Connectivity in the set of Tight Frame Wavelets (TFW)

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Abstract

We introduce new ideas to treat the problem of connectivity of wavelets. We develop a method which produces intermediate paths of Tight Frame Wavelets (TFW). Using this method we prove that a large class of TFW-s, with only mild conditions on their spectrum, are arcwise connected.

1 Introduction

In the theory of wavelets some prominence has been given to the question of connectivity. The significance of the question (as well as the realization that it is probably not an easy question) has been further emphasized in [1]. As far as we know the general question still remains open, despite strong results given in [8], [2], and [7]. Following the development of tight frame wavelets in [4], the question is naturally extended to this larger class of wavelets. The main contribution of this article is to this extended question on connectivity. We develop a new technique here, particularly suitable for the set of tight frame wavelets. As a consequence we prove the connectivity of a very large class of tight frame wavelets. And, although we do not resolve the question completely, we hope to convince the reader that we are "almost there". Let us now be more precise; in the rest of this introduction, we shall explain the necessary notions, and describe the current state of affairs and the nature of our contribution.

Following [4], we shall say that a function $\psi \in L^2(\mathbb{R})$ is a **tight frame wavelet** (TFW)

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if the collection of dyadic dilates and integer translates given by

$$\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k): \;\; j \in \mathbb{Z},\; k \in \mathbb{Z}\} \tag{1.1}$$

is a tight frame (with constant 1) for $L^2(\mathbb{R})$; that is, for all $f \in L^2(\mathbb{R})$,

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\, \psi_{j,k} = f\,, \tag{1.2}$$

unconditionally in $L^2(\mathbb{R})$. If we require more, that is, that the system $\{\psi_{j,k}(x)\}$ is an orthonormal basis for $L^2(\mathbb{R})$, then we shall say that ψ is an **orthonormal wavelet**. It turns out (see [3], Chapter 7) that (1.2) is equivalent to $\psi \in L^2(\mathbb{R})$ satisfying the following two equations:

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 \; = 1 \qquad \text{for a. e. } \xi \in \mathbb{R}, \tag{1.3}$$

and

$$t_q(\xi, \psi) = \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\,\overline{\widehat{\psi}(2^j(\xi + 2q\pi))} = 0 \qquad \text{for a. e. } \xi \in \mathbb{R}, \text{ and all } q \in 2\mathbb{Z}+1\,. \tag{1.4}$$

An easy consequence is that the set of all TFW-s is a subset of the unit ball in $L^2(\mathbb{R})$, that is, for every $\psi \in$ TFW we have $||\psi||_2 \le 1$. Moreover, a function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if ψ is a TFW and $||\psi||_2 = 1$ (see [3], Chapter 7).

As we already mentioned, two interesting open questions developed. **Is the set of all orhtonormal wavelets** (a subset of the unit sphere in $L^2(\mathbb{R})$) **connected in the $L^2(\mathbb{R})$ metric? Is the set of all TFW-s** (a subset of the unit ball in $L^2(\mathbb{R})$) **connected in the $L^2(\mathbb{R})$ metric?** In this article we are primarily concerned with the second question. Regarding the first question, let us mention only two strong results in a positive direction. It is shown in [8] that the set of MRA orthonormal wavelets is connected in the $L^2(\mathbb{R})$ metric. Secondly, D. Speegle has shown in [7] that the set of Minimally Supported Frequency (MSF) wavelets (orthonormal wavelets of the form $\psi = (\chi_K)\check{}\,$) is also connected in the $L^2(\mathbb{R})$ metric.

Until now less has been achieved for the set of TFW-s. D. Speegle's idea has been succesfully transformed into the setting of TFW-s (see [5]). On the other hand, the ideas from [8] provided only partial results for TFW-s (see section 4 in [4] for details).

In this article we provide a completely new set of ideas to treat the second question mentioned above. They are specifically tailored for TFW-s and although we do not resolve the question completely, we establish the strongest results to date in the positive direction. More precisely, we prove that the set $\mathcal{K}_\tau \cup \mathcal{K}_d$ is arcwise connected in $L^2(\mathbb{R})$. The first set $\mathcal{K}_\tau$ consists of all TFW-s ψ for which there exists $\varepsilon > 0$ such that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|)} \left|\operatorname{supp}\hat{\psi} \cap (2n\pi - \varepsilon\,,\, 2n\pi + \varepsilon)\right| < \infty\,,$$

where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$. In this paper we let $\operatorname{supp} \widehat{\psi} = \{\xi \in \mathbb{R} : \widehat{\psi}(\xi) \neq 0\}$, a set which is uniquely defined, up to a null set. The second set $\mathcal{K}_d$ consists of TFW-s ψ for which there exists $\varepsilon \in (0, \pi]$ such that

$$\left| \limsup_{n \to \infty} \frac{1}{2^n} (\operatorname{supp} \hat{\psi} \cap 2^n J_0^\varepsilon) \right| = 0$$

where $J_0^\varepsilon = (-\varepsilon, -\frac{\varepsilon}{2}] \cup [\frac{\varepsilon}{2}, \varepsilon)$. Observe that both sets contain all TFW-s which are band-limited, that is, those TFW-s ψ for which $\operatorname{supp} \hat{\psi}$ is bounded. Furthermore, we show that the entire path that we construct remains within $\mathcal{K}_\tau$ (respectively, $\mathcal{K}_d$) if the end points are from $\mathcal{K}_\tau$ (respectively, $\mathcal{K}_d$). We shall start by explaining basic ideas in Section 2. The construction of the path is given in Section 3 and the connectivity of the sets $\mathcal{K}_\tau$ and $\mathcal{K}_d$ is treated in Sections 4 and 5 respectively.

2 The basic ideas

It is easy to see from (1.3) and (1.4) that if $0 < \varepsilon \leq \pi$ the function $\psi^\varepsilon \in L^2(\mathbb{R})$ given by $\widehat{\psi^\varepsilon} = \chi_{J_0^\varepsilon}$, where $J_0^\varepsilon = (-\varepsilon, -\frac{\varepsilon}{2}] \cup [\frac{\varepsilon}{2}, \varepsilon)$, is a tight frame wavelet. To prove connectivity inside TFW it is enough to join a given $\psi \in TFW$ with ψ^ε by a continuous arc inside TFW. The idea is to move points from $K = \operatorname{supp} \hat{\psi}$ into $(-\varepsilon, \varepsilon)$ in such a way that (1.3) and (1.4) are preserved. Related to (1.3) we will often make use of the **power set** $[A]$ of a measurable set $A \subset \mathbb{R}$ which is defined as

$$[A] = \{2^j \xi : j \in \mathbb{Z}, \xi \in A\} = \bigcup_{j \in \mathbb{Z}} 2^j A \tag{2.1}$$

Related to (1.4) we shall consider the "periodized" sets $\bigcup_{n \in \mathbb{Z}, n \neq 0} (A + 2n\pi)$ and their restrictions to $K = \operatorname{supp} \hat{\psi}$:

$$\tau_K(A) = \left(\bigcup_{n \in \mathbb{Z}, n \neq 0} (A + 2n\pi) \right) \cap K \,. \tag{2.2}$$

Observe that the term $n = 0$ is not considered in $\tau_K(A)$. The interplay between dilations and translations that appears in (1.4) makes the problem of connectivity a difficult one.

We start by proving two results that contain the basic ideas. The first one shows how to modify a TFW to obtain a new one, which will be used to build the arc needed for connectivity. Given a set $E \subset \mathbb{R}$ of finite measure and a function ψ we define a new function ψ_E by

$$\widehat{\psi_E}(\xi) = \begin{cases} \widehat{\psi}(\xi) & \text{if } \xi \in \mathbb{R} \setminus [E] \\ \chi_E(\xi) & \text{if } \xi \in [E] \end{cases} \tag{2.3}$$

The next proposition gives sufficient conditions on E to show that ψ_E is a TFW when ψ is also a TFW.

Proposition 2.1. *Let ψ be a TFW. Suppose that E is a measurable subset of $\mathbb{R}$ such that*

$$i)\ \sum_{j\in\mathbb{Z}} \chi_E(2^j\xi) \le 1 \text{ for a.e } \xi \in \mathbb{R}\,, \qquad ii)\ \sum_{k\in\mathbb{Z}} \chi_E(\xi + 2k\pi) \le 1 \text{ for a.e } \xi \in \mathbb{R}\,,$$

and

$$iii)\ \tau_K(E) = \Big(\bigcup_{k\in Z,\, k\neq 0} (E + 2k\pi)\Big) \cap (supp\,\widehat{\psi}) \subset [E]\,.$$

Then, the function ψ_E defined by (2.3) is again a TFW.

Proof. Since $|E| \le 2\pi$ due to $ii)$, it is clear that $\psi_E \in L^2(\mathbb{R})$. Thus, we need to prove

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi_E}(2^j\xi)|^2 = 1 \qquad \text{for a. e. } \xi \in \mathbb{R}, \tag{2.4}$$

and

$$t_q(\xi, \psi_E) = \sum_{j=0}^{\infty} \widehat{\psi_E}(2^j\xi)\, \overline{\widehat{\psi_E}(2^j(\xi + 2q\pi))} = 0 \qquad \text{for a. e. } \xi \in \mathbb{R}, \text{ and all } q \in 2\mathbb{Z}+1\,, \tag{2.5}$$

using (1.3) and (1.4), and the properties of E.

If $\xi \notin [E]$, $2^j\xi \notin [E]$ for any $j \in \mathbb{Z}$ by the definition of the power set. Thus, if $\xi \notin [E]$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi_E}(2^j\xi)|^2 = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1 \ \text{ for a. e. } \xi \notin [E]$$

Now, if $\xi \in [E]$, $2^j\xi \in [E]$ for all $j \in \mathbb{Z}$. Thus

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi_E}(2^j\xi)|^2 = \sum_{j\in\mathbb{Z}} \chi_E(2^j\xi)\,.$$

If $\xi \in [E]$, $\xi = 2^{-j_0}\eta$ for some $j_0 \in \mathbb{Z}$ and $\eta \in E$. Thus,

$$\sum_{j\in\mathbb{Z}} \chi_E(2^j\xi) \ge \chi_E(2^{j_0}\xi) = \chi_E(\eta) = 1\,.$$

This inequality together with $i)$ shows

$$\sum_{j\in\mathbb{Z}} \chi_E(2^j\xi) = 1 \ \text{ for a. e. } \xi \in [E]$$

This proves (2.4).

We need to prove (2.5). To do this suppose, first, that $\xi \in [E]$. Then, there exists $m \in \mathbb{Z}$ such that $2^m\xi \in E$. We consider first the case $m < 0$. In this case, if $j = 0, 1, 2, \ldots$, $\widehat{\psi_E}(2^j\xi) = \chi_E(2^j\xi) = \delta_{m,j} = 0$, which gives $t_q(\xi, \psi_E) = 0$.

Consider now the case $m \ge 0$. Then

$$t_q(\xi, \psi_E) = \sum_{j=0}^{\infty} \widehat{\psi_E}(2^j\xi)\, \overline{\widehat{\psi_E}(2^j(\xi + 2q\pi))} = \overline{\widehat{\psi_E}(2^m\xi + 2^{m+1}q\pi)}\,.$$

The element $\eta = 2^m\xi + 2^{m+1}q\pi \in \bigcup_{k\in Z,\, k\neq 0}(E+2k\pi)$ since $q \neq 0$. Observe that $\eta \notin E$ since $ii)$ implies that the sets $E+2k\pi$, $k \in \mathbb{Z}$, are disjoint. If $\eta \in [E]$ we have $\widehat{\psi_E}(\eta) = \chi_E(\eta) = 0$, obtaining $t_q(\xi, \psi_E) = 0$. If $\eta \notin [E]$ it does not belong to $\operatorname{supp}\widehat{\psi}$ by $iii)$; hence in this case $\widehat{\psi_E}(\eta) = \widehat{\psi}(\eta) = 0$, obtaining, again, $t_q(\xi, \psi_E) = 0$.

Finally, we must take care of the case $\xi \notin [E]$. Given q odd, assume first that $\xi + 2q\pi \notin [E]$. Then,

$$\widehat{\psi_E}(2^j\xi) = \widehat{\psi}(2^j\xi) \qquad \text{and} \qquad \widehat{\psi_E}(2^j(\xi + 2q\pi)) = \widehat{\psi}(2^j(\xi + 2q\pi))$$

for all $j \in \mathbb{Z}$. Hence, by (1.4), $t_q(\xi, \psi_E) = t_q(\xi, \psi) = 0$.

If $\xi + 2q\pi = \xi' \in [E]$, using the equality

$$t_q(\xi, \psi) = \sum_{j=0}^{\infty} \hat{\psi}(2^j(\xi' - 2q\pi))\overline{\hat{\psi}(2^j\xi')} = \overline{t_{-q}(\xi', \psi)}\,, \tag{2.6}$$

we obtain $t_q(\xi, \psi_E) = \overline{t_{-q}(\xi', \psi_E)} = 0$ since now we have $\xi' = \xi + 2q\pi \in [E]$ and we have shown above that in this case $t_{-q}(\xi', \psi_E) = 0$ for all q odd. This finishes the proof of the Proposition. □

Our second result establishes sufficient conditions to prove the continuity, in the $L^2(\mathbb{R})$ norm, of paths of the form $\{\psi_{E_t} : 0 \le t \le 1\}$, where each ψ_{E_t} is defined as in (2.3)

Proposition 2.2. *Let $\{E_t : 0 \le t \le 1\}$ be a collection of measurable sets of finite measure such that*

$$i) \ \lim_{t\to t'} |E_t \triangle E_{t'}| = 0 \qquad \text{and} \qquad ii) \ \lim_{t\to t'} |\widetilde{E_t} \triangle \widetilde{E_{t'}}| = 0\,,$$

where for a set A we define $\widetilde{A} = [A] \cap J_0^{\varepsilon}$. Then, if $\psi \in L^2(\mathbb{R})$ and ψ_{E_t} is defined by (2.3) with E replaced by E_t, the path $\{\psi_{E_t} : 0 \le t \le 1\}$ is continuous in the $L^2(\mathbb{R})$ metric, that is

$$\lim_{t\to t'} \int_{\mathbb{R}} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi = 0\,. \tag{2.7}$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi = &\int_{[E_t]\cap[E_{t'}]} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi \\ &+ \int_{[E_t]\triangle[E_{t'}]} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi \\ &+ \int_{\mathbb{R}\setminus([E_t]\cup[E_{t'}])} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi \end{aligned} \tag{2.8}$$

For the first integral in (2.8) we have

$$[E_t]\cap[E_{t'}] = (E_t\cap E_{t'})\cup\{([E_t] \cap [E_{t'}]) \cap ((E_t \setminus E_{t'}) \cup (E_{t'} \setminus E_t))\}\cup\{([E_t] \cap [E_{t'}]) \setminus (E_t \cup E_{t'})\}$$

with disjoint union. Thus,

$$\int_{[E_t]\cap[E_{t'}]} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi = |([E_t]\cap[E_{t'}])\cap((E_t\setminus E_{t'})\cup(E_{t'}\setminus E_t))| \le |E_t\triangle E_{t'}|\,. \quad (2.9)$$

For the second integral in (2.8) we have

$$[E_t]\triangle[E_{t'}] = ([E_t]\setminus[E_{t'}])\cup([E_{t'}]\setminus[E_t])\,.$$

Since $|\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2 \le 2(|\widehat{\psi_{E_t}}(\xi)|^2 + |\widehat{\psi_{E_{t'}}}(\xi)|^2)$ we deduce

$$\begin{aligned}\int_{[E_t]\triangle[E_{t'}]} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi &\le 2\int_{[E_t]\setminus[E_{t'}]} (\chi_{E_t}(\xi) + |\widehat{\psi}(\xi)|^2)\, d\xi \\ &\quad + 2\int_{[E_{t'}]\setminus[E_t]} (|\widehat{\psi}(\xi)|^2 + \chi_{E_{t'}}(\xi))\, d\xi \\ \le 2\int_{[E_t]\setminus[E_{t'}]} |\widehat{\psi}(\xi)|^2\, d\xi + 2|E_t\setminus E_{t'}| &+ 2|E_{t'}\setminus E_t| + 2\int_{[E_{t'}]\setminus[E_t]} |\widehat{\psi}(\xi)|^2\, d\xi \\ = 2|E_t\triangle E_{t'}| + 2\int_{[E_t]\triangle[E_{t'}]} |\widehat{\psi}(\xi)|^2\, d\xi\,. \end{aligned} \quad (2.10)$$

The last integral in (2.8) is zero since for $\xi\in\mathbb{R}\setminus([E_t]\cup[E_{t'}])$ we have $\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi) = \widehat{\psi}(\xi) - \widehat{\psi}(\xi) = 0$. Thus, from (2.8), (2.9), and (2.10) we deduce

$$\int_{\mathbb{R}} |\widehat{\psi_{E_t}}(\xi) - \widehat{\psi_{E_{t'}}}(\xi)|^2\, d\xi \le 3|E_t\triangle E_{t'}| + 2\int_{[E_t]\triangle[E_{t'}]} |\widehat{\psi}(\xi)|^2\, d\xi\,. \quad (2.11)$$

The first term in (2.11) clearly tends to zero as $t\to t'$ by hypothesis *i*). It suffices to show

$$\lim_{t\to t'} \int_{[E_t]\triangle[E_{t'}]} |\widehat{\psi}(\xi)|^2\, d\xi = 0\,. \quad (2.12)$$

Since for a set A we have $[A] = [\widetilde{A}] = \bigcup_{l\in\mathbb{Z}} 2^l\widetilde{A}$ with disjoint union, we obtain

$$[E_t]\triangle[E_{t'}] = [\widetilde{E_t}]\triangle[\widetilde{E_{t'}}] = \bigcup_{l\in\mathbb{Z}} 2^l(\widetilde{E_t}\triangle\widetilde{E_{t'}}) = [\widetilde{E_t}\triangle\widetilde{E_{t'}}]$$

Thus,

$$\int_{[E_t]\triangle[E_{t'}]} |\widehat{\psi}(\xi)|^2\, d\xi = \int_{[\widetilde{E_t}\triangle\widetilde{E_{t'}}]} |\widehat{\psi}(\xi)|^2\, d\xi = \int_{\widetilde{E_t}\triangle\widetilde{E_{t'}}} \left(\sum_{j\in\mathbb{Z}} 2^j|\widehat{\psi}(2^ju)|^2\right) du\,,$$

where we have made the change of variables $\xi = 2^ju$ and used $[A] = \cup_{j\in\mathbb{Z}} 2^jA$. We observe that the function $G(u) = \sum_{j\in\mathbb{Z}} 2^j|\widehat{\psi}(2^ju)|^2$ is in $L^1(J_0^\varepsilon)$ (in fact, $\int_{J_0^\varepsilon} G(u)\, du = \int_{\mathbb{R}} |\widehat{\psi}(\xi)|^2\, d\xi$). Hypothesis *ii*) and an application of the Lebesgue dominated convergence theorem give (2.12) (see exercise 1.12 or Theorem 6.11 in [6]). This finishes the proof of Proposition 2.2. $\square$

Remark. The example $E_m = 2^{-m} J_0^\varepsilon$, $m \in \mathbb{N}$, and $E = \emptyset$ shows that $\lim_{m\to\infty} |E_m \triangle E| = \lim_{m\to\infty} |E_m| = 0$, while $\lim_{m\to\infty} |\widetilde{E_m} \triangle \widetilde{E}| = \lim_{m\to\infty} |\widetilde{E_m}| = \lim_{m\to\infty} |\widetilde{J_0^\varepsilon}| = \varepsilon$. This shows that $i)$ does not imply $ii)$ of Proposition 2.2. On the other hand, if $E_m = 2^{-m} J_0^\varepsilon$, $m \in \mathbb{N}$, and $E = J_0^\varepsilon$ we have $\lim_{m\to\infty} |\widetilde{E_m} \triangle \widetilde{E}| = 0$, while $\lim_{m\to\infty} |E_m \triangle E| = \lim_{m\to\infty}(|E_m| + |E|) = \lim_{m\to\infty}(2^{-m}\varepsilon + \varepsilon) = \varepsilon$. This shows that $ii)$ does not imply $i)$ of Proposition 2.2.

To finish this section we present a result that follows easily from Propositions 2.1 and 2.2. This is the type of result that will be extended to more general settings in the sections to come.

Corollary 2.3. *Let $\psi \in TFW$ with $K = supp\,\widehat{\psi}$ and suppose that there exists $\varepsilon \in (0, \pi)$ such that*

$$K \cap (2k\pi + (-\varepsilon, \varepsilon)) = \emptyset \qquad \text{for all} \quad k \in \mathbb{Z} \setminus \{0\}. \tag{2.13}$$

Then, there is a continuous path $\{\psi_t : 0 \le t \le 1\} \subset TFW$ such that $\psi_0 = \psi$ and $\psi_1 = \psi^\varepsilon$ in $L^2(\mathbb{R})$, where $\widehat{\psi^\varepsilon} = \chi_{J_0^\varepsilon}$.

Proof. Let $E_t = (-\frac{\varepsilon}{2}(1+t), -\frac{\varepsilon}{2}] \cup [\frac{\varepsilon}{2}, \frac{\varepsilon}{2}(1+t))$, $0 \le t \le 1$, and define $\psi_t = \psi_{E_t}$ (see (2.3)). It is clear that $\psi_0 = \psi$ and $\psi_1 = \psi_\varepsilon$. The continuity of the path follows from Proposition 2.2 since $|\widetilde{E_t} \triangle \widetilde{E_{t'}}| = |E_t \triangle E_{t'}| = \varepsilon|t - t'| \to 0$ as $t \to t'$. To prove that each ψ_t is a TFW apply Proposition 2.1 to $E = E_t$ (notice that $iii)$ follows from (2.13)). $\Box$

3 Dynamics of the construction

We show in this section how to obtain measurable sets E that satisfy the hypotheses of Proposition 2.1. For $0 < \varepsilon \le \pi$ let $J_0^\varepsilon = (-\varepsilon, -\frac{\varepsilon}{2}] \cup [\frac{\varepsilon}{2}, \varepsilon)$ and define

$$J_N^\varepsilon = 2^{-N} J_0^\varepsilon, \qquad N = 1, 2, 3, \ldots .$$

Recall that for a measurable set $A \subset \mathbb{R}$ the **power set** of A is defined by

$$[A] = \{2^l \xi : \xi \in A, l \in \mathbb{Z}\} = \bigcup_{l \in \mathbb{Z}} 2^l A$$

(as in (2.1)). Let $K \subset \mathbb{R}$ be a fixed, measurable set; recall that for a set $B \subset \mathbb{R}$ the 2π-translates of B restricted to K are defined by

$$\tau_K B = \Big(\bigcup_{k \in \mathbb{Z},\ k \neq 0} (2\pi k + B) \Big) \cap K \tag{3.1}$$

(as in (2.2)). Given a measurable set $I \subset J_0^\varepsilon$, define $I_0 = I$, $I_1 = ([\tau_K I_0] \setminus [I_0]) \cap J_1^\varepsilon$, and, recurrently

$$I_{N+1} = ([\tau_K I_N] \setminus \bigcup_{j=0}^{N} [I_j]) \cap J_{N+1}^\varepsilon, \qquad N = 1, 2, 3, \ldots \tag{3.2}$$

Finally, define

$$E_I = \bigcup_{N=0}^{\infty} I_N \,. \tag{3.3}$$

Observe that

$$[I_{N+1}] = [\tau_K I_N] \setminus \bigcup_{j=0}^{N} [I_j]\,, \qquad N = 0, 1, 2, \dots\,. \tag{3.4}$$

Hence, the sets $[I_N], N = 0, 1, 2, \dots$, are all mutually disjoint. Moreover, from (A1) in Appendix we deduce:

$$[E_I] = \bigcup_{N=0}^{\infty} [I_N] \qquad \text{(disjoint union)}. \tag{3.5}$$

Lemma 3.1. *If I is a measurable subset of J_0^ε, $0 < \varepsilon \leq \pi$, the set E_I defined by (3.3) satisfies i), ii), and iii) of Proposition 2.1*

Proof. By the definition of E_I (see (3.3)) and the disjointness of the sets $I_N, N = 0, 1, 2, \dots$ we can write

$$\sum_{j\in\mathbb{Z}} \chi_{E_I}(2^j\xi) = \sum_{N=0}^{\infty}\sum_{j\in\mathbb{Z}} \chi_{I_N}(2^j\xi) = \sum_{N=0}^{\infty}\sum_{j\in\mathbb{Z}} \chi_{2^{-j}I_N}(\xi)\,. \tag{3.6}$$

For each N fixed, the sets $2^{-j}I_N, j \in \mathbb{Z}$ are disjoint; thus

$$\sum_{j\in\mathbb{Z}} \chi_{2^{-j}I_N}(\xi) = \chi_{\cup_{j\in\mathbb{Z}}2^{-j}I_N}(\xi) = \chi_{[I_N]}(\xi)\,.$$

This last equality together with (3.6) allows us to obtain

$$\sum_{j\in\mathbb{Z}} \chi_{E_I}(2^j\xi) = \sum_{N=0}^{\infty} \chi_{[I_N]}(\xi) = \chi_{[E_I]} \leq 1\,,$$

by (3.5). This shows i) of Proposition 2.1.

The set E_I is contained in $(-\varepsilon, \varepsilon) \subset (-\pi, \pi)$, so that ii) of Proposition 2.1 is immediate. It remains to prove $\tau_K(E_I) \subset [E_I]$, where $K = \operatorname{supp}\widehat{\psi}$ for a TFW ψ. By (A6) in Appendix

$$\tau_K(E_I) = \tau_K(\bigcup_{N=0}^{\infty} I_N) = \bigcup_{N=0}^{\infty} \tau_K(I_N) \subset \bigcup_{N=0}^{\infty} [\tau_K(I_N)]\,.$$

By (3.4), $[\tau_K(I_N)] \subset [I_{N+1}] \cup (\bigcup_{j=0}^{N}[I_j]) = \bigcup_{j=0}^{N+1}[I_j]$. Thus, by (3.5):

$$\tau_K(E_I) \subset \bigcup_{N=0}^{\infty}\bigcup_{j=0}^{N+1} [I_j] = \bigcup_{N=0}^{\infty} [I_N] = [E_I]\,.$$

□

Lemma 3.1 and Proposition 2.1 provide us with a way of constructing Tight Frame Wavelets starting from a Tight Frame Wavelet ψ and a measurable subset I contained in J_0^ε. If we start with the Shannon wavelet, that is ψ is given by $\widehat{\psi} = \chi_S$ where $S = (-2\pi, -\pi) \cap (\pi, 2\pi)$, we obtain several Tight Frame Wavelets.

Corollary 3.2. *Let $I \subset J_0^\varepsilon$ measurable, $K = S$, and E_I defined in (3.3). Then $S_I = (S \setminus [E_I]) \cap E_I$ is a Tight Frame Set, that is $(\chi_{S_I})\check{}$ is a Tight Frame Wavelet.*

Proof. By definition 2.3 with $\psi = (\chi_S)\check{}$ we obtain

$$\widehat{\psi_{E_I}}(\xi) = \begin{cases} 1 & \text{if } \xi \in S \setminus [E_I] \\ \chi_{E_I}(\xi) & \text{if } \xi \in [E_I] \end{cases}$$

Thus, $\widehat{\psi_{E_I}} = \chi_{S_I}$. Apply Lemma 3.1 and Proposition 2.1. □

4 Connectivity of the set $\mathcal{K}_\tau$

The construction presented in Section 3 together with the results of Section 2 are applied to show that the set $\mathcal{K}_\tau$ (see definition below) is pathwise connected.

Definition 4.1. The set $\mathcal{K}_\tau$ is the set of all $\psi \in TFW$ such that there exists $\varepsilon \in (0, \pi]$ such that if $K = \operatorname{supp} \widehat{\psi}$,

$$\sum_{|n|\geq 1} \frac{1}{|n|} |K \cap (2n\pi + (-\varepsilon, \varepsilon))| < \infty. \tag{4.1}$$

Recall that the function ψ^ε given by $\widehat{\psi^\varepsilon} = \chi_{J_0^\varepsilon}$, where $J_0^\varepsilon = (-\varepsilon, \frac{\varepsilon}{2}] \cup [\frac{\varepsilon}{2}, \varepsilon)$, is a TFW.

Theorem 4.1. *Let $\psi \in \mathcal{K}_\tau$ and $\varepsilon \in (0, \pi]$ associated with ψ by Definition 4.1 . Then, there is a path $\{\psi_t : 0 \leq t \leq 1\} \subset \mathcal{K}_\tau$ continuous in the $L^2(\mathbb{R})$-metric and such that $\psi_0 = \psi$ and $\psi_1 = \psi^\varepsilon$. Moreover, for all $t \in [0, 1]$, supp $\widehat{\psi_t} \subset K \cup (-\varepsilon, \varepsilon)$, where $K =$ supp $\widehat{\psi}$.*

Proof. For $t \in [0, 1]$ let

$$I(t) = \left(-\frac{\varepsilon}{2}(1 + t), -\frac{\varepsilon}{2}\right] \cup \left[\frac{\varepsilon}{2}, \frac{\varepsilon}{2}(1 + t)\right) .$$

Define $E_t = E_{I(t)}$ where $E_{I(t)}$ is defined as in (3.3) starting with $I = I(t)$. We then set $\psi_t = \psi_{E_t}$ as in definition 2.3, that is

$$\widehat{\psi_t}(\xi) = \begin{cases} \widehat{\psi}(\xi) & \text{if } \xi \in \mathbb{R} \setminus [E_t] \\ \chi_{E_t}(\xi) & \text{if } \xi \in [E_t] \end{cases}$$

We claim that the family $\{\psi_t : t \in [0, 1]\}$ has the properties stated in this theorem. From Lemma 3.1 and Proposition 2.1 we know that each ψ_t is a TFW; moreover, the support of $\widehat{\psi_t}$ is contained in $(\operatorname{supp} \widehat{\psi}) \cup (-\varepsilon, \varepsilon) = K \cup (-\varepsilon, \varepsilon)$, so that $\psi_t \in \mathcal{K}_\tau$ for all $t \in [0, 1]$.

Clearly $E_0 = I(0) = \emptyset$ a.e. and $E_1 = I(1) = J_0^\varepsilon$. Therefore, $\psi_0 = \psi$ and $\psi_1 = \psi^\varepsilon$ in $L^2(\mathbb{R})$, since $[J_0^\varepsilon] = \mathbb{R}$. Thus, we only need to check the continuity of the path. This result follows if we show $i)$ and $ii)$ of Proposition 2.2 assuming we start with $\psi \in \mathcal{K}_\tau$. To do this we need the following Lemmas.

Lemma 4.2. *Let $A \subset (-\varepsilon,\varepsilon)$, $0<\varepsilon\le\pi$. Then, there exists $C>0$ such that*

$$|[\tau_K(A)] \cap J_0^\varepsilon| \le C\varepsilon \sum_{|n|\ge 1} \frac{1}{|n|} |K \cap (2n\pi + A)| \,.$$

Proof. By the definition of $\tau_K(A)$ and (A1) in Appendix we deduce

$$[\tau_K(A)] = \left[\bigcup_{n\in\mathbb{Z},\ n\neq 0} K \cap (2n\pi + A) \right] = \bigcup_{|n|\ge 1} [K \cap (2n\pi + A)] \,.$$

Notice that if $B \subset J_{-l}^\varepsilon$, then $|\tilde{B}| = |[B] \cap J_0^\varepsilon| = 2^{-l}|B|$. Now, for $|n| \ge 1$, using that $A \subset (-\varepsilon,\varepsilon)$ we have $2n\pi + A \subset J_{-l}^\varepsilon \cup J_{-l-1}^\varepsilon$ for some integer $l = l(n) \ge 1$, with $2|n|\pi \sim 2^{l(n)}\varepsilon$, that is $2^{-l(n)} \sim \frac{\varepsilon}{2|n|\pi}$. Hence, the previous observation with $B = K \cap (2n\pi + A)$ gives,

$$\begin{aligned} |[\tau_K(A)] \cap J_0^\varepsilon| &\le \sum_{|n|\ge 1} |[K \cap (2n\pi + A)] \cap J_0^\varepsilon| \\ &\le C \sum_{|n|\ge 1} 2^{-l(n)} |K \cap (2n\pi + A)| \\ &\le C'\varepsilon \sum_{|n|\ge 1} \frac{1}{|n|} |K \cap (2n\pi + A)| \end{aligned}$$

□

Let us now prove that the hypotheses *i)* and *ii)* in Proposition 2.2 are always satisfied in this setting.

Lemma 4.3. *Let $K \subset \mathbb{R}$ be a measurable set such that (4.1) holds for some $\varepsilon > 0$. Then, the map $I \longrightarrow E_I$ described before Lemma 3.1 (associated with K via τ_K) satisfies*

$$|E_I \triangle E_{I'}| \longrightarrow 0, \text{ and } |\widetilde{E_I} \triangle \widetilde{E_{I'}}| \longrightarrow 0, \text{ as } |I \triangle I'| \longrightarrow 0 \,,$$

where I and I' are measurable subsets of J_0^ε and for a set $A \subset \mathbb{R}$ we have $\widetilde{A} = [A] \cap J_0^\varepsilon$.

Proof. Since $I_N, I'_N \subset J_N^\varepsilon$ and the sets J_N^ε, $N = 0,1,2,\dots$ are disjoint we have

$$E_I \triangle E_{I'} = \bigcup_{N=0}^{\infty} I_N \triangle I'_N \qquad \text{and} \qquad \widetilde{E_I} \triangle \widetilde{E_{I'}} \subset \bigcup_{N=0}^{\infty} 2^N (I_N \triangle I'_N).$$

Hence

$$|E_I \triangle E_{I'}| \le \sum_{N=0}^{\infty} |I_N \triangle I'_N| \qquad \text{and} \qquad |\widetilde{E_I} \triangle \widetilde{E_{I'}}| \le \sum_{N=0}^{\infty} 2^N |I_N \triangle I'_N|,$$

so that it suffices to show

$$\sum_{N=0}^{\infty} 2^N |I_N \triangle I'_N| \to 0 \qquad \text{as} \qquad |I \triangle I'| \to 0 \tag{4.2}$$

Since $2^{N+1} I_{N+1}$ is contained in J_0^ε, the definition of I_{N+1} (see (3.2)) gives,

$$\begin{aligned} 2^{N+1}|I_{N+1}| = |\widetilde{I_{N+1}}| &\le |[\tau_K(I_N)] \cap J_0^\varepsilon| \\ &\le C\varepsilon \sum_{|n|\ge 1} \frac{1}{|n|} |K \cap (2n\pi + I_N)|, \end{aligned} \tag{4.3}$$

where the last inequality is due to Lemma 4.2. Using (A8), (A3), (A7), (A1), and (A3) we obtain

$$\begin{aligned} 2^{N+1}(I_{N+1} \triangle I'_{N+1}) &= \left\{ ([\tau_K I_N] \setminus \bigcup_{l=0}^{N} [I_l]) \triangle ([\tau_K I'_N] \setminus \bigcup_{l=0}^{N} [I'_l]) \right\} \cap J_0^\varepsilon \\ &\subset \left\{ [\tau_K(I_N \triangle I'_N)] \cup \left(\left[\left(\cup_{l=0}^N I_l \right) \triangle \left(\cup_{l=0}^N I'_l \right) \right] \right) \right\} \cap J_0^\varepsilon \,. \end{aligned}$$

By Lemma 4.2 applied to $I_N \triangle I'_N$ we deduce

$$\begin{aligned} 2^{N+1}|I_{N+1} \triangle I'_{N+1}| &\le C\varepsilon \sum_{|n|\ge 1} \frac{1}{|n|} |K \cap (2n\pi + (I_N \triangle I'_N)| + \sum_{l=0}^{N} 2^l |I_l \triangle I'_l| \\ &= C\varepsilon \int_{I_N \triangle I'_N} (\sum_{|n|\ge 1} \frac{1}{|n|} \chi_{K+2n\pi}(\xi))\, d\xi + \sum_{l=0}^{N} 2^l |I_l \triangle I'_l| \,. \end{aligned} \tag{4.4}$$

By condition (4.1) (notice that $I_N \triangle I'_N \subset (-\varepsilon, \varepsilon)$), induction on N, and an application of the Lebesgue dominated convergence theorem we obtain from (4.4) that

$$2^{N+1}|I_{N+1} \triangle I'_{N+1}| \longrightarrow 0 \qquad \text{as} \qquad |I \triangle I'| \longrightarrow 0, \qquad N = 0, 1, 2, \ldots \tag{4.5}$$

Moreover, condition (4.1), again, and the Lebesgue dominated convergence theorem imply that given $\eta > 0$, there exists $M \ge 0$ such that if $L_M = \bigcup_{l=M}^{\infty} J_l^\varepsilon = (-\frac{\varepsilon}{2^M}, \frac{\varepsilon}{2^M})$ a. e., then

$$\sum_{|n|\ge 1} \frac{1}{|n|} |K \cap (2n\pi + L_M)| < \eta \,. \tag{4.6}$$

By (4.3) we get that for the M that satisfies (4.6) and for any initial pair of intervals I and I' contained in J_0^ε,

$$\sum_{N=M}^{\infty} 2^N |I_N \triangle I'_N| \le \sum_{N=M}^{\infty} 2^N (|I_N| + |I'_N|) \le 2C\varepsilon \sum_{|n|\ge 1} \frac{1}{|n|} |K \cap (2n\pi + L_M)| < 2C\varepsilon\eta \,. \tag{4.7}$$

Combining (4.5) and (4.7) we obtain (4.2). □

Notice that Proposition 2.2 and Lemma 4.3 applied to $I = I(t)$ and $I' = I'(t)$ allow as to finish the proof of **Theorem 4.1**, showing the connectivity of the set $\mathcal{K}_\tau$. □

Corollary 4.4. *The set $\mathcal{K}$ of all $\psi \in TFW$ such that $K = supp\, \hat{\psi}$ has finite measure is arcwise connected.*

Proof. Each element of the path ψ_t constructed in the proof of Theorem 4.1 belongs to $\mathcal{K}$ if $\psi \in \mathcal{K}$ since $\operatorname{supp}\widehat{\psi_t} \subset (\operatorname{supp}\hat{\psi}) \cup (-\varepsilon,\varepsilon)$. Thus, we only need to prove that $\mathcal{K} \subset \mathcal{K}_\tau$. To show this observe that the sets $2n\pi + (-\varepsilon,\varepsilon)$ are disjoint so that

$$\sum_{|n|\geq 1} \frac{1}{|n|}|K \cap (2n\pi + (-\varepsilon,\varepsilon))| \leq \sum_{|n|\geq 1} |K \cap (2n\pi + (-\varepsilon,\varepsilon))|$$
$$= |K \cap (\bigcup_{|n|\geq 1} 2n\pi + (-\varepsilon,\varepsilon))| \leq |K| < \infty\,.$$

□

5 Connectivity of the set $\mathcal{K}_d$

Let $\psi \in TFW$ and $K = \operatorname{supp}\widehat{\psi}$. For $\varepsilon \in (0,\pi]$ and for $n = 1,2,3,\ldots$ define

$$K_n^\varepsilon = \frac{1}{2^n}(K \cap 2^n J_0^\varepsilon) \subset J_0^\varepsilon = \left(-\varepsilon, -\frac{\varepsilon}{2}\right] \cup \left[\frac{\varepsilon}{2}, \varepsilon\right)\,. \tag{5.1}$$

Recall that for a collection of sets K_n,

$$\limsup_{n\to\infty} K_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} K_m\right)$$

Definition 5.1. The set $\mathcal{K}_d$ is the set of all $\psi \in TFW$ for which there exists $\varepsilon \in (0,\pi]$ such that $|\limsup_{n\to\infty} K_n^\varepsilon| = 0\,.$

Remark. Let $\psi \in TFW$ and $K = \operatorname{supp}\widehat{\psi}$. If for some $\varepsilon > 0$ the condition

$$\sum_{n=1}^{\infty} \frac{|K \cap 2^n J_0^\varepsilon|}{2^n} < \infty$$

is satisfied, then $\psi \in \mathcal{K}_d$. To see this let $L_n = \bigcup_{m=n}^\infty K_m^\varepsilon$. Then, $|L_n| \leq \sum_{m=n}^\infty \frac{1}{2^m}|K \cap 2^m J_0^\varepsilon|$. Given $\eta > 0$, choose n_0 large enough so that $\sum_{m=n}^\infty \frac{1}{2^m}|K \cap 2^m J_0^\varepsilon| < \eta$ for all $n \geq n_0$. Since $L_{n+1} \subset L_n$ and $|L_n| < \infty$ ($L_n \subset J_0^\varepsilon$) it follows that $|\limsup_{n\to\infty} K_n^\varepsilon| = 0$.

In this section we prove that the set $\mathcal{K}_d$ is arcwise connected. Observe that the function ψ^ε $(0 < \varepsilon \leq \pi)$ given by $\widehat{\psi^\varepsilon} = \chi_{J_0^\varepsilon}$ belongs to $\mathcal{K}_d$. We will show that there is a continuous arc inside $\mathcal{K}_d$ joining any element $\psi \in \mathcal{K}_d$ with ψ^ε, where ε is the number associated to $\psi \in \mathcal{K}_d$ by Definition 5.1. We use the same arc described at the beginning of the proof of Theorem 4.1. Therefore, it is enough to show that *i)* and *ii)* of Proposition 2.2 hold assuming that $\psi \in \mathcal{K}_d$. To do this we need to introduce new notation and prove several Lemmas.

For a measurable set K define, for $m = 0,1,2,3,\ldots$

$$H_m = K \cap \left(\bigcup_{n=m}^{\infty} 2^n J_0^\varepsilon\right) \qquad \text{and} \qquad G_m = K \cap (-2^{m-1}\varepsilon\,, 2^{m-1}\varepsilon)\,, \tag{5.2}$$

so that $H_m \cup G_m = K$ with disjoint union.

Starting with $I \subset J_0^\varepsilon$ define $I_0(K) = I$ and $I_N(K)$, $N \geq 1$ as in (3.2). Also, if $K = G \cup H$ with disjoint union, let

$$I_N^*(G) = \left([\tau_G(I_{N-1}(K))] \setminus \bigcup_{j=1}^{N-1} [I_j(K)] \right) \cap J_N^\varepsilon \,, \qquad N \geq 1 \,, \tag{5.3}$$

and

$$I_N^*(H) = \left([\tau_H(I_{N-1}(K))] \setminus \bigcup_{j=1}^{N-1} [I_j(K)] \right) \cap J_N^\varepsilon \,, \qquad N \geq 1 \,. \tag{5.4}$$

Lemma 5.1. *If $K = G \cup H$ with disjoint union, then for all $N \geq 1$, we have*

$$I_N(K) = I_N^*(G) \cup I_N^*(H)$$

Proof. Since $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ using (5.3), (5.4), and (A1) in Appendix, we have

$$\begin{aligned} I_N(K) &= ([\tau_K(I_{N-1}(K))] \setminus \bigcup_{j=1}^{N-1} [I_j(K)]) \cap J_N^\varepsilon \\ &= \left(\left[(\bigcup_{l \in \mathbb{Z},\ l \neq 0} 2\pi l + I_{N-1}(K)) \cap (G \cup H) \right] \setminus \bigcup_{j=1}^{N-1} [I_j(K)] \right) \cap J_N^\varepsilon \\ &= \left([\tau_G(I_{N-1}(K)) \cup (\tau_H(I_{N-1}(K))] \setminus \bigcup_{j=1}^{N-1} [I_j(K)] \right) \cap J_N^\varepsilon \\ &= I_N^*(G) \cup I_N^*(H) \,. \end{aligned}$$

□

Lemma 5.2. *i) For all $m = 0, 1, 2, \ldots$, we have $2^N I_N^*(H_m) \subset [H_m] \cap J_0^\varepsilon$, $N = 1, 2, 3, \ldots$. Hence*

$$\bigcup_{N=1}^{\infty} 2^N I_N^*(H_m) \subset [H_m] \cap J_0^\varepsilon$$

.

ii) Let $L_m = \bigcup_{n=m}^{\infty} \frac{1}{2^n}(K \cap 2^n J_0^\varepsilon)$. Then $[H_m] \cap J_0^\varepsilon = L_m$ for all $m = 0, 1, 2, \ldots$.

Proof. *i*) By definition,

$$\begin{aligned} 2^N I_N^*(H_m) &= 2^N \left\{ \left([\tau_{H_m}(I_{N-1}(K))] \setminus \bigcup_{j=1}^{N-1} [I_j(K)] \right) \cap J_N^\varepsilon \right\} \\ &= \left\{ \left[\left(\bigcup_{l \in \mathbb{Z},\ l \neq 0} 2\pi l + I_{N-1}(K) \right) \cap H_m \right] \setminus \bigcup_{j=1}^{N-1} [I_j(K)] \right\} \cap J_0^\varepsilon \subset [H_m] \cap J_0^\varepsilon \end{aligned}$$

ii) After writing the definition of H_m, use (A1) in Appendix and the properties of the power set (both twice) to obtain

$$\begin{aligned}[H_m] \cap J_0^\varepsilon &= [K \cap (\bigcup_{n=m}^{\infty} 2^n J_0^\varepsilon)] \cap J_0^\varepsilon = (\bigcup_{n=m}^{\infty} [K \cap 2^n J_0^\varepsilon]) \cap J_0^\varepsilon \\ &= (\bigcup_{n=m}^{\infty} [2^{-n} K \cap J_0^\varepsilon]) \cap J_0^\varepsilon = [\bigcup_{n=m}^{\infty} (2^{-n} K) \cap J_0^\varepsilon] \cap J_0^\varepsilon \\ &= \bigcup_{n=m}^{\infty} (2^{-n} K) \cap J_0^\varepsilon = L_m \,.\end{aligned}$$

□

Lemma 5.3. *With E_I and $E_{I'}$ defined as in (3.3) we have*

i) $|\widetilde{E_I} \triangle \widetilde{E_{I'}}| \leq |\bigcup_{N=0}^{\infty} 2^N (I_N(K) \triangle I'_N(K))|$

and

ii) $|E_I \triangle E_{I'}| \leq 2 |\bigcup_{N=0}^{\infty} 2^N (I_N(K) \triangle I'_N(K))|$

Proof. The first one is clear by definition (see also the proof of Lemma 4.3). We need to prove ii). From the definition of E_I we deduce (as in Lemma 4.3)

$$E_I \triangle E_{I'} = \bigcup_{N=0}^{\infty} (I_N(K) \triangle I'_N(K)) \,. \tag{5.5}$$

For $N = 0, 1, 2, \cdots$ write $A_N(K) = I_N(K) \setminus I'_N(K)$ and $B_N(K) = I'_N(K) \setminus I_N(K)$. Then

$$I_N(K) \triangle I'_N(K) = A_N(K) \cup B_N(K) \tag{5.6}$$

For $N_1, N_2 \in \mathbb{N}$ with $N_1 \neq N_2$, we have $[I_{N_1}(K)] \cap [I_{N_2}(K)] = \emptyset$, which implies $[A_{N_1}(K)] \cap [A_{N_2}(K)] = \emptyset$. Thus, for all $j_1, j_2 \in \mathbb{Z}$, the sets $2^{j_1} A_{N_1}(K)$ and $2^{j_2} A_{N_2}(K)$ are disjoint if $N_1 \neq N_2$; similarly for the sets $2^j B_N(K), N = 0, 1, 2, \ldots, j \in \mathbb{Z}$.

Let $A(K) = \bigcup_{N=0}^{\infty} 2^N A_N(K)$ and $B(K) = \bigcup_{N=0}^{\infty} 2^N B_N(K)$. Then, by the disjointness property proved in the preceding paragraph, we deduce:

$$\begin{aligned}|A(K)| &= \left| \bigcup_{N=0}^{\infty} 2^N A_N(K) \right| = \sum_{N=0}^{\infty} 2^N |A_N(K)| \\ &\geq \sum_{N=0}^{\infty} |A_N(K)| = \left| \bigcup_{N=0}^{\infty} A_N(K) \right| ,\end{aligned} \tag{5.7}$$

and similarly for $B(K)$, that is,

$$|B(K)| \geq \left| \bigcup_{N=0}^{\infty} B_N(K) \right| . \tag{5.8}$$

From (5.5), (5.6), (5.7), and (5.8) we deduce

$$|E_I \triangle E_{I'}| = \left| \bigcup_{N=0}^{\infty} I_N(K) \triangle I'_N(K) \right| = \left| \left(\bigcup_{N=0}^{\infty} A_N(K) \right) \cup \left(\bigcup_{N=0}^{\infty} B_N(K) \right) \right|$$

$$\leq \left|\bigcup_{N=0}^{\infty} A_N(K)\right| + \left|\bigcup_{N=0}^{\infty} B_N(K)\right| \leq |A(K)| + |B(K)| \,.$$

Since

$$A(K) = \bigcup_{N=0}^{\infty} 2^N A_N(K) = \bigcup_{N=0}^{\infty} 2^N (I_N(K) \setminus I'_N(K)) \subset \bigcup_{N=0}^{\infty} 2^N (I_N(K) \triangle I'_N(K)) \,,$$

and, similarly,

$$B(K) \subset \bigcup_{N=0}^{\infty} 2^N (I_N(K) \triangle I'_N(K)) \,,$$

we deduce

$$|E_I \triangle E_{I'}| \leq 2 \left|\bigcup_{N=0}^{\infty} 2^N (I_N(K) \triangle I'_N(K))\right|$$

which is what we wanted to prove. $\square$

Lemma 5.4. *Let $\varepsilon > 0$ and $A \subset \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ be a measurable set. Then*

$$|[A] \cap J_N^\varepsilon| \leq \frac{|A|}{2^{N+1}} \qquad N = 0, 1, 2, \dots \,.$$

Proof. We have, $A = \bigcup_{j=N+1}^{\infty} A \cap 2^j J_N^\varepsilon$ with disjoint union. Let $C_j = A \cap 2^j J_N^\varepsilon$, $j = N+1, \dots, \infty$. By (A1) in Appendix,

$$[A] \cap J_N^\varepsilon = \bigcup_{j=N+1}^{\infty} [C_j] \cap J_N^\varepsilon$$

with disjoint union. Also, for $j = N+1, \dots, \infty$, since $C_j \subset 2^j J_N^\varepsilon$,

$$[C_j] \cap J_N^\varepsilon = \bigcup_{l=-\infty}^{\infty} (2^l C_j) \cap J_N^\varepsilon = 2^{-j} C_j \,.$$

Hence,

$$|[A] \cap J_N^\varepsilon| = \sum_{j=N+1}^{\infty} |[C_j] \cap J_N^\varepsilon| = \sum_{j=N+1}^{\infty} \frac{1}{2^j} |C_j| \leq \frac{1}{2^{N+1}} \sum_{j=N+1}^{\infty} |C_j| \,.$$

Since $A = \bigcup_{j=N+1}^{\infty} C_j$ with disjoint union, we deduce $|[A] \cap J_N^\varepsilon| \leq \frac{1}{2^{N+1}} |A|$, as wanted. $\square$

Lemma 5.5. *Let $G \subset K$ be two measurable subsets of $\mathbb{R}$. For any measurable set $I \subset J_0^\varepsilon$ we have*

$$|I^*_{N+1}(G)| \leq \frac{1}{2^{N+2}} |\tau_G(I_N(K))|, \qquad N = 0, 1, 2, \dots$$

Proof. By definition (5.3) we have

$$I^*_{N+1}(G) \subset [\tau_G(I_N(K))] \cap J_{N+1}^\varepsilon \,.$$

Since $(\tau_G(I_N(K))) \cap (-\varepsilon, \varepsilon) = \emptyset$ we use Lemma 5.4 to conclude

$$|I^*_{N+1}(G)| \leq |[\tau_G(I_N(K))] \cap J_{N+1}^\varepsilon| \leq \frac{|\tau_G(I_N(K))|}{2^{N+2}} \,.$$

$\square$

Lemma 5.6. *Let $G \subset K$ be two measurable subsets of $\mathbb{R}$. For any pair of measurable sets $I, I' \subset J_0^\varepsilon$ and $N = 0, 1, 2, 3, \dots,$ we have*

$$2^{N+1}|I^*_{N+1}(G) \triangle I'^*_{N+1}(G)| \leq \frac{1}{2}|\tau_G(I_N(K) \triangle I'_N(K))| + \sum_{j=0}^{N} 2^j |I_j(K) \triangle I'_j(K)|\,.$$

Proof. Let $E_N = \bigcup_{j=1}^N I_j(K)$ and $E'_N = \bigcup_{j=1}^N I'_j$. Using properties (A8), (A3), and (A7) from Appendix we obtain:

$$\begin{aligned} I^*_{N+1}(G) \triangle I'^*_{N+1}(G) &= \big\{([\tau_G(I_N(K))] \setminus [E_N]) \triangle ([\tau_K(I'_N(K))] \setminus [E'_N])\big\} \cap J^\varepsilon_{N+1} \\ &\subset \big\{\big([\tau_G(I_N(K))] \triangle [\tau_G I'_N(K)]\big) \cup \big([E_N] \triangle [E'_N]\big)\big\} \cap J^\varepsilon_{N+1} \\ &\subset \big\{[\tau_G(I_N(K)) \triangle \tau_G(I'_N(K))] \cup [E_N \triangle E'_N]\big\} \cap J^\varepsilon_{N+1} \\ &\subset \big\{[\tau_G(I_N(K) \triangle I'_N(K))] \cup [E_N \triangle E'_N]\big\} \cap J^\varepsilon_{N+1}\,. \end{aligned}$$

Since $\{\tau_G(I_N(K) \triangle I'_N(K))\} \cap (-\varepsilon, \varepsilon) = \emptyset$, by Lemma 5.4 we deduce

$$\big|[\tau_G(I_N(K) \triangle I'_N(K))] \cap J^\varepsilon_{N+1}\big| \leq \frac{|\tau_G(I_N(K) \triangle I'_N(K))|}{2^{N+2}}\,.$$

Also, by (A1) from Appendix,

$$[E_N \triangle E'_N] = \left[\bigcup_{j=0}^{\infty} I_j(K) \triangle I'_j(K)\right] = \bigcup_{j=0}^{\infty} [I_j(K) \triangle I'_j(K)]$$

and

$$\big|[I_j(K) \triangle I'_j(K)] \cap J^\varepsilon_{N+1}\big| = \frac{|I_j(K) \triangle I'_j(K)|}{2^{N+1-j}}\,.$$

All these estimates give

$$\big|I^*_{N+1}(G) \triangle I'^*_{N+1}(G)\big| \leq \frac{1}{2^{N+2}}|\tau_G(I_N(K) \triangle I'_N(K))| + \frac{1}{2^{N+1}} \sum_{j=0}^{N} 2^j |I_j(K) \triangle I'_j(K)|$$

which is the desired result. $\square$

Lemma 5.7. *Suppose that $\psi \in \mathcal{K}_d$ and $K = supp\,\widehat{\psi}$.*

i) For all $N = 0, 1, 2, 3, \dots,$ we have $|I_N(K) \triangle I'_N(K)| \to 0$ as $|I \triangle I'| \to 0$

ii) For all $N = 1, 2, 3, \dots$ and all $m \in \mathbb{Z}^+$ we have $2^N|I^*_N(G_m) \triangle I'^*_N(G_m)| \to 0$ as $|I \triangle I'| \to 0$

Proof. From Lemma 5.2 we deduce:

$$I^*_{N+1}(H_m) \subset \frac{L_m}{2^{N+1}} \qquad \text{and} \qquad I'^*_{N+1}(H_m) \subset \frac{L_m}{2^{N+1}}\,.$$

Thus, since $\psi \in \mathcal{K}_d$, given $\eta > 0$ there exists $m_0 \in \mathbb{N}$ such that $|L_{m_0}| < \frac{\eta}{4}$. Hence,

$$|I^*_{N+1}(H_{m_0})| \leq |L_m| < \frac{\eta}{4} \qquad \text{and} \qquad |I'^*_{N+1}(H_{m_0})| \leq |L_m| < \frac{\eta}{4}\,.$$

Using Lemma 5.1 and (A9) from Appendix we deduce

$$I_{N+1}(K)\triangle I'_{N+1}(K) \subset \left\{I^*_{N+1}(G_{m_0})\triangle I'^*_{N+1}(G_{m_0})\right\} \cup \left\{I^*_{N+1}(H_{m_0})\triangle I'^*_{N+1}(H_{m_0})\right\} \,,$$

so that

$$|I_{N+1}(K)\triangle I'_{N+1}(K)| \le |I^*_{N+1}(G_{m_0})\triangle I'^*_{N+1}(G_{m_0})| + \frac{\eta}{2}\,. \tag{5.9}$$

By Lemma 5.6 we deduce that for $N = 0, 1, 2, 3, \ldots$ and any $m \in \mathbb{Z}^+$,

$$2^{N+1}|I^*_{N+1}(G_m)\triangle I'^*_{N+1}(G_m)| \le \frac{1}{2}|\tau_{G_m}(I_N(K)\triangle I'_N(K))| + \sum_{j=0}^{N} 2^j |I_j(K)\triangle I'_j(K)|\,. \tag{5.10}$$

We prove $i)$ and $ii)$ by induction on N. The case $N = 0$ of $i)$ is clear since $I_0(K) = I$ and $I'_0(K) = I'$. The case $N = 1$ of $ii)$ follows from the case $N = 0$ of (5.10); to see this write

$$|\tau_{G_m}(I\triangle I')| = |\bigcup_{n\in\mathbb{Z},\ n\ne 0} (I\triangle I' + 2n\pi) \cap G_m|$$

Since G_m is a bounded set, the above union has only a finite number of terms, say $M(m)$. Thus $|\tau_{G_m}(I\triangle I')| \le M(m)|I\triangle I'| \to 0$ as $|I\triangle I'| \to 0$.

Assume that $i)$ and $ii)$ hold for $j = 0, 1, 2, 3, \ldots, N$. Since G_m is bounded an argument as above shows

$$\frac{1}{2}|\tau_{G_m}(I_N(K)\triangle I'_N(K))| \to 0 \qquad \text{as} \qquad |I\triangle I'| \to 0\,,$$

since we are assuming $|I_N(K)\triangle I'_N(K)| \to 0$. Thus, from (5.10) we deduce $ii)$ for $j = N+1$. From (5.9) we deduce $i)$ for $j = N + 1$. □

Lemma 5.8. *Suppose that $\psi \in \mathcal{K}_d$ and $K = supp\,\widehat{\psi}$. Then*

$$\left|\bigcup_{N=0}^{\infty} 2^N(I_N(K)\triangle I'_N(K))\right| \to 0 \qquad \textit{as} \qquad |I\triangle I'| \to 0\,.$$

Proof. From Lemma 5.1, (A9) in Appendix, and Lemma 5.2 we deduce for $m = 1, 2, 3, \ldots$

$$\begin{aligned}
&\bigcup_{N=0}^{\infty} 2^N(I_N(K)\triangle I'_N(K)) \\
&= (I\triangle I') \cup \bigcup_{N=1}^{\infty} 2^N \left\{(I^*_N(G_m) \cup (I^*_N(H_m))\triangle(I'^*_N(G_m) \cup (I'^*_N(H_m))\right\} \\
&\subset (I\triangle I') \cup \bigcup_{N=1}^{\infty} 2^N \{(I^*_N(G_m)\triangle(I^*_N(G_m))\} \cup \bigcup_{N=1}^{\infty} 2^N\{I^*_N(H_m)\triangle I'^*_N(H_m)\} \\
&\subset (I\triangle I') \cup \bigcup_{N=1}^{\infty} 2^N \left\{I^*_N(G_m)\triangle(I'^*_N(G_m)\right\} \cup L_m
\end{aligned}$$

(Notice that L_m does not depend on I.) Hence,

$$\left|\bigcup_{N=0}^{\infty} 2^N (I_N(K)\triangle I'_N(K))\right| \leq |I\triangle I'| + \sum_{N=1}^{\infty} 2^N |I^*_N(G_m)\triangle I'^*_N(G_m)| + |L_m|$$

$$\leq |I\triangle I'| + \sum_{N=1}^{M} 2^N |I^*_N(G_m)\triangle I'^*_N(G_m)| + \sum_{N=M+1}^{\infty} 2^N |I^*_N(G_m)|$$

$$+ \sum_{N=M+1}^{\infty} 2^N |I'^*_N(G_m)| + |L_m|\,. \tag{5.11}$$

Given $\eta > 0$, choose $m_0 \in \mathbb{N}$ such that $|L_m| < \frac{\eta}{8}$ for all $m \geq m_0$. By Lemma 5.5 applied to $G = G_m$ we have, if $m \geq m_0$,

$$\sum_{N=M+1}^{\infty} 2^N |I^*_N(G_m)| \leq \frac{1}{2} \sum_{N=M+1}^{\infty} |\tau_{G_m}(I_{N-1}(K))|$$

$$= \frac{1}{2}\left|\tau_{G_m}\left(\bigcup_{N=M+1}^{\infty} I_{N-1}(K)\right)\right| \leq \frac{1}{2}|\tau_{G_m}(-\frac{\varepsilon}{2^M}, \frac{\varepsilon}{2^M})|\,, \tag{5.12}$$

and similarly for I'^*_N. Since G_m is a bounded set, the union that appears in the definition of $\tau_{G_m}(-\frac{\varepsilon}{2^M}, \frac{\varepsilon}{2^M})$ has only a finite number of terms, say $K(m)$. Thus $|\tau_{G_m}(-\frac{\varepsilon}{2^M}, \frac{\varepsilon}{2^M})| \leq K(m)|(-\frac{\varepsilon}{2^M}, \frac{\varepsilon}{2^M})| \to 0$ as $M \to \infty$. Choose M large enough so that

$$|\tau_{G_m}(-\frac{\varepsilon}{2^M}, \frac{\varepsilon}{2^M})| < \eta/4 \tag{5.13}$$

Using (5.11), (5.12), and (5.13) we obtain

$$\left|\bigcup_{n=1}^{\infty} 2^N (I_N(K)\triangle I'_N(K))\right| \leq |I\triangle I'| + \sum_{N=1}^{M} 2^N |I^*_N(G_m)\triangle I'^*_N(G_m)| + \frac{3\eta}{8}\,. \tag{5.14}$$

The desired result follows by using *ii*) of Lemma 5.7. □

Theorem 5.9. *The set $\mathcal{K}_d$ given in Definition 5.1 is arcwise connected.*

Proof. Consider the same path as the one in the proof of Theorem 4.1. By Proposition 2.2 all we need to prove is

$$|E_I \triangle E_{I'}| \to 0 \text{ as } |I\triangle I'| \to 0 \qquad \text{and} \qquad |\widetilde{E_I} \triangle \widetilde{E_{I'}}| \to 0 \text{ as } |I\triangle I'| \to 0\,.$$

This follows from Lemmas 5.3 and 5.8 applied to $I = I(t)$ and $I' = I'(t)$. □

Final Remark.

After the work in this paper was completed, we realized of a more general condition under which the arguments on connectivity presented here still hold. The condition reads as follows:

Given a bounded TF-set J_0, a measurable set K and a measurable subset I of the negative powers of J_0, we define

$$L(I) = [\tau_K(I)] \cap J_0 \,.$$

We say that K is J_0-admissible if the map $I \to L(I)$ is continuous in measure in the sense that given ε, there exists δ so that "$|I| \le \delta \Rightarrow |L(I)| < \varepsilon$".

With this assumption one proves that any TFW with spectrum in K can be continuously connected with the one with spectrum in J_0.

It is easy to see that both, condition $\mathcal{K}_\tau$ and condition $\mathcal{K}_d$ given in this paper, imply the previous one. Also, there is a condition of uniform admissibility which gives that all the intermediate TFW's of the arc may have spectrum in the union of K and J_0. The details will appear in a forthcoming paper.

6 Appendix

(A1) If $A_N \subset \mathbb{R},\ N = 1,2,3,\dots,$ then $[\bigcup_{N=1}^{\infty} A_N] = \bigcup_{N=1}^{\infty}[A_N]$

Proof.

$$\left[\bigcup_{N=1}^{\infty} A_N\right] = \bigcup_{j\in\mathbb{Z}} 2^j \left(\bigcup_{N=1}^{\infty} A_N\right) = \bigcup_{N=1}^{\infty}\left(\bigcup_{j\in\mathbb{Z}} 2^j A_N\right) = \bigcup_{N=1}^{\infty}[A_N]\,.$$

(A2) $[A] \setminus [B] \subset [A \setminus B]$

Proof. If $\xi \in [A]\setminus[B]$, there exist $a \in A$ and $m \in \mathbb{Z}$ such that $\xi = 2^m a$. Clearly, $a \notin B$, otherwise $\xi \in [B]$. Hence, $a \in A \setminus B$ and $\xi = 2^m a \in [A \setminus B]$.

NOTE. The inclusion is strict. For $A = J_0^\varepsilon \cup J_1^\varepsilon$ and $B = J_1^\varepsilon$ we have $[A] = \mathbb{R}$ and $[B] = \mathbb{R}$, so that $[A] \setminus [B] = \emptyset$. Moreover, $[A \setminus B] = [J_0^\varepsilon] = \mathbb{R}$.

(A3) $[A]\triangle[B] \subset [A\triangle B]$

Proof. Since $[A]\triangle[B] = ([A] \setminus [B]) \cup ([B] \setminus [A])$, using (A.2) and then (A.1) we obtain $[A]\triangle[B] \subset [A \setminus B] \cup [B \setminus A] = [A\triangle B]$.

(A4) $[A \cap B] \subset [A] \cap [B]$

Proof. If $\xi \in [A\cap B]$, there exist $m \in \mathbb{Z}$ and $\eta \in A\cap B$ such that $\xi = 2^m\eta$. Since $\eta \in A$ and $\eta \in B$, $\xi \in [A]$ and $\xi \in [B]$.

NOTE. The inclusion is strict. For $A = J_0^\varepsilon$ and $B = J_1^\varepsilon$ we have $[A \cap B] = [\emptyset] = \emptyset$, but $[A] \cap [B] = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

(A5) $\tau_K(A) \setminus \tau_K(B) \subset \tau_K(A \setminus B)$

Proof. If $\xi \in \tau_K(A) \setminus \tau_K(B)$, there exist $k \in \mathbb{Z}\setminus\{0\}$ and $\eta \in A$ such that $\xi = 2k\pi + \eta \in K$. Clearly, $\eta \notin B$, otherwise $\xi \in \tau_K(B)$. Hence $\eta \in A \setminus B$ and $\xi = 2k\pi + \eta \in \tau_K(A \setminus B)$.

(A6) If $A_N \subset \mathbb{R}$, $N = 1, 2, 3, \ldots$, then $\bigcup_{N=1}^{\infty} \tau_K(A_N) = \tau_K(\bigcup_{N=1}^{\infty} A_N)$

Proof.

$$\bigcup_{N=1}^{\infty} \tau_K(A_N) = \bigcup_{N=1}^{\infty} \left(\bigcup_{l\in\mathbb{Z},\, l\neq 0} (A_N + 2l\pi) \cap K \right) = \bigcup_{l\in\mathbb{Z},\, l\neq 0} \left(\bigcup_{N=1}^{\infty} (A_N + 2l\pi) \right) \cap K$$
$$= \bigcup_{l\in\mathbb{Z},\, l\neq 0} \left((\bigcup_{N=1}^{\infty} A_N) + 2l\pi \right) \cap K = \tau_K(\bigcup_{N=1}^{\infty} A_N) .$$

(A7) $\tau_K(A) \triangle \tau_K(B) \subset \tau_K(A \triangle B)$

Proof. Follows from (A5) and (A6):

$$\tau_K(A) \triangle \tau_K(B) = (\tau_K(A) \setminus \tau_K(B)) \cup (\tau_K(B) \setminus \tau_K(A))$$
$$\subset \tau_K(A \setminus B)) \cup \tau_K(B \setminus A) = \tau_K\left((A \setminus B) \cup (B \setminus A)\right) = \tau_K(A \triangle B)$$

(A8) $(A \setminus A_1) \triangle (B \setminus B_1) \subset (A \triangle B) \cup (A_1 \triangle B_1)$

Proof.

$$\begin{aligned}
&(A \setminus A_1) \triangle (B \setminus B_1) \\
&= \{(A \cap A_1^c) \cap (B \cap B_1^c)^c\} \cup \{(B \cap B_1^c) \cap (A \cap A_1^c)^c\} \\
&= \{(A \cap A_1^c) \cap (B^c \cup B_1)\} \cup \{(B \cap B_1^c) \cap (A^c \cup A_1)\} \\
&= (A \cap A_1^c \cap B^c) \cup (A \cap A_1^c \cap B_1) \cup (B \cap B_1^c \cap A^c) \cup (B \cap B_1^c \cap A_1) \\
&= ((A \setminus B) \cap A_1^c) \cup ((B \setminus A) \cap B_1^c) \cup ((B_1 \setminus A_1) \cap A) \cup ((A_1 \setminus B_1) \cap B) \\
&\subset (A \setminus B) \cup (B \setminus A) \cup (B_1 \setminus A_1) \cup (A_1 \setminus B_1) \\
&= (A \triangle B) \cup (A_1 \triangle B_1)
\end{aligned}$$

(A9) $\left(\bigcup_{j\in\mathbb{Z}} A_j\right) \triangle \left(\bigcup_{j\in\mathbb{Z}} B_j\right) \subset \bigcup_{j\in\mathbb{Z}} (A_j \triangle B_j)$. In particular

$$(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$$

Proof. If $x \in \left(\bigcup_{j\in\mathbb{Z}} A_j\right) \triangle \left(\bigcup_{j\in\mathbb{Z}} B_j\right)$ we have $x \in \left(\bigcup_{j\in\mathbb{Z}} A_j\right) \setminus \left(\bigcup_{j\in\mathbb{Z}} B_j\right)$ or $x \in \left(\bigcup_{j\in\mathbb{Z}} B_j\right) \setminus \left(\bigcup_{j\in\mathbb{Z}} A_j\right)$. In the first case, $x \in A_{j_0}$ for some $j_0 \in \mathbb{Z}$, but $x \notin B_j$ for all $j \in \mathbb{Z}$. Then, $x \in A_{j_0} \setminus B_{j_0}$, and consequently $x \in \bigcup_{j\in\mathbb{Z}} (A_j \triangle B_j)$. The proof is similar in the second case.

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