21st Century Mathematics in the Classroom

Analysis using ultrasmall numbers

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The Dawn of Analysis

$$f : x \mapsto x^2$$

Seventeenth century:

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The Critic

Bishop Berkeley (1734)

“ And what are these evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?”

¿Y qué son estos incrementos evanescentes? No son ni cantidades finitas ni cantidades infinitamente pequeñas, ni tampoco son nada. ¿No podríamos acaso llamarlos fantasmas de cantidades difuntas?

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  Disadvantages
  - technically complicated mastering of order of quantifiers
  - "reverse" method: error on output determines input

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- Leibniz' concept of x being "infinitely close" to a.
- Newton's concept of x "moving" towards a.

"arbitrarily close to"

manipulation of adverbs

The Nonstandard answers

The Compactness Theorem

"A set of first-order sentences has a model if and only if every finite subset of it has a model. "

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Let $\mathcal{S}_n$ be the sentence (for $n \in \mathbb{N}$)

$$(\exists x) \left(0 < x < \frac{1}{n}\right)$$

and $A = \{\mathcal{S}_n \mid n \in \mathbb{N}\}$

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There is a model in which

$$(\exists x)(\forall n \in \mathbb{N})\left(0 < x < \frac{1}{n}\right)$$

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$$\mathbb{Q} \subset {}^*\mathbb{Q}$$

$$\mathbb{R} \subset {}^*\mathbb{R}$$

- Every function $f : \mathbb{R} \to \mathbb{R}$ has a unique extension

$$^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$$

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Problem solved?

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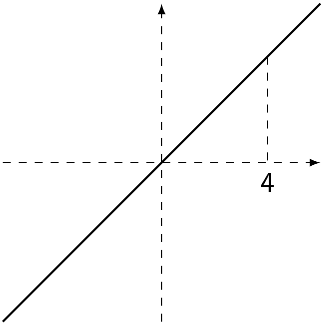

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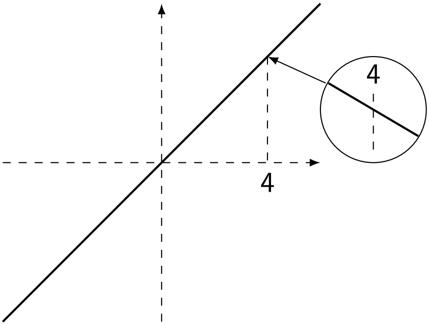

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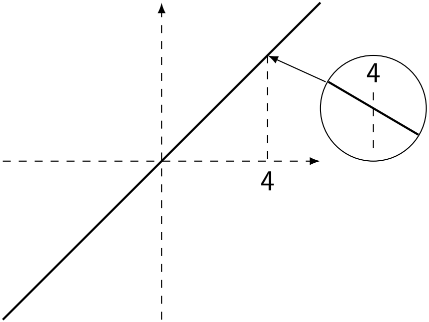

very nasty...

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Yes, if we want the integral to be the sum of infinitely many infinitely thin slices.

Internal view

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Difficulties 1 and 2 remain.

Many levels

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Hrbacek (2004)
FRIST
Simplifies and extends the power of Péraire's approach.

Introductory Analysis

Hrbacek Lessmann O'Donovan

Adaptation of FRIST to high school teaching.

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Since 2006: used in at least two Geneva Colleges by up to 10 teachers.

ANALYSIS
WITH
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Let x, y and z be real numbers.

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The **context** of a property, function or set is the list of parameters used in its definition.
When observability is mentioned in some property, it is relative to its context.

Closure Principle

Numbers, sets or functions, defined without reference to observability are always observable.
If a number, set or function, satisfies a given property, then there is an observable number satisfying that property.

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$f(a)$ is as observable as f and a

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Relative to any number, there exist ultrasmall real numbers.

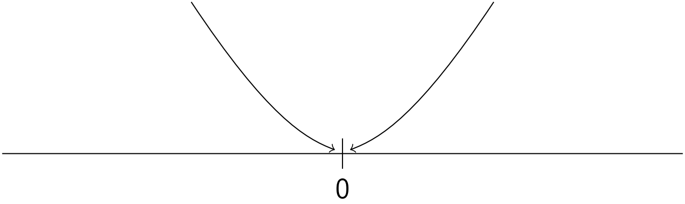
A real number is **ultralarge** if it is larger in absolute value than any strictly positive observable number

Let a, b be real numbers. We say that a is **ultraclose** to b, written

$$a \simeq b,$$

if $b - a$ is ultrasmall or if $a = b$.

With respect to a given number
ultrasmall numbers are somewhere here



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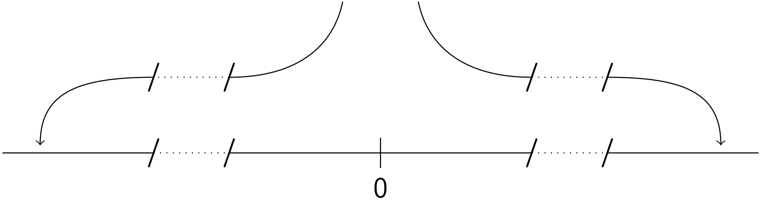

Principle of the observable neighbour

Relative to a context, any real number x which is not ultralarge can be written in the form $a + h$ where a is observable and $h \simeq 0$.

Contextual Notation

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These properties are internal. Both difficulties are solved here.

A context is *extended* if parameters are added to the list.

A property is not changed if the context is extended.

Essential properties

Relative to a context containing $a \neq 0$ and let ε and δ be ultrasmall, then

1. $a \cdot \varepsilon$ is ultrasmall.
2. $\varepsilon + \delta \simeq 0$
3. $\varepsilon \cdot \delta$ is ultrasmall
4. $\dfrac{a}{\varepsilon}$ is ultralarge

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Proof that $a \cdot \varepsilon \simeq 0$:
wlog $a > 0$ and $\varepsilon > 0$

By contradiction: assume there is an observable $b > 0$ such that $a \cdot \varepsilon > b > 0$.

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Then $\varepsilon > \frac{b}{a} > 0$
By closure $\frac{b}{a}$ is observable hence $\varepsilon \not\simeq 0$.

□

Immediate consequence:

Relative to a context containing a and b with $a \simeq x$ and $b \simeq y$, then

1. $a + b \simeq x + y$
2. $a - b \simeq x - y$
3. $a \cdot b \simeq x \cdot y$
4. If also $b \neq 0$, $\frac{1}{b} \simeq \frac{1}{y}$.

Derivative

A real function f defined on an interval containing a is ***differentiable at*** *a if there is an observable value D such that, for any dx*

$$\frac{f(a+dx)-f(a)}{dx} \simeq D$$

Then $D = f'(a)$ is the ***derivative*** *of f at a.*

$dx \simeq 0$ and $dx \neq 0$ by definition of dx, but it can be positive or negative.

Student's presentation

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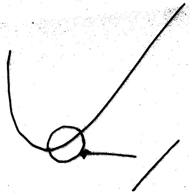

$$\frac{\Delta f(x)}{dx} = \frac{f(x+dx) - f(x)}{dx} \approx f'(x)$$

$\Downarrow$

context
neighbour

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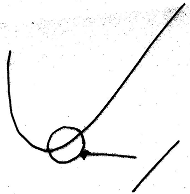

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Example x^2+1

$$\Delta f(x) = (x+dx)^2 + 1 - (x^2+1)$$

$$= x^2 + 2x\,dx + dx^2 + 1 - x^2 - 1$$

$$\frac{\Delta f(x)}{dx} = 2x \left[+ \underbrace{dx}_{\simeq 0} \right]$$

$$f'(x) = 2x$$

chain rule: student's presentation

$$(f \circ g)'$$

$$\frac{\Delta f(g(a))}{dx} = \frac{f(g(a+dx)) - f(g(a))}{dx}$$

$$= \frac{f(g(a) + \Delta g(a)) - f(g(a))}{dx}$$

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$$\left(\begin{array}{l} g(a) = y \\ f(g(a)) = f(y) \\ f'(y) \simeq \dfrac{\Delta f(y)}{dy} \\ \dfrac{\Delta f(y)}{dy} = f'(y) + \varepsilon \\ \Delta f(y) = \boxed{f'(y)\,dy + \varepsilon\, dy} \end{array} \right)$$

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$$\left(\begin{aligned} g(a) &= y \\ f(g(a)) &= f(y) \\ f'(y) &\simeq \frac{\Delta f(y)}{dy} \\ \frac{\Delta f(y)}{dy} &= f'(y) + \varepsilon \\ \Delta f(y) &= f'(y)\,dy + \varepsilon\,dy \end{aligned} \right)$$

$$= \frac{f'(y)dy + \varepsilon dy}{dx} = \frac{f'(g(a))\Delta g(a)}{dx} + \underbrace{\frac{\varepsilon \Delta g(a)}{dx}}_{\simeq 0}, \qquad \frac{\Delta g(a)}{dx} \simeq g'(a)$$

$$\simeq f'(g(a)) \cdot g'(a) = (f\circ g)'(a)$$

Continuity

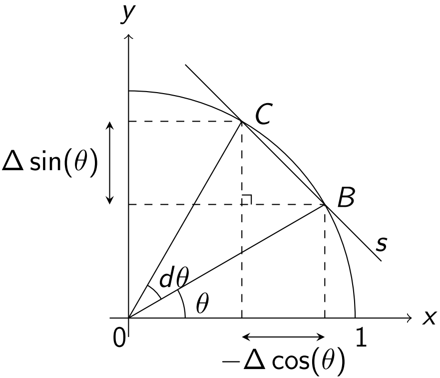
Let f be a real function defined around a. We say that f ***is continuous at*** *a if (for any x)*

$$x \simeq a \Rightarrow f(x) \simeq f(a).$$

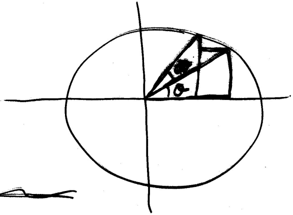
Also written

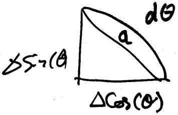
$$f(a + dx) \simeq f(a)$$

Continuity of sine and cosine



Student's proof





$$\underbrace{\Delta \sin^2(\theta)}_{\simeq 0} + \underbrace{\Delta \cos^2(\theta)}_{\simeq 0} = \underbrace{a^2}_{\simeq 0}$$

$$\Delta \sin^2(\theta) = a^2 - \cos$$

$$a \simeq d\theta$$

Area under a curve: student's presentation

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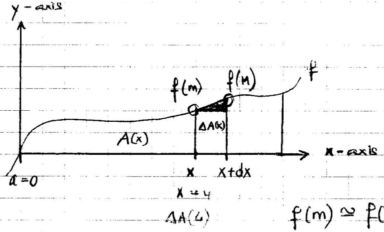
$f : x \longmapsto \frac{1}{10} x^3 + 2$



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ΔA(x) is the variation of the area.

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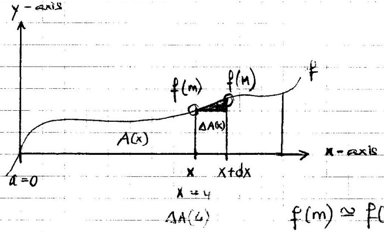

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$\Delta A(x)$ is the variation of the area.

$$f(m)\,dx \leq \Delta A(x) \leq f(M)\,dx$$

$$f(m) \leq \frac{\Delta A(x)}{dx} \leq f(M) \simeq f(x)$$

$$A'(x) = f(x)$$

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we do not define continuity at x_0 by the limit
• the limit of a function at x_0 is the value that f should take at x_0 to be continuous

$$\lim_{x \to a} f(x) = L$$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x) \quad (|x - a| \leq \delta \Rightarrow |f(x) - L| \leq \varepsilon)$$

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$$(\forall x) \quad x \simeq a \Rightarrow f(x) \simeq L$$

$$\lim_{x \to a} f(x) = L$$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x) \quad (|x - a| \leq \delta \Rightarrow |f(x) - L| \leq \varepsilon)$$

the left hand part of the implication has no meaning without the right hand part.

$$(\forall x) \quad x \simeq a \Rightarrow f(x) \simeq L$$

f and a determine the context.

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f and a determine the context.
$x \simeq a$ is defined independently.
then $f(x) \simeq L$ is verified algebraically.

thank you!