STABILITY OF CYLINDERS IN $\mathbb{E}(\kappa,\tau)$ HOMOGENEOUS SPACES

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Abstract. We extend the classical Plateau-Rayleigh instability criterion in the $\mathbb{E}(\kappa,\tau)$ spaces. We prove the existence of a positive number $L_0 > 0$ such that if a truncated circular cylinder of radius ρ in $\mathbb{E}(\kappa,\tau)$ has length $L > L_0$ then it is unstable. This number L_0 depends on κ, τ and ρ. The value L_0 is sharp under axially-symmetric variations of the surface. We also extend this result for the partitioning problem in $\mathbb{E}(\kappa,\tau)$.

1. Introduction and statement of results

In capillary theory [10], the Plateau-Rayleigh instability criterion asserts that a truncated piece of a circular cylinder of radius $\rho > 0$ in $\mathbb{R}^3$ is unstable if its length L satisfies

$$L > 2\pi\rho. \tag{1.1}$$

Circular cylinders are surfaces with constant mean curvature (cmc to abbreviate) and the inequality (1.1) can be derived from a well-established theory of stability of cmc surfaces going back, at least, to the initial works of Barbosa, do Carmo and Eschenburg [3, 4]. An analogous bound for cmc cylinders in the hyperbolic 3-space has been recently exhibited by the authors [5], and similar instability criteria for cylindrical liquids have been obtained in other contexts of the capillary theory: see e.g. [2, 6, 12, 13, 15] and references therein.

In this paper, we consider the stability of truncated cylinders in the family of simply-connected homogeneous 3-dimensional spaces whose isometry group has dimension 4. These spaces can be parametrized by two real parameters κ, τ and are known as the $\mathbb{E}(\kappa,\tau)$ spaces. The $\mathbb{E}(\kappa,\tau)$ spaces complete the classification of the Thurston geometries along with the space forms $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{S}^3$, whose isometry group are of dimension 6, and the space *Sol*, whose isometry group is of dimension 3. The $\mathbb{E}(\kappa,\tau)$ spaces admit a Riemannian submersion $\pi : \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$ onto the 2-dimensional space form $\mathbb{M}^2(\kappa)$ and their bundle curvature is τ. If $\tau = 0$, then $\mathbb{E}(\kappa, 0)$ is one the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$. If $\tau \neq 0$, then $\mathbb{E}(\kappa,\tau)$ is the Heisenberg space if $\kappa = 0$; the universal cover of the special linear group if $\kappa < 0$; and the Berger spheres if $\kappa > 0$.

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A local model for the $\mathbb{E}(\kappa,\tau)$ spaces is the following. If $r>0$, let $\mathbb{D}(r)=\{(x,y)\in\mathbb{R}^2\colon x^2+y^2<r^2\}$ be the disk of radius $r>0$. Let $\mathcal{R}(\kappa,\tau)$ be the space $\mathbb{R}^3$ if $\kappa\geq 0$ or $\mathbb{D}(2/\sqrt{-\kappa})\times\mathbb{R}$ if $\kappa<0$. Let us endow $\mathcal{R}(\kappa,\tau)$ with coordinates (x,y,z) and the metric

$$g=\sigma^2(dx^2+dy^2)+(\sigma\tau(ydx-xdy)+dz)^2,\quad \sigma=\frac{4}{4+\kappa(x^2+y^2)}.$$

Then, $(\mathcal{R}(\kappa,\tau),g)$ is isometric to the $\mathbb{E}(\kappa,\tau)$ space. The Riemannian submersion is isomorphic to the projection onto the first two coordinates, $\pi\colon\mathcal{R}(\kappa,\tau)\to\mathbb{R}^2$ if $\kappa\geq 0$ and $\pi\colon\mathcal{R}(\kappa,\tau)\to\mathbb{D}(2/\sqrt{-\kappa})$ if $\kappa<0$. Notice that the base space in this submersion is the two-dimensional space form $\mathbb{M}^2(\kappa)$, equipped with the metric $\sigma^2(dx^2+dy^2)$. When $\kappa\leq 0$ this model for the $\mathbb{E}(\kappa,\tau)$ spaces is global but if $\kappa>0$, then this model omits one fiber. Indeed, in the case $\kappa>0,\tau>0$ of the Berger spheres $\mathbb{S}^3_b$, if we regard $\mathbb{S}^3_b=\{(z,w)\in\mathbb{C}^2\colon |z|^2+|w|^2=1\}$ then an explicit isometry between $\mathcal{R}(\kappa,\tau)$ and $\mathbb{S}^3_b-\{(e^{i\theta},0)\colon\theta\in\mathbb{R}\}$ is

$$\Psi(x,y,z)=\frac{1}{\sqrt{1+\frac{\kappa}{4}(x^2+y^2)}}\left(\frac{\sqrt{\kappa}}{2}(x+iy)e^{i\frac{\kappa}{4\tau}z},e^{i\frac{\kappa}{4\tau}z}\right).$$

With this isometry we see that two points (x,y,z) and $(x,y,z+\frac{8\tau\pi}{\kappa})$ are identified to the same point in $\mathbb{S}^3_b$. Finally, if $\kappa=\tau=0$ in this model, then $\mathbb{E}(0,0)$ is simply the Euclidean space $\mathbb{R}^3$, whose isometry group is of dimension 6.

In the last decades, the theory of cmc surfaces in the $\mathbb{E}(\kappa,\tau)$ spaces has received the attention of many researchers since the extension of Hopf's theorem by Abresch and Rosenberg [1]. This produced a vast literature which, without aiming to collect it, we refer the reader to [7, 8, 9] and references therein. One the most relevant topics in the $\mathbb{E}(\kappa,\tau)$ spaces is the study of the stability of cmc surfaces [11, 14, 16, 20]. A cmc surface is said to be *stable* if it is a second order minimizer of the area functional under the preservation of the volume. In case that we drop the volume preserving condition, the surface is said *strongly stable*.

Our aim in this paper is the extension of the Plateau-Rayleigh estimate (1.1) in the context of $\mathbb{E}(\kappa,\tau)$ spaces, and for that matter we need to generalize the analogue of the circular cylinders. The natural notion of a cylinder is the lifting in $\mathbb{E}(\kappa,\tau)$ of a circle of $\mathbb{M}^2(\kappa)$, where by a circle we mean a closed curve of $\mathbb{M}^2(\kappa)$ with constant geodesic curvature κ_g. In particular, if $\kappa\geq 0$ then κ_g is any positive constant and if $\kappa<0$, then $\kappa_g^2>-\kappa$.

Definition 1.1. Given a curve $\alpha\colon I\subset\mathbb{R}\to\mathbb{M}^2(\kappa)$, the *vertical cylinder* over α is defined as $\mathsf{C}_\alpha=\pi^{-1}(\alpha(I))$. The vertical cylinder is called *circular* if α is a circle. In such a case, the radius of C_α is the radius of α.

Notice that a vertical cylinder C_α has constant mean curvature if and only if κ_g is constant because the mean curvature of C_α is $\kappa_g/2$. It was proved in [14] that a cmc vertical cylinder C_α is strongly stable if and only if $\kappa_g^2+\kappa\leq 0$. In particular this implies $\kappa\leq 0$. If $\kappa=0$, then α is a geodesic ($\kappa_g=0$), but if $\kappa<0$ there are three possibilities: α is a horocycle ($\kappa_g^2=-\kappa$), an equidistant curve ($0<\kappa_g^2<-\kappa$) or a geodesic ($\kappa_g=0$).

From the result of [14], the only cmc vertical cylinders that are not strongly stable are the circular ones. Following with the spirit of the classical Plateau-Rayleigh instability criterion, we want to establish conditions of stability of compact pieces of circular cylinders in the $\mathbb{E}(\kappa,\tau)$ spaces in terms of their length. We precise the terminology. Given a circular cylinder C_α and $L>0$, we define the *truncated circular cylinder of length* L as the set $\mathsf{C}_\alpha(L)=\{(x,y,z)\in\mathsf{C}_\alpha\colon 0\leq z\leq L\}$. In other words, $\mathsf{C}_\alpha(L)$ is the compact piece of C_α between the planes of equations $z=0$ and $z=L$. Since the translations $(x,y,z)\mapsto(x,y,z+t)$ are isometries in $\mathbb{E}(\kappa,\tau)$, the fact to fix the z-coordinate between 0 and L does not lose generality in this definition.

The first result in this paper is the extension of the classical Plateau-Rayleigh result in the $\mathbb{E}(\kappa,\tau)$ spaces.

Theorem 1.2. *Let C_α be a circular cylinder of radius ρ in $\mathbb{E}(\kappa,\tau)$. If L_0 is given by*

$$L_0=\begin{cases}\frac{2\pi}{-\kappa}\sinh(\rho\sqrt{-\kappa})\sqrt{-\kappa+4\tau^2\tanh^2(\frac{\rho\sqrt{-\kappa}}{2})} & \kappa<0,\\ 2\pi\rho\sqrt{1+\tau^2\rho^2} & \kappa=0,\\ \frac{2\pi}{\kappa}\sin(\rho\sqrt{\kappa})\sqrt{\kappa+4\tau^2\tan^2(\frac{\rho\sqrt{\kappa}}{2})} & \kappa>0,\end{cases} \tag{1.2}$$

then if $L>L_0$ the truncated cylinder $\mathsf{C}_\alpha(L)$ is not stable. If $\kappa>0$ and $\tau\neq 0$ we assume, in addition, $L<8\tau\pi/\kappa$.

Notice that if $\kappa=\tau=0$, then $L_0=2\pi\rho$ and Thm. 1.2 rediscovers the classical Plateau-Rayleigh instability criterion (1.1) in $\mathbb{R}^3$. In the case $\kappa>0$ and $\tau\neq 0$ corresponding to the Berger spheres, the assumption $L<8\tau\pi/\kappa$ comes from the fact that if $L\geq 8\tau\pi/\kappa$ then $\mathsf{C}_\alpha(L)$ is identified as a torus by the periodicity of the fibers. We will also prove that this inequality is optimal in case that we consider axially-symmetric variations of the surface (Cor. 4.1).

In the second result of this paper we study the stability of vertical cmc cylinders when regarded as solutions of the partitioning problem in the $\mathbb{E}(\kappa,\tau)$ spaces. The general setting is analogous to the Euclidean case and 3-manifolds in general [17, 19, 21]. Given a domain $W\subset\mathbb{E}(\kappa,\tau)$ with smooth boundary ∂W, a surface Σ with $\mathrm{int}(\Sigma)\subset\mathrm{int}(W)$ and $\partial\Sigma\subset\partial W$ is a *capillary surface* if is a critical point of the area functional among all surfaces in these conditions that separate W into two domains of prescribed volumes. Any capillary surface is characterized by the fact that has constant mean curvature and the contact angle that Σ makes with ∂W along $\partial\Sigma$ is constant. In this context, we have similar notions of stability.

In the $\mathbb{E}(\kappa,\tau)$ spaces, we investigate the stability in the partitioning problem of truncated pieces of vertical cmc cylinders between two planes. To be precise, let $\Pi_c=$

$\{(x,y,z)\in\mathbb{E}(\kappa,\tau)\colon z=c\}$, $c\in\mathbb{R}$. Let C_α be a vertical cmc cylinder and $\mathsf{C}_\alpha(L)=\{(x,y,z)\in\mathsf{C}_\alpha : 0\leq z\leq L\}$. Notice that we are now including the case that $\kappa_g^2+\kappa\leq 0$, that is, α is not circle. If α is a circle, then the intersection of $\mathsf{C}_\alpha(L)$ with the support planes $\Pi_0\cup\Pi_L$ is orthogonal, hence $\mathsf{C}_\alpha(L)$ is a capillary surface. The result that we prove is the following.

Theorem 1.3. *Let $\mathsf{C}_\alpha(L)$ be a truncated circular cylinder of radius ρ and length L in $\mathbb{E}(\kappa,\tau)$, supported on the planes $\Pi_0\cup\Pi_L$. If $L>L_0$, where L_0 is given by* (1.2) *then $\mathsf{C}_\alpha(L)$ is not stable in the partitioning problem. If $\kappa,\tau>0$, we assume moreover $L<8\tau\pi/\kappa$.*

In case that $\kappa_g^2+\kappa\leq 0$, the vertical cmc cylinder $\mathsf{C}_\alpha(L)$ makes constant contact angle with $\Pi_0\cup\Pi_L$ only if the ambient space is $\mathbb{H}^2(\kappa)\times\mathbb{R}$. In such a case, we will prove in Thm. 5.1 that $\mathsf{C}_\alpha(L)$ is strongly stable regardless of the value L. Theorem 5.1 is analogous to the result proved in [14] but in the context of the partitioning problem.

The organization of the paper is the following. In Sect. 2 we investigate the model $\mathcal{R}(\kappa,\tau)$ of the $\mathbb{E}(\kappa,\tau)$ spaces. In Sect. 3 we introduce the two variational problems considered in this paper. In both cases, a self-adjoint elliptic operator is defined and the stability problem can be reformulated in terms of the eigenvalues of this operator under suitable boundary conditions. For Thm. 1.2 the eigenvalue problem has Dirichlet conditions, while in the case of Thm. 1.3 are of Neumann type. The proof of Thm. 1.2 is given in Sect. 4 and the proof of Thm. 1.3 is exhibited in Sect. 5.

2. Vertical cylinders in $\mathbb{E}(\kappa,\tau)$

For each $\mathbb{E}(\kappa,\tau)$ space, we consider the model $\mathcal{R}(\kappa,\tau)$ described in Sect. 1. In this model, the vector field $E_3=\partial_z$ is a Killing vector field and τ is characterized by the property $\overline{\nabla}_X E_3=\tau X\times E_3$ for all $X\in\mathfrak{X}(\mathbb{E}(\kappa,\tau))$, where $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{E}(\kappa,\tau)$. Using this model, an orthonormal basis of the tangent space is given by $\{E_1,E_2,E_3\}$, where

$$E_1=\frac{1}{\sigma}\partial_x-\tau y\partial_z,\quad E_2=\frac{1}{\sigma}\partial_y+\tau x\partial_z,\quad E_3=\partial_z.$$

Hence, we have

$$\partial_x=\sigma(E_1+\tau yE_3),\quad \partial_y=\sigma(E_2-\tau xE_3),\quad \partial_z=E_3. \tag{2.1}$$

The Levi-Civita connection $\overline{\nabla}$ is determined by the relations

$$\begin{array}{lll}\overline{\nabla}_{E_1}E_1=-\dfrac{\sigma_y}{\sigma^2}E_2, & \overline{\nabla}_{E_1}E_2=\dfrac{\sigma_y}{\sigma^2}E_1+\tau E_3, & \overline{\nabla}_{E_1}E_3=-\tau E_2,\\ \overline{\nabla}_{E_2}E_1=\dfrac{\sigma_x}{\sigma^2}E_2-\tau E_3, & \overline{\nabla}_{E_2}E_2=-\dfrac{\sigma_x}{\sigma^2}E_1, & \overline{\nabla}_{E_2}E_3=\tau E_1,\\ \overline{\nabla}_{E_3}E_1=-\tau E_2, & \overline{\nabla}_{E_3}E_2=\tau E_1, & \overline{\nabla}_{E_3}E_3=0,\end{array} \tag{2.2}$$

Here, σ_x and σ_y stand for the partial derivatives of σ with respect to x and y, respectively.

Let C_α be a vertical cylinder, where $\alpha\colon I \to \mathbb{M}^2(\kappa)$ is a regular curve. If α is parametrized by arc-length, a basis of the tangent plane on C_α is $\{\alpha', E_3\}$ where α' refers to the horizontal lift of α' through the submersion π. Let N be the unit normal vector on C_α chosen so $\alpha'\times E_3 = N$. The matricial expression of the second fundamental form A of C_α with respect to $\{\alpha', E_3\}$ is

$$A = \begin{pmatrix} \langle \overline{\nabla}_{\alpha'}\alpha', N\rangle & \langle \overline{\nabla}_{\alpha'}E_3, N\rangle \\ \langle \overline{\nabla}_{E_3}\alpha', N\rangle & \langle \overline{\nabla}_{E_3}E_3, N\rangle \end{pmatrix} = \begin{pmatrix} \kappa_g & \tau \\ \tau & 0 \end{pmatrix},$$

where κ_g is the geodesic curvature of α as a curve in $\mathbb{M}^2(\kappa)$. The following result compiles some properties of the vertical cylinders: see [14, Appendix].

Lemma 2.1. *Let C_α be a vertical cylinder. Then,*

(1) *The Gauss curvature of C_α is $K = 0$ and the mean curvature is $H = \kappa_g/2$.*
(2) *The norm of the second fundamental form A is $|A|^2 = \kappa_g^2 + 2\tau^2$.*
(3) *The Ricci curvature along the direction of N is $\mathrm{Ric}(N) = \kappa - 2\tau^2$.*

Since the metric on $\mathbb{M}^2(\kappa)$ is $\sigma^2(dx^2+dy^2)$, the circle α can be parametrized, up to an isometry in $\mathbb{M}^2(\kappa)$, by

$$\alpha(s) = \left(r\cos\frac{s}{\sigma r}, r\sin\frac{s}{\sigma r}\right), \qquad r>0,\ \sigma = \frac{4}{4+\kappa r^2}.$$

The center of α is the origin $(0,0)$ of $\mathbb{R}^2$ if $\kappa \geq 0$ or $\mathbb{D}(2/\sqrt{-\kappa})$ if $\kappa<0$. The radius ρ of α is given by

$$\rho = \begin{cases} \frac{2}{\sqrt{-\kappa}}\operatorname{arctanh}\frac{r\sqrt{-\kappa}}{2} & \kappa<0, \\ r & \kappa = 0, \\ \frac{2}{\sqrt{\kappa}}\arctan\frac{r\sqrt{\kappa}}{2} & \kappa>0. \end{cases}$$

The geodesic curvature κ_g of α in $\mathbb{M}^2(\kappa)$ is constant and given by

$$\kappa_g = \frac{4-\kappa r^2}{4r} = \begin{cases} \sqrt{-\kappa}\coth(\rho\sqrt{-\kappa}) & \kappa<0, \\ \dfrac{1}{\rho} & \kappa=0, \\ \sqrt{\kappa}\cot(\rho\sqrt{\kappa}) & \kappa>0. \end{cases} \tag{2.3}$$

Let

$$R = r\sigma = \frac{4r}{4+\kappa r^2} = \begin{cases} \frac{1}{\sqrt{-\kappa}}\sinh(\rho\sqrt{-\kappa}) & \kappa<0, \\ \rho & \kappa=0, \\ \frac{1}{\sqrt{\kappa}}\sin(\rho\sqrt{\kappa}) & \kappa>0. \end{cases}$$

The lifting of α via the submersion provides a parametrization of C_α, namely,

$$\psi(s,t) = \left(r\cos\frac{s}{R}, r\sin\frac{s}{R}, t\right), \quad s,t\in\mathbb{R}. \tag{2.4}$$

We compute the first fundamental form (g_{ij}). Using (2.1), we obtain

$$\psi_s = \frac{1}{\sigma}(-\sin\frac{s}{R}, \cos\frac{s}{R}, 0) = -\sin\frac{s}{R}E_1 + \cos\frac{s}{R}E_2 - \tau r E_3,$$
$$\psi_t = (0,0,1) = E_3.$$

Thus (g_{ij}) and its inverse (g^{ij}) are the matrices

$$(g_{ij}) = \begin{pmatrix} 1+r^2\tau^2 & -r\tau \\ -r\tau & 1 \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & r\tau \\ r\tau & 1+r^2\tau^2 \end{pmatrix}. \tag{2.5}$$

Finally, by Lem. 2.1, the mean curvature H of C_α is constant with $H = \kappa_g/2$.

3. Stability of cmc surfaces in $\mathbb{E}(\kappa,\tau)$ spaces

In this section, we recall the notions of stability and index of a cmc surface in the contexts of fixed boundary (Thm. 1.2) and the partitioning problem (Thm. 1.3). We begin with the first case. Let Σ be an oriented surface in $\mathbb{E}(\kappa,\tau)$ and let $\{\Sigma_t : t \in (-\epsilon,\epsilon)\}$ be a compactly supported variation of Σ. If we define the functionals

$$\mathcal{A}(t) = \text{Area}(\Sigma_t), \qquad \mathcal{V}(t) = \text{Volume}(\Sigma_t),$$

where $\mathcal{V}(t)$ is the volume enclosed by Σ_t, it is known that Σ is a critical point of $\mathcal{A}$ for all volume preserving variations if and only the mean curvature H of Σ is constant. In such a case, Σ is said to be *stable* if $\mathcal{A}''(0) \geq 0$ for all compactly supported normal variations that preserve the volume of Σ. If we drop the volume preserving condition in the variations of Σ, we say that Σ is *strongly stable*. Stability of Σ is equivalent to

$$\mathcal{A}''(0) = -\int_\Sigma u(\Delta u + |A|^2 u + \text{Ric}(N)u) \geq 0, \tag{3.1}$$

for all $u \in C_0^\infty(\Sigma)$ such that $\int_\Sigma u = 0$. Here Δ is the Laplacian operator on Σ, A is the second fundamental form of Σ, N is the unit normal vector field of Σ and Ric the Ricci curvature of $\mathbb{E}(\kappa,\tau)$. The mean zero integral $\int_\Sigma u = 0$ comes from the condition that the variations preserve the volume of Σ. In consequence, Σ is strongly stable if $\mathcal{A}''(0) \geq 0$ for all $u \in C_0^\infty(\Sigma)$.

The parenthesis in (3.1) defines the *Jacobi operator* by

$$\mathcal{L} = \Delta + |A|^2 + \text{Ric}(N), \tag{3.2}$$

which is a self-adjoint elliptic operator. Since $\mathcal{L}$ is self-adjoint, we define the quadratic form Q by

$$Q[u] = -\int_\Sigma u \cdot \mathcal{L}[u],$$

in the space $\mathcal{V} = \{u \in C_0^\infty(\Sigma) \colon \int_\Sigma u = 0\}$. The *weak Morse index* of Σ, denoted by $\text{index}_w(\Sigma)$, is defined as the maximum dimension of any subspace of $\mathcal{V}$ on which Q is negative definite. In a certain sense, the weak index measures the ways to reduce the area of Σ, up to second order, preserving the volume of Σ. Thus Σ is stable if and only if $\text{index}_w(\Sigma)=0$.

Suppose Σ is compact. Then $\text{index}_w(\Sigma)$ coincides with the number of negative eigenvalues λ of the eigenvalue problem

$$\begin{cases} \mathcal{L}[u]+\lambda u=0 \text{ in } \Sigma,\\ u=0 \text{ in } \partial\Sigma,\\ u\in\mathcal{V}. \end{cases} \tag{3.3}$$

Since the condition $\int_\Sigma u=0$ is difficult to work with, instead of (3.3), we consider the eigenvalue problem

$$\begin{cases} \mathcal{L}[u]+\lambda u=0 \text{ in } \Sigma,\\ u=0 \text{ in } \partial\Sigma,\\ u\in C_0^\infty(\Sigma). \end{cases} \tag{3.4}$$

By the ellipticity of $\mathcal{L}$, it is well-know that the eigenvalues of (3.4) (also of (3.3)) are ordered as a discrete spectrum $\lambda_1<\lambda_2\leq\lambda_3\cdots\nearrow\infty$ counting multiplicity. The *Morse index* of Σ, denoted $\text{index}(\Sigma)$, is the number of negative eigenvalues of (3.4). Since in (3.4) it is only required that u belongs to $C_0^\infty(\Sigma)$, we have that $\text{index}(\Sigma)=0$ if and only if Σ is strongly stable or equivalently, $\lambda_1\geq 0$. Both indexes are related by the inequalities

$$\text{index}_w(\Sigma)\leq\text{index}(\Sigma)\leq\text{index}_w(\Sigma)+1. \tag{3.5}$$

The instability criterion in the Plateau-Rayleigh estimate (1.1) and in the results of this paper (Thms. 1.2 and 1.3) are obtained once we could show that $\lambda_2<0$ in the eigenvalue problem (3.4). In such a case, the inequalities (3.5) imply $\text{index}_w(\Sigma)\geq 1$. This proves that Σ is not stable.

When Σ is not compact, the definition of the index of Σ is given by taking an exhaustion $\Sigma_1\subset\Sigma_2\subset\ldots\subset\Sigma$ by bounded subdomains of Σ. Then the weak Morse index and the Morse index of Σ are defined by

$$\text{index}_w(\Sigma)=\lim_{n\to\infty}\text{index}_w(\Sigma_n),\quad \text{index}(\Sigma)=\lim_{n\to\infty}\text{index}(\Sigma_n). \tag{3.6}$$

These definitions are independent of the choice of the exhaustion of Σ. Both numbers can be infinite, but if they are finite, then the relation (3.5) holds too.

Stability in the partitioning problem is defined similarly. Let S be a surface that separates $\mathbb{E}(\kappa,\tau)$ into two domains and name W to one of them, having $\partial W=S$. Let Σ be an orientable surface with non-empty boundary $\partial\Sigma$ such that $\text{int}(\Sigma)\subset\text{int}(W)$ and $\partial\Sigma\subset S$. Assume that $\text{int}(\Sigma)$ separates W into two connected components, each having as boundary the union of $\text{int}(\Sigma)$ and a domain in S. Fix one of these components, say D, and let $\Omega=\partial D\cap S$. An *admissible variation* of Σ is a variation $\{\Sigma_t\colon t\in(-\epsilon,\epsilon)\}$

such that $\text{int}(\Sigma_t) \subset \text{int}(W)$ and $\partial\Sigma_t \subset S$. By denoting $\Omega(t)$ to the domain bounded by $\partial\Sigma_t$ in S, we define the energy functional

$$\mathcal{E}(t) = \text{Area}(\Sigma_t) - \cos\gamma\,\text{Area}(\Omega(t)),$$

where $\gamma \in (0,\pi)$. A surface Σ is a critical point of $\mathcal{E}$ for all volume preserving variations of Σ if and only if the mean curvature H of Σ is constant and the angle between Σ and S along $\partial\Sigma$ is constant and it coincides with γ. In such a case, we say that Σ is a *capillary surface*. The angle γ is the angle formed by the unit normal vectors N of Σ and $\widetilde{N}$ of S along $\partial\Sigma$, that is $\cos\gamma = \langle N, \widetilde{N}\rangle$. The vector N points into Ω whereas $\widetilde{N}$ points outwards Ω. See [18, 19] for details. The second variation of $\mathcal{E}$ is

$$\mathcal{E}''(0) = -\int_\Sigma u\cdot\mathcal{L}[u] + \int_{\partial\Sigma} u\Big(\frac{\partial u}{\partial\nu} - \mathbf{q}u\Big),$$

where

$$\mathbf{q} = \frac{1}{\sin\gamma}\widetilde{A}(\widetilde{\nu},\widetilde{\nu}) + \frac{\cos\gamma}{\sin\gamma}A(\nu,\nu). \tag{3.7}$$

Here $\widetilde{A}$ is the second fundamental form of S with respect to $-\widetilde{N}$. The vectors ν and $\widetilde{\nu}$ are the exterior unit conormal vectors of $\partial\Sigma$ on Σ and on S respectively. Associated to the quadratic form $\mathcal{E}''(0)$ we also have the notions of weak Morse index and Morse index. The eigenvalue problems (3.3) and (3.4) are the same but replacing the condition $u = 0$ in $\partial\Sigma$ by the so-called Robin condition

$$\frac{\partial u}{\partial\nu} - \mathbf{q}u = 0 \quad \text{in } \partial\Sigma. \tag{3.8}$$

4. Proof of Theorem 1.2.

Under the hypothesis of Thm. 1.2, we know $\kappa_g^2 + \kappa > 0$ and C_α is parametrized by (2.4). By Lem. 2.1, the Jacobi operator is

$$\mathcal{L} = \Delta + \kappa_g^2 + \kappa.$$

For the computation of the Laplacian Δ, we use its expression in local coordinates $\psi = \psi(s,t)$, namely,

$$\Delta u = \frac{1}{\sqrt{\det(g_{ij})}}\sum_{i,j=1}^2 \partial_i\left(\sqrt{\det(g_{ij})}\,g^{ij}\partial_j u\right),$$

where $\partial_1 = \partial_s$ and $\partial_2 = \partial_t$. Notice that $g^{11} = 1$, $g^{12} = r\tau$, $g^{22} = 1 + r^2\tau^2$ and $\det(g_{ij}) = 1$. If $u = u(s,t)$, then it is immediate

$$\Delta u = u_{ss} + 2r\tau u_{st} + (1 + r^2\tau^2)u_{tt}. \tag{4.1}$$

We now compute the eigenvalues of the eigenvalue problem (3.4) for the truncated cylinders $\mathsf{C}_\alpha(L)$. Let us observe that the domain of u is the rectangle $[0,2\pi R]\times[0,L]$ in the (s,t)-plane and u is $2\pi R$-periodic in the s-variable. Consider separation of variables $u(s,t)=f(s)g(t)$, where $f=f(s)$ and $g=g(t)$ are smooth functions of one variable and f is $2\pi R$-periodic. Using that α is a closed curve and (4.1), the eigenvalue problem (3.4) is

$$\begin{cases} f''g+2r\tau f'g'+(1+r^2\tau^2)fg''+\left(\kappa_g^2+\kappa+\lambda\right)fg=0, & \text{in } [0,2\pi R]\times[0,L]\\ g(0)=g(L)=0. \end{cases} \tag{4.2}$$

Here it is understood that the derivative $(\cdot)'$ is with respect to each one of the variables of f and g. Dividing the first equation by fg, we have

$$\frac{f''}{f}+2r\tau\frac{f'}{f}\frac{g'}{g}+(1+r^2\tau^2)\frac{g''}{g}+\kappa_g^2+\kappa+\lambda=0.$$

Differentiating with respect to s and next with respect to t, we obtain

$$\left(\frac{f'}{f}\right)'\left(\frac{g'}{g}\right)'=0.$$

This equation implies that either f'/f or g'/g are constant functions. If $g'/g=c\in\mathbb{R}$, then either $g(t)$ is a constant function or $g(t)=e^{ct}$. The boundary condition $g(0)=g(L)=0$ implies that if g is constant then $g=0$ and hence $u(s,t)=0$, which it is not possible. In case that $g(t)=e^{ct}$, then the boundary condition is not fulfilled.

Consequently $f'/f=c$ for some constant $c\in\mathbb{R}$. We have two possibilities.

(1) Case $c\neq 0$. Then $f(s)=e^{cs}$ but this case is not possible because f is a periodic function.

(2) Case $c=0$. Then $f(s)$ is a non-zero constant function since otherwise $u(s,t)=0$. Then Eq. (4.2) reduces to

$$(1+r^2\tau^2)g''+\left(\kappa_g^2+\kappa+\lambda\right)g=0. \tag{4.3}$$

The solution of this equation depends on the sign of $\kappa_g^2+\kappa+\lambda$.

(a) Case $\kappa_g^2+\kappa+\lambda<0$. If

$$\delta^2=-\frac{\kappa_g^2+\kappa+\lambda}{1+r^2\tau^2},$$

then the solution of (4.3) is $g(t)=A\cosh(\delta t)+B\sinh(\delta t)$, $A,B\in\mathbb{R}$. Imposing the boundary condition (4.2), we arrive $A=B=0$ and g would be 0, which it is not possible.

(b) Case $\kappa_g^2+\kappa+\lambda=0$. Then the solution of (4.3) is $g(t)=At+B$, $A,B\ n\mathbb{R}$. Again the boundary conditions (4.2) imply $A=B=0$. This case is not possible.

(c) Case $\kappa_g^2+\kappa+\lambda>0$. Let

$$\mu^2=\frac{\kappa_g^2+\kappa+\lambda}{1+r^2\tau^2},\quad \mu>0.$$

The solutions of (4.2) are $g(t)=A\sin(\mu t)+B\cos(\mu t)$, where $A,B\in\mathbb{R}$. Since $g(0)=g(L)=0$, then $B=0$ and $\mu L\in\{n\pi : n\in\mathbb{N}\}$. Therefore the eigenvalues $\lambda=\lambda_n$ are given by

$$\lambda_n=\frac{n^2\pi^2}{L^2}(1+r^2\tau^2)-(\kappa_g^2+\kappa), \tag{4.4}$$

while the eigenfunctions are $g_n(t)=\sin(\frac{n\pi}{L}t)$, $n\in\mathbb{N}$.

We have seen in Sect. 2 that if the second eigenvalue λ_2 is negative, then the surface is not stable. The condition $\lambda_2<0$ is fulfilled whenever L satisfies

$$L>2\pi\sqrt{\frac{1+r^2\tau^2}{\kappa_g^2+\kappa}}. \tag{4.5}$$

This is just (1.2) after some manipulations and we conclude the proof.

Theorem 1.2 gives a sufficient criterion of instability for truncated pieces $\mathsf{C}_\alpha(L)$ of circular cylinders because we have restricted to find eigenfunctions u that are of type $u(s,t)=f(s)g(t)$. Using separation of variables, the eigenfunctions are of type $u_n(s,t):=\sin(\frac{n\pi}{L}t)$. This implies that the variation of $\mathsf{C}_\alpha(L)$ associated to u_n does not depend on s. Thus the variation of the surface is axially symmetric with respect to the axis of the cylinder, in this case, the z-axis of $\mathcal{R}(\kappa,\tau)$. In consequence, if we are studying stability of $\mathsf{C}_\alpha(L)$ only for those variation that are axially-symmetric all the above computations provide sharp estimates in the stability/instability criterion. By simplicity in the statement, we express the critical length in terms of r and κ_g given by (4.4).

Corollary 4.1. *Let $\mathsf{C}_\alpha(L)$ be a truncated piece of a circular cylinder of length $L>0$. Then $\mathsf{C}_\alpha(L)$ is:*

1. *strongly stable for axially-symmetric variations if and only if $L\leq L_0/2$.*
2. *stable for axially-symmetric variations if and only if $L\leq L_0$.*

Proof. The statement (1) is immediate from the expression of λ_1. For the statement (2), notice that the eigenvalues of (3.3) are those λ_n that satisfy that the corresponding eigenfunction belongs to $\mathcal{V}$. The first eigenfunction $u_1(s,t)=\sin(\frac{\pi}{L}t)$ does not satisfy the mean zero integral, so λ_1 is not an eigenvalue of (3.3). The next eigenvalue is λ_2 where the eigenfunction $u_2(s,t)=\sin(\frac{2\pi}{L}t)$ has mean zero integral because

$$\int_{\mathsf{C}_\alpha(L)}u_2=\int_0^{2\pi}\int_0^L\sin\left(\frac{2\pi}{L}t\right)dsdt=0,$$

where we have used that $\det(g_{ij}) = 1$. Thus λ_2 is the first eigenvalue of (3.3). Then the result is immediate from the expression of λ_2. □

We finish this section by a remark about the case $\kappa_g^2+\kappa \leq 0$, where κ_g is constant. In this case, it was proved in [14] that C_α is strongly stable. This can be deduced directly from the expression of the quadratic form Q defined in (3.1). Indeed, by integration by parts, for all $u \in C_0(\mathsf{C}_\alpha)$ we have

$$Q[u] = \int_{\mathsf{C}_\alpha} |\nabla u|^2 - (\kappa_g^2 + \kappa)u^2 \geq 0,$$

because $\kappa_g^2 + \kappa \leq 0$. On the other hand, if one still wants to calculare the Morse index, the arguments in the proof of Thm. 1.2 fail because now the curve α is not closed. However, it is possible to compute explicitly the weak index of C_α and deduce that this index is 0, proving strongly stability when $\kappa_g^2 + \kappa \leq 0$. To show an example, consider the case $\kappa < 0$ and $\kappa_g = 0$. Then α is a geodesic of the hyperbolic plane $\mathbb{H}^2(\kappa)$. A parametrization of the corresponding cylinder is C_α is $\psi(s,t) = (s,0,t)$, with $s \in (-s_1, s_1)$, where $s_1 = 2/\sqrt{-\kappa}$. Since α is not compact, we need to take an exhaustion of C_α by considering pieces of cylinders of type $\psi([-s_0,s_0]\times[-L,L])$, with $0 < s_0 < s_1$, and letting $s_0 \to s_1$ and $L \to \infty$. Now $\psi_s = \sigma E_1$ and $\psi_t = E_3$ and the eigenvalue problem (4.2) becomes

$$\begin{cases} \dfrac{1}{\sigma^2} f''g + fg'' + (\kappa+\lambda)\, fg = 0, \\ g(-L) = g(L) = 0, \\ f(-s_0) = f(s_0) = 0. \end{cases}$$

Making a similar reasoning, we obtain $g(t) = \sin(\frac{k\pi}{L}t)$, $k \in \mathbb{N}$, and

$$\frac{1}{\sigma^2}\frac{f''}{f} + \kappa + \lambda = \frac{k^2\pi^2}{L^2},$$

or equivalently,

$$f'' + \sigma^2\left(\kappa + \lambda - \frac{k^2\pi^2}{L^2}\right) f = 0.$$

The solutions of this equation depend whether the parenthesis is negative, zero or positive. By the boundary conditions $f(-s_0) = f(s_0) = 0$, the first two cases are discarded and necessarily $\kappa + \lambda - \frac{k^2\pi^2}{L^2} = \delta^2$, for some $\delta > 0$. Then it is immediate to deduce $\delta = \frac{\pi n}{2s_0}$, $n \in \mathbb{N}$. In particular, we have

$$\lambda = \lambda_{k,n} = \frac{1}{\sigma^2}\left(\delta^2 + \frac{k^2\pi^2}{L^2} - \kappa\right) > 0$$

because $\kappa < 0$. Thus all compact pieces $\psi([-s_0, s_0] \times [-L, L])$ are strongly stable. If now $s_0 \to s_1$ and $L \to \infty$, we deduce that C_α is strongly stable.

5. Proof of Theorem 1.3

Now we prove Thm. 1.3 in the setting of the partitioning problem. With the notation of Sec. 3, the surface S is the union of two planes Π_c of equation $z = c$, specifically $S = \Pi_0 \cup \Pi_L$, and the domain W is the domain in $\mathcal{E}(\kappa, \tau)$ having $\Pi_0 \cup \Pi_L$ as boundary. Hence, the cylinder C_α is included in W and its boundary are two circles, one included in Π_0 and the other in Π_L. Consequently, if we fix D as the bounded domain in W enclosed by C_α, then Ω are the two disks in Π_0 and Π_L bounded by the circular components of the boundary of C_α.

Once we have identified all the geometric objects in the partitioning problem, we first see that C_α intersects orthogonally each plane Π_c and, in consequence, $\mathsf{C}_\alpha(L)$ is a capillary surface on $\Pi_0 \cup \Pi_L$ with $\gamma = \pi/2$. A parametrization of Π_c is $\phi(x, y) = (x, y, c)$, $x, y \in \mathbb{R}$. Using (2.1) we have

$$\begin{aligned} \phi_x &= \sigma(E_1 + \tau y E_3), \\ \phi_y &= \sigma(E_2 - \tau x E_3), \\ \widetilde{N} &= \frac{1}{\sqrt{1 + \tau^2(x^2 + y^2)}}(-\tau y E_1 + \tau x E_2 + E_3). \end{aligned} \tag{5.1}$$

In the cylinder C_α and thanks to the computation of ψ_s and ψ_t of the parametrization (2.4), the unit normal N of C_α is

$$N = \cos\frac{s}{R} E_1 + \sin\frac{s}{R} E_2.$$

Thus, along $\partial\Sigma$, we have $\langle N, \widetilde{N}\rangle = 0$. This means that the intersection is orthogonal and that the contact angle is $\gamma = \pi/2$.

As a consequence, the function $\mathbf{q}$ in (3.7) reduces to $\mathbf{q} = \widetilde{A}(\widetilde{\nu}, \widetilde{\nu})$. For the computation of $\widetilde{A}(\widetilde{\nu}, \widetilde{\nu})$, we calculate the second fundamental form $\widetilde{A}$ with respect to the basis $\{\phi_x, \phi_y\}$. We know that

$$\widetilde{A} = \begin{pmatrix} \langle \overline{\nabla}_{\phi_x}\phi_x, \widetilde{N}\rangle & \langle \overline{\nabla}_{\phi_x}\phi_y, \widetilde{N}\rangle \\ \langle \overline{\nabla}_{\phi_y}\phi_x, \widetilde{N}\rangle & \langle \overline{\nabla}_{\phi_y}\phi_y, \widetilde{N}\rangle \end{pmatrix}.$$

Using (2.2), we have

$$\begin{aligned} \overline{\nabla}_{\phi_x}\phi_x &= \sigma_x E_1 - (\sigma_y + 2\tau^2 y \sigma^2) E_2 + \sigma_x \tau y E_3, \\ \overline{\nabla}_{\phi_x}\phi_y &= (\sigma_y + \tau^2\sigma^2 y) E_1 + (\sigma_x + \sigma^2\tau^2 x) E_2 + \tau(-\sigma_x x - \sigma + \sigma^2) E_3, \\ \overline{\nabla}_{\phi_y}\phi_x &= (\sigma_y + \tau^2\sigma^2 y) E_1 + (\sigma_x + \sigma^2\tau^2 x) E_2 + \tau(\sigma_y y + \sigma - \sigma^2) E_3, \\ \overline{\nabla}_{\phi_y}\phi_y &= -(\sigma_x + 2\tau^2\sigma^2 x) E_1 + \sigma_y E_2 - \sigma_y \tau x E_3. \end{aligned}$$

Thus the matrix of $\widetilde{A}$ is

(5.2)
$$\frac{\tau}{\sqrt{1+\tau^2(x^2+y^2)}}\begin{pmatrix} -x(\sigma_y+2\tau^2 y\sigma^2) & -\sigma+(1+(x^2-y^2)\tau^2)\sigma^2-y\sigma_y \\ \sigma-(1-(x^2-y^2)\tau^2)\sigma^2+x\sigma_x & y(2x\tau^2\sigma^2+\sigma_x) \end{pmatrix}.$$

We calculate $\widetilde{\nu}$. We have

$$\widetilde{\nu}=\widetilde{N}\times\frac{\psi_s}{|\psi_s|}=N=-\frac{1}{r}(xE_1+yE_2)=-\frac{1}{r\sigma}(x\phi_x+y\phi_y).$$

It is now immediate from (5.2) that $\widetilde{A}(\widetilde{\nu},\widetilde{\nu})=0$ and consequently $\mathbf{q}=0$. This implies that the Robin condition (3.8) in the eigenvalue problem is simply

$$\frac{\partial u}{\partial\nu}=0 \quad \text{in } \partial\mathsf{C}_\alpha(L).$$

The computation of ν gives

$$\nu=N\times\frac{\psi_s}{|\psi_s|}=\frac{1}{\sqrt{1+r^2\tau^2}}(-y\tau E_1+x\tau E_2+E_3).$$

Since $\nabla u=f'g\psi_s+fg'\psi_t$, we have

$$\frac{\partial u}{\partial\nu}=\langle\nabla u,\nu\rangle=\frac{fg'}{\sqrt{1+r^2\tau^2}}. \tag{5.3}$$

Again, consider separation of variables as in Thm. 1.2. Now the eigenvalue problem is (4.2) together with (5.3), which leads to

$$\begin{cases} f''g+2r\tau f'g'+(1+r^2\tau^2)fg''+\left(\kappa_g^2+\kappa+\lambda\right)fg=0, \\ g'(0)=g'(L)=0. \end{cases} \tag{5.4}$$

As in the proof of Thm. 1.2, we divide by fg and a similar argument allows to deduce $f'(s)=cf(s)$. Again $c=0$ because $f(s)=e^{cs}$ is not possible by the periodicity of f. Since $c=0$, f is a non-zero constant function and (5.4) reduces to

$$\begin{cases} g''+\dfrac{\kappa_g^2+\kappa+\lambda}{1+r^2\tau^2}g=0, \\ g'(0)=g'(L)=0. \end{cases}$$

If $\kappa_g^2+\kappa+\lambda\leq 0$ then g does not fulfill the boundary conditions at $t=0$ and $t=L$. Thus $\kappa_g^2+\kappa+\lambda$ must be positive. Let

$$\mu^2=\frac{\kappa_g^2+\kappa+\lambda}{1+r^2\tau^2},\quad \mu>0.$$

The boundary condition $g'(0) = 0$ implies $g(t) = A\cos(\mu t)$ for some constant $A \neq 0$. The other boundary condition $g'(L) = 0$ yields $\mu = n\pi/L$, $n \in \mathbb{N}$. Consequently, the eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{L^2}(1 + r^2\tau^2) - (\kappa_g^2 + \kappa).$$

These eigenvalues coincide with the ones deduced in (4.4). In order to achieve instability we impose $\lambda_2 < 0$ which yields

$$L > 2\pi\sqrt{\frac{1 + r^2\tau^2}{\kappa_g^2 + \kappa}}.$$

This is again (4.5) and we conclude the proof.

Theorem 1.3 considers the stability in the partitioning problem of circular cylinders. We now treat vertical cmc cylinders when $\kappa_g^2 + \kappa \leq 0$ and its stability in the partitioning problem. Notice that by Lem. 2.1, the mean curvature of C_α is constant but this is not sufficient to be a capillary surface because of the condition of contact angle. In fact, the intersection of C_α with the planes Π_c is not orthogonal unless that $\tau = 0$. The condition $\kappa_g^2 + \kappa \leq 0$ implies that the curve α is not closed. We have two cases.

(1) Case $\kappa_g = 0$, that is, α is a geodesic. Without loss of generality, we can assume that α is parametrized by $\alpha(s) = (s, 0, 0)$ and C_α by $\psi(s,t) = (s, 0, t)$. Then $\psi_s = \sigma E_1$ and $\psi_t = E_3$. This gives $N = E_2$. On the other hand, from (5.1) we have

$$\widetilde{N} = \frac{1}{\sqrt{1 + s^2\tau^2}}(s\tau E_2 + E_3)$$

along $\mathsf{C}_\alpha \cap \Pi_c$. This gives $\langle N, \widetilde{N}\rangle = s\tau/\sqrt{1 + s^2\tau^2}$ which it is only constant if $\tau = 0$. This case corresponds when $\mathbb{E}(\kappa,\tau)$ is the product space $\mathbb{H}^2(\kappa) \times \mathbb{R}$.

(2) Case $\kappa_g \neq 0$. Then necessarily $\kappa < 0$. Now α is a horocycle or an equidistant curve. The parametrization of α is $\alpha(s) = (r\cos\frac{s}{R}, r\sin\frac{s}{R} - y_0)$ where $y_0 \in (0, \frac{2}{\sqrt{-\kappa}}]$ and $r > \frac{2}{\sqrt{-\kappa}} - y_0$. Since the parametrization of C_α is $\psi(s,t) = (\alpha(s), t)$, using (2.1) we have

$$\begin{aligned}\psi_s &= \frac{r\sigma}{R}\left(-\sin\frac{s}{R}E_1 + \cos\frac{s}{R}E_2 + (r\tau + y_0\sin\frac{s}{R})E_3\right),\\ \psi_t &= E_3.\end{aligned}$$

Thus

$$N = \cos\frac{s}{R}E_1 + \sin\frac{s}{R}E_2.$$

Now along $\mathsf{C}_\alpha \cap \Pi_c$, we have

$$\langle N, \widetilde{N}\rangle = \frac{y_0\tau\cos\frac{s}{R}}{\sqrt{1 + \tau^2(r^2 + y_0^2 - 2ry_0\sin\frac{s}{R})}}.$$

As a consequence, the contact angle along the intersection curve is constant if and only if $\tau = 0$ and, in such a case, $\gamma = \pi/2$.

Summarizing, the case $\kappa_g^2 + \kappa \leq 0$ only occurs if $\mathbb{E}(\kappa,\tau)$ is $\mathbb{H}^2(\kappa)\times\mathbb{R}$. In the following result, we prove that C_α is strongly stable. This extends to the partitioning problem the analogous situation of the case proved in [14].

Theorem 5.1. *Let C_α be a vertical cmc cylinder in $\mathbb{H}^2(\kappa)\times\mathbb{R}$ such that $\kappa_g^2+\kappa<0$. Then, for any $L>0$ the truncated cylinder $\mathsf{C}_\alpha(L)$ is strongly stable on $\Pi_0\cup\Pi_L$ in the partitioning problem.*

Proof. The proof is a direct computation of the quadratic form $\mathcal{E}_p''(0)$. Let $u\in C_0(\mathsf{C}_\alpha(L))$. By an integration by parts, we have

$$\mathcal{E}_p''(0) = -\int_{\mathsf{C}_\alpha(L)} u\left(\Delta u + (\kappa_g^2+\kappa)u\right) + \int_{\partial\mathsf{C}_\alpha(L)} u\frac{\partial u}{\partial\nu}ds = \int_{\mathsf{C}_\alpha(L)} |\nabla u|^2 - (\kappa_g^2+\kappa)u^2 \geq 0$$

because $\kappa_g^2+\kappa\leq 0$. □

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