Analysis
with ultrasmall numbers

Higher Level

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Infinity itself looks flat and uninteresting. [...] The chamber [...] was anything but infinite, it was just very very very big, so big that it gave the impression of infinity far better than infinity itself.
(Douglas Adams: The Hitchhiker's Guide to the Galaxy)

Contents

1
Velocity and Position

Exercise 1

Suppose the velocity [1] of a car is constant and equal to $60km/h$.

(1) Let f be the function which describes the position of the car with respect to time.

Draw the graph f for t ranging from 0 to 3 hours.

(2) Let v be the function which describes the velocity of the car with respect to time.

Draw the graph of v for t ranging from 0 to 3 hours.

(3) Given the position graph, how can one find the velocity of the car at any given time?

(4) Given the velocity graph, how can one find the position of the car after any given time?

⚠ Note the difference: velocity (deduced from position) is *local*. It is possible to give the velocity *at* a given time. Position (deduced from velocity) is *global*. It is only possible to find the *variation* of the position over an *interval* of time.

Exercise 2

The velocity of a car (in km/h) is given by the following function with respect to time (in h): (decimal division of hours for simplicity)

$$v : t \mapsto \begin{cases} 60 & \text{if } 0 \leq t \leq 0.5 \\ 120 & \text{if } 0.5 < t \leq 2 \\ 80 & \text{if } 2 < t \leq 2.5 \\ 60 & \text{if } 2.5 < t \leq 3 \end{cases}$$

Calculate the positions at $t = 1, \quad t = 2$ and $t = 3$.

Draw the velocity graph and indicate *on the velocity graph* where the position at $t = 2$ can be drawn.

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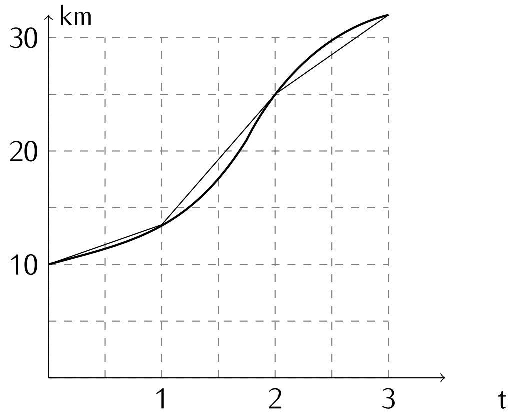
[1] The velocity is speed with a direction. Speed is always positive (or zero); velocity can be negative.

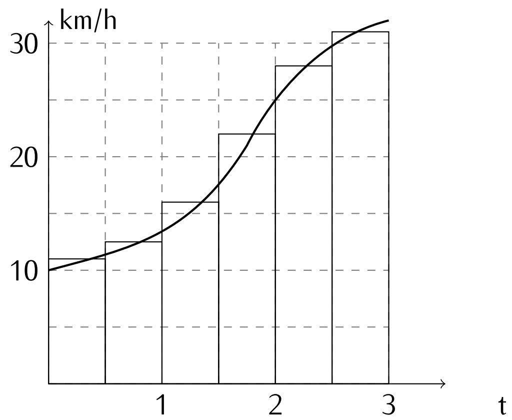
Exercise 3

The following curve can be approximated by a piecewise linear function whose slope is easily calculated by pieces. If this curve represents the position function of a moving body, use the linear pieces to give a (graphical) representation of the velocity function. (The exact values are not important here, approximate relative values will do.)



Exercise 4

The following curve can be approximated by a "staircase" function whose area is calculated by adding the areas of the rectangles. If this curve represents the velocity function of a moving body, use the rectangles to give a (graphical) representation of the position function. (The exact values are not important here, approximate relative values will do.)



The main goal of the subject called **mathematical analysis** will be to check when and how to approximate a curve by pieces of straight lines and when and how to approximate areas by rectangles and to understand what these can be used to calculate. Intuitively, it should seem clear that in order for the approximation to be good, the pieces of straight lines or the rectangles must be small – or that the number of pieces is large. The crucial questions are: How small? and How large?

2
Basic Principles

Exercise 5

Hold a pencil in your hand. Do not move.
Now drop the pencil.

First the pencil was motionless. Then it was in motion.
How did the motion start? How is the transition from "not moving" to "moving"?

Exercise 6

If δ is a positive value which is extremely small (even smaller than that!),

(1) what can you say about the size of δ^2, $2 \cdot \delta$ and $-\delta$?

(2) what can you say about $2 + \delta$ and $2 - \delta$?

(3) what can you say about $\frac{1}{\delta}$?

Exercise 7

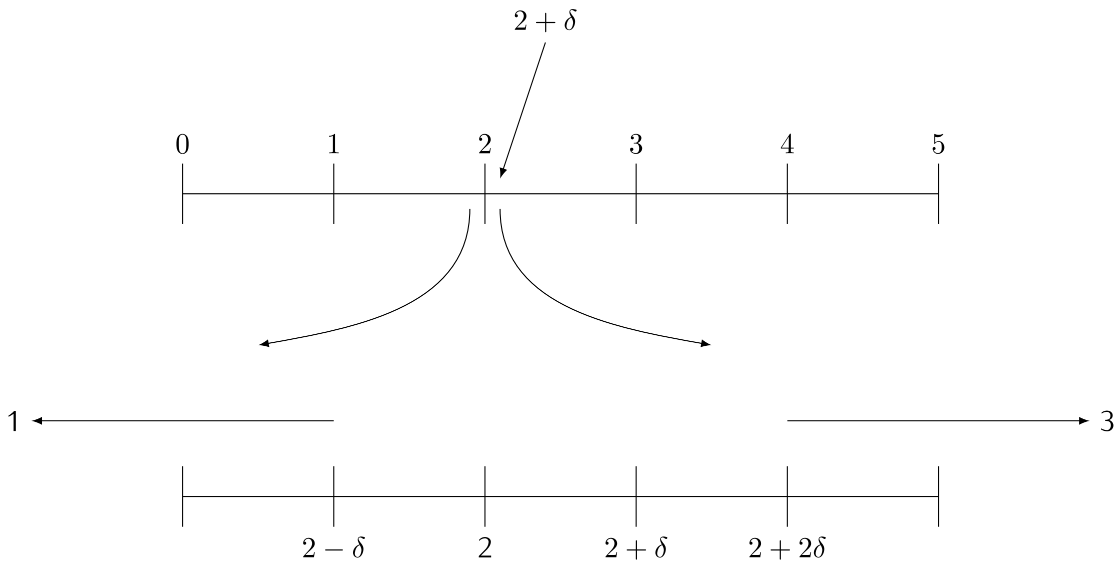
If N is a positive huge number (really very huge!),

(1) what can you say about N^2, $2N$ and $-N$?

(2) what can you say about $N + 2$ and $N - 2$?

(3) what can you say about $\frac{1}{N}$?

(4) what can you say about $\frac{N}{2}$?



Zooming in

Exercise 8

Let $f : x \mapsto x^2$, and let δ be "vanishingly small" and positive.

(1) Draw the result of a zoom on f centred on $\langle 2; 4\rangle$ so that δ becomes visible.
Show, on the drawing, the values 2 and $f(2)$, $2+\delta$ and $f(2+\delta)$, $2-\delta$ and $f(2-\delta)$.
What does the curve look like?

(2) For the same function, draw the result of a zoom centred on $\langle 1; 1\rangle$
Show, on the drawing, the values 1 and $f(1)$, $1+\delta$ and $f(1+\delta)$, $1-\delta$ and $f(1-\delta)$.

(3) Similar question for a zoom centred on $\langle 0; 0\rangle$.

Exercise 9

Draw the result of zooms so that δ becomes visible for
$g : x \mapsto x^3$, and $h : x \mapsto |x|$
For g: centres are $\langle 1; 1\rangle$, $\langle 2; 8\rangle$ and $\langle 0; 0\rangle$
For h: centres are $\langle 1; 1\rangle$, $\langle 2; 2\rangle$ and $\langle 0; 0\rangle$

Exercise 10

Draw a zoom centred on $\langle 0; 0\rangle$ and another zoom centred on $\langle 0; -1\rangle$ for

$$k : x \mapsto \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Observability and Closure

Imagine levels of resolution as with microscopes. Some numbers are **always observable** – observable without microscope. One can always observe all the numbers you have encountered so far which can be defined, described or computed without using the concept of observability, such as

$$1 \ ; \ 0,2 \ ; \ 0 \ ; \ 3/4 \ ; \ \sqrt{2} \ ; \ \pi; \ \ldots.$$

If we zoom in, they remain observable (even though when we zoom we see a shorter range of numbers)

Some numbers are ultrasmall relative to these.

Consider objects that can be seen with the unaided eye. You will need optical tools such as microscopes and telescopes to see more objects. With this "optical" level of resolution, we discover smaller and larger objects. But this is not the end of the story: there are ultrasmall numbers relative to this finer level of resolution. We will need radio-telescopes and electronic microscopes. Then huge and tiny objects become observable... and so on and so forth!

⚠ When we use a finer level of resolution we do not lose sight of the numbers which were already observable: a number *observable* at a given level of resolution is observable at all finer levels of resolution.

This means that if y is less observable than x, then x remians observable when y is observable.

We also use that every real number is observable at *some* level of observation.

These properties are summarised here:

Properties of Observability Let x, y and z be real numbers.

(1) x is as observable as x.

(2) If y is less observable than x, then x is observable when y is observable..

(3) If y is less observable than x and if z is less observable than y, then z is less observable than x.

Definition 1
*The **context** of a property, function or set is the list of parameters used in its definition.*

The word "observable" refers to a context whether it is explicitly mentioned or not. If a number is observable whenever any other number is observable, we say that it is *always* observable. It is thus meaningless to use the concept of observability if it is not possible to determine the context. An informal way to define the context is: the context is the parameters, sets and functions the statement is about. Therefore to determine the context of a statement, one must be able to understand it and describe what it says and about what it says something.

Closure Principle
Numbers, sets or functions, defined without reference to observability are always observable. If a number, set or function, satisfies a given property, then there is an observable number satisfying that property

In the last sentence the context of observability is given by the property.

The closure principle tells us that all "familiar" numbers such as $1;\ 3;\ 10^{10};\ \sqrt{2}$ or π are always observable

It also tells us that if a number is uniquely calculated using some parameters, the resulting number will be observable. Non observable results do not show up unless explicitly summoned. We also have

$$\boxed{f(a) \quad \text{is observable}}$$

This refers to the context, by the word "observable". The only parameters of this property are f and a and since a function has a unique output, it is observable

The sets of whole numbers ($\mathbb{N}$), of integers ($\mathbb{Z}$), of rationals ($\mathbb{Q}$) and real numbers ($\mathbb{R}$) are defined without reference to observability, hence are always observable.

The interval $[a, b]$ satisfies the property $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ which has parameters a and b and $\mathbb{R}$. Hence it is as observable as a, b and $\mathbb{R}$. Since $\mathbb{R}$ is always observable it is as observable as a and b, and in fact will be as visible as the least visible of a and b.

Example: Let $f : x \mapsto x^2 + 3$, The parameters of f are 2 and 3 which are always observable. The number $f(4)$ is thus also always observable.

Definition 2
*A real number is **ultrasmall** if it is nonzero and strictly smaller in absolute value than any strictly positive observable number*

This definition makes an implicit reference to a context.

⚠ Note that 0 is not ultrasmall.

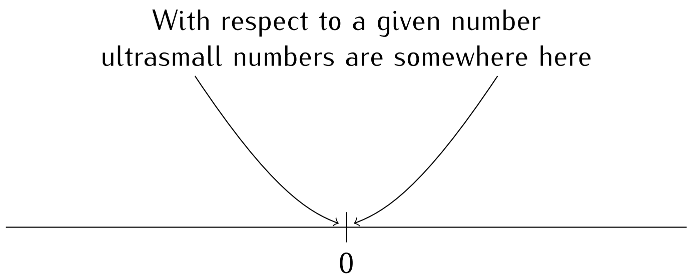
Principle of ultrasmallness
Relative to any number, there exist ultrasmall real numbers.

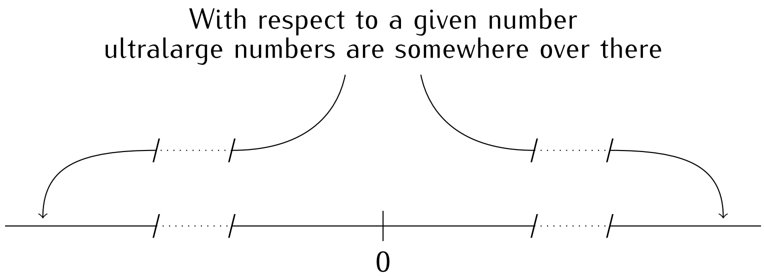
Exercise 11
Consider a context such that ε is ultrasmall.
Explain why ε is not observable.

Exercise 12
Consider a context such that a is observable and ε is ultrasmall.
Explain why $a + \varepsilon$ is not observable.

Definition 3
*A real number is **ultralarge** if it is larger in absolute value than any strictly positive observable number*

⚠ Note the asymmetry: if h is ultrasmall relative to an observable x, then h is not observable. But then x is observable relative to h (property 2 of observability), hence x **is not** ultralarge relative to h.





Definition 4
Let a, b be real numbers. We say that a is ***ultraclose*** *to b, written*

$$a \simeq b,$$

if $b - a$ is ultrasmall or if $a = b$.

This definition makes an implicit reference to a context.
In particular, $x \simeq 0$ if x is ultrasmall or zero.

If $a \simeq b$ then a and b are said to be neighbours. If (relative to some context) a is a neighbour of b and is observable then a is the observable neighbour of b.

A rational number may have an observable neighbour which is not rational. The number $\sqrt{2}$ is always observable because it is completely and uniquely defined by the parameter 2. Relative to this context consider an ultralarge N and take the first N digits of $\sqrt{2}$. This is a rational number which is not observable (it depends on N and is therefore as observable as N). Yet it is ultraclose to an observable number which is $\sqrt{2}$.

The existence of an observable neighbour is given by the following.

Principle of the observable neighbour
Relative to a context, any real number x which is not ultralarge can be written in the form $a + h$ where a is observable and $h \simeq 0$.

Theorem 1
Relative to a context:
If a and b are observable and if $a \simeq b$, then $a = b$

Exercise 13

Show that if x has an observable neighbour, then it is unique. This is in fact equivalent to proving theorem 1.

This unique number is **the observable neighbour** of x.

Exercise 14

Prove the following:

Theorem 2
Let a and b be the context. If $x \in [a;b]$ then the observable neighbour of x is in $[a;b]$.

Exercise 15

Prove the following:

(1) If ε is ultrasmall relative to x then $\frac{1}{\varepsilon}$ is ultralarge relative to x.

(2) If M is ultralarge relative to x then $\frac{1}{M}$ is ultrasmall relative to x.

Exercise 16

Prove the following theorems (together they give all the rules needed for analysis and will be referred to by "ultracomputation" or "ultracalculus"):

Theorem 3
Let ε and δ be ultrasmall relative to a context and let a be observable.

(1) Then: $a \cdot \varepsilon$ is ultrasmall.

(2) Then: $\varepsilon + \delta \simeq 0$

(3) Then: $\varepsilon \cdot \delta$ is ultrasmall

(4) If $a \neq 0$ Then: $\dfrac{a}{\varepsilon}$ is ultralarge

Theorem 4 (Ultracomputation)
Relative to a context: If a and b are observable and if $a \simeq x$ and $b \simeq y$,

(1) $a + b \simeq x + y$

(2) $a - b \simeq x - y$

(3) $a \cdot b \simeq x \cdot y$

(4) If also $b \neq 0$, $\dfrac{a}{b} \simeq \dfrac{x}{y}$.

For the last item of theorem 4, it is enough to show

Theorem 5
Relative to a context. If b is observable and $b \neq 0$ and if $b \simeq y$ then $\dfrac{1}{b} \simeq \dfrac{1}{y}$

and use item 3 to conlcude.

Practice exercise 1 Answer page 14

Consider a context.

(1) Give an example of x and y such that $x \simeq y$ but $x^2 \not\simeq y^2$.

(2) Give an example of x and y such that $x \simeq y$ but $\frac{1}{x} \not\simeq \frac{1}{y}$.

Practice exercise 2 Answer page 14

Relative to a context.

In the following, assume that ε, δ are positive ultrasmall and H, K positive ultralarge numbers. Determine whether the given expression yields an ultrasmall number, an ultralarge number or a number in between.

(1) $1 + \dfrac{1}{\varepsilon}$

(2) $\dfrac{\sqrt{\delta}}{\delta}$

(3) $\sqrt{H+1} - \sqrt{H-1}$

(4) $\dfrac{H+K}{H \cdot K}$

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7}$

(6) $\dfrac{\sqrt{1+\varepsilon} - 2}{\sqrt{1+\delta}}$

Practice exercise 3 Answer page 15

Relative to a context find ultrasmall ε and δ (or the relation between them) such that $\dfrac{\varepsilon}{\delta}$ is:

(1) not ultralarge and not ultrasmall,

(2) ultralarge,

(3) ultrasmall.

⚠ The previous exercise show that if no relation is known between ultrasmall numbers ε and δ, their quotient can be of any possible magnitude.

Contextual Notation
The only acceptable properties are those that do not refer to observability or those that use the symbol "$\simeq$" understood as relative to the context of the property in question.

A context is *extended* if parameters are added to the list.

Since observable numbers remain observable if we zoom further in, a property is not changed if the context is extended.

Answers to practice exercises

Answers to practice exercice 1, page 13

(1) Let $x = N$ be ultralarge, and $y = N + \frac{1}{N}$ so $x \simeq y$ but $x^2 = N^2 \not\simeq N^2 + 2 + \frac{1}{N^2} = y^2$.

(2) Let h be ultrasmall, then let $x = h$ and $y = h^2$. Then $x \simeq 0$ and $y \simeq 0$ hence $x \simeq y$. Then $\frac{1}{h}$ and $\frac{1}{h^2}$ are both ultralarge and $\frac{1}{h^2} - \frac{1}{h} = \frac{1}{h}(\frac{1}{h} - 1)$. By ultracomputation, this is ultralarge, hence $\frac{1}{x} \not\simeq \frac{1}{y}$.

Answers to practice exercice 2, page 13

The terms ultrasmall or ultralarge all refer to a given context.

(1) As $\frac{1}{\varepsilon}$ is ultralarge $1 + \frac{1}{\varepsilon}$ is ultralarge.

(2) We have $\frac{\sqrt{\delta}}{\delta} = \frac{1}{\sqrt{\delta}}$ which is ultralarge.

(If $\delta < c$ for any observable c, then $\sqrt{\delta} < \sqrt{c}$ and $\sqrt{\delta} \simeq 0$ hence $\frac{1}{\sqrt{\delta}}$ is ultralarge.)

(3) Maybe surprisingly, this is ultrasmall. To see this we multiply and divide by the conjugate:

$$\begin{aligned}
\sqrt{H+1} - \sqrt{H-1} &= \frac{(\sqrt{H+1} - \sqrt{H-1})(\sqrt{H+1} + \sqrt{H-1})}{\sqrt{H+1} + \sqrt{H-1}} \\
&= \frac{(H+1) - (H-1)}{\sqrt{H+1} + \sqrt{H-1}} \\
&= \frac{2}{\sqrt{H+1} + \sqrt{H-1}}.
\end{aligned}$$

H is assumed positive, its square root (plus or minus 1) is also a positive ultralarge. The sum of 2 positive ultralarge numbers is ultralarge hence the quotient is ultrasmall.

(4) $\dfrac{H+K}{HK} = \dfrac{1}{K} + \dfrac{1}{H}$ is ultrasmall.

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7} = \dfrac{35 + 7\varepsilon - 35 - 5\delta}{49 + 7\delta} = \dfrac{\overbrace{7\varepsilon - 5\delta}^{\simeq 0}}{\underbrace{49 + 7\delta}_{\simeq 49}}$ is ultrasmall or zero.

(6) $\dfrac{\overbrace{\sqrt{1+\varepsilon} - 2}^{\simeq -1}}{\underbrace{\sqrt{1+\delta}}_{\simeq 1}} \simeq -1$, hence not ultralarge and not ultrasmall.

Answers to practice exercice 3, page 13

(1) Take $\varepsilon = \delta$ then $\dfrac{\varepsilon}{\delta} = 1$.

(2) Take $\delta = \varepsilon^2$, then $\dfrac{\varepsilon}{\delta} = \dfrac{1}{\varepsilon}$ is ultralarge.

(3) Take $\varepsilon = \delta^2$, then $\dfrac{\varepsilon}{\delta} = \delta$ is ultrasmall.

3
Derivatives

We will often use dx to indicate an ultrasmall *increment*[1] of the variable x. It may be positive or negative but will never – by definition – 0.

Exercise 17
Let
$$f : x \mapsto x^2$$
The graph of this function is a curve (a parabola). Zoom in on the point $\langle 2, 4\rangle$. 2 and 4 are always observable. Consider the value of the function at $2+dx$, and draw a straight line passing through $\langle 2, 4\rangle$ and $\langle 2+dx, f(2+dx)\rangle$.

- What is the slope of this straight line?
- What is the observable neighbour of this slope?

Definition 5
*A real function f defined on an interval containing a is **differentiable at** a if there is an observable value D such that, for any dx*
$$\frac{f(a+dx)-f(a)}{dx} \simeq D$$
*Then $D = f'(a)$ is the **derivative** of f at a.*

The "for any dx" means that the value of D must not depend on the choice of the ultrasmall dx, in particular, whether it is positive or negative.

When the derivative exists, it is the observable neighbour of $\dfrac{f(a+dx)-f(a)}{dx}$.

⚠ This is a statement about f at a, hence the context is the list of parameters of f and a.

Metaphorically, finding the derivative can be described by: Zoom in. If what you see is indiscernible from a straight line, then measure the slope of that line. Zoom out. Drop what you cannot see anymore.

[1] increment: a positive or negative change in a variable. The term is generally used to mean a small change.

Exercise 18

Using definition 5 calculate the derivatives (if they exist) of the following:

(1) $f : x \mapsto 3x^2 + x - 5$ at $x = -2$ and $x = 2$.

(2) $g : x \mapsto 2x^3 - 2$ at $x = 1$ and $x = 0$.

(3) $h : x \mapsto |x|$ at $x = 2$, $x = -2$ and at $x = 0$.

Exercise 19

Let $f : x \mapsto x^3 - x - 6$. Check that 2 is a root of f. Are there other roots?

At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

Exercise 20

Let $f : x \mapsto 2x^3 - 4x^2 + 2x$. At what values of x is the function equal to zero? At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x de we have $f'(x) > 0$ and at wha

Use all this information to make a rough sketch of the function.

Practice exercise 4 Answer page 29

Calculate the derivative of the following:

(1) $f : x \mapsto 5x^2 - 10x$ at $x = 2$

(2) $g : x \mapsto 5(x - 10)^2$ at $x = 3$

(3) $h : x \mapsto x^4 + x^3 + x^2 + x + 1$ at $x = 1$

(4) $k : x \mapsto 5x^2 + 10$ at $x = 2$

Exercise 21

Consider the derivative at x (general case) of the function

$$f : x \mapsto x^2 + 3x.$$

Show that it is differentiable for all x and that $f'(x) = 2x + 3$.

Notice that in a derivative, the division is **always** between two ultrasmall numbers. They <u>cannot</u> be replaced by 0 since $\frac{0}{0}$ is not defined.

If a function is differentiable for all x in an interval, then f is said to be differentiable on the interval.

Definition 6
*If $f'(x)$ exists for all x in I **the derivative function** is*

$$\begin{aligned} f' : I &\to \mathbb{R} \\ x &\mapsto f'(x) \end{aligned}$$

If $f'(a) = 0$, then in an ultrasmall neighbourhood of a the function is **stationary** – on an ultrasmall neighbourhood $[a - dx; a + dx]$ its variation is of the form $\varepsilon \cdot dx$ for ultrasmall ε – its graph is indistinguishable from a horizontal line.

Exercise 22
Differentiate $f : x \mapsto x^2$ and $g : x \mapsto x^3$ at general x.

Notation: Let dx be ultrasmall relative to f and x. We write

$$\Delta f(a) = f(a + dx) - f(a) \text{ or } f(a + dx) = f(a) + \Delta f(a).$$

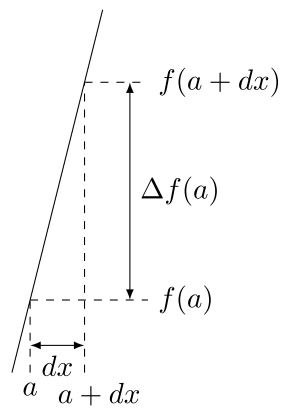
Hence, we have:

$$\frac{\Delta f(a)}{dx} \simeq f'(a).$$

Notation: A "$\simeq$" symbol may be replaced by a "$=$" symbol by adding a value ultraclose to zero on one of the sides i.e., $A \simeq B \Rightarrow A = B + \varepsilon$ where $\varepsilon \simeq 0$. Sometimes working with equality is safer.

Hence

$$\frac{\Delta f(a)}{dx} = f'(a) + \varepsilon \text{ with } \varepsilon \simeq 0$$



Note: drawings involving ultrasmall or ultralarge values are not meant to be to scale nor be a correct representation. Their purpose – as all drawings used in mathematics – is merely to help the mind.

Practice exercise 5 Answer page 29

Using definition 5, give the derivative functions of the following functions:

(1) $f : x \mapsto 3x + 2$

(2) $g : x \mapsto 2x^2 - x$

(3) $h : x \mapsto 5x^3 + 2x^2 - x$

(4) $k : x \mapsto 5x^3 + 2x^2 + 3x + 2$

In some cases, the slope to the right of a point is not the same as the slope to the left of that point. The derivative is the slope when it is the same on both sides.

Exercise 23

Using definition 5 calculate the derivatives (if they exist) of the following:

(1) $g : x \mapsto \begin{cases} x^2 - 1 & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$ at $x = -3$, $x = 1$ and $x = 0$.

(2) $h : x \mapsto \begin{cases} x^2 - 1 & \text{if } x < 0.5 \\ x - 1.25 & \text{if } x \geq 0.5 \end{cases}$ at $x = 0.5$.

(3) $k : x \mapsto \begin{cases} x^2 & \text{if } x > 0 \\ -x^3 & \text{if } x \leq 0 \end{cases}$ at $x = 0$.

(4) $f : x \mapsto \begin{cases} x^2 - 1 & \text{if } x < 0.5 \\ x + 1 & \text{if } x \geq 0.5 \end{cases}$ at $x = 0.5$.

Exercise 24

Let $f : x \mapsto ax + b$.

Show that the slope of f is a.

Theorem 6 (Derivative at a maximum or a minimum.)
Let f be a real function defined on an open interval $]a; b[$ differentiable at $c \in]a; b[$.
If $f(c)$ is a maximum (or a minimum) then $f'(c) = 0$.

Exercise 25

Prove theorem 6. (Hint, consider the variation $\Delta f(c)$.)

Exercise 26

Prove the theorem 7.

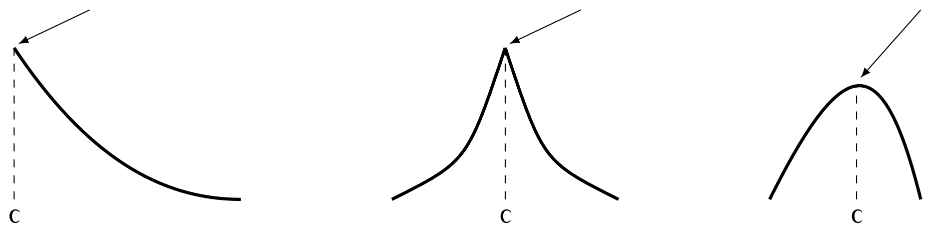
Theorem 7 (Critical Point Theorem)

Let f be a continuous function on I and suppose that c is a point in I and f has either a maximum or a minimum at c. Then one of the following three things must happen:

(1) c is an end point of I.

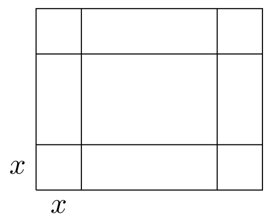
(2) $f'(c)$ is undefined.

(3) $f'(c) = 0$



Exercise 27

A factory wants to make cardboard boxes (with no top) out of sheets of $30cm \times 16cm$



The volume will be a function of x. The dimensions of the base are $30 - 2x$ and $16 - 2x$ (in centimetres). The height is x. What value(s) of x give(s) the maximum volume for the box?

Suppose f is differentiable at x_0. We observe that through a microscope, the curve of a function f at x_0 is indistinguishable from a straight segment. This straight segment meets the function at $\langle x_0; f(x_0)\rangle$ and there is a (unique) line which extends this segment with slope equal to the derivative. This line is the tangent line.

Definition 7

Let f be differentiable at x_0. The tangent line T_{x_0} is a line through $\langle x_0; f(x_0)\rangle$ with slope $f'(x_0)$.

The tangent line satisfies $T(x_0) = f(x_0)$ and $T'(x_0) = f'(x_0)$.

Exercise 28

Let $f : x \mapsto x^2$. Find the equation of the straight line tangent to f at $x = 3$.

Exercise 29

Show that

$$T_{x_0} : x \mapsto f'(x_0)(x - x_0) + f(x_0).$$

Exercise 30

Give the equation of the line tangent to $x \mapsto x^3 - 3 \cdot x^2$ at $x = 2$. For which values of x is this tangent horizontal?

Exercise 31

(1) Find the slope of the curve given by $y = 5x^3 - 25x^2$ at $x = 3.5$.

Equivalent statement: compute $f'(x)\Big|_{x=3.5}$

(2) Find the equation of the line tangent to the curve at $x = 1$.

Exercise 32

(1) For $f : x \mapsto x^2 + 5$ and the point $A\langle 0; 0\rangle$, what is the equation of the line passing through A, and tangent to f?

(2) If $g : x \mapsto ax^2 + b$, what values must a and b take to make $g(x)$ tangent to $t : x \mapsto 3x - 2$ at $x = 5$? What are the coordinates of the contact point?

Exercise 33

Differentiate

(1) $f : x \mapsto \frac{1}{x}$ for $x = 1$ and $x = 2$.

(2) $g : x \mapsto \frac{1}{3x+2}$ for $x = 0$ and $x = 1$.

(3) $h : x \mapsto \frac{1}{x^2}$ for $x = 1$ and $x = -1$.

Linearity of the derivative

Theorem 8
Let f and g be real functions differentiable at a. Then the function $f+g$ is differentiable at a and
$$(f+g)'(a) = f'(a) + g'(a).$$

Theorem 9
Let $c \in \mathbb{R}$ and f be a real function differentiable at a. Then the function $c \cdot f$ is differentiable at a and
$$(c \cdot f)'(a) = c \cdot f'(a).$$

Exercise 34
Prove theorems 8 and 9

Theorem 10
Let $c \in \mathbb{R}$ and $f : x \mapsto c$, for $x \in \mathbb{R}$
$$f'(x) = 0.$$

Exercise 35
Prove theorem 10

Antiderivatives

Definition 8 (Antiderivative)
*If f' is the derivative function of f, then f is the **antiderivative** function of f'.*

Exercise 36
The velocity of an object is given by the derivative of its position (variation of position divided by variation of time).

The acceleration is given by the derivative of the velocity (variation of velocity divided by variation of time).

On earth, the acceleration of a falling body is constant (when there is no air friction) and approximately equal to $9.81\frac{m}{s^2}$, written g.

(1) Find the formula for the velocity with respect to time.

(2) Given the formula for velocity, find the formula for the position of a falling body with respect to time.

Exercise 37

Show that if F is an antiderivative of f, then for any constant C, $F + C$ is also an antiderivative of f.

Exercise 38

Considering previous exercise, reconsider your answers for exercise 36. Think in terms of units to determine what the constants could represent.

Exercise 39

Find the antiderivatives for the following:

(1) $x \mapsto 3x$

(2) $x \mapsto x^2$

(3) $x \mapsto 5$

(4) $t \mapsto 3t + 5$

(5) $u \mapsto u^2 + 3u + 5$

(6) $v \mapsto v^3$

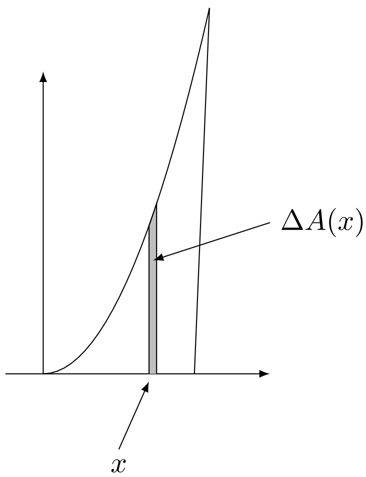
Check your results by differentiating them.

Area under the curve of $x \mapsto x^2$

Exercise 40

To find the area under $f : x \mapsto x^2$ between $x = 0$ and $x = 2$, the idea is to consider the *variation* of the area in order to find the area itself.

Assume that the area under f, between 0 and x is given by a function $A(x)$. Consider the variation $\Delta A(x)$, for ultrasmall variation of x noted dx.



Even though the exact value of $\Delta A(x)$ may not be directly seen, it can be shown to be between two values, m and M calculated by rectangles.

$$m < \Delta A(x) < M$$

- Give a formula for m, using x and f.
- Give a formula for M, using x and f.
- Divide all terms by dx.
- Show that all resulting quotients are ultraclose.
- Conclude that the area is given by a function which is the derivative of a known function.

THINGS TO LOOK OUT FOR

$f'(a)$ is NOT equal to $\frac{\Delta f(a)}{dx}$.

The relation is one of ultracloseness.

$$f'(a) \simeq \frac{\Delta f(a)}{dx}$$

Practice exercise 6 Answer page 29

Find the derivative of each of the following functions and specify its domain, starting from the definition.

(1) $a : x \mapsto 1$

(2) $b : x \mapsto |x|$

(3) $c : x \mapsto x$

(4) $d : x \mapsto x^2$

(5) $e : x \mapsto |x^2|$

(6) $f : x \mapsto x^3$

(7) $g : x \mapsto |x^3|$

(8) $h : x \mapsto \dfrac{1}{x}$

(9) $i : x \mapsto \dfrac{1}{x^2}$

Practice exercise 7 Answer page 29

Find the derivative of each of the following functions and specify its domain, using linearity and the results from the previous exercise.

(1) $a : x \mapsto 2x^2 - 4x + 5$

(2) $b : x \mapsto \dfrac{x^3 + 2x}{7}$

(3) $c : x \mapsto 3x^3 - \dfrac{2}{x}$

(4) $d : x \mapsto \dfrac{x^2 - 2x + 5}{x}$

(5) $e : x \mapsto 5x^3 - 7|x| + 8$

Practice exercise 8 Answer page 30

Find all the antiderivatives of each of the following functions, using linearity and the results from the exercise 1.

(1) $a : x \mapsto 10x$

(2) $b : x \mapsto x^2$

(3) $d : x \mapsto \dfrac{x}{|x|}$

(4) $e : x \mapsto 3x - 4$

(5) $f : x \mapsto x^2 - 2x + 4$

(6) $g : x \mapsto \dfrac{1}{x^2}$

(7) $h : x \mapsto 2x^2 - \dfrac{1}{2x^2}$

Practice exercise 9 Answer page 30

Let

$$f : x \mapsto \frac{1}{3}x^3 + \frac{7}{2}x^2 + 12x$$

Calculate its derivative, find where the derivative is positive, where it is negative and where it is equal to zero.

Calculate the intercepts of f and sketch the graph of f.

Practice exercise 10 Answer page 31

Consider the functions differentiated above:

(1) $a : x \mapsto 2x^2 - 4x + 5$

(2) $b : x \mapsto \dfrac{x^3 + 2x}{7}$

For a, give the equation the line tangent to the curve at $x = -2$

For b, give the equation the line tangent to the curve at $x = 1$

Answers to practice exercises

Answers to practice exercice 4, page 18

(1) $f'(2) = 10$

(2) $g'(3) = -70$

(3) $h'(1) = 10$

(4) $k'(2) = 20$

Answers to practice exercice 5, page 20

(1) $f'(x) = 3$

(2) $g'(x) = 4x - 1$

(3) $h'(x) = 15x^2 + 4x - 1$

(4) $k'(x) = 15x^2 + 4x + 3$

Answers to practice exercice 6, page 26

(1) $a'(x) = 0$ Domain=$\mathbb{R}$

(2) $b'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ Domain=$\mathbb{R} \setminus \{0\}$

(3) $c'(x) = 1$ Domain=$\mathbb{R}$

(4) $d'(x) = 2x$ Domain=$\mathbb{R}$

(5) $e'(x) = 2x$ Domain=$\mathbb{R}$

(6) $f'(x) = 3x^2$ Domain=$\mathbb{R}$

(7) $g'(x) = \begin{cases} 3x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$ Domain=$\mathbb{R}$

(8) $h'(x) = \dfrac{-1}{x^2}$ Domain=$\mathbb{R}$

(9) $i'(x) = \dfrac{-2}{x^3}$ Domain=$\mathbb{R}$

Answers to practice exercice 7, page 26

(1) $a'(x) = 4x - 4$ Domain=$\mathbb{R}$

(2) $b'(x) = \dfrac{3x^2 + 2}{7}$ Domain=$\mathbb{R}$

(3) $c'(x) = 9x^2 + \frac{2}{x^2}$ Domain=$\mathbb{R} \setminus \{0\}$

(4) $d'(x) = 1 - \frac{5}{x^2}$ Domain=$\mathbb{R} \setminus \{0\}$

(5) $e'(x) = \begin{cases} 15x^2 - 7 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ 15x^2 + 7 & \text{if } x < 0 \end{cases}$ Domain=$\mathbb{R} \setminus \{0\}$

Answers to practice exercice 8, page 26

(1) $A(x) = 5x^2 + C$ for any $C \in \mathbb{R}$

(2) $B(x) = \frac{x^3}{3} + C$ for any $C \in \mathbb{R}$

(3) $D(x) = C$ for any $C \in \mathbb{R}$ (function undefined at $x = 0$)

(4) $E(x) = \frac{3}{2}x^2 - 4x + C$ for any $C \in \mathbb{R}$

(5) $F(x) = \frac{x^3}{3} - x^2 + 4x + C$ for any $C \in \mathbb{R}$

(6) $G(x) = -\frac{1}{x} + C$ for any $C \in \mathbb{R}$

(7) $H(x) = \frac{2}{3}x^3 + \frac{1}{2x} + C$ for any $C \in \mathbb{R}$

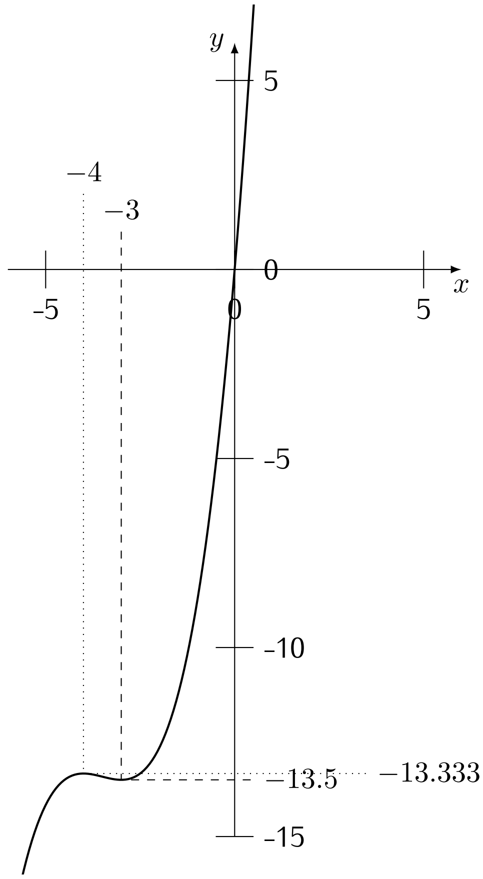
Answers to practice exercice 9, page 27

$f(x) = x\left(\frac{1}{3}x^2 + \frac{7}{2}x + 12\right)$

$\mathcal{S} = \{0\}$

$f'(x) = x^2 + 7x + 12 = (x + 3)(x + 4)$

$\mathcal{S}' = \{-3, -4\}$



Answers to practice exercice 10, page 27

(1) $t_a : x \mapsto -12x - 3$

(2) $t_b : x \mapsto \dfrac{5}{7}x - \dfrac{2}{7}$

4
Continuity

Informally: a function is continuous at $x = a$ if it is where you would expect it to be by observing where it is in the neighbourhood of a.

Definition 9 (Continuity)
*Let f be a real function defined around a. We say that f **is continuous at** a if (for any x)*

$$x \simeq a \Rightarrow f(x) \simeq f(a).$$

The continuity of f at a is a property of f and a. Hence the context is given by f and a.
The definition of continuity can also be interpreted in the following ways:

Definition 10 (Continuity: equivalent definition)
*Let f be a real function defined around a. We say that f **is continuous at** a if*

$$f(a + dx) \simeq f(a) \text{ not depending on } dx.$$

(As for the derivative, the context is f and a.)

Exercise 41
Show that $f : x \mapsto x^3$ is continuous at $a = 2$.

Exercise 42
Show whether $f : x \mapsto \dfrac{x}{x^2 + 1}$ is continuous for all values of x.

Exercise 43

(1) Show that $f : x \mapsto |x|$ is continuous at $x = 0$, at $x = 1$, at $x = -1$ and at x in general.

(2) Show that $g : x \mapsto \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$ and at x in general.

(3) Show that $g : x \mapsto \begin{cases} x^2 & \text{if } x \geq -1 \\ x^3 & \text{if } x < -1 \end{cases}$ is not continuous at $x = -1$ but is continuous for all other values of x.

Exercise 44

Prove the following theorem:

Theorem 11
If a real function f is differentiable at a then f is continuous at a.

(1) Give a direct proof.

(2) Give a proof by contrapositive.

Exercise 45

Use an induction proof to show that $x \mapsto x^n$ is continuous for all n.

Exercise 46

Use an induction proof to show that $x \mapsto a_0 + \sum_{k=1}^{n} a_k x^k$ is continuous for all n.

Exercise 47

Prove the following theorem:

Theorem 12
Let f and g be two real functions continuous at a. Then

(1) $f \pm g$ is continuous at a.

(2) $f \cdot g$ is continuous at a.

(3) $\dfrac{f}{g}$ is continuous at a if $g(a) \neq 0$.

Exercise 48

Prove the following theorem:

Theorem 13
Let f and g be two real functions. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Definition 11 (Continuity on an Interval)

*(1) Let f be a real function defined on the open interval $]a; b[$. Then f is **continuous on** $]a; b[$ if f is continuous at all $x \in]a; b[$.*

*(2) Let f be a real function defined on the closed interval $[a; b]$. Then f is **continuous on** $[a; b]$ if f is continuous at all $x \in]a; b[$ and if f continuous on the right at a and on the left at b.*

Informally: a function is continuous on an interval if its curve can be drawn without lifting the pencil, or if the function is where you expect it to be if it is hidden by a vertical line.

Exercise 49

Determine whether $f : x \mapsto x^2$ is continuous on its domain.

Clearly, if f and g are continuous on an interval I then the sum, difference, product and quotient (if $g(x) \neq 0$) are continuous on I. Moreover, if g is continuous on an interval containing $f(I)$ then $g \circ f$ is continuous on I.

Exercise 50

Show, using the definition of continuity, whether the following functions are continuous on the given intervals.

(1) $f_1 : x \mapsto \frac{1}{3}x + \sqrt{2}$ on $\mathbb{R}$

(2) $f_2 : x \mapsto x^2 - 3x - 1$ on $\mathbb{R}$

(3) $f_3 : x \mapsto \dfrac{x+2}{x-1}$ on $]1; +\infty[$

Exercise 51

Determine whether $f : x \mapsto \frac{1}{x}$ is continuous on its domain.

Exercise 52

(1) If at 8 am a thermometer indicated $15°C$ and indicated $20°C$ a 12 o'clock, is it possible to assert that there was at least one moment between 8 am and noon when the thermometer indicated $18°C$?

If yes, is it possible to say at what time?

(2) A person in a car starts then drives $100km$ in 1 hour. Is it possible to assert that at least at one instant during their trip their speedometer indicated $80\frac{km}{h}$? $100\frac{km}{h}$? $120\frac{km}{h}$?

Theorem 14 (Intermediate Value theorem)
Let f be a real function continuous on $[a; b]$. Let d be a real number between $f(a)$ and $f(b)$. Then there exists c in $[a; b]$ such that $f(c) = d$.

This theorem does not tell us how to find the root or the value c such that $f(c) = d$. It only asserts the *existence* of such a number. For specific functions where we can calculate the roots explicitly this theorem is not really necessary but, when proving theorems about continuous functions in general, it is the only way to know that there is a root.

Exercise 53

Give an example of a function f discontinuous on $[a; b]$ with $f(a) < 0$ and $f(b) > 0$ such that there is no c in the interval $[a; b]$ such that $f(c) = 0$.

Exercise 54

Proving theorem 14.

Let f be continuous on an interval $[a;b]$.

Assume $d = 0$ and $f(a) < 0 < f(b)$.

The context is f, a, b and 0. Take an ultralarge positive integer N and partition $[a;b]$ into N even parts, each of ultrasmall length $dx = \frac{b-a}{N}$. We thus have $x_0 = a,\ x_1 = x_0 + dx,\ \ldots,\ x_N = b$.

Call x_j the first point of the partition such that $f(x_j) \geq 0$. Hence $f(x_{j-1}) < 0$.

(1) How close are $f(x_j)$ and $f(x_{j-1})$?

(2) Let c be the observable part of x_j. Is it the observable part of x_{j-1}?

(3) Is $f(c)$ observable?

(4) How close is $f(c)$ from $f(x_j)$ and $f(x_{j-1})$?

(5) What is the value of $f(c)$?

(For $d \neq 0$ the theorem would hold for $g(x) = f(x) + d$; for $f(a) > f(b)$, reverse all inequality symbols.)

Definition 12

*A function has **maximum** (respectively **minimum**) on an interval I if there is a $c \in I$ such that for any $x \in I$ we have $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$).*
*If a point is either a maximum or a minimum, it is an **extremum**.*

Theorem 15 (Extreme value)

Let f be a continuous function on $[a;b]$. Then it has a maximum and a minimum on $[a;b]$.

Exercise 55

Without loss of generality, we consider the case of a maximum (for the minimum replace f by $-f$). Context is f, a and b.

We proceed similarly to exercise 54.

Let f be continuous on an interval $[a;b]$.

Take an ultralarge positive integer N and partition $[a;b]$ into N even parts, each of length $dx = \frac{b-a}{N}$. We thus have $x_0 = a,\ x_1 = x_0 + dx,\ \ldots,\ x_N = b$.

Call x_j the first point of the partition such that $f(x_j) \geq f(x_i)$ for any i between 0 and N.

(1) Call c the observable part of x_j. Is $f(c)$ observable?

(2) Let x be observable. Then there is an i such that $x_i \leq x \leq x_{i+1}$. Using continuity, conclude that $f(x) \leq f(x_j)$.

(3) By the closure principle, conclude that $f(c)$ is the maximum.

Continuity and Differentiability

Theorem 16 (Rolle)
Let f be a real function continuous on $[a;b]$ and differentiable on $]a;b[$. If $f(a) = f(b)$, then there is a $c \in]a;b[$ such that

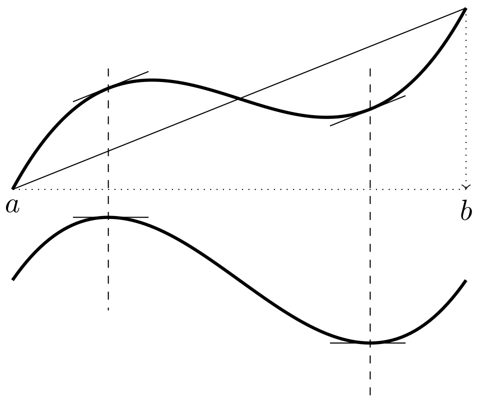
$$f'(c) = 0.$$

Exercise 56
Prove Rolle's theorem.

Theorem 17 (Mean Value)
Let f be a real function continuous on $[a;b]$ and differentiable on $]a;b[$. Then there is a $c \in]a;b[$ such that

$$f(b) - f(a) = f'(c) \cdot (b - a).$$

Exercise 57
Consider g which is obtained by subtracting the line $\ell(x)$ through $(a, f(a))$ and $(b, f(b))$ from the function f i.e., $g(x) = f(x) - \ell(x)$.



Show that g satisfies Rolle's theorem and conclude with the mean value theorem.

Variation

We now make the link between global variation and derivative.

Definition 13
Let f be a real function defined on an interval I.

(1) *The function f is* ***increasing on*** *I if $f(x) \leq f(y)$, whenever $x < y$ in I.*

(2) *The function f is* ***decreasing on*** *I if $f(x) \geq f(y)$, whenever $x < y$ in I.*

If the inequalities are strict, then we say that the function is strictly increasing or strictly decreasing.

Theorem 18 (Variation and Derivative)
Let f be a real function differentiable on an interval I. Then

(1) If $f'(x) \geq 0$ (> 0) whenever $x \in I$ then f is (resp. strictly) increasing on I.

(2) If $f'(x) \leq 0$ (< 0) whenever $x \in I$ then f is (resp. strictly) decreasing on I.

(3) If $f'(x) = 0$ whenever $x \in I$ then f is constant on I.

The converse is obvious: if f is increasing at a, then $f'(a) > 0$.

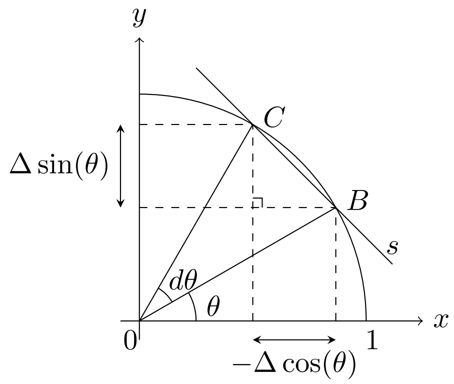
Exercise 58
Prove theorem 18 using the mean value theorem.

Exercise 59
Prove that following theorem:

Theorem 19
The antiderivative of a function – when it exists – is unique up to an additive constant i.e., for any constant C

$$f' = g' \Rightarrow f = g + C$$

Exercise 60
Consider the trigonometric circle. The chord BC is shorter than the arc BC. Trigonometric functions are defined as vertical and horizontal coordinates of a point on the unit circle, thus without reference to observability and are therefore always observable.



Show that sine and cosine are continuous functions.

Practice exercise 11 Answer page 39
Let f be continuous and positive on $[a; b]$
Assuming the area function under f is given by A. Show how A can be bounded above and below. Show that there is a value $c \in [a; b]$ such that $A = f(c) \cdot (b - a)$.

Answers to practice exercice 11, page 38

Since f is continous on the interval, it has a minimum $\langle m, f(m)\rangle$ and a maximum $\langle M, f(M)\rangle$

Then $f(m)\cdot(b-a) \leq A \leq f(M)\cdot(b-a)$. So $f(m) \leq \frac{A}{b-a} \leq f(M)$. By the intermediate value theorem (f is continuous) there is $c \in [a;b]$ such that $f(c) = \frac{A}{b-a}$ hence that $A = f(c)\cdot(b-a)$.

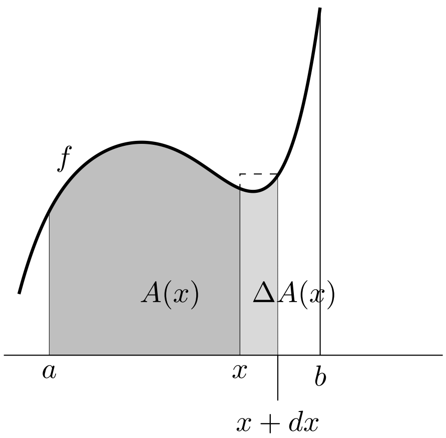
5

Integrals

Area under a curve

Consider a nonnegative function f continuous on a closed interval $[a;b]$. Note $A(x)$ the area between the curve of f and the horizontal x-axis.

The variation between x and $x+dx$ is $\Delta A(x)$.



Exercise 61

Using the drawing above, consider $f : x \mapsto 3x^2 + x$ between 2 and $2+dx$.

(1) Write the formula for the variation of the area $\Delta A(2)$ or at least for upper and lower bounds to $\Delta A(2)$.

(2) Determine the equation of A.

Theorem 20

Let f be a non-negative function continuous on $[a;b]$. Then the function

$$A : x \mapsto A(x),$$

where $A(x)$ is the area under the curve of f between a and x, has the following properties

(1) $A'(x) = f(x)$, whenever $x \in [a;b]$.

(2) $A(a) = 0$.

Exercise 62

Prove theorem 20.

Reread exercises 40 and 61 and generalise the proof. At one point you will need the extreme value theorem (theorem 15).

Exercise 63

Calculate the area under $f : x \mapsto 5x^3 - 2x^2 + x - 2$ between $x = 1$ and $x = 4$.

Use $A' = f$ and $A(1) = 0$.

Exercise 64

Consider the area under f between a and b. Show that if $A' = f$ and $A(a) = 0$, then $A(x) + C$ leads to $C = -A(a)$.

Hence the area is calculated by $A(b) - A(a)$.

Notation

$$A(b) - A(a) \text{ is written } A(x)\Big|_a^b$$

A $\int$um of $\int$lices

Exercise 65

Let $g : x \mapsto x^2$, $a = 0$ and $b = 5$.

(1) Cut the interval $[a; b]$ into an ultralarge number N of pieces. Put all these pieces together again – add all their lengths. What is the result?

Write this using the symbol for a sum i.e., sum for $k = 0$ to $N - 1$.

(2) For each $dx = \frac{b-a}{N}$ there is a corresponding Δy. Add all the Δy between $f(a)$ and $f(b)$. Find the result.

(3) Use the microscope equation to express Δy in terms of y or y'. Add all these terms. Find the result.

The (vertical) variation of f between a and b is written $f(x)\Big|_a^b$

Fundamental Theorem of Calculus

Definition 14
*Let f be a real function defined on $[a;b]$. Let n be a positive integer. Let $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \ldots, n$. We say that f **is integrable on** $[a;b]$ if there is an observable I such that for any ultralarge integer n with $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \ldots, n$, we have*

$$\sum_{i=0}^{n-1} f(x_i) \cdot dx \simeq I.$$

If such an I exists, it is called the integral of f between a and b; written

$$\int_a^b f(x) \cdot dx.$$

Note that this sum is defined whether f is positive or not.

Theorem 21
Let f be an function continuous on $[a;b]$. Let $\frac{1}{N} \simeq 0$, $dx = \frac{b-a}{N}$ and $x_k = a + k \cdot dx$, then there exists a point $c \in [a;b]$ such that

$$f(c) \cdot (b-a) = \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

Exercise 66
Prove theorem 21. To prove this, use that since f is continuous on $[a;b]$ there is a minimum $f(m)$ and a maximum $f(M)$ of f on $[a;b]$. Replace all $f(x_k)$ one by $f(m)$ then by $f(M)$ to conclude that the sum is not ultralarge and then use continuity again to conclude the proof.

Theorem 22 (Additivity of the integral)
Let f be a real integrable function continuous on $[a;c]$ and $b \in [a;c]$. Then

$$\int_a^b f(x) \cdot dx + \int_b^c f(x) \cdot dx = \int_a^c f(x) \cdot dx.$$

Exercise 67
Prove theorem 22.

Theorem 23
If f is continuous an $[a;b]$ then f is integrable on $[a;b]$

Exercise 68

Difficult!

To prove theorem 23, you must show that for $\frac{1}{N} \simeq 0$ and $\frac{1}{M} \simeq 0$ with $du = \frac{b-a}{N}$ and $u_k = a + k \cdot du$ and also $dv = \frac{b-a}{M}$ and $v_j = a + j \cdot dv$ then

$$\sum_{k=0}^{N-1} f(u_k) \cdot du \simeq \sum_{j=0}^{N-1} f(v_j) \cdot dv$$

This can be done by using $\sum_{i=0}^{N \cdot M-1} f(w_i) \cdot dw$ with $dw = \frac{b-a}{M \cdot M}$ and comparing each sum with this one.

By symmetry, it is enough to show that

$$\sum_{k=0}^{N-1} f(u_k) \cdot du \simeq \sum_{i=0}^{N \cdot M-1} f(w_i) \cdot dw$$

Consider an interval $[u_\ell; u_{\ell+1}]$ and *the same interval* $[w_{M \cdot \ell}; w_{M \cdot \ell + M}]$, this interval of length du is one step in the sum of $f(u_k) \cdot dx$ and M steps in the sum of $f(w_i) \cdot dw$.

Show that

$$f(u_\ell) \cdot du \simeq \sum_{i=M \cdot \ell}^{M \cdot \ell + M - 1} f(w_i) \cdot dw$$

and conclude the proof.

Theorem 24

If f is a continuous function on $[a, b]$ then

$$F(x) = \int_a^x f(t) \cdot dt$$

is an antiderivative of f on $]a, b[$ and the only one satisfying $F(a) = 0$.

Exercise 69

Prove theorem 24 starting with the definition of the derivative applied to the integral. By theorem 23, it is integrable.

Theorem 25 (Fundamental theorem of Calculus)

Let f be a function continuous on $[a; b]$. Let F be an antiderivative of f on $[a; b]$. Then

$$\int_a^b f(x) \cdot dx = F(b) - F(a).$$

The method used in the proof can also be seen as looking at the link between the global variation of a function F and its derivative f.

Exercise 70

Consider the variation of F between a and b.
Let $n \in \mathbb{N}$ such that $1/N \simeq 0$ and $dx = \frac{b-a}{N}$ and $x_k = a + k \cdot dx$.
Then clearly, we have

$$F(b) - F(a) = \sum_{k=0}^{N-1} \Delta F(x_k)$$

Here the context is f, a, b – not necessarily any given x_j!

(1) On each interval $[x_k, x_{k+1}]$ (which is also in the form $[x_k, x_k + dx]$) there is a c such that

$$F(x_k + dx) - F(x_k) = f(c) \cdot dx,$$

Why is this? By what theorem?

(2) Explain why we have $f(c) \simeq f(x_k)$.

(3) Conclude by explaining why:

$$\sum_{k=0}^{N-1} F(x_k + dx) - F(x_k) = \sum_{k=0}^{N-1} f(x_k) \cdot dx + \sum_{k=0}^{N-1} \varepsilon_k \cdot dx.$$

Exercise 71

Explain and prove that

$$\left(\sum_{k=0}^{N-1} \varepsilon_k \cdot dx \right) \simeq 0$$

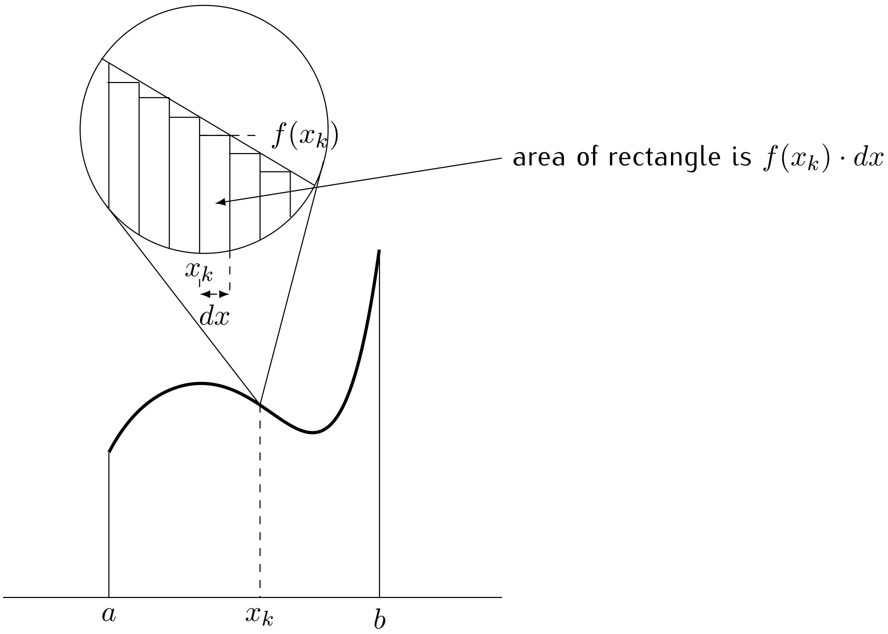
hence that

$$F(b) - F(a) \simeq \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

Hence, the global variation of F between a and b is, up to an ultrasmall value, the sum of $F'(x_i) \cdot dx$ provided F' is continuous on $[a, b]$.

Page 41 we looked at one slice of the area under a positive function. Now we show that if we sum up all slices on the area under a curve, the antiderivative gives the answer. Hence we have

$$\text{area} \simeq \sum_{i=0}^{N-1} f(x_k) \cdot dx.$$



⚠ The drawing can be misleading. It is only a specific case. A continuous function does not necessarily appear as a straight line under magnification. The extreme value theorem ensures that it has a maximum and minimum on the interval.

Notation: we write

$$F(x)\Big|_a^b = F(b) - F(a).$$

If bounds are given, the integral represents a value: it is a **definite integral**. If no bounds are given, it represents an antiderivative: it is an **indefinite integral**.

Exercise 72

Show that for a definite integral, it does not matter which antiderivative is chosen.

Exercise 73

What conditions would a function need to satisfy in order to be non-integrable? Give such a function.

Exercise 74

A constant function $f : x \mapsto C$ from a to b defines a rectangle. Check that the area under f is the "usual" formula: $(b-a) \cdot C$

Exercise 75

The function $y = x$ defines a triangle. Show that the area of the triangle from 0 to a yields the "usual" result for the area of a triangle.

Exercise 76

(1) Calculate the area between the curve and the x-axis for $y = x^2$ from $x = -5$ to $x = 5$.

(2) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = 0$ to $x = 3$.

(3) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -2$ to $x = 0$.

(4) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -10$ to $x = 10$.

Notice that the integral can be a negative value. If f represents the velocity of an object, a negative integral means that the distance is becomming smaller. If the integral is equal to zero, the object is back where it started.

So far we have assumed that an area function exists. Now we can give a definition.

Definition 15 (Area)
The area between a positive continuous function and the x-axis, on an interval $[a; b]$ is given by the integral of the function on $[a; b]$.

Exercise 77
Calculate the mean value of $x \mapsto x^2$ on $[-4; 4]$.

Linearity

Theorem 26 (Linearity of the integral)
Let f and g be real functions continuous on $[a; b]$. Let λ, μ be real numbers. Then

(1)

$$\int_a^b (\lambda \cdot f(x)) \cdot dx = \lambda \cdot \int_a^b f(x) \cdot dx$$

(2)

$$\int_a^b (f(x) + g(x)) \cdot dx = \int_a^b f(x) \cdot dx + \int_a^b g(x) \cdot dx.$$

Note that if f and g are integrable then all linear combinations of f and g are integrable.

Theorem 27 (Monotonicity of the integral)
Let f be a real function continuous on $[a;b]$.

(1) If $f(x) \geq 0$ (resp. > 0) for each $x \in [a;b]$ then

$$\int_a^b f(x) \cdot dx \geq 0 \quad (\textit{resp.} > 0).$$

(2) If $f(x) = 0$ for each $x \in [a;b]$ then

$$\int_a^b f(x) \cdot dx = 0.$$

(3) If $f(x) \leq 0$ (resp. < 0) for each $x \in [a;b]$ then

$$\int_a^b f(x) \cdot dx \leq 0 \quad (\textit{resp.} < 0).$$

Exercise 78

Prove theorems 26 and 27.

6
Differential Calculus

For the following rules, the proofs proceed by steps:

(1) Definition of the derivative.

(2) Definition of Δ.

(3) Definition of operations on functions.

(4) Expansion of $f(a+dx)$ as $f(a)+\Delta f(a)$.

(5) Division by dx.

(6) Algebra.

(7) Definition of the antiderivative for the inverse rule about integration.

Exercise 79
Prove the following theorem:

Theorem 28
Let f and g be two real functions differentiable at a. Then the function $f \cdot g$ is differentiable at a and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

Exercise 80
Using the derivatives of $f : x \mapsto x^2$ and $g : x \mapsto x^3$, calculate the derivative of $h : x \mapsto x^5$ $(= x^2 \cdot x^3)$.

Exercise 81
Prove :

Theorem 29

$$(x^n)' = n \cdot x^{n-1}$$

by induction.

Exercise 82

Calculate the area between $y = 5x^4 - 3x^3 + 2x^2 - 10$ and the x-axis from $x = -1$ to $x = 1$.

Exercise 83

Sketch the curve of $f : x \mapsto x^2$ and $g : x \mapsto x^3$. Calculate the points where $f(x) = g(x)$
Calculate the closed geometric area of the surface between the two curves.

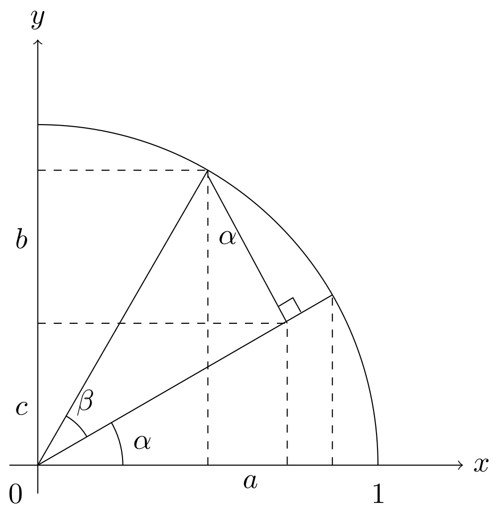
Circular functions

Exercise 84

Observe the following drawing where the angle β has been drawn on top of the angle α.

(1) Explain why the angle right at the top is equal to α

(2) Express the lengths of a, b and c in terms of $\sin(\alpha), \cos(\alpha), \sin(\beta)$ and $\cos(\beta)$.



Exercise 85

Finish the proof of

Theorem 30

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Exercise 86

Use the definition of the derivative and theorem 30 to expand $\Delta \sin(a)$

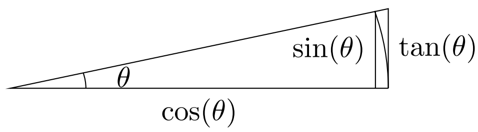
Exercise 87

To continue, you will need to prove theorem 31:

Theorem 31

$$\frac{\sin(d\theta)}{d\theta} \simeq 1.$$

Suppose first that $\theta > 0$ is in the first quadrant.



Comparing the area of the sector with that of the inside and outside triangles, we obtain

$$\text{inside triangle} \leq \text{sector} \leq \text{outside triangle.}$$

Rewrite this chain of inequalities replacing the areas by the corresponding formulae.

By using $-\theta$ if θ is negative, we see that the same inequalities are true for negative θ (in the fourth quadrant).

Let θ be ultrasmall. By continuity, $\cos(\theta) \simeq 1$. Then conclude the proof of the theorem.

Exercise 88

Show that

$$\frac{1 - \cos(d\theta)}{d\theta} \simeq 0.$$

Hint: multiply above and below by $(1 + \cos(d\theta))$

Exercise 89

Using theorem 31 and previous exercise, find the derivative of $\sin(x)$ and of $\cos(x)$.

These results are summarised here:

Theorem 32

(1) $\sin'(\theta) = \cos(\theta)$

(2) $\cos'(\theta) = -\sin(\theta)$

Exercise 90

Prove theorem 33.

Theorem 33 (Integration by parts)
Let f and g be real functions continuous on $[a;b]$ such that f' and g' are continuous on $[a;b]$. Then

$$\int_a^b f'(x)\cdot g(x)\cdot dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f(x)\cdot g'(x)\cdot dx.$$

Example: Consider the integral

$$\int_0^{\pi/2} x\cdot\sin(x)\cdot dx.$$

To integrate by parts, use $f' : x \mapsto \sin(x)$ et $g : x \mapsto x$. We have $f(x) = -\cos(x)$ and $g'(x) = 1$, hence

$$\int_0^{\pi/2} x\cdot\sin(x)\cdot dx = -x\cdot\cos(x)\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos(x)\cdot dx = \sin(x)\Big|_0^{\pi/2} = 1.$$

We also deduce that

$$\int x\cdot\sin(x)\cdot dx = -x\cdot\cos(x) + \sin(x) + C.$$

Exercise 91

Use integration by parts to compute the following integrals:

(1) $\displaystyle\int x\cdot\cos(x)\cdot dx$

(2) $\displaystyle\int (\cos(x))^2\cdot dx$

(3) $\displaystyle\int x^2\cdot\sin(x)\cdot dx$

(4) $\displaystyle\int \sin(x)\cdot\cos(x)\cdot dx$

Exercise 92

Prove:

Theorem 34
Let f and g be two real functions differentiable at a and $g(a)\neq 0$. Then the function $\dfrac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}.$$

Exercise 93

Calculate $\tan'(x)$ using $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

Exercise 94

Find the slope of $f : x \mapsto \dfrac{x^2 - 2x + 1}{x^3 + x^2}$ at $x = 1$.

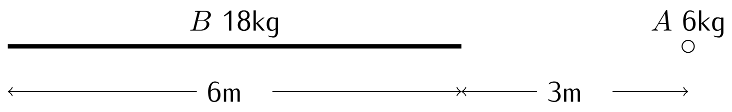
Exercise 95

Show that for $m \in \mathbb{Z}$

$$(x^m)' = m \cdot x^{m-1}.$$

Exercise 96

Given that the gravitational force between two masses is $F = G\dfrac{m_1 \cdot m_2}{d^2}$ (where d is the distance between the two masses and G the universal constant of gravitation), what is the force between objects A and B in the following situation? (For simplicity, the linear mass will be considered to have no width and the other will be considered reduced to a point.)



Practice exercise 12 Answer page 65

Differentiate the following for general x:

(1) $f : x \mapsto 5x^4 + x^3 - 2x^2 + 25$

(2) $g : x \mapsto 5\sqrt{3}\,x^2 - 100$

(3) $h : x \mapsto \dfrac{x^2 + 2x - 1}{x^3 - 5}$

(4) $j : x \mapsto 5x^4 + \dfrac{1}{3x^2 - 2x + \pi}$

(5) $k : x \mapsto (5x + 2) \cdot \dfrac{1}{5x + 2}$

(6) $l : x \mapsto \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4}$

(7) $m : x \mapsto \dfrac{1 + x}{1 + \frac{1+x}{x^2}}$

Practice exercise 13 Answer page 65

Sketch the curve of $y = -(x - 3)(x + 1)(x - 1)$.

Practice exercise 14 Answer page 65

Let $y = \dfrac{10x}{x^2 + 1}$. Sketch the curve and give the equation of the line tangent to the curve at $x = 3$.

Practice exercise 15 Answer page 66

Consider each of the following as a function f, find the corresponding derivative function f'.

(1) $x^3 + x^2 + 2x - 4$

(2) $-x^3 + 2x^2 - 2x + 1$

(3) $\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$

(4) $\frac{1}{3}(x-2)^3$

(5) $\dfrac{x^2}{x+2}$

(6) $x - 1 + \dfrac{9}{x+1}$

(7) $\dfrac{4x^2+4x+5}{4x+2}$

(8) $\dfrac{-x^2-2x-1}{x+3}$

(9) $|x-2|$

(10) $\dfrac{x^2}{|x|+2}$

(11) $x + 2 - \dfrac{1}{x+1}$

(12) $|x^3 - 6x^2 + 11x - 6|$

Exercise 97

Find the derivative of the following functions. Since they are piecewise defined, the answer will be in 3 parts – one special point is the meeting point for both rules.

(1)

$$f : x \mapsto \begin{cases} x^2 & \text{if } x \geq 1 \\ 2x - 1 & \text{if } x < 1 \end{cases}$$

(2)

$$g : x \mapsto \begin{cases} x^2 & \text{if } x > 2 \\ x + 2 & \text{if } x \leq 2 \end{cases}$$

(3)

$$h : x \mapsto \begin{cases} x^2 & \text{if } x \geq 3 \\ 2x & \text{if } x < 3 \end{cases}$$

Exercise 98

For each of the following functions, find an antiderivative:

(1) $f : t \mapsto 3t^2 + 1$

(2) $f : t \mapsto 4 - 3t^3$

(3) $f : s \mapsto 7s^{-3}$

(4) $f : x \mapsto (x-6)^2$

(5) $f : y \mapsto y^{\frac{3}{2}}$

(6) $f : x \mapsto |x|$

(7) $f : u \mapsto u^2 + u^{-2}$

(8) $f : x \mapsto 4$

(9) $f : t \mapsto t$

(10) $f : z \mapsto \dfrac{2}{z^2}$

Check your results by differentiating them.

Exercise 99

(1) If $F'(x) = x + x^2$ for all x, find $F(1) - F(-1)$.

(2) If $F'(x) = x^4$ for all x, find $F(2) - F(1)$.

(3) If $F'(t) = t^{\frac{1}{3}}$ for all t, find $F(8) - F(10)$.

Exercise 100

The following computation may seem correct: $\int_{-1}^{1} x^{-2}dx = -\frac{1}{x}\Big|_{-1}^{1} = -2$ yet there is no $x \in [-1,1]$ such that $f(x) < 0$. By theorem 27 we should therefore have a positive value for the integral. Why is this not so?

Chain rule

Exercise 101

Prove the following theorem, assuming that $\Delta g(a) \neq 0$:

Theorem 35 (Chain Rule)
Let f and g be real functions such that g is differentiable at a and f is differentiable at $g(a)$. The the function $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Exercise 102

Prove that the theorem holds also if $\Delta g(a) = 0$.

Exercise 103

Give the derivatives of the following functions:

(1) $f : x \mapsto (x^3 + 2x)^4$

(2) $g : x \mapsto (5x^3 + 3x^2)^{13}$

Exercise 104

Use $(\sqrt{x})^2 = x$ and theorem 35 to find the derivative of $y = \sqrt{x}$ (for $x > 0$) – assuming it exists.

Exercise 105

Give the derivatives of the following functions:

(1) $f : x \mapsto (\sqrt{x}+1)^4$

(2) $g : x \mapsto \sqrt{5x^3+3x^2}$

(3) $h : x \mapsto \sqrt{x^2}$

Practice exercise 16 Answer page 66

Find the derivatives of the following:

(1) $f_1 : x \mapsto \sqrt{3x^3+2x+1}$

(2) $f_2 : x \mapsto (x^2+3)^5$

(3) $f_3 : x \mapsto (ax+b)^n$

(4) $f_4 : x \mapsto \sqrt{x^3+1}$

(5) $f_5 : x \mapsto \sin(x^2+3x)$

(6) $f_6 : \theta \mapsto \cos^2(3\theta)$

(7) $f_7 : u \mapsto \sin(\sin(u))$

(8) $f_8 : x \mapsto \tan^2(\tan^2(x^2))$

(9) $f_9 : v \mapsto \dfrac{\sin(v)}{\tan(v)}$

(10) $f_{10} : x \mapsto \sin^2(x)+\cos^2(x)$

Theorem 36 (Integration with inside derivative)
Let f and g be real functions differentiable on $[a;b]$ such that f' and g' are continuous on $[a;b]$. Then

$$\int_a^b f'(g(x)) \cdot g'(x) \cdot dx = f(g(x))\Big|_a^b .$$

Exercise 106

Prove theorem 36.

Exercise 107

Compute the following integrals:

(1) $\displaystyle\int 2x \cdot \sin(x^2) \cdot dx$

(2) $\displaystyle\int x^2 \cdot (x^3+1) \cdot dx$

(3) $\displaystyle\int \sin(x) \cdot \cos(\cos(x)) \cdot dx$

(4) $\displaystyle\int \sin(x) \cdot \cos^2(x) \cdot dx$

The differential

Definition 16
Let f be a real function differentiable on an interval around a. Let dx be ultrasmall. The ***differential of f at a****, written $df(a)$, is*

$$df(a) = f'(a) \cdot dx.$$

Thus

$$\frac{df(a)}{dx} = f'(a)$$

or still (if we use $y = f(a)$)

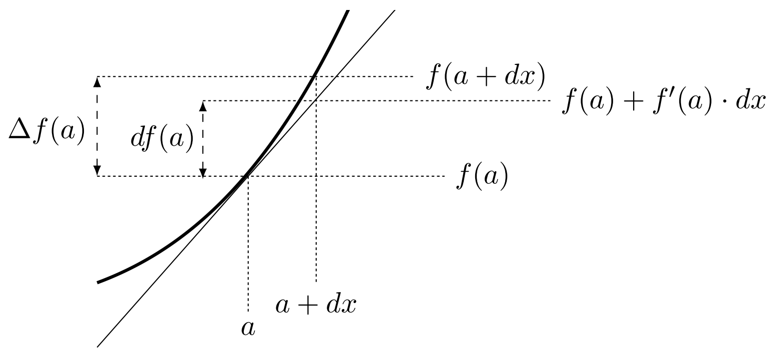
$$\frac{dy}{dx} = y'$$

.

If f is differentiable the following holds:

$$\frac{\Delta f(a)}{dx} \simeq \frac{df(a)}{dx}$$

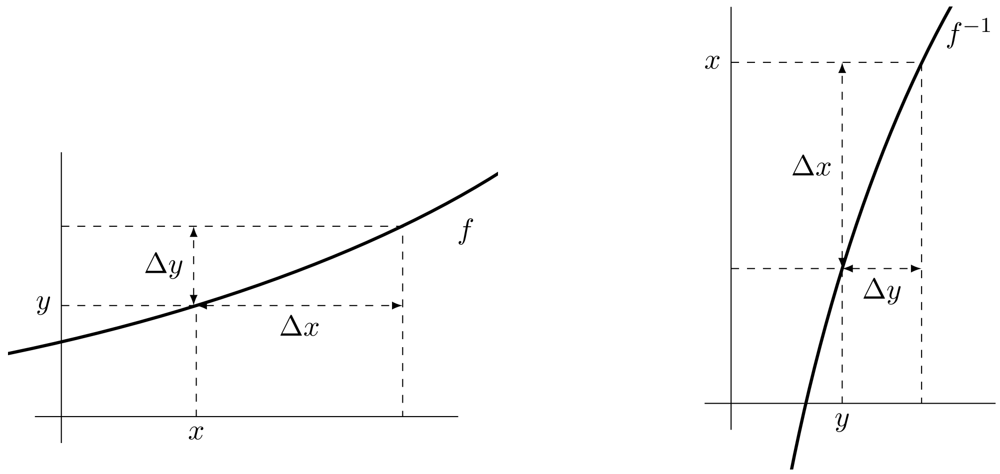
Whereas $\Delta f(a)$ is the variation of the function, the differential is the variation along the tangent line.



Let f be a function. Recall that the inverse function of f, if it exists, is written f^{-1} and is such that $f^{-1}(f(x)) = x$ amd if we write $f(x) = y$ then we also have $f(f^{-1}(y)) = y$.

⚠ $f^{-1}(x)$ is <u>not</u> $\dfrac{1}{f(x)}$.

A function has an inverse if the image of its curve by a symmetry through the $y = x$ axis is the curve of a function.



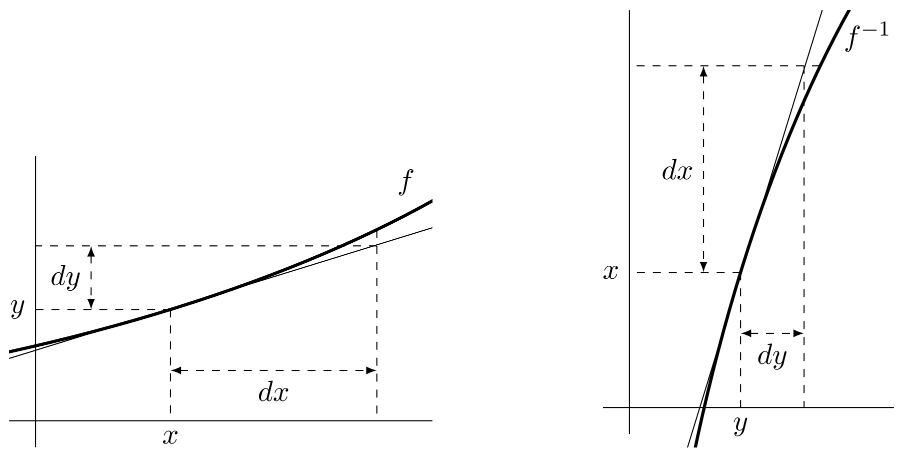
Theorem 37 (Derivative of the Inverse)
If $f : I \to J$ is a function, differentiable on I and has an inverse f^{-1}, and $f'(a) \neq 0$ then this inverse is differentiable at $b = f(a) \in J$ and

$$\frac{df^{-1}(b)}{dy} = \frac{1}{f'(a)}.$$

In general form:

$$\frac{df^{-1}(y)}{dy} = \frac{1}{f'(x)}$$

You may also use the following drawing to observe that the slope of the tangent of the inverse is the reciprocal of the slope of the original tangent.



Exercise 108
Find the derivative of $y = x^{\frac{1}{n}}$.

Exercise 109

Find the derivative of $y = x^{\frac{m}{n}}$.

This shows that the rule in exercise 81 holds also for rational n.

Exercise 110

Use $\mid x \mid = \sqrt{x^2}$ to find an expression for the derivative of $\mid x \mid$.

Exercise 111

Difficult exercise!

Let h be ultrasmall relative to 1.

$$H : x \mapsto \begin{cases} 0 & \text{if } x \leq -h \\ \frac{1}{2h}\,(x+h) & \text{if } -h < x < h \\ 1 & \text{if } x \geq h \end{cases}$$

(1) What is the context of the function?

(2) Calculate $H'(x)$.

(3) Sketch H, first with horizontal scale $[-2; 2]$ and vertical scale $[0; 1]$ then, for same vertical scale, take a horizontal scale $[-2 \cdot h; 2 \cdot h]$.

Exercise 112

For the inverse functions, it is convenient to use the differential.

Prove the following theorem:

Hint: Suppose that $\arcsin(x) = y$ i.e., $\sin(y) = x$. Then $\arcsin'(x) = \frac{dy}{dx} = \frac{dy}{d\sin(x)}$.

Theorem 38

(1) $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$

(2) $\arccos'(x) = -\dfrac{1}{\sqrt{1-x^2}}$

(3) $\arctan'(x) = \dfrac{1}{1+x^2}$

Exercise 113

Let ε be ultrasmall relative to 1. Consider the function

$$H : x \mapsto \frac{1}{2} + \frac{1}{\pi} \cdot \arctan\left(\frac{x}{\varepsilon}\right).$$

Calculate $H'(x)$ and sketch the curves of H and H'.

Exercise 114

(1) Show that $x \mapsto \cos\left(\frac{1}{x}\right)$ cannot be extended continuously at $x = 0$.

(2) Show that

$$x \mapsto \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$ but that its derivative $x \mapsto g'(x)$ is not continuous at 0.

Exercise 115

Compute the derivatives of the following:

(1) $f : x \mapsto \sin^2(3x + \pi)$

(2) $g : x \mapsto x \cdot \sin(x^2 + 1)$

(3) $h : x \mapsto \sin^2\left(\dfrac{x}{x^2+1}\right) + \cos^2\left(\dfrac{x}{x^2+1}\right)$

(4) $j : x \mapsto 1 + \tan^2(x)$

Exercise 116

(1) Show that $f : x \mapsto \sin^6(x) + \cos^6(x) + 3\sin^2(x)\cos^2(x)$ is a constant function.

(Hint: use the derivative…)

(2) At what values does $f : x \mapsto \sin(x) + \cos(x)$ have stationary points?

(3) What is the equation of the straight line tangent to $y = \sin^2(x)$ at $x = \frac{\pi}{4}$?

Variable substitution

Consider $\displaystyle\int_a^b f(x) \cdot dx$.

If x is a function of u written $x = g(u)$ then $dx = g'(u) \cdot du$,

$f(x)$ becomes $f(g(u))$ and the limits must be changed to a_1 and b_1 so that $g(a_1) = a$ and $g(b_1) = b$

Example: Let

$$\int_0^1 \sqrt{1 + \sqrt{x}} \cdot dx.$$

Consider the variable change $u = 1 + \sqrt{x}$. Then $x = (u-1)^2 = g(u)$, the derivative of g is continuous. If $x = 0$ then $u = 1$ and if $x = 1$ then $u = 2$. Moreover $f(g(u)) = \sqrt{u}$ and

$$dx = 2 \cdot (u-1) \cdot du.$$

Replacing all terms we obtain

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx = 2\int_1^2 \sqrt{u} \cdot (u-1) \cdot du = 2\int_1^2 \left(u^{3/2} - u^{1/2}\right) \cdot du$$

so that

$$2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Bigg|_1^2 = \frac{8+8\sqrt{2}}{15}.$$

As g has an inverse which is $x \mapsto 1 + \sqrt{x}$ and is differentiable (except at $x = 0$), we can revert to the variable x and find an antiderivative:

$$\int \sqrt{1+\sqrt{x}} \cdot dx = \frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5 - \frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3 + C.$$

Exercise 117

Calculate

$$\int_0^1 \sqrt{5x+2} \cdot dx.$$

Use $u = 5x + 2$. Calculate du, change the bounds, calculate the integral.

Same integral. Use $v = \sqrt{5x+2}$

The difficulty is usually to find which variable substitution is best.

Exercise 118

Use variable substitution to evaluate the following:

(1) $\displaystyle\int_0^{10} \frac{1}{(2x+2)^2} \cdot dx$

(2) $\displaystyle\int (3-4z)^6 \cdot dz$

(3) $\displaystyle\int_{-1}^{1} 2t\sqrt{1-t^2} \cdot dt$

(4) $\displaystyle\int_a^b \sqrt{3y+1} \cdot dy$

(5) $\displaystyle\int \frac{4y}{(2+3y^2)^2} \cdot dy$

(6) $\displaystyle\int_{-2}^{2} x(4-5x^2)^2 \cdot dx$

(7) $\displaystyle\int (1-x)^{\frac{3}{2}} \cdot dx$

Practice exercise 17 Answer page 67

(1) $\displaystyle\int_0^1 \frac{u}{\sqrt{1-u^2}} \cdot du$

(2) $\displaystyle\int_1^2 \frac{u}{\sqrt{1-u^2}} \cdot du$

(3) $\displaystyle\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx$

(4) $\displaystyle\int_0^{10} t(t^2+3)^{-2} \cdot dt$

(5) $\displaystyle\int_{\sqrt{6}}^5 x(x^2+2)^{\frac{1}{3}} \cdot dx$

(6) $\displaystyle\int_{-1}^1 \frac{x^2}{(4-x^3)^2} \cdot dx$

(7) $\displaystyle\int_1^2 \frac{1}{t^2\sqrt{1+\frac{1}{t}}} \cdot dt$

Variable substitution is formalised in the following theorem.

Theorem 39 (Integration by variable substitution)
Let f be a real function continuous on $[a;b]$. Let g be a function whose derivative is continuous and such that for $e, d \in \mathbb{R}$ we have $g(d) = a$ and $g(e) = b$. Then

$$\int_a^b f(x) \cdot dx = \int_d^e f(g(u)) \cdot g'(u) \cdot du.$$

This formula looks probably quite difficult, but hopefully, the exercises done above show that it amounts to a systematic procedure.

A simplified writing can be used: we have already used the writing $y = f(x)$ where y is a dependent variable and x the independent variable. When several functions are used, we can write $u = f(x)$ and $v = g(x)$, then we have (for constant c and for $U' = u$ and $V' = v$):

- $c' = 0$
- $(c \cdot u)' = c \cdot u'$
- $(u + v)' = u' + v'$
- $(u \cdot v)' = u' \cdot v + u \cdot v'$
- $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$
- $(u \circ v)' = u' \cdot v'$ (in this case, u depends on v which depends on x).
- $(x^n)' = nx^{n-1}$
- $\sin'(x) = \cos(x)$
- $\cos'(x) = -\sin(x)$
- $\tan'(x) = 1 + \tan^2(x) = \frac{1}{\cos^2(x)}$
- $\int c \cdot u \cdot dx = c \cdot U + k$
- $\int (u + v) \cdot dx = U + V + k$
- $\int u(v) \cdot v' \cdot dx = U(v) + k$
- $\int u' \cdot v \cdot dx = u \cdot v - \int u \cdot v' \cdot dx$

Optimisation Problems

Exercise 119

A $1l$ milk pack is made of cardboard. Its sides can only be rectangles. The height is twice one of the other two dimensions. The area of the outside of the pack must be minimal.

What are the dimensions of the pack?

Exercise 120

Imagine you want to protect a part of a rectangular garden against a wall. You have 100m of fence. (No fence is needed against the wall.)

What is the biggest area that you can protect?

Exercise 121

A cylindrical jar has a volume defined by its radius and its height. If it contains one litre (1dm^3), what are the dimensions that will make it have the least outside area?

Exercise 122

Find the length and width of the rectangle inscribed within the ellipse given by the formula $4x^2 + y^2 = 16$ (sides parallel to the coordinate axes) such that its area is maximal.

Exercise 123

Let $\mathcal{P}$ be the parabola given by $x \mapsto x^2$ and A be the point $\langle 0; 5 \rangle$.

Find the point(s) on the parabola $\mathcal{P}$ such that its (their) distance to A is minimal.

Answers to practice exercises

Answers to practice exercice 12, page 53

(1) $f'(x) = 20x^3 + 3x^2 - 4x$

(2) $g'(x) = 10\sqrt{3}x$

(3) $h'(x) = -\dfrac{x^4 + 4x^3 - 3x^2 + 10x + 10}{(x^3 - 5)^2}$

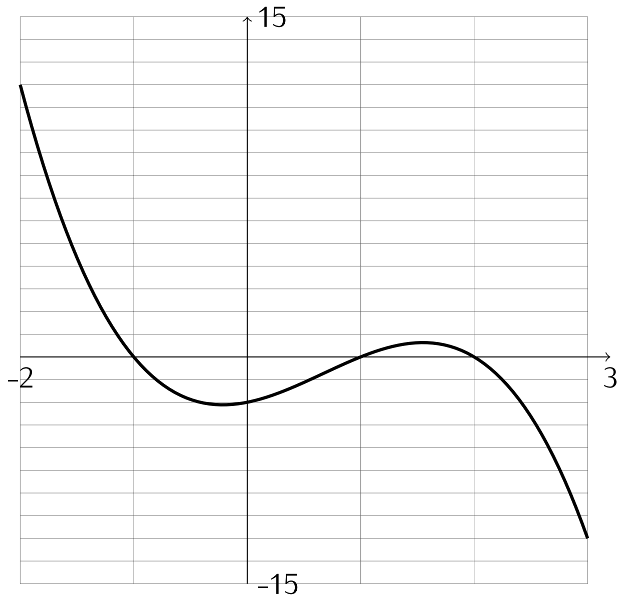
(4) $j'(x) = 20x^3 - \dfrac{6x - 2}{(3x^2 - 2x + \pi)^2}$

(5) $k'(x) = 0$

(6) $l'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} - \dfrac{3}{x^4} - \dfrac{4}{x^5}$

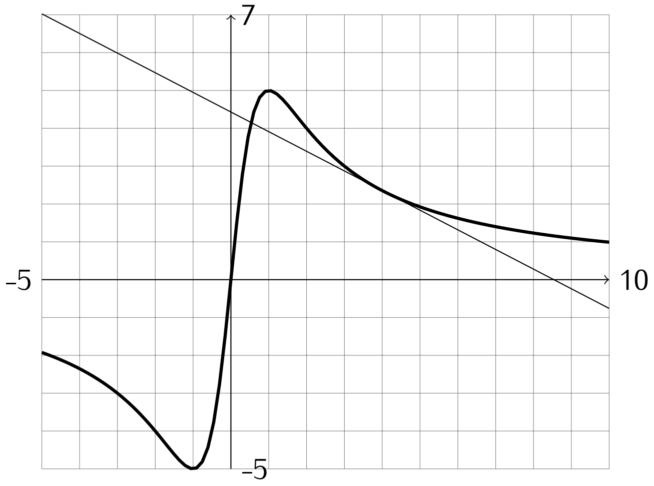
(7) $m'(x) = \dfrac{(x^2 + x + 1)(3x^2 + 2x) - (x^3 + x^2)(2x + 1)}{(x^2 + x + 1)^2} = \dfrac{x(x^3 + 2x^2 + 4x + 2)}{(x^2 + x + 1)^2}$

Answers to practice exercice 13, page 53



Answers to practice exercice 14, page 53

Tangent line is $y = -\dfrac{4}{5}x + \dfrac{27}{5}$



Answers to practice exercice 15, page 54

(1) $3x^2 + 2x + 2$

(2) $-3x^2 + 4x - 2$

(3) $x^2 - 5x + 6$

(4) $(x-2)^2$

(5) $\dfrac{x(x+4)}{(x+2)^2}$

(6) $\dfrac{x^2+2x-8}{(x+1)^2}$

(7) $\dfrac{4x^2+4x-3}{(2x+1)^2}$

(8) $-\dfrac{x^2+6x+5}{(x+3)^2}$

(9) $\begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \\ \text{not differentiable} & \text{if } x = 2 \end{cases}$

(10) $\begin{cases} \dfrac{x(x+4)}{(x+2)^2} & \text{if } x \geq 0 \\ \\ \dfrac{-x(x-4)}{x-2)^2} & \text{if } x \leq 0 \end{cases}$

(11) $\dfrac{x^2+2x+2}{(x+1)^2}$

(12) $\begin{cases} 3x^2 - 12x + 11 & \text{if } x \in]1;2[\cup]3;\infty[\\ -3x^2 + 12x - 11 & \text{if } x \in]-\infty;1[\cup]2;3[\\ \text{not differentiable} & \text{if } x \in \{1;2;3\} \end{cases}$

Answers to practice exercice 16, page 56

(1) $f_1' : x \mapsto \dfrac{9x^2+2}{2\sqrt{3x^3+2x+1}}$

(2) $f_2' : x \mapsto 10x \cdot (x^2+3)^4$

(3) $f_3' : x \mapsto an \cdot (ax+b)^{n-1}$

(4) $f_4' : x \mapsto \dfrac{3x^2}{2\sqrt{x^3+1}}$

(5) $f_5' : x \mapsto \cos(x^2+3x) \cdot (2x+3)$

(6) $f_6' : \theta \mapsto -6\cos(3\theta) \cdot \sin(3\theta)$

(7) $f_7' : u \mapsto \cos(\sin(u)) \cdot \cos(u)$

(8) $f_8' : x \mapsto 8x\tan(\tan^2(x^2)(1+\tan^2(\tan^2(x^2))(\tan(x^2)(1+\tan^2(x^2))$

(9) $f_9 : v \mapsto -\sin(v)$

(10) $f_{10}' : x \mapsto 0$

Answers to practice exercice 17, page 62

(1) 1 Use $x = 1 - u^2$.

(2) undefined – for $u > 1$ we have the square root of a negative number.

(3) $\frac{8(\sqrt{2}+1)}{15}$ Use $u = 1 + \sqrt{x}$

(4) $\frac{50}{309}$ Use $u = t^2 + 3$

(5) $\frac{195}{8}$ Use $u = x^2 + 2$

(6) $\frac{2}{45}$ Use $u = 4 - x^3$

(7) $-\sqrt{6} + 2\sqrt{2}$ Use $u = 1 + \frac{1}{t}$

7 More on integration

Exercise 124

Prove the following:

Theorem 40
Let f be a function continuous on $[a, b]$ and let $g(x)$ be defined by the following:

$$g : x \mapsto \int_a^x f(t)dt$$

Then g is an antiderivative of f.

Definition 17
The ∞ symbol in the bounds of an integral indicates a limit.

$$\int_a^{\infty} f(x) \cdot dx = \lim_{n \to \infty} \int_a^n f(x) \cdot dx$$

This is calculated by taking ultralarge N in $\int_a^N$ and taking the observable part of the result (if it exists and is independent of N).

Exercise 125

Check that an derivative of $x \mapsto \dfrac{x}{x+1}$ is $x \mapsto \dfrac{1}{(x+1)^2}$.

Sketch the curve of $f : x \mapsto \dfrac{1}{(x+1)^2}$ for $x > 0$.

Calculate the area under f between 0 and 10.

Calculate the area under f between 0 and $+\infty$

Exercise 126

Do infinitely long objects have a finite area?

(1) Calculate the area under $f : x \mapsto \frac{1}{x^2}$ between $x = 1$ and $x = \infty$, i.e: show that this area does not depend on which ultralarge is chosen.

(2) Without any calculation, explain why the total length of both sides (the curve above and the straight line below) is infinite.

(3) Does this prove that a finite amount of paint would be enough to cover the area but not enough to paint the border lines?

Definition 18
If the function to integrate is not defined at one of the bounds, then

$$\int_a^b f(x) \cdot dx = \lim_{u \to a_+} \int_u^b f(x) \cdot dx$$

or

$$\int_a^b f(x) \cdot dx = \lim_{u \to b_-} \int_a^u f(x) \cdot dx$$

Exercise 127
Evaluate the integrals:

(1) $\displaystyle\int_0^1 2x^{-2} \cdot dx$

(2) $\displaystyle\int_{-2}^3 u^{-3} \cdot du$

(3) $\displaystyle\int_{-1}^2 -5(t+1)^{-1/4} \cdot dt$

(4) $\displaystyle\int_0^4 \frac{1}{2\sqrt{x}} \cdot dx$

Exercise 128
In the following problems an object moves along the y axis. Its velocity varies with respect to the time. Find how far the object moves between the given times t_0 and t_1.

(1) $v = 2t + 5$ $\quad t_0 = 0 \quad t_1 = 2$

(2) $v = 4 - t$ $\quad t_0 = 1 \quad t_1 = 4$

(3) $v = 3$ $\quad t_0 = 2 \quad t_1 = 6$

(4) $v = 3t^2$ $\quad t_0 = 1 \quad t_1 = 3$

(5) $v = 10t^{-2}$ $\quad t_0 = 1 \quad t_1 = 100$

Antiderivative of $x \mapsto \dfrac{1}{x}$

Let n be a positive integer. From $(x^{n+1})' = (n+1) \cdot x^n$ we can deduce

$$\int x^n \cdot dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1.$$

Hence an antiderivative of $x \mapsto \dfrac{1}{x}$ is not a particular case of this formula.

Exercise 129

Let f be an antiderivative of $x \mapsto \frac{1}{x}$ (why is there one?). Then f is strictly increasing (why?) and so it has an inverse, call it g. Show that this implies $g'(x) = g(x)$.

Exercise 130

Let $a, b > 0$. Use the substitution $u = \frac{t}{a}$ to show that (considering f to be the antiderivative of $\frac{1}{x}$.)

$$\int_a^{a \cdot b} \frac{1}{t} \cdot dt = \int_1^b \frac{1}{u} \cdot du.$$

Deduce that $f(a \cdot b) = f(a) + f(b)$.

Exercise 131

Let $a > 0$ and b a rational number. Show that (considering f to be the antiderivative of $\frac{1}{x}$.)

$$f(a^b) = b \cdot f(a).$$

(To find the substition, consider the transformation of the bounds.)

Exercise 132

What kind of function has the properties $f(a \cdot b) = f(a) + f(b)$ and $f(a^b) = b \cdot f(a)$?

Theorem 41

The antiderivative f of $\frac{1}{x}$ satisfies the following limits:

$$\lim_{x \to 0^+} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to +\infty} f(x) = +\infty.$$

Exercise 133

Prove theorem 42. Hint: for ultralarge x use ultralarge N such that $2^N \leq x$.

Definition 19

The ***natural logarithm*** *is the function* $\ln :]0; +\infty[\to \mathbb{R}$ *defined by*

$$x \mapsto \int_1^x \frac{1}{t} \cdot dt.$$

Definition 20

We define e to be the unique number such that

$$\ln(e) = 1.$$

e is an irrational number whose first digits are

$$e = 2.71828\ldots$$

Definition 21
*The **exponential** function* $\exp : \mathbb{R} \longrightarrow]0; +\infty[$ *is defined as the inverse of* ln.

We have, for rational x, that $a^x = \exp(x\ln(a))$, hence $e^x = \exp(x)$. For irrational x, we **define** a^x to be $\exp(x\ln(a))$ hence also $e^x = \exp(x)$ for all x.

We also have $\ln(a^y) = y \cdot \ln(a)$ for all y. Writing $x = a^y$ we get $\ln(x) = \log_a(x) \cdot \ln(a)$ so $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

The following property makes it a remarkable function.

Theorem 42

$$(\exp(x))' = \exp(x).$$

(this was proven by exercise 136).

Theorem 43

(1) Let $b \in \mathbb{R}$. *The function* $x \mapsto x^b$ *is differentiable on its domain and* $(x^b)' = b \cdot x^{b-1}$, *for all* $x \in \mathbb{R}$.

(2) Let $a > 0$. *The base* a *exponential is differentiable on its domain and* $(a^x)' = \ln(a) \cdot a^x$, *for* $x > 0$.

(3) Let $a > 0$. *The base* a *logarithm is differentiable and* $(\log_a(x))' = \frac{1}{\ln(a) \cdot x}$.

Exercise 134
Prove theorem 44.

Exercise 135
Let f be a positive real function whose derivative is continuous. Calculate:

$$\int \frac{f'(x)}{f(x)} \cdot dx$$

Exercise 136
Calculate

$$\int \tan(x) \cdot dx$$

Exercise 137
Let f be a positive real function whose derivative is continuous. Calculate:

$$\int f'(x) \cdot e^{f(x)} \cdot dx$$

Exercise 138

Using $\ln(x) = 1 \cdot \ln(x)$, use integration by parts to compute $\int \ln(x)dx$.

Exercise 139

(1) Differentiate $\ln(x)$.

(2) Differentiate e^x.

(3) Integrate $x \mapsto e^x$.

(4) Differentiate the function $x \mapsto \ln(\ln(x))$.

(5) Differentiate the function $x \mapsto \ln(x^a)$ (Note that a is not the variable!)

(6) Differentiate the function $x \mapsto \ln(a^x)$.

(7) Differentiate $x \mapsto e^{x^2}$.

(8) Using the fact that $u = e^{ln(u)}$ (if $u > 0$) differentiate $x \mapsto a^x$ (for $a > 0$ and $x > 0$).

(9) Same idea: Differentiate the function $x \mapsto x^x$.

Exercise 140

Differentiate $\ln(|x|)$.

This proves the following extension:

Theorem 44

The antiderivative of $\frac{1}{x}$ is $\ln(|x|) + K$ for some constant K.

Mean value of a function

The mean value is unambiguous when we consider n points, where n is a positive integer. We now show that defining the mean value of a continuous function on $[a;b]$ as

$$\frac{1}{b-a}\int_a^b f(x) \cdot dx$$

is a natural extension of this concept.

Consider a continuous function f and the interval $[a;b]$. Context is a, b and f. Let N be a positive unlimited integer. Let $dx = (b-a)/N$ and $x_i = a + i \cdot dx$, for $i = 1, \ldots, N$. Then the mean value of the function can be approximated by the mean value of the N points $f(x_i)$, $i = 0, \ldots, N-1$. But

$$\frac{\sum_{i=0}^{N-1} f(x_i)}{N} = \frac{dx}{b-a} \sum_{i=0}^{N-1} f(x_i) = \frac{1}{b-a} \sum_{i=0}^{N-1} f(x_i) \cdot dx \simeq \frac{1}{b-a} \int_a^b f(x) \cdot dx,$$

since f is continuous on $[a;b]$.

The mean is the part of this number which is observable i.e., the integral. We therefore define:

Definition 22
*The **mean value** of a function f continuous on $[a;b]$ is*

$$\frac{1}{b-a} \int_a^b f(x) \cdot dx.$$

The mean value is a number μ such that the area under the curve is equal to $\mu \cdot (b-a)$, i.e., the height of a rectangle of basis $(b-a)$ whose (oriented) area is equal to the integral.

Theorem 45
If f is a function continuous on $[a;b]$, then there exists a point $c \in [a;b]$ such that $f(c)$ is the mean value of the function on $[a;b]$.

Note that theorem 46 is a restatement of theorem 21 which is the mean value theorem, for the antiderivative of f. When we claim that there is a $c \in [a;b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \cdot dx,$$

we are in fact asserting that there is a $c \in [a;b]$ such that

$$f(c) \cdot (b-a) = \int_a^b f(x) \cdot dx = F(b) - F(a),$$

and as $F'(x) = f(x)$, we conclude that there is a $c \in [a;b]$ such that $F'(c) \cdot (b-a) = F(b) - F(a)$.

Exercise 141
Calculate the mean value of $x \mapsto x^2$ on $[-4;4]$.

Exercise 142
Calculate the mean value of $x \mapsto x^3$ on $[-4;4]$.

Exercise 143
Let $f : x \mapsto x^2$ and the interval $[0;t]$. Find the value of t such that the mean value of f over the interval is equal to π.

Exercise 144

An object falling on earth satisfies the equation $d(t) = \frac{1}{2}gt^2$ where $g \approx 9.81[m/s^2]$, t is the time in seconds and $d(t)$ is the vertical distance.

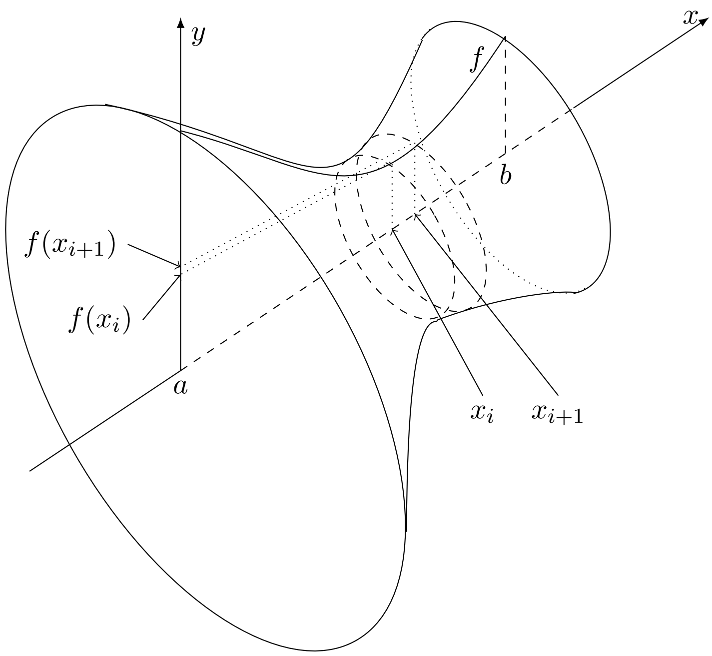
If an object falls for $10s$, what is its average distance from its initial point?

Exercise 145

An object falling on earth satisfies the equation $d(t) = \frac{1}{2}gt^2$ where $g \approx 9.81[m/s^2]$, t is the time in seconds and $d(t)$ is the vertical distance.

If an object falls for $10s$, what is its average distance from its initial point?

Solid of Revolution



Exercise 146

An area is calculated by approximating the surface by ultrasmall rectangles. To find the formula for the volume of a solid of revolution, proceed in the same manner: consider that the solid is ultraclose to an ultralarge number of ultrathin disks. Find the formula for the volume of a solid of revolution given by a function f.

Exercise 147

Evaluate the volume of the solid of revolution of $y = \dfrac{1}{x}$ around the x-axis between $x = 1$ and $x = 10$.

Exercise 148

Evaluate the volume of the solid of revolution of $y = \dfrac{1}{x}$ around the x-axis between $x = 1$ and $x = +\infty$ i.e: take unlimited N then show that the result does not depend on the choice of N.

Arc length

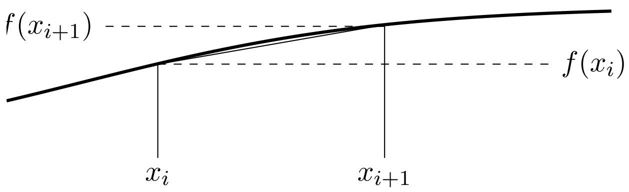
Exercise 149

Approximating the length of a curve by ultrasmall straight lines leads to the following definition. Explain why it is a reasonable definition (using the drawing).

Definition 23

Let $f : [a; b] \to \mathbb{R}$ be smooth. Then the graph of f has length

$$L = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx.$$



Exercise 150

Find the lengths of the following curves:

(1) $y = 2x^{3/2} \quad 0 \leq x \leq 1$

(2) $y = \dfrac{2}{3}(x + 2)^{\frac{3}{2}} \quad 0 \leq x \leq 3$

Practice exercise 18 Answer page 95

Find the antiderivatives of the following functions:

- $f_a : x \mapsto 5x^4 - 2x + 4$
- $f_b : x \mapsto x^3 - 5x^2 + 3x - 2$
- $f_c : x \mapsto 2x - 1$
- $f_d : x \mapsto \frac{5}{4}x^4 - \frac{3}{4}x^2 + \frac{5}{2}x + \frac{3}{2}$
- $f_e : x \mapsto 2x + 1 - \frac{1}{x^2}$
- $f_f : x \mapsto 3 + \frac{2}{x^2} - \frac{5}{x^3}$
- $f_g : x \mapsto x^3 + \frac{1}{x^2}$
- $f_h : x \mapsto \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
- $f_i : x \mapsto \frac{1}{\sqrt{x}} + \sqrt{x}$
- $f_j : x \mapsto (x+1)^2$
- $f_k : x \mapsto 15(3x-2)^4$
- $f_l : x \mapsto (2x+1)^3$
- $f_m : x \mapsto (3-x)^{11}$
- $f_n : x \mapsto (3-4x)^4$
- $f_o : x \mapsto \sqrt{3x-2}$
- $f_p : x \mapsto \frac{1}{\sqrt{x-1}}$
- $f_q : x \mapsto 4x(3-x^2)^5$
- $f_r : x \mapsto (2x-3)(x^2-3x+1)^4$
- $f_s : x \mapsto (3x^2-4x+1)(x^3-2x^2+x+3)^2$
- $f_t : x \mapsto (4x^2-5x)^2(16x-10)$
- $f_u : x \mapsto (3x-1)(3x^2-2x+5)^3$
- $f_v : x \mapsto \frac{2x}{(x^2+1)^2}$
- $f_w : x \mapsto \frac{2x+1}{(x^2+x+3)^2}$
- $f_x : x \mapsto x\sqrt{x^2+1}$
- $f_y : x \mapsto \frac{3x^2}{\sqrt{9+x^3}}$
- $f_z : x \mapsto (3x^2+1)\sqrt{x^3+x+2}$
- $f_A : x \mapsto e^{2x}$
- $f_B : x \mapsto \frac{1}{e^{3x}}$
- $f_C : x \mapsto xe^{-x^2}$
- $f_D : x \mapsto 2^{-x}$
- $f_E : x \mapsto e^{2x}\sqrt{1+e^{2x}}$
- $f_F : x \mapsto x^2e^x$
- $f_G : x \mapsto e^x \sin(x)$
- $f_H : x \mapsto \frac{e^x}{1+e^{2x}}$
- $f_I : x \mapsto \frac{1}{2x+3}$
- $f_J : x \mapsto \frac{2x}{x-1}$
- $f_K : x \mapsto \frac{x-1}{x+1}$
- $f_L : x \mapsto (\ln(x))^2$
- $f_M : x \mapsto \frac{\cos(x)}{1+\sin(x)}$
- $f_N : x \mapsto \ln(x)$
- $f_O : x \mapsto \frac{x}{x+1}$
- $f_P : x \mapsto \frac{1}{x\ln(x)}$

8 limits

A function f is defined on the left of a (resp. on the right) if $f(x)$ is defined for all $x \simeq a$ with $x < a$ (resp. $x > a$). It is clear that f is defined around a if and only if f is defined on the right and on the left of a.

Definition 24 (One sided Continuity)
Let f be a real function and $a \in \mathbb{R}$.

*(1) Suppose that f is defined on the left of a. Then f is **continuous on the left at** a if $x < a$ and $x \simeq a \implies f(x) \simeq f(a)$.*

*(2) Suppose that f is defined on the right of a. Then f is **continuous on the right at** a if $x > a$ and $x \simeq a \implies f(x) \simeq f(a)$.*

It is immediate that f is continuous at a if and only if it is continuous on the right and on the left at a.

We now extend the concept of continuity at a point to continuity on an interval.

Exercise 151

Prove directly that $x \mapsto \sqrt{x}$ is continuous on its domain i.e, for any value $x = a$ in the domain.

Hint: start by the definition, then multiply and divide by $(\sqrt{a+dx} + \sqrt{a}$.

If we want to study the behaviour of f in the neighbourhood of a, the function f must be defined *around* a, but not necessarily at a. If the function is defined in a neighbourhood of a, by closure, it is possible to use a neighbourhood defined by observable bounds. Hence $f(x)$ must exist for $x \simeq a$ but $f(a)$ does not necessarily exist. Context is f and a.

Definition 25
*A **deleted interval of** a is an interval around a not containing a.*

The limit of f at a is the value that f should take in order to be continuous at a.

Definition 26
*Let f be a real function defined on a deleted interval of a. Context is f and a. We say that f **has a limit at** a if there exists an observable number L such that if we had $f(a) = L$ then f would be continuous at a,*

In other terms, if there is an observable number L such that

$$x \simeq a \Longrightarrow f(x) \simeq L.$$

Of course, by this definition, if f *is* continuous at a, then the limit of f at a is $f(a)$.

$$\boxed{\text{The limit of } f \text{ at } a \text{ is the observable value of } f(x) \text{ when } x \simeq a}$$

The definition of limit can also be interpreted in the following way:

If f has a limit at a then it is the observable neighbour of $f(a + dx)$.
If L is the limit of f at a we write

$$f(a + dx) \simeq L$$

or

$$\lim_{x \to a} f(x) = L,$$

or

$$\lim_{h \to 0} f(a + h) = L.$$

Exercise 152
Calculate

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3}.$$

Show that it is equal to

$$\lim_{h \to 0} \frac{2(3 + h)^2 - 7(3 + h) + 3}{(3 + h) - 3}.$$

Exercise 153
Consider the signum function sgn, defined by

$$\text{sgn} : x \mapsto \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

Check that sgn is defined around 0. Does it have a limit at 0?

One Sided Limits

A function is defined **on the left** (respectively **on the right**) of a, if $f(x)$ exists for $x \simeq a$, $x < a$ (respectively $x \simeq a$, $x > a$).

Definition 27
*Let f be a real function defined on the left of a. The function f **has a limit on the left of** a if there is an obervable number L such that*

$$x \simeq a \text{ and } x < a \implies f(x) \simeq L.$$

If the limit on the left exists it is unique (it is the observable neighbour of $f(x)$). We write:

$$\lim_{x \to a_-} f(x) = L, \quad \text{or} \quad x \simeq a_- \Rightarrow f(x) = L.$$

The symbol a_- indicates that we choose numbers less than a.
Similarly we define the **limit on the right of** a and write:

$$\lim_{x \to a_+} f(x) = L, \quad \text{or} \quad x \simeq a_+ \Rightarrow f(x) = L.$$

The symbol a_+ indicates that we choose numbers greater than a.

Exercise 154
Consider f defined by

$$f : x \mapsto \sin(1/x), \quad \text{for } x > 0.$$

Check that f is defined on the right of 0.
Does it have a limit on the right of zero?

Using limits, the derivative may be re-defined in the following way:

Let f be a real function defined on an interval containing a. **The derivative of** f **at** a is the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists. If it exists, it is noted $f'(a)$. It is the derivative of f at a and f is said to be **differentiable** at a.

> The limit is only a rewriting. The "equal" sign used is there to say that the limit is the value that the function can be ultraclose to.
> When a limit appears in a problem, the first thing to do is to rewrite it in terms of ultracloseness.

We extend the definition of limit to the cases where the function reaches ultralarge values.

Introducing a new symbol: if relative to a context, we consider ultralarge values of x or ultralarge values of $f(x)$, the infinity symbol "∞" is used. But **no value can ever be equal to** ∞.

⚠ The ∞ symbol cannot be used in operations, because it is not a number.

Definition 28
*Let f be a real function defined on a deleted interval of a. The context is f and a. We say that f **tends to plus infinity** ($+\infty$) (resp. minus infinity ($-\infty$)) at a if $f(x)$ is positive ultralarge (resp. negative ultralarge) whenever $x \simeq a \quad x \neq a$*
written

$$\lim_{x\to a} f(x) = \infty$$

The definition for one-sided limits is similar.

Similarly

$$\lim_{x\to\infty} f(x) = L$$

stands for: there is an observable L such that $f(x) \simeq L$ whenever x is ultralarge.

Theorem 46 (Rule of de l'Hospital for $0/0$)
Let f and g be differentiable functions at a. Suppose that $f(a) = g(a) = 0$, but that $g'(a) \neq 0$. Then

$$\frac{f(a+dx)}{g(a+dx)} \simeq \frac{f'(a)}{g'(a)}$$

(provided $f'(a)$ and $g'(a)$ exist).

Exercise 155
Prove theorem 40.

The rule of de l'Hospital also holds for the case where a is ultralarge. And more generally

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if $lim_{x\to a} g'(x) \neq 0$.

Exercise 156
Evaluate using de L'Hospital's rule.

$$\frac{x-1}{\sqrt{x^2-1}}$$

for $x \simeq 1$.

Exercise 157
Evaluate using de L'Hospital's rule.

(1) $\dfrac{1/t-1}{t^2-2t+1}$ for $t \simeq 1$ (with $(t > 1)$).

(2) $\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ for $x \simeq 1$.

(3) $\dfrac{x^2}{\sqrt{2x+1}-1}$ for $x \simeq 0$.

(4) $\dfrac{2+1/t}{3-2/t}$ for $t \simeq 0$.

(5) $\dfrac{x+5-2x^{-1}-x^{-3}}{3x+12-x^{-2}}$ for ultralarge x

(6) $\left(t+\dfrac{1}{t}\right)((4-t)^{3/2}-8)$ for $t \simeq 0$.

(7) $\dfrac{u+u^{-1}}{1+\sqrt{1-u}}$ for ultralarge u.

Practice exercise 19 Answer page 95

Calculate the following limits. The answer should be a number, $+\infty$, $-\infty$ or "does not exist"

(1) $\lim\limits_{x\to\infty} \dfrac{6x-4}{2x+5}$

(2) $\lim\limits_{x\to\infty} x^3-10x^2-6x-2$

(3) $\lim\limits_{x\to\infty} \dfrac{x^2-x+4}{3x^2+2x-3}$

(4) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+2}}{\sqrt{3x+1}}$

(5) $\lim\limits_{x\to\infty} x-\sqrt{x}$

(6) $\lim\limits_{x\to\infty} \sqrt[3]{x+2}$

(7) $\lim\limits_{x\to 0_-} 1+\dfrac{1}{x}$

(8) $\lim\limits_{x\to 0} \dfrac{1}{x^2}-\dfrac{1}{x}$

(9) $\lim\limits_{x\to 0} \dfrac{1+2x^{-1}}{7+x^{-1}-5x^{-2}}$

(10) $\lim\limits_{x\to 2} \dfrac{1-x}{2-x}$

(11) $\lim\limits_{x\to 3_+} \dfrac{x+1}{(x-2)(x-3)}$

(12) $\lim\limits_{x\to 3} \dfrac{x+1}{(x-2)(x-3)}$

(13) $\lim\limits_{x\to 1} \dfrac{3x^2+4}{x^2+x-2}$

(14) $\lim\limits_{x\to 2_+} \dfrac{x^2+4}{x^2-4}$

(15) $\lim\limits_{x\to\infty} \sqrt{x^2+1}-x$

(16) $\lim\limits_{x\to-\infty} \sqrt{x^2+1}-x$

(17) $\lim\limits_{x\to\infty} \sqrt{x^2-3x+2}-\sqrt{x^2+1}$

(18) $\lim\limits_{x\to\infty} \sqrt[3]{x+4}-\sqrt[3]{x}$

Practice exercise 20 Answer page 95

Evaluate using de L'Hospital's rule.

(1) $\lim\limits_{x\to 0} \dfrac{\sqrt{9+x}-3}{x}$

(2) $\lim\limits_{x\to 2} \dfrac{2-\sqrt{x+2}}{4-x^2}$

(3) $\lim\limits_{u\to\infty} \dfrac{\sqrt{u+1}+\sqrt{u-1}}{u}$

(4) $\lim\limits_{x\to 0} \dfrac{(1-x)^{1/4}-1}{x}$

(5) $\lim\limits_{t\to 0_+} \left(\dfrac{1}{t}+\dfrac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$

(6) $\lim\limits_{u\to 1} \dfrac{(u-1)^3}{u^{-1}-u^2+3u-3}$

(7) $\lim\limits_{u\to 0_+} \dfrac{1+5/\sqrt{u}}{2+1/\sqrt{u}}$

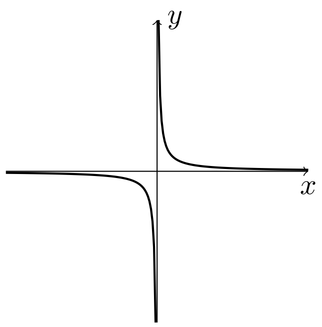
(8) $\lim\limits_{x\to\infty} \dfrac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$

(9) $\lim\limits_{t\to\infty} \dfrac{1-t/(t-1)}{1-\sqrt{t/(t-1)}}$

9
Asymptotes

Exercise 158

Consider the function $f : x \mapsto \frac{1}{x}$.



(1) What is the domain of this function? Specify the context.

(2) What happens to the curve close to the vertical axis i.e., for values of x close to 0? Consider ultrasmall values of x.

(3) What happens to the curve close to the horizontal axis? i.e., for very large values of x? Consider ultralarge values of x (positive or negative).

(4) Draw this function for a horizontal range of $[-100; 100]$ and a vertical range of $[-100; 100]$.

(5) Does f have a limit at 0?

Informally: For a given function f, a straight line is **an asymptote** of the function f if it is ultraclose to the function when either

- x tends to $\pm\infty$ (horizontal or oblique asymptote).
- y (or $f(x)$) tends to $\pm\infty$ (vertical asymptote).

Definition 29
*A real function f **has a vertical asymptote at** $x = a$ if $f(x)$ is positive or negative ultralarge for $x \simeq a$, x being less than a or x being greater than a.*
If it is the case for x greater than a, we write

$$x \simeq a_+ \Rightarrow f(x) \text{ is ultralarge}$$

or

$$\lim_{x \to a_+} f(x) = \pm\infty$$

If it is the case for x less than a, we write

$$x \simeq a_- \Rightarrow f(x) \text{ is ultralarge}$$

or

$$\lim_{x \to a_-} f(x) = \pm\infty$$

Example: The function $f : x \mapsto 1/x$ has a vertical asymptote at 0. The only parameter of the function is 1, always observable. If dx is a positive ultrasmall number then $f(dx)$ is positive ultralarge. Hence

$$\frac{1}{dx} \text{ is ultralarge}$$

We also extend properties of limits to cases where x is positive ultralarge or negative ultralarge, written $x \to +\infty$ or $x \to -\infty$

Definition 30
*A real function f defined on an interval of the form $[b, +\infty[$ or $]-\infty, b]$ has a **horizontal asymptote at** $+\infty$ (resp. $-\infty$) if there is an observable number L such that*

$$x \to \infty \Rightarrow f(x) \simeq L.$$

(the same holds for $-\infty$)

A context is f and b, but it is always possible to consider an observable b relative to f hence a context is given by f, and x is ultralarge relative to that context. When this situation occurs, we say that L is the limit of f at plus infinity (resp. minus infinity), or that the limit of f is L when x tends to infinity.

We write that f has **a horizontal asymptote** $y = L$ at plus infinity if

$$\lim_{x \to +\infty} f(x) = L.$$

(Similarly for negative infinity.)

Example: Consider the limit

$$\lim_{x \to +\infty} \frac{x^2 - 3x + 1}{x^2 + 1}.$$

This means: consider the fraction for an ultralarge value of x.

The function $f : x \mapsto \dfrac{x^2 - 3x + 1}{x^2 + 1}$ is defined on $\mathbb{R}$. $1, 2$ and 3 are always observable. Let x be ultralarge. Then

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 + 1} = \frac{x^2(2 - \frac{3}{x} + \frac{1}{x^2})}{x^2(1 + \frac{1}{x^2})} = \frac{2 - \overbrace{\dfrac{3}{x}}^{\simeq 0} + \overbrace{\dfrac{1}{x^2}}^{\simeq 0}}{1 + \underbrace{\dfrac{1}{x^2}}_{\simeq 0}} \simeq \frac{2}{1} = 2,$$

hence f has a horizontal asymptote $y = 2$ at $\pm\infty$.

We now define the oblique asymptote

Definition 31
*A real function f has an **oblique asymptote at** $+\infty$ (resp. $-\infty$) if there exist observable numbers a, b (context is f) such that*

$$x \to +\infty \Rightarrow [f(x) - (ax + b)] \simeq 0 \quad (\textit{resp. } x \to -\infty \Rightarrow [f(x) - (ax + b)] \simeq 0).$$

*The line $y = ax + b$ is the **oblique asymptote of** f (at $\pm\infty$).*

The existence of an oblique asymptote is a property of f hence the context is f.

This is equivalent to saying that $f(x) \simeq ax + b$ whenever x is ultralarge.

Example: Consider

$$f : x \mapsto \frac{x^3 + 2x^2 + x - 1}{x^2 + 1}$$

defined on $\mathbb{R}$. Using long division we have

$$f(x) = x + 2 - \frac{3}{x^2 + 1}.$$

Let x be ultralarge. We have

$$f(x) - (x + 2) = \frac{-3}{x^2 + 1} \simeq 0,$$

because $x^2 + 1$ is ultralarge. Hence f has an oblique asymptote at $y = x + 2$ (at $\pm\infty$), i.e., $a = 1$ and $b = 2$.

Exercise 159
Find the asymptotes (if any) of

(1) $f : x \mapsto \dfrac{x}{2x^2 + 1}$

(2) $g : x \mapsto \dfrac{2x^2 + 1}{x}$

(3) $h : x \mapsto \dfrac{x^3 + 2}{2x^2 - 1}$

(4) $i : x \mapsto \dfrac{x^2 + 2x + 1}{x + 1}$

(5) $j : x \mapsto \dfrac{3x^3 + 2x^2 - x + 12}{x^2 + 8}$

For functions which are not rational functions, where the polynomial long division does not apply, we have the following:

Theorem 47
Let f be a real function and let a and b be observable (context is f). Then f has an oblique asymptote at $y = ax + b$ at $+\infty$ (resp. $-\infty$) if and only if

$$\lim_{x\to+\infty} \frac{f(x)}{x} = a \quad \textit{and} \quad \lim_{x\to+\infty} [f(x) - ax] = b.$$

$$\textit{(resp.} \lim_{x\to-\infty} \frac{f(x)}{x} = a \quad \textit{and} \quad \lim_{x\to-\infty} [f(x) - ax] = b.)$$

Remark: If $a = 0$ the line $y = ax + b$ becomes $y = b$ i.e., a horizontal asymptote.

Exercise 160
Use the definition of limit to rewrite the previous theorem without any reference to limits.

Exercise 161
Prove the previous theorem.

Example: Consider $f : x \mapsto \sqrt{x^2 + 1}$ defined on $\mathbb{R}$. Let x be positive ultralarge. Then

$$\frac{f(x)}{x} = \frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2(1+1/x^2)}}{x} = \frac{|x| \overbrace{\sqrt{1+1/x^2}}^{\simeq 1}}{x} \simeq \begin{cases} 1 & \text{it } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Moreover:

$$f(x) - x = \sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1} - x)\cdot(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \frac{1}{\sqrt{x^2+1} + x} \simeq 0.$$

Hence f has an oblique asymptote at $y = x$ at $+\infty$.
At $-\infty$ the function has an oblique asymptote at $y = -x$.

Exercise 162
Find the asymptotes at infinity (if any) of

(1) $f : x \mapsto \dfrac{\sin(x)}{x}$

(2) $g : x \mapsto \dfrac{x^2 + \sin(x)}{x}$

(3) $h : x \mapsto \dfrac{x^2 + \sin(x)}{\sqrt{x}}$

(4) $i : x \mapsto x^{\frac{3}{2}}$

Exercise 163

Consider a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. Reminder: the order (or degree) of a polynomial function is the value of the highest exponent of the variable.

(1) In which cases will there be a vertical asymptote?

(2) In which cases will be there be a horizontal asymptote?

(3) In which cases will there be a horizontal asymptote at $y = 0$?

(4) In which cases will there be an oblique asymptote?

Practice exercise 21 Answer page 96

Find all asymptots of the following functions.

(1) $f_1 : x \mapsto \dfrac{x^2 - x}{x - 1}$

(2) $f_2 : x \mapsto \dfrac{4x^3 + 2x^2 - 5}{3x^3 - 4x^2}$

(3) $f_3 : x \mapsto \sqrt{x^2 + x}$

(4) $f_4 : x \mapsto \dfrac{\sqrt{x^5 + x}}{\sqrt{3x^5 - x}}$

(5) $f_5 : x \mapsto \dfrac{x^2 + 2x}{\sin(x)}$

(6) $f_6 : x \mapsto \dfrac{\sin(x)}{x^2 - x}$

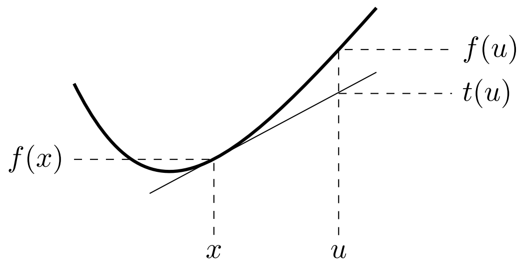
(7) $f_7 : x \mapsto \dfrac{10^x}{10^x + 1}$

10
Bending

Definition 32
Let f be differentiable on an interval I. The curve of f is ***bending upwards on*** I *if for every $x, u \in I$, $f(u)$ is above the line tangent to f at $(x, f(x))$, i.e.,*

$$f(u) \geq f'(x)(u - x) + f(x).$$

The curve of f is ***bending downwards on*** I *if $(-f)$ is bending upwards.*



For ultrasmall $(u - x)$ this can be rephrased in the following manner:

Definition 33
A differentiable function f is bending upwards at a if

$$f(a + dx) \geq f(a) + f'(a) \cdot dx.$$

Theorem 48 (Bending and Second Derivative)
Let f be twice differentiable on an interval I. Then

(1) If $f''(x) \geq 0$ whenever $x \in I$ then f is bending upwards on I.

(2) If $f''(x) \leq 0$ whenever $x \in I$ then f is bending downwards on I.

Exercise 164
Use the mean value theorem to prove theorem 48.

11 Curve Sketching

Curve sketching needs the following steps:

- Find the domain.
- Find the roots and the intercept (if any).
- Find the asymptotes (if any).
- Find the derivative (if any).
- Find the roots of the derivative (if any).
- Find the second derivative (if any).
- Find the roots of the second derivative (if any).
- Determine the maximums and minimums and bending direction.
- Put all these values in a table.
- Draw arrows which indicate the general direction of the function:
- Use this information to choose a convenient scale.
- Sketch the function.

Practice exercise 22 Answer page 97

Sketch the following:

(1) $f_1 : x \mapsto \dfrac{x^2}{x+2}$

(2) $f_2 : x \mapsto x - 1 + \dfrac{9}{x+1}$

(3) $f_3 : x \mapsto \dfrac{-x^2 - 2x - 1}{x+3}$

(4) $f_4 : x \mapsto x + 3 + \dfrac{1}{2x+1}$

(5) $f_5 : x \mapsto \dfrac{x^2 - 4x + 6}{(x-2)^2}$

(6) $f_6 : x \mapsto \dfrac{2x^2 - 3}{x^2 - 1}$

(7) $f_7 : x \mapsto \dfrac{x^2 + 3x - 4}{x^2 - x - 2}$

(8) $f_8 : x \mapsto \dfrac{x^3 + 2}{2x}$

(9) $f_9 : x \mapsto \dfrac{x^3 - 1}{x^2}$

(10) $f_{10} : x \mapsto \dfrac{2x - 1}{\sqrt{x^2 + 2}}$

(11) $f_{11} : x \mapsto \dfrac{\sqrt{x^2+1}}{x+1}$

(12) $f_{12} : x \mapsto \dfrac{\sqrt{x^2-4x+3}}{x+1}$

Practice exercise 23 Answer page 98

Sketch the following

- $g_1 : x \mapsto x\ln(x)$
- $g_2 : x \mapsto \dfrac{x}{\ln(x)}$
- $g_3 : x \mapsto \dfrac{e^x}{\ln(x)}$
- $g_4 : x \mapsto \dfrac{\sin(\sqrt{x})}{e^x}$
- $g_5 : x \mapsto \sin(\cos(x))$
- $g_6 : x \mapsto \cos(\sin(x))$
- $g_7 : x \mapsto \dfrac{e^x}{1+e^x}$
- $g_8 : x \mapsto \dfrac{1}{1+e^x}$
- $g_9 : x \mapsto \ln(x^2+1)$
- $g_{10} : x \mapsto \dfrac{e^x}{x-2}$
- $g_{11} : x \mapsto e^{-x^2}$
- $g_{12} : x \mapsto \dfrac{x \cdot e^x}{\ln(x)}$

Answers to practice exercises

Answers to practice exercice 18, page 73

(1) 3
(2) ∞
(3) $1/3$
(4) $1/\sqrt{3}$
(5) ∞
(6) ∞
(7) $-\infty$
(8) ∞
(9) 0
(10) does not exist
(11) ∞
(12) does not exist
(13) does not exist
(14) ∞
(15) 0
(16) ∞
(17) 0
(18) $-3/2$

Answers to practice exercice 19, page 73

(1) $1/6$
(2) $1/16$
(3) 0
(4) $-1/4$
(5) $1/2$
(6) -1
(7) 5
(8) ∞
(9) 2

Answers to practice exercice 20, page 83
(Integration constant to be added)

- $F_a : x \mapsto x^5 - x^2 + 4x$
- $F_b : x \mapsto \dfrac{1}{4}x^4 - \dfrac{5}{3}x^3 + \dfrac{3}{2}x^2 - 2x$
- $F_c : x \mapsto x^2 - x$
- $F_d : x \mapsto \dfrac{1}{4}x^5 - \dfrac{1}{4}x^3 + \dfrac{5}{4}x^2 + \dfrac{3}{2}x$
- $F_e : x \mapsto x^2 + x + \dfrac{1}{x}$
- $F_f : x \mapsto 3x - \dfrac{2}{x} + \dfrac{5}{2x^2}$
- $F_g : x \mapsto \dfrac{x^4}{4} - \dfrac{1}{x}$
- $F_h : x \mapsto \dfrac{3}{4}\sqrt[3]{x^4} + \dfrac{3}{2}\sqrt[3]{x^2}$
- $F_i : x \mapsto 2\sqrt{x} + \dfrac{2}{3}\sqrt{x^3}$
- $F_j : x \mapsto \dfrac{1}{3}(x+1)^3$
- $F_k : x \mapsto (3x-2)^5$
- $F_l : x \mapsto \dfrac{1}{8}(2x+1)^4$
- $F_m : x \mapsto -\dfrac{1}{12}(3-x)^{12}$
- $F_n : x \mapsto -\dfrac{1}{20}(3-4x)^5$
- $F_o : x \mapsto \dfrac{2}{9}\sqrt{(3x-2)^3}$
- $F_p : x \mapsto 2\sqrt{x-1}$
- $F_q : x \mapsto -\dfrac{1}{3}(3-x^2)^6$
- $F_r : x \mapsto \dfrac{1}{5}(x^2-3x+1)^5$
- $F_s : x \mapsto \dfrac{1}{3}(x^3-2x^2+x-3)^3$
- $F_t : x \mapsto \dfrac{2}{3}(4x^2-5x)^3$

- $F_u : x \mapsto \frac{1}{8}(3x^2 - 2x + 5)^4$
- $F_v : x \mapsto -\frac{1}{x^2+1}$
- $F_w : x \mapsto -\frac{1}{x^2+x+3}$
- $F_x : x \mapsto \frac{1}{3}\sqrt{(x^2+1)^3}$
- $F_y : x \mapsto 2\sqrt{9+x^3}$
- $F_z : x \mapsto \frac{2}{3}(x^3+x+2)\sqrt{x^3+x+2}$
- $F_A : x \mapsto \frac{e^{2x}}{2}$
- $F_B : x \mapsto -\frac{1}{3e^{3x}}$
- $F_C : x \mapsto -\frac{e^{-x^2}}{2}$
- $F_D : x \mapsto -\frac{1}{\ln(2)}2^{-x}$
- $F_E : x \mapsto \frac{1}{3}(e^{2x}+1)^{\frac{3}{2}}$
- $F_F : x \mapsto e^x(x^2-2x+2)$
- $F_G : x \mapsto \frac{e^x}{2}(\sin(x)-\cos(x))$
- $F_H : x \mapsto \arctan(e^x) - \frac{\pi}{2}$
- $F_I : x \mapsto \frac{\ln(x+\frac{3}{2})}{2}$
- $F_J : x \mapsto 2x + 2\ln(x-1)$
- $F_K : x \mapsto x - 2\ln(x+1)$
- $F_L : x \mapsto 2x\left(\frac{\ln(x)^2}{2} - \ln(x) + 1\right)$
- $F_M : x \mapsto \ln(\sin(x)+1)$
- $F_N : x \mapsto x\ln(x) - x$
- $F_O : x \mapsto x - \ln(x+1)$
- $F_P : x \mapsto \ln(\ln(x))$

Answers to practice exercice 21, page 89

Vertical asymptote of the form $x = c$, horizontal asymptote of the form $y = b$, oblique asymptote of the form $y = ax + b$.

(1) $y = x$

(2) $y = 1$, $x = 0$, $x = 4/3$

(3) $\begin{cases} y = x & \text{if } x > 0 \\ y = -x & \text{if } x < 0 \end{cases}$

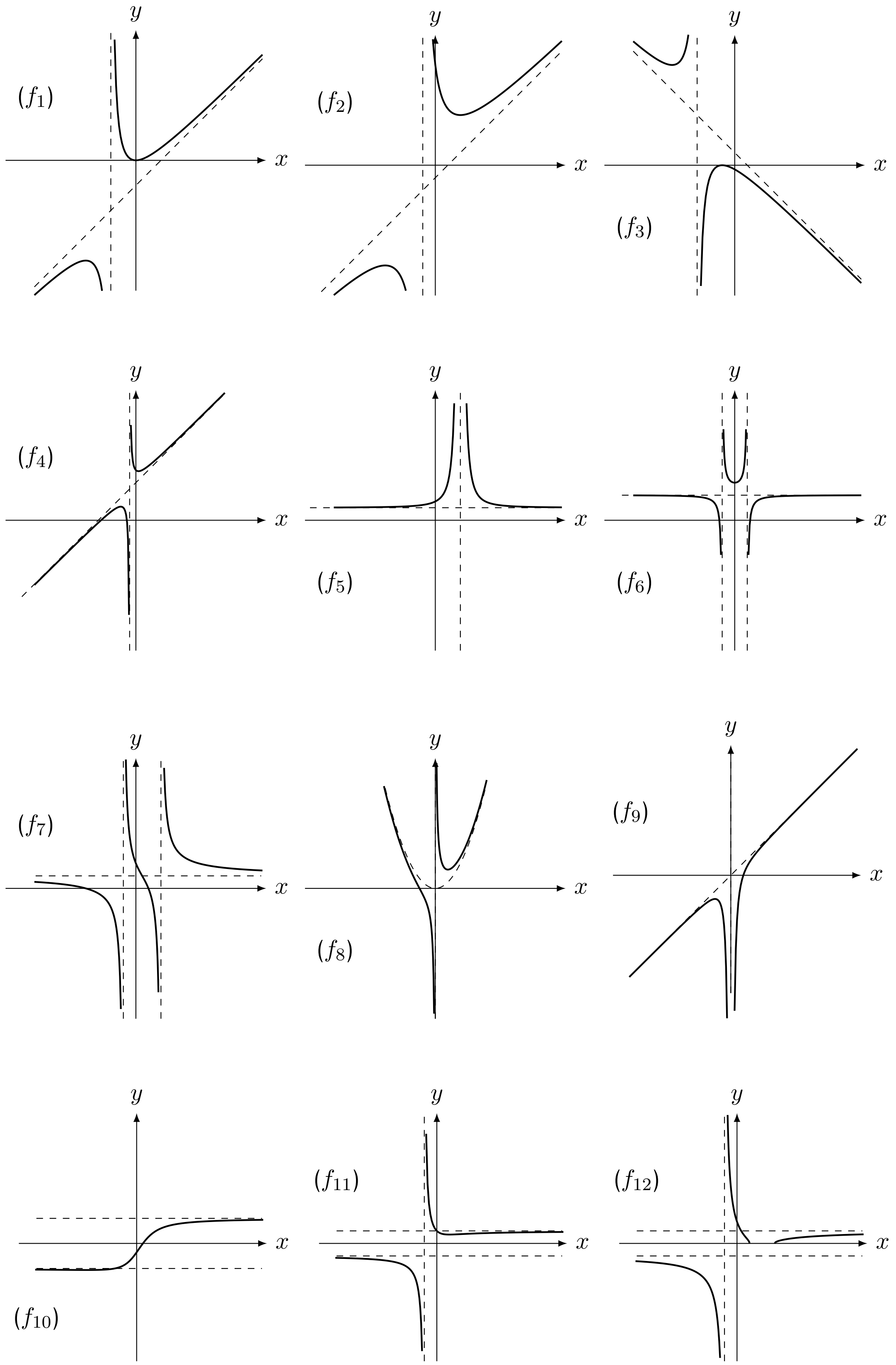
(4) $y = \sqrt{1/3}$, $x = \sqrt[4]{1/3}$

(5) $x = k \cdot \pi \quad k \in \mathbb{Z}$

(6) $y = 0$, $x = 2$

(7) $\begin{cases} y = 0 & \text{if } x < 0 \\ y = 1 & \text{if } x > 0 \end{cases}$

Answers to practice exercice 22, page 93



Answers to practice exercice 23, page 94