}e^{-\frac{\gamma q}{q-2}(y|\xi_j)}\Delta^{\nu-\frac{n}{r}}(y)dy\right)^{\frac{q-2}{2}}\\ &= C_{\nu,\gamma,q}\Delta^{\nu}(\xi_j)\int_{\Omega}\left(\int_{B_j}|\hat{\phi}_y(\xi)|^2 d\xi\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy.\end{aligned}$$

Thus,

$$\begin{aligned}\|T\phi\|^q_{b^q_{\nu}} &\le C_{\nu,\gamma,q}\int_{\Omega}\sum_j\left(\int_{B_j}|\hat{\phi}_y(\xi)|^2 d\xi\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy\\ &\le C_{\nu,\gamma,q}\int_{\Omega}\left(\sum_j\int_{B_j}|\hat{\phi}_y(\xi)|^2 d\xi\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy,\end{aligned}$$

because if $\rho > 1$, every sequence of positive numbers $\{a_j\}$ satisfies

$$\sum_j a_j^{\rho} \le \left(\sum_j a_j\right)^{\rho}.$$

Here, $r = q/2 > 1$. Next, by the finite overlapping property and by the Plancherel theorem,

$$\begin{aligned}\|T\phi\|^q_{b^q_{\nu}} &\le C'_{\nu,\gamma,q}\int_{\Omega}\left(\int_{\Omega}|\hat{\phi}_y(\xi)|^2 d\xi\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy\\ &= C'_{\nu,\gamma,q}\int_{\Omega}\left(\int_{\mathbf{R}^n}|\phi_y(x)|^2 dx\right)^{\frac{q}{2}}\Delta^{\nu-\frac{n}{r}}(y)dy\\ &= C_{\nu,\gamma,q}\|\phi\|^q_{L^{2,q}_{\nu}},\end{aligned}$$

as we wished to show. □

We can now prove the following result.

COROLLARY 4.22. *If $Q'_\nu < q < Q_\nu$, then P_ν extends to a bounded operator from $L^{2,q}_\nu$ to $A^{2,q}_\nu$.*

PROOF: Without loss of generality, we may assume that $2 < q < Q_\nu$. By Theorems 3.29 and 4.12, $\mathcal{L}$ is a bounded isomorphism from b^q_ν to $A^{2,q}_\nu$. Then, it follows from Lemma 4.21 that $P_\nu = \mathcal{L}T$ extends to a bounded operator from $L^{2,q}_\nu$ to $A^{2,q}_\nu$. □

4.5. $L^{p,q}_\nu$ boundedness for the weighted Bergman projector P_ν.

If we interpolate the $L^{\infty,q}_\nu$ or $L^{1,q}_\nu$ estimates obtained in Theorem 4.10 with the $L^{2,q}_\nu$ estimates established in Corollary 4.22, we obtain the next theorem which generalizes Part 3 of Theorem 4.3.

THEOREM 4.23. *The weighted Bergman projector P_ν extends to a bounded projector from $L^{p,q}_\nu$ to $A^{p,q}_\nu$ if*

$$\frac{1}{q_\nu p'} < \frac{1}{q} < 1 - \frac{1}{q_\nu p'}.$$

PROOF: For a fixed value of ν, we have the following picture:

$units < 1.2cm, 1.2cm > xfrom0to8, yfrom0to6 < 10pt > [.3, .5]from{-}.50to60 <$

$A < 0pt, 5pt > at2 - .51\frac{}{p}$ ¡5pt,0pt¿ at -.5 5.5 D ¡0pt,5pt¿ at 1.1157 .6557 $\frac{1}{q}$ ¡0pt,5pt¿ at 5.5 -.5 F ¡5pt,0pt¿ at -.5 3.1 B ¡0pt,0pt¿ at .8 2.7 C ¡0pt,3pt¿ at 2 5.2 E ¡0pt,0pt¿ at 1.6586 1.4286

FIGURE 1. $D=\left(\frac{\frac{n}{r}-1}{\nu+\frac{2n}{r}-1},\frac{\frac{n}{r}-1}{\nu+\frac{2n}{r}-1}\right)$, $E=\left(\frac{\frac{n}{r}-1}{2(\frac{n}{r}-1)+\nu},\frac{\frac{n}{r}-1}{2(\frac{n}{r}-1)+\nu}\right)$, $F=\left(0,\frac{\frac{n}{r}-1}{\frac{n}{r}}\right)$,

By interpolation, P_ν is bounded on $L^{p,q}_\nu$ for $(\frac{1}{q},\frac{1}{p})$ in the interior of the light-shaded hexagon of vertices

$$B = \left(\frac{1}{2\left(1+\frac{\nu}{\frac{n}{r}-1}\right)},\frac{1}{2}\right), \qquad A = \left(\frac{1}{\left(1+\frac{\nu}{\frac{n}{r}-1}\right)},0\right), \qquad C = \left(\frac{1}{\left(1+\frac{\nu}{\frac{n}{r}-1}\right)},1\right)$$

and their symmetric points with respect to $(\frac{1}{2},\frac{1}{2})$. □

On the other hand, P_ν does not extend to a bounded operator on $L^{p,q}_\nu$ on the dark-shaded regions of the figure, as the next result shows. This result generalizes part (2) of Theorem 4.3.

Theorem 4.24. *P_ν extends to a bounded operator on $L^{p,q}_\nu$ only if*

$$\frac{n}{r}-1<\frac{n}{rp}+\frac{\nu+\frac{n}{r}-1}{q}<\nu+\frac{n}{r}.$$

Proof: Recall that P_ν is a self-adjoint operator and hence, P_ν is bounded on $L^{p,q}_\nu$ if and only if P_ν is bounded on $L^{p',q'}_\nu$. Apply P_ν to the function $f(z)=\Delta^{-(\nu-\frac{n}{r})}(y)\chi_{b(i\mathbf{e})}(z)$, where $z=x+iy$ and $b(i\mathbf{e})$ is an Euclidean ball with centre $i\mathbf{e}$ relatively compact in T_Ω. It is clear that $f\in\bigcap_{1\le p,q\le\infty}L^{p,q}_\nu$. Moreover, by the mean value property, there is a positive constant $C=C(n)$ such that for every $z\in T_\Omega$,

$$P_\nu f(z)=C\Delta^{-\nu-\frac{n}{r}}\left(\frac{z+i\underline{e}}{i}\right).$$

Now, by Lemma 3.20, $P_\nu f$ belongs to $L^{p,q}_\nu\cap L^{p',q'}_\nu$ only if $\nu+\frac{n}{r}>\frac{n}{rp}+\frac{\nu+\frac{n}{r}-1}{q}$ and $\nu+\frac{n}{r}>\frac{n}{rp'}+\frac{\nu+\frac{n}{r}-1}{q'}$. The conclusion follows. $\square$

Remark 4.25. At this time, the problem of determining whether P_ν is bounded on $L^{p,q}_\nu$ for $(\frac{1}{p},\frac{1}{q})$ in the blank region in the above figure is open.

5. Applications

In this section we give some applications of our main results, that is Theorem 4.23 and Theorem 4.24. For the particular case of the Lorentz cone, these applications were described in [2]. We will not give details of the proofs, which will appear somewhere else.

5.1. Transfer of L^p_ν estimates for the Bergman projector to bounded symmetric domains of tube type.

First of all, it is well known (cf. e.g. Chapter X of [11]) that every tube domain T_Ω over a symmetric cone Ω can be realized via a biholomorphic mapping as a bounded symmetric domain D. "Symmetric" means that every point of D is an isolated fixed point of an involutive automorphism of D and this property implies the homogeneity of the domain. Such a bounded symmetric domain is said to be of tube type. In one complex variable, the upper half-plane is realized as the unit disc via the linear fractional transformation

$$\Phi(z)=i\frac{1+z}{1-z}.$$

The biholomorphic transformations from T_Ω to D which generalize Φ are known as *Cayley transformations*. We assume that the bounded domain D is the Harish-Chandra realization (cf. [11], p. 189) of the tube domain T_Ω. In this case, we shall call D a standard bounded symmetric domain of tube type. In particular, D is starlike around 0 and circular, that is, $e^{i\theta}z \in D$ if $\theta \in \mathbf{R}$ and $z \in D$.

THEOREM 5.1. *Let D be a realization of a tube domain T_Ω over a symmetric cone Ω as a standard bounded symmetric domain D. The conclusions of Theorems 4.23 and 4.24 are valid with T_Ω replaced by D.*

As an example, Theorems 4.23 and 4.24 for the tube in $\mathbf{C}^n$, over the Lorentz cone Λ_n ($n \geq 3$) are also valid for the Lie ball $\widetilde{\Omega}$ of $\mathbf{C}^n$ defined by

$$\widetilde{\Omega} = \left\{ z \in \mathbf{C}^n : \left| \sum_{j=1}^{n} z_j^2 \right| < 1,\ 1 - 2|z|^2 + \left| \sum_{j=1}^{n} z_j^2 \right|^2 > 0 \right\}.$$

The proof of Theorem 5.1 is based on a transfer principle using the explicit form of the Cayley transformation and some homogeneity arguments (see, [1]).

5.2. Duality $(A_\nu^{p,q}, A_\nu^{p',q'})$.

THEOREM 5.2. *Let $p, q \in [1, \infty)$ and $\nu > \frac{n}{r} - 1$. Assume that the weighted Bergman projector P_ν extends to a bounded projector from $L_\nu^{p,q}$ to $A_\nu^{p,q}$. Then the topological dual $(A_\nu^{p,q})'$ of $A_\nu^{p,q}$ identifies with $A_\nu^{p',q'}$ by means of the map*

$$G \in A_\nu^{p',q'} \mapsto L_G(F) = \int_\Omega F(z)\overline{G(z)}\Delta^{\nu-\frac{n}{r}}(y)dy. \tag{5.3}$$

PROOF: By Hölder's inequality, it is clear that given $G \in A_\nu^{p',q'}$, L_G is a bounded linear functional on $A_\nu^{p,q}$ with $\|L_G\| \leq \|G\|_{A_\nu^{p',q'}}$. Conversely, let $L \in (A_\nu^{p,q})'$. By the Hahn-Banach theorem, L extends to a bounded linear functional on $L_\nu^{p,q}$ with the same operator norm. Since $(L_\nu^{p,q})'$ identifies with $L_\nu^{p',q'}$ via the standard $L_\nu^{2,2}$ duality pairing, there exists a function $\phi \in L_\nu^{p',q'}$ satisfying $\|L\| = \|\phi\|_{L_\nu^{p',q'}}$ such that for every $F \in A_\nu^{p,q}$,

$$L(F) = \int_{T_\Omega} F(z)\overline{\phi(z)}\Delta^{\nu-\frac{n}{r}}(y)dxdy.$$

But, $P_\nu F = F$ and P_ν is a self-adjoint operator. Hence,

$$L(F) = \int_{T_\Omega} F(z)\overline{P_\nu\phi(z)}\Delta^{\nu-\frac{n}{r}}(y)dxdy.$$

Under our hypotheses, $\phi \in L_\nu^{p',q'}$ implies $P_\nu\phi \in A_\nu^{p',q'}$. This proves that $L = L_G$ with $G = P_\nu\phi \in A_\nu^{p',q'}$. $\square$

5.3. Sampling and atomic decomposition for functions in weighted Bergman spaces.

We first recall the definition of the Bergman distance on T_Ω. Define a matrix function $\{g_{j,k}\}_{1\le j,k\le n}$ on Ω by

$$g_{j,k}(z) = \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log B(z,z)$$

where B is the unweighted Bergman kernel of T_Ω. The map $z \in T_\Omega \mapsto \mathcal{H}_z$ with

$$\mathcal{H}_z(u,v) = \sum_{1\le j,k\le n} g_{j,k}(z) u_j \overline{v}_k \qquad \big(u = (u_1, ..., u_n),\ v = (v_1, ..., v_n) \in \mathbf{C}^n\big),$$

defines a Hermitian metric on $\mathbf{C}^n$, called the *Bergman metric*. The *Bergman length* of a smooth path $\gamma : [0,1] \to T_\Omega$ is given by

$$l(\gamma) = \int_0^1 \{H_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\}^{\frac{1}{2}} dt$$

and the Bergman distance $d(z_1, z_2)$ between two points z_1, z_2 of T_Ω is

$$d(z_1, z_2) = \inf_\gamma l(\gamma)$$

where the infimum is taken over all smooth paths $\gamma : [0,1] \to T_\Omega$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$.

Recall that the Bergman distance d is equivalent to the Euclidean distance on the compact sets of $\mathbf{C}^n$ contained in T_Ω and the Bergman balls in T_Ω are relatively compact in T_Ω. Next, let $\mathbf{R}^n$ be the group of translations by vectors in $\mathbf{R}^n$ and let H again denote the simply transitive group of automorphisms of the symmetric cone Ω defined in Section 2. Observe that the group $\mathbf{R}^n \times H$ acts simply transitively on T_Ω and recall that the Bergman distance d is invariant under automorphisms of $\mathbf{R}^n \times H$.

The following Whitney decomposition of the tube domain T_Ω can be proved exactly in the same way as the dyadic decomposition of the symmetric cone Ω (Theorem 2.27 and Corollary 2.44).

THEOREM 5.4. *Given $\delta \in (0,1]$, there exists a sequence $\{z_j\}$ of points of T_Ω such that if $B_j = B_\delta(z_j)$, $B'_j = B_{\frac{\delta}{3}}(z_j)$,*

(i) *the balls B'_j are pairwise disjoint;*
(ii) *the balls B_j form a cover of T_Ω;*
(iii) *there exists a positive integer $N = N(\Omega)$ (independent of δ) such that every point of T_Ω belongs to at most N balls B_j.*

The sequence of points $\{z_j\}$ is called a $\delta-$*lattice* in T_Ω.

To establish the sampling theorem for functions in A^p_ν, we need the next result.

Proposition 5.5. *There exists a positive constant C such that for every holomorphic F in T_Ω and for every $\delta \in (0,1)$, the following properties hold:*

(i) $|F(z)|^p \leq \delta^{-n} \int_{d(z,w)<\delta} |F(w)|^p \frac{dudv}{\Delta^{\frac{2n}{r}}(v)}$;

(ii) *if $d(z,\zeta) < \delta$, then*

$$|F(z) - F(\zeta)|^p \leq C\delta^p \int_{d(z,w)<1} |F(w)|^p \frac{dudv}{\Delta^{\frac{2n}{r}}(v)}.$$

Proof: We recall that the measure $\frac{dudv}{\Delta^{\frac{2n}{r}}(v)}$ is invariant under automorphisms of T_Ω. Therefore, it suffices to prove that

$$|F(i\mathbf{e})|^p \leq \delta^{-n} \int_{d(i\mathbf{e},w)<\delta} |F(w)|^p \frac{dudv}{\Delta^{\frac{2n}{r}}(v)};$$

and that, if $d(i\mathbf{e},\zeta) < \delta$, then

$$|F(i\mathbf{e}) - F(\zeta)|^p \leq C\delta^p \int_{d(i\mathbf{e},w)<1} |F(w)|^p \frac{dudv}{\Delta^{\frac{2n}{r}}(v)}.$$

The first inequality follows from the mean value property and the equivalence between d and the Euclidean distance in a neighborhood of $i\mathbf{e}$. The second inequality follows from the equality

$$F(i\mathbf{e}) - F(\zeta) = \int_{[\zeta, i\mathbf{e}]} \nabla F(w) \cdot dw$$

and from Cauchy estimates

$$\begin{aligned} |\nabla F(w)| &\leq C \int_{B(i\mathbf{e},1)} |F(s+it)| dsdt \\ &\leq C \left(\int_{B(i\mathbf{e},1)} |F(s+it)|^p \frac{dsdt}{\Delta^{\frac{2n}{r}}(t)} \right)^{\frac{1}{p}}. \end{aligned}$$

□

We can now prove the sampling theorem.

Theorem 5.6. *Let $\{z_j\}$ be a $\delta-$lattice in T_Ω, $\delta \in (0,1)$, with $z_j = x_j + iy_j$.*

(i) *There exist a positive constant C_δ such that every $F \in A^p_\nu$ satisfies*

$$\sum_j |F(z_j)|^p \Delta^{\nu + \frac{n}{r}}(y_j) \leq C_\delta \|F\|_{A^p_\nu}.$$

(ii) *Conversely, if δ is small, there is a positive constant C_δ such that every $F \in A^p_\nu$ satisfies*

$$\|F\|_{A^p_\nu} \leq C_\delta \sum_j |F(z_j)|^p \Delta^{\nu + \frac{n}{r}}(y_j).$$

Proof: (i) By Proposition 5.5 (i), for every j,

$$|F(z_j)|^p \leq \delta^{-n} \int_{B'_j} |F(w)|^p \frac{dudv}{\Delta^{\frac{2n}{r}}(v)}.$$

On B'_j, the function $\Delta(v)$ is almost constant; therefore,

$$\begin{aligned}\sum_j |F(z_j)|^p \Delta^{\nu+\frac{n}{r}}(y_j) &\leq C_p\delta^{-n} \sum_j \Delta^{\nu+\frac{n}{r}}(y_j) \int_{B'_j} |F(w)|^p \frac{dudv}{\Delta^{\frac{2n}{r}}(v)} \\ &\leq C_p\delta^{-n} \sum_j \int_{B'_j} |F(w)|^p \Delta^{\nu-\frac{n}{r}}(v) dudv \\ &\leq C_p\delta^{-n} \|F\|_{A^p_\nu}\end{aligned}$$

because the balls B'_j are pairwise disjoint.

(ii) We have

$$\begin{aligned}&\int_{T_\Omega} |F(z)|^p \Delta^{\nu-\frac{n}{r}}(y) dxdy \\ &\leq C_p \sum_j \Delta^{\nu+\frac{n}{r}}(y_j) \int_{B_j} |F(z)|^p \frac{dxdy}{\Delta^{\frac{2n}{r}}(y)} \\ &\leq C'_p \sum_j \Delta^{\nu+\frac{n}{r}}(y_j) \int_{B_j} \left(|F(z_j)|^p + |F(z)-F(z_j)|^p\right) \frac{dxdy}{\Delta^{\frac{2n}{r}}(y)} \\ &\leq C''_p \left(\sum_j \Delta^{\nu+\frac{n}{r}}(y_j) |F(z_j)|^p + \sum_j \Delta^{\nu+\frac{n}{r}}(y_j) \int_{B_j} |F(z)-F(z_j)|^p \frac{dxdy}{\Delta^{\frac{2n}{r}}(y)} \right);\end{aligned}$$

since the invariant measure of B_j is independent of j (and equal to the invariant measure of $B_\delta(\mathbf{e})$). Now, by Proposition 5.5 (ii), we obtain

$$\begin{aligned}&\int_{T_\Omega} |F(z)|^p \Delta^{\nu-\frac{n}{r}}(y) dxdy \\ &\leq C_p \left(\sum_j \Delta^{\nu+\frac{n}{r}}(y_j) |F(z_j)|^p + \delta^p \sum_j \int_{B_j} \int_{d(z,w)<1} |F(w)|^p \frac{dudv}{\Delta^{\frac{2n}{r}}(v)} \frac{dxdy}{\Delta^{\frac{2n}{r}}(y)} \right)\end{aligned}$$

If we show that the sum of the second term above is bounded by a constant C, independent of δ, times the left hand side, then we can choose δ small enough, and conclude the proof.

Notice that, by the finite overlapping property of the balls B_j, for fixed w,

$$\sum_j \int_{z\in B_j, d(z,w)<1} \frac{dxdy}{\Delta^{\frac{2n}{r}}(y)} \leq N \int_{d(z,w)<1} \frac{dxdy}{\Delta^{\frac{2n}{r}}(y)} < C$$

for some universal constant C, by invariance of the distance and of the measure.

Using this fact, and switching the integration order in the second term on the right hand side above, we obtain the desired estimate. □

It is easy to deduce the atomic decomposition from the sampling theorem for values of p for which the weighted Bergman projection P_ν is bounded. More precisely, we have the following theorem (cf. [9]).

THEOREM 5.7. *Assume that P_ν is bounded on L^p_ν and let $\{z_j\}$ be a δ-lattice in T_Ω. Then the following assertions hold:*

(i) *For every complex sequence $\{\lambda_j\}$ such that*

$$\sum_j |\lambda_j|^p \Delta^{\nu+\frac{n}{r}}(y_j) < \infty, \tag{5.8}$$

the series $\sum_j \lambda_j B_\nu(z, z_j)\Delta^{\nu+\frac{n}{r}}(y_j)$ is convergent in A^p_ν. Moreover, its sum F satisfies the inequality

$$\|F\|^p_{A^p_\nu} \leq C \sum_j |\lambda_j|^p \Delta^{\nu+\frac{n}{r}}(y_j).$$

(ii) *For δ small enough, every function $F \in A^p_\nu$ may be written as*

$$F(z) = \sum_j \lambda_j B_\nu(z, z_j)\Delta^{\nu+\frac{n}{r}}(y_j),$$

with

$$\sum_j |\lambda_j|^p \Delta^{\nu+\frac{n}{r}}(y_j) \leq C\|F\|_{A^p_\nu}. \tag{5.9}$$

PROOF: We call l^p_ν the space of complex sequences $\{\lambda_j\}$ which satisfies (5.8).

(i) From part (i) of the sampling theorem (Theorem 5.6), we deduce that the linear operator

$$\begin{array}{rlll} R: & A^p_\nu & \to l^p_\nu & \\ & F & \mapsto & RF = \{F(z_j)\} \end{array}$$

is bounded. Hence its adjoint $R^* : l^{p'}_\nu \to A^{p'}_\nu$ is also bounded. The conclusion follows because

$$R^*\left(\{\lambda_j\}\right)(z) = \sum_j \lambda_j B_\nu(z, z_j)\Delta^{\nu+\frac{n}{r}}(y_j).$$

(ii) From Theorem 5.6 (ii), for δ small enough, we obtain that

$$\|F\|_{A^{p'}_\nu} \leq C\|\{F(z_j)\}\|_{l^{p'}_\nu}.$$

This implies that $R^* : l^p_\nu \to A^p_\nu$ is onto. Moreover, if $\mathcal{N}$ denotes the subspace of l^p_ν consisting of all sequences $\{\lambda_j\}$ such that the sum

$$\sum_j \lambda_j B_\nu(z, z_j)\Delta^{\nu+\frac{n}{r}}(y_j)$$

is identically zero, then the linear map

$$\begin{array}{ll} l^p_\nu/\mathcal{N} & \to A^p_\nu \\ \{\lambda_j\} & \mapsto \sum_j \lambda_j B_\nu(z,z_j)\Delta^{\nu+\frac{n}{r}}(y_j) \end{array}$$

is a bounded isomorphism. The continuity of its inverse gives estimates (5.9).
□

Finally, Theorem 5.7 gives the solution of a so-called Cartan B problem which we now describe. To keep matters simple, we assume that T_Ω is the tube domain in $\mathbf{C}^3$ over the Lorentz cone $\Omega = \Lambda_3$. Thus, $n = 3$ and $r = 2$. Again, we denote by $\mathbf{H}$ the upper half-plane of the complex plane $\mathbf{C}$. For all $p \in [1,\infty)$, it is easy to show that the restriction f of $F \in A^p_\nu(T_\Omega)$ to $\mathbf{H}^2 = \mathbf{H}\times\mathbf{H}$ given by

$$R(F)(z_1,z_2) = F(z_1+z_2, z_1-z_2, 0)$$

belongs to the weighted Bergman space $A^p_\nu(\mathbf{H}^2)$. If dV denotes the Lebesgue measure on $\mathbf{H}^2$, the latter space is the subspace of $L^p(\mathbf{H}^2, (y_1y_2)^{\nu-2}dV(z_1,z_2))$ consisting of holomorphic functions. Moreover, the restriction map[1]

$$R : A^p_\nu(T_{\Lambda_3}) \to A^p_\nu(\mathbf{H}^2)$$

is continuous. We are interested in the range of p for which R is onto. It has been proved in [7] that this is the case when $p \in [1, 2\nu+1)$. Theorem 5.7 leads to an extension of the result to the range $p \in [2\nu+1, 2\nu+2)$. Moreover, there exists a linear extension map.

This application may be extended to all tube domains over symmetric cones. If the rank of the cone is r, $\mathbf{H}^2 = \mathbf{H}\times\mathbf{H}$ should be replaced by $\mathbf{H}^r = \mathbf{H}\times\cdots\times\mathbf{H}$ (r times).

6. Final remarks

6.1. Hardy's inequality, boundary values and Besov spaces.

In this subsection, we report briefly on the generalization of the three problems solved in Section in the upper half-plane. We refer to [4] and [8]. Throughout the subsection, $\{\xi_j\}$ will be a fixed $\frac{1}{2}$-lattice in Ω. We construct a smooth partition of the unity associated with the covering $B_j = B_1(\xi_j)$. For this purpose, we choose a function $\phi_0 \in \mathcal{C}^\infty_c(B_2(\mathbf{e}))$ such that

$$0 \le \phi_0 \le 1 \ \text{ and } \ \phi_0|_{B_1(\mathbf{e})} \equiv 1.$$

[1]It is called a restriction map since it is actually given by a restriction when considering the spherical cone instead of the Lorentz cone.

For every j, we also write $\xi_j = g_j\mathbf{e}$ for some $g_j \in G$. Then we can define $\phi_j(\xi_j) = \phi_0(g_j^{-1}\xi)$ so that

$$\phi_j \in \mathcal{C}_c^\infty(B_2(\xi_j)),\ \ 0 \le \phi_j \le 1 \ \text{ and } \ \phi_j|_{B_j} \equiv 1.$$

We assume that $\xi_0 = \mathbf{e}$ so that there is no ambiguity of notations. Further, by the finite overlapping property, there exists a constant $C > 1$ such that if we define $\Phi(\xi) = \sum_j \phi_j(\xi)$,

$$\frac{1}{C} \le \Phi(\xi) \le C.$$

We also define $\psi_j \in \mathcal{S}$ by $\psi_j = \phi_j/\Phi$.

Proposition 6.10. *The following properties hold:*

(1) $\hat{\psi}_j \in \mathcal{C}_c^\infty(B_2(\xi_j))$;

(2) $0 \le \hat{\psi}_j \le 1$ and $\displaystyle\sum_j \hat{\psi}_j(\xi) = 1\ \forall\ \xi \in \Omega$;

(3) *the functions ψ_j are uniformly bounded in $L^1(\mathbf{R}^n)$, so that there exists a positive constant C such that for all $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$ and for all j,*

$$\|f * \psi_j\|_p \le C\|f\|_p.$$

We introduce a new family of Besov-type spaces $B_\nu^{p,q}$, $1 \le p, q \le \infty$, $\nu \in \mathbf{R}$. They are defined as equivalence classes of tempered distributions by means of the semi-norm

$$\|f\|_{B_\nu^{p,q}} = \left(\sum_j \sum_j \Delta^{-\nu}(\xi_j)\|f * \psi_j\|_p^q\right)^{\frac{1}{q}}, \quad f \in \mathcal{S}'(\mathbf{R}^n).$$

The Besov space $B_\nu^{p,q}$ is a Banach space and does not depend on the choice of $\{\xi_j\}$ and $\{\psi_j\}$.

On the other hand, we introduce a *generalized wave operator* $\Box = \Delta\left(\frac{1}{i}\frac{d}{dx}\right)$ on the cone Ω. That is the differential operator of degree r defined by the equality

$$\Delta\left(\frac{1}{i}\frac{d}{dx}\right)\left(e^{ix\cdot\xi}\right) = \Delta(\xi)e^{ix\cdot\xi}, \quad \xi \in \mathbf{R}^n,$$

which corresponds in cones of rank 1 and 2 to

$$\Box = \frac{1}{i}\frac{d}{dx} \ \text{ in } \ \Omega = (0,\infty)$$

and

$$\Box = \frac{1}{4}\left(-\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) \ \text{ in } \ \Omega = \Lambda_n.$$

The following theorem is proved in [4]:

Theorem 6.11. *Let $\nu > \frac{n}{r} - 1$ and $1 \le p < \infty$. With the notations*

$$q_\nu = 1 + \frac{\nu}{\frac{n}{r} - 1}, \quad q_{\nu,p} = \begin{cases} \frac{\frac{n}{r}-1}{\frac{n}{rp'}-1} q_\nu & \text{if } \frac{n}{r} > p' \\ \infty & \text{otherwise} \end{cases},$$

assume that $2 \le q < q_{\nu,p}$. The following properties are equivalent:

(1) *P_ν extends to a bounded projector from $L^{p,q}_\nu$ to $A^{p,q}_\nu$;*
(2) *the Laplace operator $\mathcal{L}$ is a bounded isomorphism from $B^{p,q}_\nu$ to $A^{p,q}_\nu$;*
(3) *for m large enough, $\Box^m : A^{p,q}_\nu \longrightarrow A^{p,q}_{\nu+mq}$ is a bounded isomorphism.*

In this theorem, assertion (3) generalizes Hardy's inequality for Bergman spaces (subsection 1.5) while assertion (2) implies that the space of boundary value functions of $A^{p,q}_\nu$ functions is the Besov space $B^{p,q}_\nu$, i.e. (2) is a generalization of results of subsection 1.7. For $p = 2$, we have proved part (2) in Theorems 3.29 and 4.12 (see also Lemma 3.27). Moreover, in Corollary 4.22, under the assumption $1 \le q < Q_\nu$, we showed the implication (2) $\Rightarrow$ (3). For more details, the reader should consult [3]. Using Theorem 6.11, four of the authors [4] were able to find other necessary condtions on p, q for the $L^{p,q}_\nu$ boundedness of P_ν. This allows to color in dark parts of the blank regions in the previous figure.

6.2. Projections to Hardy spaces.

It is natural to ask whether the projection P_0, which is the orthogonal projection onto the Hardy space $H^2(T_\Omega)$, which identifies with a closed sub-space of $L^2(\mathbf{R}^n)$, extends to a bounded operator on $L^p(\mathbf{R}^n)$, i.e. under which assumptions on p Theorem 1.55 extends to several variables. The answer has been known for thirty years:

Theorem 6.12. ([12],[22]) *The operator P_0 extends to a bounded operator on $L^p(\mathbf{R}^n)$ only if $p = 2$.*

It is a consequence of the fact that the characteristic function of the unit ball is not a Fourier multiplier of $L^p(\mathbf{R}^n)$ when $p2$. The Bergman projection, that we have studied all along these notes, has a better behavior than the Szegö projection P_0. It still has some mystery, as we have shown, at least for us.

In the one dimensional case, we have seen that Hardy spaces are in some way the limit of weighted Bergman spaces. It is no more true in higher rank.

Indeed, recall that the condition $\nu > \frac{n}{r} - 1$ for Bergman spaces A^p_ν is imposed so that the weight $\Delta^{\nu - \frac{n}{r}}(y) dx dy$ is locally integrable near the topological boundary of T_Ω. We know that $A^p_\nu = \{0\}$ when $\nu \le \frac{n}{r} - 1$. Also, by the Paley-Wiener Theorem we see that the Hardy space in T_Ω should correspond to the value of the parameter $\nu = 0$. It is natural to ask what it is limit space appearing when we let $\nu \to \frac{n}{r} - 1^+$ (compare with the 1-dimensional case in the last paragraph of §1.7). The surprising answer was found by M. Vergne and H. Rossi in the case $p = 2$ (see [11, p. 270]). Namely, we obtain a new holomorphic function space with a norm of Hardy type

$$H^p_\mu = \{F \in \mathcal{H}(T_\Omega) \mid \sup_{y\in\Omega} \int_{\partial\Omega} \int_{\mathbf{R}^n} |F(x + i(y + t))|^p \, dx \, d\mu(t) < \infty\},$$

where μ is a measure supported in the boundary of the cone. For the light-cone Λ_n such measure is explicitly given by:

$$\int_{\partial\Omega} f(t) \, d\mu(t) = \int_{\mathbf{R}^{n-1}} f(|t'|, t') \, \frac{dt'}{|t'|}, \quad f \in C_c(\mathbf{R}^n).$$

In general, the measure μ is a particular case of the so-called *positive Riesz distributions*, and is obtained as the distributional limit:

$$d\mu(t) = \lim_{\nu \to \frac{n}{r} - 1^+} (\nu - \tfrac{n}{r} + 1) \, \Delta^{\nu - \frac{n}{r}}(t) \, \chi_\Omega(t) \, dt.$$

For more information about such Hardy-type spaces, see [14].

Let us mention that, for these new spaces, the behavior of the projector is completely unknown.

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Department of Mathematics, Faculty of Science P.O. Box 812, University of Yaoundé I, Yaoundé, Cameroon
E-mail address: dbekolle@uycdc.uninet.cm

Université d'Orléans, Laboratoire MAPMO B.P. 6759, 45067 Orléans Cedex 2, France
E-mail address: Aline.Bonami@labomath.univ-orleans.fr

Departemento de Matematicás, Universidad Autonoma de Madrid, 28049 Madrid, Spain
E-mail address: gustavo.garrigos@uam.es

Department of Mathematics, Faculty of Science P.O. Box 812, University of Yaoundé I, Yaoundé, Cameroon
E-mail address: cnana@uycdc.uninet.cm

Dipartimento di Matematica, Politecnico di Torino, 10129 Torino, Italy
E-mail address: peloso@calvino.polito.it

Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail address: fricci@sns.it