A NEW FAMILY OF TRANSLATING SOLITONS IN HYPERBOLIC SPACE

ANTONIO BUENO, RAFAEL LÓPEZ

Abstract. If ξ is a Killing vector field of the hyperbolic space $\mathbb{H}^3$ whose flow are parabolic isometries, a surface $\Sigma \subset \mathbb{H}^3$ is a ξ-translator if its mean curvature H satisfies $H = \langle N, \xi\rangle$, where N is the unit normal of Σ. We classify all ξ-translators invariant by a one-parameter group of rotations of $\mathbb{H}^3$, exhibiting the existence of a new family of grim reapers. We use these grim reapers as barriers to prove a half-space theorem of the non-existence of ξ-translators included in vertical half-spaces.

1. Introduction and results

Let Σ be an orientable smooth surface and $\Psi : \Sigma \to \mathbb{H}^3$ an isometric immersion in hyperbolic space $\mathbb{H}^3$. The mean curvature flow (MCF in short) is a differentiable map $\Psi : \Sigma \times [0, T) \to \mathbb{H}^3$ such that if $\Psi_t = \Psi(-, t)$, then $\Psi_0 = \Psi$ and $\frac{\partial \Psi_t}{\partial t} = H(\Psi_t)N(\Psi_t)$, where $H(\Psi_t)$ and $N(\Psi_t)$ are the mean curvature and the unit normal of Ψ_t respectively [1, 4]. Our interest are those surfaces whose shapes evolve along the MCF by translations along a direction of $\mathbb{H}^3$. These surfaces are called translating solitons of the MCF, or translators for short. In Euclidean space $\mathbb{R}^3$, translators Σ along a direction $\mathbf{v} \in \mathbb{R}^3$ are characterized by $H = \langle N, \mathbf{v}\rangle$, where H and N are the mean curvature and the unit normal of Σ, respectively. Translators appear in the singularity theory of the MCF after a blow-up near type II singularities, according to Huisken and Sinestrari [12].

In the hyperbolic space, the same notion of translator can be stated by considering the flow of isometries of $\mathbb{H}^3$ determined by a Killing vector field $X \in \mathfrak{X}(\mathbb{H}^3)$. Geometrically, the shape of a translator does not change during the evolution along the one-parameter group of isometries generated by X. In general, we call this surface a X-translator to emphasize the vector field X. In contrast with the Euclidean space, the hyperbolic space is richer in terms of Killing vector fields since the flow of isometries generated by such fields have different geometric properties. Next, we recall the different isometries of $\mathbb{H}^3$ following [7]. In $\mathbb{H}^3$ there are spherical rotations (a geodesic is pointwise fixed), hyperbolic translations (two points of the ideal boundary $\mathbb{H}^3_\infty$ are fixed) and

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parabolic translations (a point of $\mathbb{H}^3_\infty$ is fixed). We remark that Do Carmo and Dajczer referred to all of these three isometries as rotations, but in our setting and commonly in the literature, by rotations we mean spherical rotations and by translations we refer to either hyperbolic or parabolic ones. A surface invariant by spherical rotations (resp. hyperbolic or parabolic translations) is said to be spherical (resp. hyperbolic or parabolic).

Translators of the MCF whose shape evolves by spherical rotations and by hyperbolic translations have been recently studied in [15]. In this paper, we investigate translators that evolve by parabolic translations, which have not been previously studied in the literature. In order to handle these translators, consider the upper half-space model of $\mathbb{H}^3$, that is $(\mathbb{R}^3_+, \langle,\rangle)$, where $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 \colon z > 0\}$, $\langle,\rangle$ is the hyperbolic metric

$$\langle,\rangle = \frac{1}{z^2}\langle,\rangle_e,$$

and $\langle,\rangle_e = dx^2 + dy^2 + dz^2$ is the Euclidean metric of $\mathbb{R}^3$. The ideal boundary $\mathbb{H}^3_\infty$ is the one-compactification of the plane of equation $z = 0$. In this model, the vector field

$$\xi = a\partial_x + b\partial_y, \qquad a, b \in \mathbb{R}. \tag{1}$$

is a Killing vector field whose flow of isometries are parabolic translations of $\mathbb{H}^3$. Geometrically, these parabolic translations correspond to horizontal Euclidean translations in the direction of the horizontal vector $(a, b, 0)$.

Definition 1.1. Let $\xi \in \mathfrak{X}(\mathbb{H}^3)$ be a Killing vector field given by (1). An immersed surface Σ in $\mathbb{H}^3$ is a ξ-translator if its mean curvature H satisfies

$$H = \langle N, \xi\rangle. \tag{2}$$

The aim of this paper is to initiate the study of ξ-translators and to obtain non-existence and uniqueness results. The first step is the classification of ξ-translators invariant by a one-parameter group of rotations or translations of $\mathbb{H}^3$. A classification in all its generality of invariant ξ-translators under *any* translation or rotation seems hopeless. When we turn our attention to an analog situation in $\mathbb{R}^3$, non-trivial translators of the MCF invariant by rotations only appear when the rotation axis is parallel to the translation direction. This also happens in other geometric frameworks such as in the product spaces $\mathbb{M}^2 \times \mathbb{R}$ [2, 17] or the Lorentz-Minkowski space [14]. Finally, in the work [15] of considering the Killing vector field of hyperbolic translations, the authors also restrict to specific isometries that have a geometric relation with their vector field in order to study invariant examples. Inspired by these situations, in this work we will restrict ourselves to the following translations and rotations that have a geometric relation with the Killing vector field ξ:

1. (Hyperbolic isometries) Group of hyperbolic translations along the vertical geodesic orthogonal to ξ;

(2) (Parabolic isometries) Group of parabolic translations that are also horizontal Euclidean translations;
(3) (Spherical isometries) Group of spherical rotations about an axis orthogonal to ξ.

In order to save notation and for clarity reasons in the exposition of the results, hereinafter we simply say parabolic or hyperbolic translations and spherical rotations, and they should be understood as one of the particular aforementioned isometries.

The first result is the classification of hyperbolic and spherical ξ-translators.

Theorem 1.2. (1) *Let Σ be a complete ξ-translator invariant by hyperbolic translations. Then, Σ is a totally geodesic plane containing the two points at $\mathbb{H}^3_\infty$ fixed by the hyperbolic translations.*
(2) *There are no spherical rotational ξ-translators.*

At this point, it is natural to consider the group generated by the composition of a spherical rotation and a hyperbolic translation, which leads to the notion of a helicoidal motion and to investigate ξ-translators invariant by this group. In Euclidean space, helicoidal translators of the mean curvature flow were classified in [10]. In hyperbolic space, helicoidal spherical and hyperbolic translators were studied in [15] where, in addition, the authors proved that both families of surfaces coincide. However, we prove that there do not exist ξ-translators of helicoidal type. This extends (2) of Thm. 1.2.

Theorem 1.3. *There are no helicoidal ξ-translators.*

We remark that according to the program established in this paper, the helicoidal motions are the composition of specific spherical rotations and hyperbolic translations.

As a consequence of Thms. 1.2 and 1.3, it remains to study ξ-translators invariant under parabolic translations. After an isometry of $\mathbb{H}^3$, any surface invariant by parabolic translations can be realized as a cylindrical surface of $\mathbb{R}^3_+$ whose rulings are horizontal lines. By analogy with the Euclidean case, we give the following definition.

Definition 1.4. A ξ-grim reaper is a ξ-translator invariant by parabolic translations.

Our first main result is a classification of the ξ-grim reapers. Up to a rotation about the z-axis, which only changes the constants a and b in (1), we suppose that the direction of the rulings of a parabolic surface is $(0,1,0)$. In such a case, the surface can be parametrized by

$$\Psi(s,t) = (x(s), t, z(s)), \quad s \in I \subset \mathbb{R}, t \in \mathbb{R}, \tag{3}$$

where s is the Euclidean arc-length parameter of the planar curve $\alpha(s) = (x(s), 0, z(s))$. The classification of the ξ-grim reapers is given in the following result. See Fig. 1.

Theorem 1.5. *Let be $\xi = a\partial_x + b\partial_y$, $a, b \in \mathbb{R}$, and Σ a complete ξ-grim reaper parametrized by* (3)*.*

(1) *If $a = 0$, then $H = 0$ and Σ is either a vertical plane parallel to ξ, or belongs to a one-parameter family, parametrized in terms of the maximum height to the x-axis. The generating curve of each example is a strictly concave graph on the x-axis and intersecting orthogonally such an axis at two points.*

(2) *If $a \neq 0$, then Σ belongs to a one-parameter family, parametrized in terms of the maximum height to the x-axis. The generating curve of each example is a bi-graph on the x-axis and both graphical components converge to the x-axis as $x \to \infty$.*

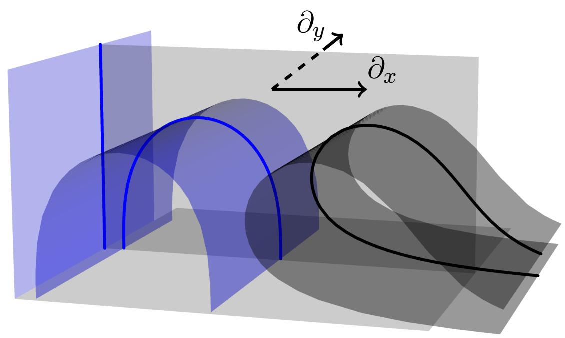

FIGURE 1. The two types of generating curves of ξ-grim reapers. Left: the vector field is $\xi = \partial_y$. Right: the vector field is $\xi = \partial_x + b\partial_y$. The model of $\mathbb{H}^3$ is the half-space model.

The last part of the paper concerns to obtain uniqueness and non-existence results of ξ-translators. For that matter, we take advantage from the fact that locally ξ-translators are solutions to a quasilinear elliptic PDE and the maximum principle applies. From this fact we use the ξ-grim reapers constructed in Thm. 1.5 as barriers to derive the desired results. A first result is inspired by the celebrated half-space theorem for minimal surfaces of Hoffman and Meeks [11], that establishes that planes are the only properly immersed minimal surfaces in $\mathbb{R}^3$ contained in a half-space. We extend this result for ξ-translators of $\mathbb{H}^3$. Since the vector field ξ given in (1) depends on the parameters a and b, and because the ξ-grim reapers theirselves are contained in a half-space, it is expectable that the sign of a or b has to be taken in account to fix what is the vertical half-space that we are considering. On the other hand, it is natural to consider vertical half-spaces because of the definition of the vector field ξ.

A similar situation appears with an analog vertical half-space theorem in the Heisenberg group [6].

Theorem 1.6. *Let $\xi = a\partial_x + b\partial_y$ and $\Pi \subset \mathbb{R}^3_+$ be the vertical plane of equation $x = x_0$. Then there are no properly immersed ξ-translators contained in the vertical half-space*

(1) $\Pi^- = \{(x, y, z) \in \mathbb{R}^3_+ : x < x_0\}$, *if* $a > 0$*;*
(2) $\Pi^+ = \{(x, y, z) \in \mathbb{R}^3_+ : x > x_0\}$, *if* $a < 0$.

This paper is organized as follows. In Sec. 2 we prove Thms. 1.2 and 1.3, while in Sec. 3 we prove Thm. 1.5. It is of special relevance the proof of the case $a \neq 0$, for which we study the qualitative properties of the generating curve of a ξ-grim reaper by means of a phase plane analysis. In Sec. 4, we prove Thm. 1.6 together other results of non-existence. To be precise, we prove non-existence of properly immersed ξ-translators in the convex side of a Killing cylinder (Thm. 4.4), and give conditions on the existence of compact graphical ξ-translators over a domain in a vertical plane (Props. 4.5 and 4.6). Due to the variety of translators and grim reapers that can be defined in $\mathbb{H}^3$, an Appendix has been added to summarize the literature concerning it.

2. Proof of Theorems 1.2 and 1.3

In the upper half-space model of $\mathbb{H}^3$ there is a relation between the mean curvature H of a surface Σ and its Euclidean mean curvature H_e, when Σ is regarded as surface isometrically immersed in $\mathbb{R}^3_+$. This relation is

$$H(x, y, z) = zH_e(x, y, z) + (N^e)_3(x, y, z), \quad (x, y, z) \in \Sigma, \tag{4}$$

where N^e is the Euclidean unit normal of Σ and the subindex $(\cdot)_3$ denotes the third coordinate of the vector.

2.1. **Proof of Theorem 1.2.** We distinguish between hyperbolic and spherical rotational surfaces.

First, assume that Σ is a hyperbolic surface. Consider the group of hyperbolic translations along the vertical geodesic orthogonal to the vector field ξ. Then such a hyperbolic translation is an Euclidean homothety from a point of the ideal boundary of $\mathbb{H}^3$, which can be assumed to be the origin O. Therefore the corresponding group is

$$\mathcal{H} = \{(x, y, z) \mapsto t(x, y, z) : t > 0\}.$$

Consequently, Σ can be parametrized as the radial graph of a curve $\alpha(s) = (x(s), y(s), 1)$, that is,

$$\Psi(s, t) = t\alpha(s) = t(x(s), y(s), 1), \quad s \in I \subset \mathbb{R},\ t \in \mathbb{R}.$$

Suppose that s is the Euclidean arc-length parameter. The Euclidean mean curvature H_e and unit normal N^e are

$$H_e = \frac{|\alpha|_e^2 \langle \alpha' \times \alpha, \alpha'' \rangle_e}{2t(|\alpha|_e^2 - \langle \alpha', \alpha \rangle_e^2)^{3/2}}, \quad N^e = \frac{\alpha' \times \alpha}{(|\alpha|_e^2 - \langle \alpha', \alpha \rangle_e^2)^{1/2}}.$$

Since $N=zN^e$, then $\langle N,\xi\rangle=\frac{1}{t}\langle N^e,\xi\rangle_e$. Thus (2) is

$$\frac{|\alpha|_e^2\langle\alpha'\times\alpha,\alpha''\rangle_e}{2(|\alpha|_e^2-\langle\alpha',\alpha\rangle_e^2)}+(\alpha'\times\alpha)_3=\frac{\langle\alpha'\times\alpha,\xi\rangle_e}{t}. \tag{5}$$

As this equation holds for any $t\in\mathbb{R}$, we deduce that the left hand-side of (5) vanishes identically, that is, $H=0$. The same occurs for the right hand-side of (5), so $\langle\alpha'\times\alpha,\xi\rangle_e=0$, which implies that α is a horizontal line parallel to the vector $(a,b,0)$. This proves that Σ is a plane of $\mathbb{R}^3$ through O. Therefore Σ is a totally geodesic vertical plane if the curve $s\mapsto(x(s),y(s),0)$ passes through O, or Σ is an equidistant plane in other case. Because the mean curvature of an equidistant plane is not zero, the result holds.

Now, we assume that Σ is a spherical surface. As stated in the Introduction, the spherical rotations $\mathcal{S}$ considered are those that leave pointwise fixed a geodesic orthogonal to ξ; we assume such a geodesic to be the z-axis. Thus, the elements of $\mathcal{S}$ are simply Euclidean rotations about the z-axis, being

$$\mathcal{S}=\{(x,y,z)\mapsto(x\cos t-y\sin t,x\sin t+y\cos t,z):t\in\mathbb{R}\}.$$

Thus, a parametrization of a spherical rotational surface Σ is

$$\Psi(s,t)=(x(s)\cos t,x(s)\sin t,z(s)),\quad s\in I\subset\mathbb{R},t\in\mathbb{R},$$

where $\alpha(s)=(x(s),0,z(s))$ is the generating curve. In the sequel we will indistinctly say rotation axis or z-axis. We assume again that s is the Euclidean arc-length parameter, hence $\alpha'(s)=(\cos\theta(s),0,\sin\theta(s))$ for some smooth function $\theta=\theta(s)$. A straightforward computation shows that Eq. (2) writes as

$$\frac{z}{2}\left(\theta'+\frac{z'}{x}\right)+x'=-\frac{z'}{z}(a\cos t+b\sin t).$$

Since the functions $\{1,\cos t,\sin t\}$ are linearly independent and a,b are not both identically zero, we deduce that the left hand-side is zero, that is, $H=0$, and from the right hand-side, we have $z'(s)=0$ for all $s\in I$. This implies that α is a horizontal Euclidean straight line and Σ is a horosphere. However, the mean curvature of a horosphere is $H=1$, obtaining a contradiction.

2.2. **Proof of Theorem 1.3.** The proof is by contradiction. First, we need a suitable parametrization of a helicoidal surface. Here we follow [15, Sect. 4]. Recall that a helicoidal surface about the z-axis is a surface invariant by the one-parameter subgroup of rigid motions of $\mathbb{H}^3$ which are the composition of a rotation around the z-axis with a hyperbolic translation, being such isometries elements of $\mathcal{H}$ and $\mathcal{S}$ as defined in the proof of Thm. 1.2. A parametrization of a helicoidal surface Σ is given by

$$\Psi(s,t)=e^{ht}(e^{it}\alpha(s),1),\quad s\in I\subset\mathbb{R},\ t\in\mathbb{R},$$

where $\alpha(s)=(x(s),y(s),1)$, $s\in I$, is a curve contained in the horosphere $z=1$ and parametrized by arc-length. The constant $h>0$ is called the pitch of the surface. Then the mean curvature H and the unit normal of Σ are given by

$$H = \frac{\rho}{h}\left(h\frac{\kappa((h^2+1)r^2+h^2)-(h\tau-\mu)}{2(h^2+(\tau+h\mu)^2)} - (\tau+h\mu)\right),$$
$$N = \rho\left(e^{it}\mathbf{n}, -\frac{\tau+h\mu}{h})\right).$$

Here κ and $\mathbf{n}$ are the curvature and the unit normal of α, respectively,

$$\tau = \langle \alpha, \alpha'\rangle_e, \quad \mu = \langle \alpha, \mathbf{n}\rangle_e, \quad \rho = \frac{h}{\sqrt{h^2+(\tau+h\mu)^2}},$$

and $r^2 = \tau^2+\mu^2$. Since $\mathbf{n}(s) = -y'(s)\partial_x + x'(s)\partial_y$, it follows

$$\begin{aligned}\langle N, \xi\rangle &= e^{-2ht}\rho\langle e^{it}\mathbf{n}, a\partial_x + b\partial_y\rangle_e \\ &= e^{-2ht}\rho\left((-ay'+bx')\cos t - (ax'+by')\sin t\right).\end{aligned}$$

Thus, equation (2) becomes

$$\frac{H}{\rho}e^{2ht} + (ay'-bx')\cos t + (ax'+by')\sin t = 0.$$

Since the functions $\{e^{2ht}, \cos t, \sin t\}$ are linearly independent, then the coefficients must vanish on its domain $I \subset \mathbb{R}$. In particular, we conclude $H = 0$ and $\langle N, \xi\rangle = 0$. From $\langle N, \xi\rangle = 0$ we deduce that Σ is a vertical plane orthogonal to ξ, which it is also minimal. However, this surface is not helicoidal. This contradiction finishes the proof.

3. Proof of Theorem 1.5

In this section we prove Thm. 1.5 obtaining a full classification of the ξ-grim reapers of $\mathbb{H}^3$. Let Σ be a parabolic surface in $\mathbb{H}^3$. As stated in the Introduction, the group of parabolic translations $\mathcal{P}$ considered are those whose flow of isometries are also horizontal Euclidean translations. After a rotation about the z-axis we can assume that the group is determined by the horizontal direction $(0,1,0) \in \mathbb{R}^3$, that is,

$$\mathcal{P} = \{(x,y,z) \mapsto (x,y,z) + t(0,1,0) : t \in \mathbb{R}\}.$$

Hence, a surface invariant by $\mathcal{P}$ is a ruled surface of $\mathbb{R}^3_+$ whose all rulings are horizontal straight-lines parallel to $(0,1,0)$.

We can parametrize Σ by (3) where $\alpha(s) = (x(s), 0, z(s))$ is parametrized by the Euclidean arc-length. Then $x' = \cos\theta$, $z' = \sin\theta$ for some smooth function $\theta = \theta(s)$. The Euclidean mean curvature and unit normal of Σ are $H_e = \theta'/2$ and $N^e = (-z', 0, x')$. If Σ is a ξ-grim reaper, according to (4), equation (2) is

$$z\frac{\theta'}{2} + x' = -a\frac{z'}{z}. \tag{6}$$

At this point, we distinguish if a vanishes or not.

3.1. **Case** $a = 0$**.** Equation (6) implies $H = 0$. Parabolic surfaces of $\mathbb{H}^3$ with $H = 0$ were classified in [7, 9]. For completeness, we describe these surfaces. If $x' = 0$ at some point, then the solution of (6) is $\theta(s) = 0$, α is a vertical line and Σ is a totally geodesic plane parallel to $(0, 1, 0)$. Suppose now $x'(s) \neq 0$ for all s. Then α is a graph $z = z(x)$ on the x-axis, and (6) is

$$\frac{z''}{1 + z'^2} = -\frac{2}{z}. \tag{7}$$

In particular, the curve α is a strictly concave graph. Multiplying (7) by z' and integrating, there is a positive constant $c > 0$ such that $1 + z'^2 = cz^{-4}$. Hence, it can be deduced that α is symmetric about a vertical line and that α intersects orthogonally the x-axis at two points. See Fig. 1, left. We denote this ξ-grim reaper by $\mathcal{G}^0(z_0)$ indicating the maximum height z_0 of α to the x-axis.

Remark 3.1. Equation (7) appears in the context of singular minimal surfaces. More exactly, solutions of (7) are -2-catenaries following the terminology of [16, Prop. 1] and their shapes are well known. See also [5].

3.2. **Case** $a \neq 0$**.** After a reflection about a vertical plane and a hyperbolic translation we assume $a = 1$, hence $\xi = \partial_x + b\partial_y$. From (6) the following system is fulfilled

$$\begin{cases} x' = \cos\theta \\ z' = \sin\theta \\ \theta' = -\dfrac{2}{z^2}\,(\sin\theta + z\cos\theta). \end{cases} \tag{8}$$

Note that the first equation of (8) can be obtained from the second and third ones, and that the parameter b does not appear. We define

$$\mathcal{R} = \{(z, \theta) \colon z > 0, \theta \in \mathbb{R}\}.$$

Denote the orbits of (8) as $\gamma = (z, \theta)$, indicating the height z and the angle θ. By uniqueness, two distinct orbits in $\mathcal{R}$ cannot intersect, and the Cauchy problem of (8) ensures the existence of an orbit passing through each initial solution $(z_0, \theta_0) \in \mathcal{R}$. We point out that every orbit defines a generating curve α of a ξ-grim reaper and backwards. This correspondence allows to translate some geometric properties of α to the analytic behavior of its corresponding orbit γ, and vice-versa. The following result is straightforward and its proof is omitted.

Proposition 3.2. *The following properties hold.*

(1) *If* $\gamma(s) = (z(s), \theta(s))$ *is an orbit, then* $\overline{\gamma}(s) = (z(-s), \theta(-s) - \pi)$ *is also an orbit. In particular, any orbit can be moved by translations* $(z, \theta) \mapsto (z, \theta + k\pi)$, $k \in \mathbb{Z}$. *Consequently, we will restrict the coordinate* θ *to lie in* $\theta \in (-\pi/2, \pi)$.

(2) *The curve* $\Gamma := \mathcal{R} \cap \{z = \Gamma(\theta)\}$, *where* $\Gamma(\theta) = -\tan\theta$, *corresponds to points of* α *with vanishing Euclidean curvature. It only appears for* $\theta \in (-\pi/2, 0] \cup (\pi/2, \pi]$ *and has the lines* $\theta = -\pi/2$ *and* $\theta = \pi/2$ *as asymptotes.*

(3) *By periodicity, we define the phase plane of* (8) *as*

$$\Theta = \{(z,\theta) \in \mathcal{R} \colon z > 0,\ \theta \in (-\pi/2, -\arctan z)\}. \tag{9}$$

(4) *The lower component of* Γ *and the line* $\theta = 0$ *divide* Θ *into monotonicity regions where each coordinate function of an orbit is strictly monotonous (Fig. 2, left). In particular, we conclude by periodicity and monotonicity that we can restrict the study of the orbits to those that intersect the line* $\theta = 0$.

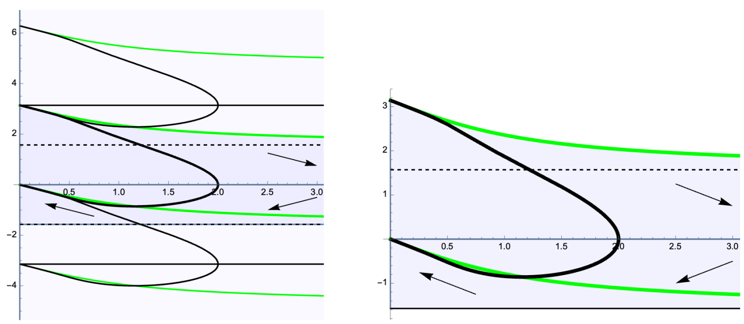

FIGURE 2. Left: the set $\mathcal{R}$ and the same orbit up to discrete translations in the θ-direction. The phase plane Θ has been highlighted in blue. Right: the phase plane and the orbit passing through $(2, 0)$, whose corresponding generating curve α has maximum height 2 and was depicted in Fig. 1, right.

The first result depicts the global behavior of the orbits and of the corresponding generating curves.

Proposition 3.3. *If* $z_0 > 0$ *then there exists a unique orbit* γ_{z_0} *passing through* $(z_0, 0)$, *which converges to* $z = 0$ *as* $|s|$ *increases. The corresponding generating curve* α_{z_0} *converges to* $z = 0$ *and has Euclidean height* z_0.

Proof. Given $z_0 > 0$, consider the solution α_{z_0} of (8) with initial conditions $x(0) = 0$, $z(0) = z_0$ and $\theta(0) = 0$, and let γ_{z_0} denote the corresponding orbit. The initial conditions yield that at $s = 0$ the height function of α_{z_0} attains a local maximum. Indeed, $\theta'(0) = -2/z_0^2 < 0$, $z'(0) = 0$ and for $s > 0$ close enough to $s = 0$ the orbit γ_{z_0} lies in the region $\theta < 0$ and $z > -\tan\theta$, hence $\theta'(s) < 0$. By continuity, γ_{z_0} ends up intersecting Γ at some instant $s_0 > 0$ where $\theta'(s_0) = 0$, hence α_{z_0} has vanishing curvature at $s = s_0$. Then, $\theta'(s) > 0$ for $s > s_0$ since γ_{z_0} lies in the region $z < -\tan\theta$. Consequently, γ_{z_0} ends up converging to the boundary component $z = 0$, and the motion of the orbits in the phase plane forbids γ_{z_0} to converge to $(0, -\pi/2)$. On the other hand, for $s < 0$ close enough to $s = 0$, γ_{z_0} lies in the region $\theta \in (0, \pi/2)$ and when $s < 0$ further decreases it can behave in two ways: either $\gamma_{z_0}(s) \to (0, \theta_*)$ with $\theta_* \in (0, \pi/2]$, or γ_{z_0} intersects the line $\theta = \pi/2$ at some $s_1 < 0$ and then ends up converging to $(0, \theta_*)$ with $\theta_* \in (\pi/2, \pi]$.

In the first case, α_{z_0} is always a graph on the x-axis because $x' = \cos\theta$ never vanishes. In the second case, α_{z_0} fails to be a graph precisely at $s = s_1$ because $x'(s_1) = 0$, but it can be expressed as a vertical bi-graph whose both components are smoothly joined at $x(s_1)$. In both cases, the maximum height of α_{z_0} is z_0 at $z = 0$. □

After this proposition we know that when s diverges, at least one of the ends of any orbit of Θ must end up converging to some $(0, \theta_*)$. The following result restricts the possible values of θ_* and the behavior of the parameter s.

Proposition 3.4. *Let $\gamma(s)$ be an orbit and assume that $\gamma(s) \to (0, \theta_*)$ as $s \to s_*$. Then $\theta_* = 0$ and $s_* = \infty$, or $\theta_* = \pi$ and $s_* = -\infty$.*

Proof. Let γ_{z_0} be the orbit passing through $(z_0, 0)$ at $s = 0$. Arguing by contradiction, assume that γ_{z_0} converges to some $(0, \theta_*)$ with $\theta_* \neq 0, \pi$.

First, assume that s increases from $s = 0$, hence $\theta_* \in (-\pi/2, 0)$. Thus $z'(s) = \sin\theta(s) \to \sin\theta_* < 0$ as $s \to s_* \leq \infty$. In fact, $s_* < \infty$ because otherwise α_{z_0}, which is arc-length parametrized, would eventually cross the line $z = 0$, a contradiction. Therefore, $\alpha_{z_0}(s) \to (x_*, 0)$ as $s \to s_*$, with $x_* = x(s_*)$ and $z(s_*) = 0$. Fix some $\hat{x} \in (0, x_*)$ with $x_* - \hat{x} < 1$ and let $\hat{s}$ such that $\hat{x} = x(\hat{s})$. In the interval $(\hat{s}, s_*)$, we write $\alpha_{z_0}(s)$ as a graph $x \mapsto (x, 0, u(x))$ with $x \in (\hat{x}, x_*)$. Then, Eq. (2) becomes

$$\frac{u''}{1+u'^2} = -\frac{2}{u} - \frac{2u'}{u^2}. \tag{10}$$

Integrating from $\hat{x}$ to x, we have

$$\arctan u'(x) = -2\int_{\hat{x}}^{x} \frac{1}{u(t)}dt + \frac{2}{u(x)} + A(\hat{x}), \qquad A(\hat{x}) = \arctan u'(\hat{x}) - \frac{2}{u(\hat{x})}.$$

By the mean value theorem, there is $c_x \in (\hat{x}, x)$ such that

$$\arctan u'(x) = -\frac{2}{u(c_x)}(x - \hat{x}) + \frac{2}{u(x)} + A(\hat{x}). \tag{11}$$

Now, let $x_n \nearrow x_*$, $x_n \in (\hat{x}, x_*)$ and name $c_n = c_{x_n}$. Bearing in mind that $u(c_n) > u(x_n)$ because u is strictly decreasing, we get from (11)

$$\arctan u'(x_n) = \frac{-2}{u(c_n)}(x_n - \hat{x}) + \frac{2}{u(x_n)} + A(\hat{x}) > \frac{-2(x_n - \hat{x}) + 2}{u(x_n)} + A(\hat{x}).$$

Letting $x_n \to x_*$, we obtain

$$\arctan u'(x_*) > \frac{-2(x_* - \hat{x}) + 2}{u(x_*)} + A(\hat{x}),$$

which is a contradiction since the left-hand side is negative while the right-hand side diverges to ∞. This contradiction ensures that $\gamma_{z_0}(s) \to (0,0)$ as s increases. In fact, $s \to \infty$ since otherwise $s \to s_* < \infty$ and then $\alpha_{z_0}(s) \to (x_*, 0)$, arriving to the same contradiction.

To prove that $\gamma_{z_0}(s) \to (0,\pi)$ as $s \to -\infty$, we follow a similar argument. If $\theta_* \in (0, \pi/2]$ then $x' = \cos\theta \neq 0$, and consequently there is $x_* < 0$ such that α_{z_0} is a graph $x \mapsto (x, 0, u(x))$ for $x \in (x_*, 0)$, with $u''(x) < 0$ and $u'(x) > 0$. Note that if $\theta_* = \pi/2$, then α_{z_0} would intersect orthogonally $z = 0$, failing to be a graph at this intersection point. However, we just need to express α_{z_0} as a graph in the open interval $x \in (x_*, 0)$. An integration of (10) from x to 0 and the mean value theorem for integrals yields

$$\arctan u'(x) = -\frac{2x}{u(c_x)} + \frac{2}{u(x)} - \frac{2}{u(0)}, \qquad c_x \in (x, 0).$$

The left-hand side is a bounded function. However, if $x \to x_*$ the right-hand side diverges to ∞, a contradiction. If $\theta_* \in (\pi/2, \pi)$, now α_{z_0} is a bi-graph whose lower component is again a strictly convex graph that converges to $z = 0$ at a finite point. The contradiction is the same as the one exposed in the proof of $\theta_* \in (-\pi/2, 0)$, concluding that $\gamma_{z_0}(s) \to (0, \pi)$ as $s \to -\infty$. This concludes the proof of Prop. 3.3. □

Now we are in position to prove the case $a \neq 0$ in Thm. 1.5. From Props. 3.3 and 3.4 we know the configuration of any orbit in the phase plane. Given $z_0 > 0$ there is a unique curve α_{z_0} whose Euclidean height is z_0 at $z = 0$ and α_{z_0} can be expressed as a bi-graph whose components converge to $z = 0$ as $x \to \infty$. The corresponding parabolic surface $\mathcal{G}(z_0)$ is a ξ-grim reaper fulfilling all the properties stated in Thm. 1.5. See Fig. 1, right.

Remark 3.5. We finish this section with some observations concerning the proof of Thm. 1.5.

(1) In order to indicate the dependence on a, let $\xi_a = a\partial_x + b\partial_y$ and let $\mathcal{G}^a(z_0)$ be the corresponding ξ_a-grim reaper according to the notation of Thm. 1.5. Fix $a_0 > 0$ and take $z_0 > 0$. Then $\mathcal{G}^a(z_0)$ varies continuously between $\mathcal{G}^0$ and $\mathcal{G}^{a_0}$ for $a \in [0, a_0]$. See Fig. 3.

(2) If $a < 0$ the corresponding ξ-grim reapers have the similar properties, but now they converge to $z = 0$ as $x \to -\infty$.

(3) We know that the generating curve of a ξ-grim reaper is a bi-graph over $z = 0$ and has a point of vertical tangent, p_0, where both bi-graphs are smoothly glued together. This point is unique and it is characterized for having the smallest x-coordinate. This provides an alternative parametrization of the family of ξ-grim reapers by the height of p_0.

4. Results of non-existence of ξ-translators

In this section, we will use the properties of ξ-grim reapers to prove the non-existence of certain ξ-translators. For this, we will employ the maximum

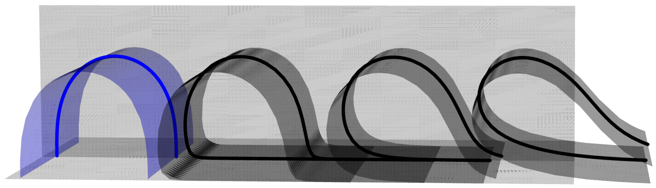

FIGURE 3. The 1-parameter family of ξ_a-grim reapers as $a \in [0, a_0]$. The ends of each $\mathcal{G}^a(z_0)$ tend to be concave graphs converging orthogonally to the ideal boundary $z = 0$.

principle of solutions of elliptic equations for Eq. (2). In contrast to the Euclidean space, it is not known if ξ-translators are minimal surfaces in a weighted space in the sense of Ilmanen [13] (a similar situation occurs for χ-translators, see [15]).

Lemma 4.1 (Tangency principle)**.** *Let Σ_1 and Σ_2 be two connected ξ-translators and assume that they are tangent at some $p \in \Sigma_1 \cap \Sigma_2$, around. If Σ_1 lies at one side of Σ_2, then $\Sigma_1 = \Sigma_2$ in the largest neighborhood of p in $\Sigma_1 \cap \Sigma_2$.*

Proof. If we express locally the two surfaces as graphs of two functions u_1 and u_2 on the (x, y)-plane, the difference function $u = u_1 - u_2$, $u = u(\bar{x})$, satisfies

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = -\frac{1}{u^2\sqrt{1+|Du|^2}}(u + au_x + bu_y).$$

Writing this equation as $\sum a_{ij}(\bar{x}, u, Du)D_{ij}u + \mathbf{b}(\bar{x}, u, Du) = 0$, we have

$$\mathbf{b}(\bar{x}, u, Du) = \frac{1+|Du|^2}{2u^2}(u + au_x + bu_y).$$

Since $\mathbf{b}(\bar{x}, u, Du)$ is non-increasing on the variable u for each fixed $(\bar{x}, Du)$, the maximum principle of quasilinear equations can be applied ([8]). □

Remark 4.2. Let be $\xi_a = a\partial_x + b\partial_y$, denote by $\mathcal{G}^a$ a ξ_a-grim reaper for $a \neq 0$ and $\mathcal{G}^0$ a grim reaper for $a = 0$ (and hence of $H = 0$). In Fig. 1, or Fig. 3, we can move $\mathcal{G}^a$ towards $\mathcal{G}^0$ by a translation in the direction of the x-axis and arrive to an interior tangency point. Nonetheless, $\mathcal{G}^0$ and $\mathcal{G}^a$ are solutions to different PDE's and hence the tangency principle is not violated.

We use the ξ-grim reapers as barriers and the tangency principle to prove Thm. 1.6.

Proof of Theorem 1.6. Without loss of generality, we assume $a > 0$ because the arguments are similar if $a < 0$. After dilations of $\mathbb{H}^3$, let $a = 1$. Consider the plane $\Pi = \{x = x_0\}$ and Π^- the half-space $x < x_0$. The proof is by contradiction, so let Σ be a ξ-translator contained in the vertical half-space

Π^-. Fix some $p_0 \in \Sigma$, take its third coordinate z_0 and bearing in mind Rem. 3.5 (3), consider the ξ-grim reaper $\mathcal{G}$ whose points of vertical tangent planes are at Euclidean height z_0. Consider $\mathcal{G}_t = \mathcal{G} + t(1,0,0)$, which are ξ-translators. By properness, Σ has no boundary, nor accumulate at a finite point, hence Σ can diverge to $z \to \infty$ or converge to $z = 0$, or even both at the same time. By increasing $t \nearrow \infty$ there is some $t_0 > 0$ such that $\mathcal{G}_{t_0}$ is contained in the half-space Π^+. Let $t \searrow -\infty$ until $t = t_1$ such that Σ and $\mathcal{G}_{t_1}$ touch for the first time. The existence of t_1 is assured because the line $\{(x_0 + t, y_0, z_0) \colon t \in \mathbb{R}\}$ intersects necessarily $\mathcal{G}$ at some interior point. The tangency principle implies that $\mathcal{G}_{t_1}$ and Σ agree, which it is a contradiction because $\mathcal{G}_{t_1}$ is not contained in any half-space Π^-. This proves Thm. 1.6. □

As a consequence of this result we find the non-existence of closed (compact and without boundary) ξ-translators.

Corollary 4.3. *There do not exist closed ξ-translators.*

In the following result, we prove non-existence of ξ-translators contained in Killing cylinders. Recall that a Killing cylinder $\mathcal{C}_L$ around a geodesic L is the set of points that lie at a fixed distance of L. The convex side of $\mathcal{C}_L$ is the component of $\mathbb{H}^3 - \mathcal{C}_L$ that contains L.

Theorem 4.4. *If $\xi = a\partial_x + b\partial_y$, then there do not exist properly immersed ξ-translators in $\mathbb{H}^3$ contained in the convex side of a Killing cylinder $\mathcal{C}_L$.*

Proof. By contradiction, suppose that Σ is a properly immersed ξ-translator, contained in the convex side of a Killing cylinder $\mathcal{C}_L$. After a rigid motion of $\mathbb{H}^3$, we can assume that L is the z-axis. In such a case, $\mathcal{C}_L$ is an Euclidean cone with vertex the origin of $\mathbb{R}^3$ and whose axis is the z-line. Its convex side is its Euclidean convex side.

Consider the case $a \neq 0$ in (1). Fix some $z_0 > z_* = \inf_{p\in\Sigma} z(p)$ and let $\mathcal{G}(z_0)$ the ξ-grim reaper given by Thm. 1.5 whose maximum height is z_0. Using a similar notation that in the proof of Thm. 1.6, for t sufficiently large, we have that $\mathcal{G}_t(z_0)$ does not intersect the convex side of $\mathcal{C}_L$. Let now $t \searrow -\infty$. Then there is a first time t_1 of contact between Σ and $\mathcal{G}_{t_1}(z_0)$ at some interior point because Σ is proper and has points with height less than z_0. Consequently the tangency principle implies that $\mathcal{G}_{t_1}(z_0)$ and Σ coincide because both surfaces are complete. This is a contradiction because $\mathcal{G}_{t_1}(z_0)$ is not contained in the convex side of $\mathcal{C}_L$.

Suppose now $a = 0$ in (1). Notice that there must be points $p \in \Sigma$ with $x(p) \neq 0$ since on the contrary, Σ would be the vertical plane $x = 0$, which it is not contained in the convex side of $\mathcal{C}_L$. Let $p^* \in \Sigma$ be a point with $x(p^*) \neq 0$, say $x(p^*) > 0$, and let $x_1 = x(p^*)/2$. Consider the solutions $z_\lambda = z_\lambda(x)$ of (7) where $z(x_1) = z(x_1 + \lambda) = 0$, $\lambda > 0$, in particular, $z'(\frac{x_1+\lambda}{2}) = 0$. If λ is sufficiently small, then the graphic of z_λ is outside $\mathcal{C}_L$, or equivalently, $\mathcal{G}^0(z(x_\lambda)) \cap \mathcal{C}_L = \emptyset$. Letting $\lambda \nearrow \infty$ the graph of z_λ is asymptotic to the vertical plane of equation $x = x_1$ (see Rem. 3.1; also [5]). Since $x(p^*) > x_1$, there is a first time λ_0 such that $\mathcal{G}^0(z(x_{\lambda_0}))$ touches Σ. The tangency principle implies that both surfaces coincide, a contradiction.

□

We finish this paper with further results about non-existence of compact ξ-translators. Recall that any compact ξ-translator must have non-empty boundary in virtue of Cor. 4.3. We distinguish the cases $a \neq 0$ and $a = 0$ in (1). First, assume $a \neq 0$. To simplify the notation, we will assume that $a = 1$.

Proposition 4.5. *Let $\xi = \partial_x + b\partial_y$. If Σ is a ξ-translator, then the coordinate function $x_{|\Sigma}$ cannot attain a local maximum at some interior point. In particular, if Σ is a compact ξ-translator whose boundary lies contained in the plane of equation $x = x_0$, then int(Σ) is contained in the vertical half-space $x < x_0$.*

The same result holds if $a < 0$ but this time $x_{|\Sigma}$ cannot attain a local minima at an interior point.

Proof. The proof is by contradiction. Instead to use the tangency principle comparing with ξ-grim reapers, we only use the ellipticity of Eq. (2). Let $p = (x_0, y_0, z_0)$ be an interior point of Σ such that the function $x_{|\Sigma}$ attains a local maximum. In particular, in a neighborhood of p, the surface Σ is a graph on the plane of equation $x = 0$. Let express Σ as the graph of a function $v = v(y, z)$. A straightforward computation implies that Eq. (2) becomes

$$\text{div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) = \frac{2(1-bv_y)}{z^3\sqrt{1+|Dv|^2}} + \frac{2v_z}{z\sqrt{1+|Dv|^2}}. \tag{12}$$

Since the function $x_{|\Sigma}$ is just the function $v = v(y, z)$, we know that v_y and v_z vanish at (y_0, z_0). Then Eq. (12) is now simply

$$(v_{yy} + v_{zz})(y_0, z_0) = \frac{2}{z_0^3}.$$

However, the fact that v is a local maximum implies $(v_{yy} + v_{zz})(y_0, z_0) \leq 0$ obtaining a contradiction because $z_0 > 0$. □

For the case $a = 0$, that is $\xi = \partial_y$, the above argument does not provide a contradiction because now the right hand-side in (12) is 0. We now employ the tangency principle.

Proposition 4.6. *Let $\xi = \partial_y$ and Σ be a ξ-translator. If the coordinate function $x_{|\Sigma}$ attains a local maximum (or minimum) at some interior point, then Σ is contained in a vertical plane parallel to the plane of equation $x = 0$. In particular, if Σ is a compact ξ-translator whose boundary lies contained in the plane Π of equation $x = x_0$, then $\Sigma \subset \Pi$.*

.

Proof. If $p = (x_0, y_0, z_0)$ be an interior point of Σ such that the function $x_{|\Sigma}$ attains a local maximum (or minimum), then a neighborhood $U \subset \Sigma$ of p is a graph on the vertical plane Π of equation $x = x_0$ and U lies on one side of Π. However Π is another ξ-translator, hence the tangency principle implies $\Sigma \subset \Pi$. □

We point that from Thm. 4.4 or from the first parts of Props. 4.5 and 4.6, we also conclude the non-existence of closed ξ-translators (Cor. 4.3).

5. Appendix: grim reapers in $\mathbb{H}^3$

In hyperbolic space there are different vector fields X to consider in (2) as well as the corresponding notions of grim reapers. Due to this variety of situations, in this section we summarize the recent developments achieved in the literature about grim reapers of $\mathbb{H}^3$. In this paper we have studied the case when X is a Killing vector field whose flow of isometries are parabolic translations. When the flow of isometries are spherical rotations and hyperbolic translations, the study was carried out in [15]. Another vector fields of interest are the conformal Killing vector fields, as for example the vector fields ∂_z, [3], and $-\partial_z$, [18]. These vector fields are of interest because ∂_z-translators and $-\partial_z$-translators are the analogs of the Euclidean self-shrinkers and self-expanders, respectively. In contrast to the vector fields ξ and χ, ∂_z-translators and $-\partial_z$-translators are minimal surfaces in a density space in the sense of Ilmanen [13].

To precise the notion of grim reaper, a X-grim reaper is said to be a X-translator if it is invariant by parabolic or hyperbolic translations. To simplify the notation, we write (P, X)-grim reaper or (H, X)-grim reaper, respectively. We summarize all types of grim reapers in $\mathbb{H}^3$.

Theorem 5.1. *The X-grim reapers in $\mathbb{H}^3$ are the following.*

(1) *(H, ξ)-grim reapers and (P, ξ)-grim reapers have been classified in this paper: see Thms. 1.2 and 1.5 respectively.*
(2) *If χ is the Killing vector field whose flow of isometries is formed by spherical rotations or hyperbolic translations, then (H, χ)-grim reapers and (P, χ)-grim reapers are classified in* [15].
(3) *(H, ∂_z)-grim reapers and (P, ∂_z)-grim reapers are classified in* [3].
(4) *$(P, -\partial_z)$-grim reapers are classified in* [3] *and $(H, -\partial_z)$-grim reapers in* [18].

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References

[1] Andrews, B., Chen, X.: Curvature flow in hyperbolic spaces. J. Reine Angew. Math. 729, 29–49 (2017)
[2] Bueno, A.: Translating solitons of the mean curvature flow in the space $\mathbb{H}^2 \times \mathbb{R}$, J. Geom. 109 (2018).
[3] Bueno, A., López, R.: Horo-shrinkers in hyperbolic space. arXiv:2402.05527 [math.DG] (2024)
[4] Cabezas-Rivas, E., Miquel, V.: Volume preserving mean curvature flow in the hyperbolic space. Indiana Univ. Math. J. 56, 2061–2086 (2007)
[5] Dierkes, U., López, R.: Cylindrical singular minimal surfaces. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. (2023)
[6] Daniel, B., Hauswirth, L.: Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group. Proc. Lond. Math. Soc. 98, 445–470 (2009)
[7] Do Carmo, M. Dajczer, M.: Rotation hypersurfaces in spaces of constant curvature. Trans. Amer. Math. Soc. 277, 685–709 (1983)
[8] Gilbarg, D., Trudinger, N. S.: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, Heidelberg (1977)
[9] Gomes, J. M.: Sobre hypersurficies com curvature média constante no espaǫ hiperbólico. Informes de matemática. Instituto de Matemática Pura e Aplicada. Série F-011-Jun/85. Rio de Janeiro (1985)
[10] Halldorsson, H. P.: Helicoidal surfaces rotating/translating under the mean curvature flow. Geom. Dedicata 162, 45–65 (2013)
[11] Hoffman, D., Meeks III, W. H.: The strong halfspace theorem for minimal surfaces. Invent. Math. 101, 373–377 (1990).
[12] Huisken, G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. 183, 45–70 (1999)
[13] Ilmanen, T.:Elliptic regularization and partial regularity for motion by mean curvature. Mem. Amer. Math. Soc. 108 (1994)
[14] M. A. Lawn, M. Ortega, Translating Solitons in a Lorentzian Setting, Submersions and Cohomogeneity One Actions, *Mediterr. J. Math.* **19** (2022)
[15] de Lima, R. F., Ramos, A. K., dos Santos, J. P.: Solitons to mean curvature flow in the hyperbolic 3-space. arXiv:2307.14136 [math.DG] (2023)
[16] López, R.: Invariant singular minimal surfaces. Ann. Glob. Anal. Geom. 53, 521–541 (2018)
[17] R. López, M. Munteanu, Solitons of the mean curvature flow in $\mathbb{S}^2 \times \mathbb{R}$. Differ. Geom. Appl. 99, 102243, (2025).
[18] Mari, L., Rocha de Oliveira, J., Savas-Halilaj, A., Sodré de Sena, R.: Conformal solitons for the mean curvature flow in hyperbolic space. arXiv:2307.05088 [math.DG] (2023)

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