Pointwise convergence of Poisson integrals associated with the Hermite operator

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Classical Fatou theorems in $\mathbb{R}^{n+1}_+$

The Dirichlet problem $\begin{cases} u_{tt} + \Delta_x u = 0 & t > 0,\ x \in \mathbb{R}^n \\ u(0,x) = f(x) \end{cases}$

has formal solution

$$u(t,x) = P_t * f(x) \quad \text{with} \quad P_t(y) = \frac{c_n\, t}{(t^2 + |y|^2)^{\frac{n+1}{2}}}.$$

Q: When does it hold

$$P_t * f(x_0) \longrightarrow f(x_0), \quad \text{as } t \searrow 0$$

1. at a.e. $x_0 \in \mathbb{R}^n$ if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$
2. at every Lebesgue point x_0 of $f \in L^1(dy/(1+|y|)^{n+1})$, ie
$$\lim_{r \searrow 0} \fint_{B_r(x_0)} |f(y) - f(x_0)|\, dy = 0 \qquad (x_0 \in \mathcal{L}_f)$$
3. when
$$\lim_{r \searrow 0} \fint_{B_r(x_0)} f(y)\, dy = f(x_0) \qquad (x_0 \in \mathcal{S}_f)$$

[Fatou, 1906] when $n = 1$ and $\mathbb{T}$

Some remarks:

- $x_0 \in \mathcal{L}_f$ actually implies non-tang conv

$$\lim_{|x-x_0|<\alpha t \searrow 0} P_t * f(x) = f(x_0), \quad \forall\, \alpha > 0$$

- $x_0 \in \mathcal{S}_f$ only implies *vertical* convergence (in general)

$$\lim_{t \searrow 0} P_t * f(x_0) = f(x_0) \qquad \textbf{(v)}$$

- In fact, when $f \geq 0$ it holds $\quad x_0 \in \mathcal{S}_f \iff \textbf{(v)}$
  [Loomis'43], [Rudin'78], [Ramey-Ullrich'88], ...
- the classical proof of "$x_0 \in \mathcal{S}_f \Rightarrow \textbf{(v)}$" uses
  1. $P_t(x-y)$ is a convolution kernel
  2. $u \mapsto P_t(u)$ is radially decreasing
  3. $\int P_t = 1$
- ... so, it does not easily generalize to other settings...
- Example [Rudin'80]: "$\mathcal{S}_f \;\not\Longrightarrow\; \textbf{(v)}$" for $\mathbf{P}_t(z,w)$ in $\mathbb{B}^{\mathbb{C}^n}$

Proof sketch for "$x_0 \in \mathcal{S}_f \Rightarrow$ (**v**)"

More generally, for μ complex measure we have

$$\lim_{r\searrow 0} \frac{\mu(B_r(x_0))}{|B_r|} = \ell \quad \Longrightarrow \quad \lim_{t\to 0^+} P_t * \mu(x_0) = \ell$$

- Assume $x_0 = 0$, $\mu =$ real, Supp $\mu \subset B_\delta$
- ... if $\mu_1 := \mu|_{B_\delta^\complement}$ then $P_t * |\mu_1|(0) = O(t)$
- let $\varrho(r) := \mu(\overline{B}_r)$, and $m_\varrho\big((a,b]\big) = \varrho(b) - \varrho(a)$
- using "polar coordinates"

$$\begin{aligned}
P_t * \mu(0) &= \int_{B_\delta} P_t \, d\mu = \int_{[0,\delta]} P_t(r)\, dm_\varrho(r) \\
\text{(parts)} \quad &= P_t(\delta)\varrho(\delta) + \int_0^\delta \varrho(r) \left|\frac{dP_t}{dr}\right| dr \\
\text{(*)} \quad &\leq (1+\varepsilon)\,\ell \left[P_t(\delta)\,|B_\delta| + \int_0^\delta |B_r| \left|\frac{dP_t}{dr}\right| dr\right] \\
\text{(undo)} \quad &= (1+\varepsilon)\,\ell \left(P_t * \mathbb{1}_{B_\delta}\right)(0) \longrightarrow (1+\varepsilon)\,\ell
\end{aligned}$$

- Thus, $\limsup_{t\to 0} P_t * \mu(0) \leq \ell$, and by symmetry $\liminf_{t\to 0} P_t * \mu(0) \geq \ell$

The Poisson-Hermite setting

- $\mathbf{L} = -\Delta + |x|^2$ in $\mathbb{R}^n$
- $\mathbb{P}_t = e^{-t\sqrt{\mathbf{L}}} = \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-\frac{t^2}{4\tau}} \, e^{-\tau \mathbf{L}} \frac{d\tau}{\tau^{3/2}}$
- $u(t,x) = \mathbb{P}_t f(x)$ solves the PDE $\quad u_{tt} = \mathbf{L}u, \; u(0) = f$
- $e^{-\tau\mathbf{L}}(x,y) =$ Mehler formula $\quad \longrightarrow$ explicit!!
- so $\mathbb{P}_t(x,y)$ is partly explicit

$$\mathbb{P}_t(x,y) = c_n t \int_0^1 e^{-\frac{t^2}{\ell(s)}} \frac{(1-s^2)^{\frac{n}{2}-1}}{s^{n/2}} \, e^{-\frac{|x-y|^2}{4s} - \frac{s}{4}|x+y|^2} \frac{ds}{\ell(s)^{3/2}}$$

with $\ell(s) = 2\log\frac{1+s}{1-s}$ ▸ 1 ▸ 2

- if $x \neq 0 \longrightarrow$ not decreasing in $|y|$...
- **Q:** find (minimal) conditions on f st $\lim_{t\to 0} \mathbb{P}_t f(x_0) = f(x_0)$
- so far only results when $x_0 \in \mathcal{L}_f$... Muck'69, Stempak-Torrea'03, Pineda-Urbina'08,...

THEOREM 1:

If $n \in \{1,2,3\}$ and $f \in L^1_c(\mathbb{R}^n)$ then

$$\lim_{r \searrow 0} \fint_{B_r(x_0)} f = f(x_0) \quad \Longrightarrow \quad \lim_{t \to 0^+} \mathbb{P}_t f(x_0) = f(x_0) \qquad (*)$$

If $n \geq 4$ (and $x_0 \neq 0$) then $\exists\, f$ st $(*)$ fails

The example is given by

$$f(y) = \frac{\mathrm{sign}\,\big((x_0 - y) \cdot x_0\big)}{|x_0 - y|^\gamma}\, \mathbb{1}_{(0,1)}(|x_0 - y|)$$

which has $\fint_{B_r(x_0)} f = 0$, and one can show that

$$\mathbb{P}_t f(x_0) \gtrsim t^{3-\gamma} \to +\infty, \quad \text{if } \gamma \in (3, n)$$

Theorem 2:

If $n \in \{4, 5, 6\}$ and $f \in L^1_c(\mathbb{R}^n)$ then $(*)$ holds if $x_0 \in \mathcal{S}_f$ and additionally

$$\lim_{r \searrow 0} r^3 \fint_{B_r} |f(x_0 + h) - f(x_0 - h)| \, dh = 0, \qquad (1)$$

and when $n \geq 7$ if it also holds

$$\lim_{r \searrow 0} r^6 \fint_{B_r} |f(x_0 + h) + f(x_0 - h) - 2f(x_0)| \, dh = 0. \qquad (2)$$

Remarks:

- (1) is void if f is even at x_0
- (2) is void if f is odd at x_0
- by example, theorem fails if (1) is removed
- so far, do not know if theorem fails if (2) is removed...

Optimal growth of f when $|y| \to \infty$

Q: Can replace $f \in L^1_c(\mathbb{R}^n)$ by a less restritive condition?
This was considered in an earlier paper [GHSTV, TAMS'16]

- the Poisson kernel satisfies

$$c_1(t,x)\,\Phi(y) \le \mathbb{P}_t(x,y) \le c_2(t,x)\,\Phi(y), \quad y \in \mathbb{R}^n$$

  with

$$\Phi(y) = \frac{e^{-|y|^2/2}}{(1+|y|)^{\frac{n}{2}}\,[\log(e+|y|)]^{\frac{3}{2}}}$$

▸ $\mathbb{P}_t(x,y)$

- so $\mathbb{P}_t|f|(x) < \infty$ is well-defined $\forall\, x \in \mathbb{R}^n \iff f \in L^1(\Phi)$
- Using estimates from [GHSTV'16], the non-local part $f_1 = f \cdot \mathbb{1}_{B_\delta(x_0)^\complement}$ satisfies

$$\mathbb{P}_t|f_1|(x_0) \le c_\delta\, t \int_{\mathbb{R}^n} |f|\,\Phi = O(t)$$

- so below we restrict to the local part $f \cdot \mathbb{1}_{B_\delta(x_0)}$, for small $\delta > 0$

Let $y = x_0 + h$. Search for a decomposition

$$\mathbb{P}_t(x,y) = \mathbb{P}_t(x_0, x_0 + h) = K_t(|h|) + \mathbb{R}_t(h)$$

st K_t is radial decreasing and $\lim_{t\to 0} \mathbb{R}_t f(x_0) = 0$

- Our attempt is based on

▸ $\mathbb{P}_t(x,y)$

$$\begin{aligned} e^{-\frac{s}{4}|x_0+y|^2} &= e^{-s|x_0|^2}\, e^{-\frac{s}{4}|h|^2}\, e^{-s\, x_0\cdot h} \\ &= e^{-s|x_0|^2}\, e^{-\frac{s}{4}|h|^2} \left[\mathrm{ch}\,(s\, x_0 \cdot h) - \mathrm{sh}\,(s\, x_0\cdot h)\right] \\ &= e^{-s|x_0|^2}\, e^{-\frac{s}{4}|h|^2} \left[1 + \mathcal{O}^{\mathrm{ev}}(s^2\, |x_0\cdot h|^2) + \mathcal{O}^{\mathrm{odd}}(s\, x_0\cdot h)\right] \end{aligned}$$

- This gives

$$\mathbb{P}_t(x_0, x_0 + h) = K_t(|h|) + \mathbb{R}^2_t(h) + \mathbb{R}^1_t(h)$$

Some reductions on f

- Assume Supp $f \subset B_\delta(x_0)$
- May also assume $f(x_0) = 0$ since

$$\begin{aligned}\mathbb{P}_t f(x_0) - f(x_0) &= \mathbb{P}_t\Big(f - f(x_0)\Big)(x_0) + f(x_0)\,\Big(\mathbb{P}_t(\mathbb{1}_{B_\delta(x_0)})(x_0) - 1\Big) \\ &= \mathbb{P}_t \tilde{f}(x_0) \;+\; \mathcal{O}(1)\end{aligned}$$

- Use the decomposition

$$f(x_0 + h) \;=\; f_{\rm odd}(h) + f_{\rm ev}(h) = \frac{f(x_0+h) - f(x_0-h)}{2} + \frac{f(x_0+h) + f(x_0-h)}{2}$$

- Then must show
  1. $\mathbb{R}^1_t f_{\rm odd}(x_0) \to 0$
  2. $\mathbb{R}^2_t f_{\rm ev}(x_0) \to 0$
  3. $K_t * f_{\rm ev}(0) \to 0$

THE ODD PART

Recall

$$\mathbb{P}_t(x_0, x_0+h) = c_n t \int_0^{1/2} e^{-\frac{t^2}{\ell(s)}} \frac{(1-s^2)^{\frac{n}{2}-1}}{s^{n/2}} e^{-\frac{|h|^2}{4s}} e^{-\frac{s}{4}|2x_0+h|^2} \frac{ds}{\ell(s)^{3/2}} + \mathcal{O}(t)$$

- Using $\ell(s) \approx s$ we have

$$\begin{aligned} \left|\mathbb{R}_t^1(h)\right| &\lesssim t \int_0^{1/2} \frac{e^{-\frac{ct^2+|h|^2}{4s}} s\,|x_0 \cdot h|}{s^{(n+1)/2}} \frac{ds}{s} \\ &\approx \frac{t\,|x_0 \cdot h|}{(t+|h|)^{n-1}} \lesssim \frac{t}{(t+|h|)^{n-2}} \qquad (\text{or } \log(t+|h|), \text{ if } n=1) \end{aligned}$$

- Thus,

$$\left|\mathbb{R}_t^1 f_{\rm odd}(x_0)\right| \lesssim \int_{|h|<\delta} \frac{t\,|f_{\rm odd}(h)|}{(t+|h|)^{n-2}}\,dh \longrightarrow 0 \quad \text{if } n \le 3 \text{ by DCT}$$

- If $n \geq 4$ we must use instead

$$r^3 \fint_{B_r} |f_{\rm odd}(h)|\,dh \le \varepsilon, \quad r \in (0,\delta)$$

- Bounds for $\left|\mathbb{R}_t^2(h)\right|$ are similar (actually better)...

Now

$$K_t(|h|) = c_n t \int_0^{1/2} e^{-\frac{t^2}{\ell(s)}} \frac{(1-s^2)^{\frac{n}{2}-1}}{s^{n/2}} e^{-s|x_0|^2} e^{-(s+\frac{1}{s})\frac{|h|^2}{4}} \frac{ds}{\ell(s)^{3/2}} + \mathcal{O}(t)$$

- arguing as before can prove $K_t(|h|) \lesssim \frac{t}{(t+|h|)^{n+1}}$
- mimicking the classical Fatou proof, if $\varrho(r) := \int_{|h|<r} f_{\rm ev} = \int_{B_r(x_0)} f$

$$\begin{aligned}
\int_{|h|<\delta} K_t(|h|)\, f_{\rm ev}(h)\, dh &= \int_0^\delta K_t(r)\, dm_\varrho(r) \\
\text{(parts)} \quad &= K_t(\delta)\varrho(\delta) + \int_0^\delta \varrho(r)\, \big|K_t'(r)\big|\, dr \\
\text{(*)} \quad &\leq \varepsilon \Big[K_t(\delta)\, |B_\delta| + \int_0^\delta |B_r|\, \big|K_t'(r)\big|\, dr \Big] \\
\text{(undo)} \quad &= \varepsilon \int_{|h|<\delta} K_t(|h|)\, dh \lesssim \varepsilon
\end{aligned}$$

- Thus $\limsup_{t\to 0} K_t * f_{\rm ev}(0) \leq 0$, and by symmetry $\lim_{t\to 0} K_t * f_{\rm ev}(0) = 0$

Further comments

- Proof can be adapted to $\mathbb{P}_t\mu$, with μ complex measure
- If $n \in \{1, 2, 3\}$ and $f \geq 0$, it seems likely that

$$\lim_{t\to 0} \mathbb{P}_t f(x_0) = \ell \quad \Longrightarrow \quad \lim_{r\searrow 0} \fint_{B_r(x_0)} f = \ell$$

... but so far we do not have a proof...

- Proof strategy may be exportable to other settings...

THANKS/GRAZIE