The Björling problem for prescribed mean curvature surfaces in $\mathbb{R}^3$

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Abstract

In this paper we prove existence and uniqueness of the Björling problem for the class of immersed surfaces in $\mathbb{R}^3$ whose mean curvature is given as an analytic function depending on its Gauss map. As an application, we prove the existence of surfaces with the topology of a Möbius strip for an arbitrary large class of prescribed functions. In particular, we use the Björling problem to construct the first known examples of self-translating solitons of the mean curvature flow with the topology of a Möbius strip in $\mathbb{R}^3$.

1 Introduction

A classical problem in minimal surface theory in $\mathbb{R}^3$ is the *Björling problem* [Bjo]. This problem was posed in 1844 by Björling and asks the following:

> *Given a regular analytic curve $\beta(s)$ in $\mathbb{R}^3$ and an analytic distribution of oriented planes $\Pi(s)$ along $\beta(s)$ such that $\beta'(s) \in \Pi(s)$, find* all *minimal surfaces in $\mathbb{R}^3$ containing $\beta(s)$ and such that the tangent plane distribution along $\beta(s)$ is given by $\Pi(s)$.*

In 1890 Schwarz [Sch] solved this problem via an integral representation formula, using holomorphic data. This problem can be applied in other generic situations: for instance, to study surfaces with certain symmetries [DHKW], and to solve global problems in the minimal surface theory [AlMi, GaMi1]; see also [ACG, GaMi2, GaMi3] and

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references therein for an outline on the development of the *geometric Cauchy problem*. The Björling problem has been also studied when the mean curvature is a non-vanishing constant, see [BrDo].

Our objective in this paper is to prove existence and uniqueness of the Björling problem for a certain class of *prescribed mean curvature surfaces* immersed in $\mathbb{R}^3$. Specifically, let be Σ an oriented, immersed surface in $\mathbb{R}^3$ and $\mathcal{H} \in C^1(\mathbb{S}^2)$. We say that Σ has *prescribed mean curvature* $\mathcal{H}$ (in short, Σ is an $\mathcal{H}$-surface) if the mean curvature H_Σ of Σ satisfies

$$H_\Sigma(p) = \mathcal{H}(\eta_p), \qquad \forall p \in \Sigma, \tag{1.1}$$

where $\eta : \Sigma \to \mathbb{S}^2$ is the *Gauss map* of Σ.

In general, the study of hypersurfaces in $\mathbb{R}^{n+1}$ defined by a prescribed curvature function in terms of the Gauss map goes back, at least, to the famous Christoffel and Minkowski problems for ovaloids, see e.g. [Chr]. In particular, the existence and uniqueness of ovaloids with prescribed mean curvature (1.1) was studied among others by Alexandrov and Pogorelov [Ale, Pog] but the global geometry of these surfaces remained largely unexplored. In [GaMi4] the authors studied uniqueness of immersed $\mathcal{H}$-spheres obtaining a Hopf-type theorem for this class of immersed surfaces, and in the making they proved a standing conjecture by Alexandrov. The global properties of $\mathcal{H}$-hypersurfaces immersed in $\mathbb{R}^{n+1}$ have been recently developed in [BGM], where the authors studied several topics such as classification of rotational $\mathcal{H}$-hypersurfaces, a priori height estimates and a structure theorem for properly embedded $\mathcal{H}$-surfaces in $\mathbb{R}^3$, and curvature estimates for stable $\mathcal{H}$-surfaces immersed in $\mathbb{R}^3$.

This paper is organized as follows: in **Section 2** we prove existence and uniqueness of the Björling problem for the class of analytic functions $\mathcal{H} \in C^\omega(\mathbb{S}^2)$ by applying Cauchy-Kovalevskaya theorem. This theorem has been used in other works to prove existence and uniqueness of the Björling problem for minimal surfaces in three-dimensional Riemannian and Lorentzian Lie groups via a Weierstrass-type representation formula, see e.g. [CMO, MMP, MeOn].

In **Section 3** we restrict ourselves to the class of analytic functions $\mathcal{H} \in C^\omega(\mathbb{S}^2)$ satisfying the symmetry condition $\mathcal{H}(-x) = -\mathcal{H}(x)$, $\forall x \in \mathbb{S}^2$. This condition on $\mathcal{H}$ ensures us that Equation (1.1) is independent of the orientation chosen on the $\mathcal{H}$-surface. Bearing this in mind, we use the existence and uniqueness of the Björling problem for adequate Björling data to construct non-orientable $\mathcal{H}$-surfaces with the topology of a Möbius strip. A particular analytic function with this symmetry condition is the one given by $\mathcal{H}(x) = \langle x, e_3 \rangle$, $\forall x \in \mathbb{S}^2$. The $\mathcal{H}$-surfaces arising for this prescribed function are the *self-translating solitons of the mean curvature flow* in $\mathbb{R}^3$, a well studied class of surfaces in the past decades. See e.g. [CSS, Hui, MSHS] and references therein for relevant works regarding this topic. As an application, we construct self-translating solitons of the mean curvature flow in $\mathbb{R}^3$ with the topology of a *Möbius strip*. After a detailed search in the literature, we can assert that these *translating Möbius strips* are the first known examples of self-translating solitons with non-orientable topology.

Finally, in **Section 4** we use the solution of the Björling problem to construct further examples of $\mathcal{H}$-surfaces for analytic prescribed functions. In particular, we construct *helicoidal* $\mathcal{H}$-surfaces, and $\mathcal{H}$-surfaces similar to Enneper's classical minimal surface.

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2 The Björling problem for $\mathcal{H}$-surfaces

In this paper we will denote by $C^{\omega}(\mathbb{S}^2)$ to the class of *analytic functions* defined on the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

Definition 1 *Let be $I \subset \mathbb{R}$ an open interval. A pair of* Björling data *for $\mathcal{H}$-surfaces in $\mathbb{R}^3$ is a regular analytic curve $\beta : I \to \mathbb{R}^3$ and an analytic vector field $B : I \to \mathbb{R}^3$ along $\beta(s)$ such that $|\beta'(s)| - |B(s)| = \langle \beta'(s), B(s)\rangle = 0,\ \forall s \in I$.*

From this definition, we get two obvious consequences: First, given $\mathcal{H} \in C^{\omega}(\mathbb{S}^2)$, a regular analytic curve $\beta(s)$ and an oriented distribution of planes $\Pi(s)$ along $\beta(s)$, we get a pair of Björling data by just defining $B(s) = J\beta'(s)$, where J denotes the $\pi/2$-rotation in the tangent plane $\Pi(s)$. And second, let us denote by ∇ to the Riemannian connection of Σ and suppose that $\beta(s)$ is parametrized by arc-length. Moreover, suppose that $\beta(s)$ is not a straight line. Then there exists $s_0 \in \mathbb{R}$ such that $|\beta''(s_0)| \neq 0$. Let I_0 be the largest subinterval of I containing s_0 such that $|\beta''(s)| \neq 0$ for all $s \in I_0$. If we define $V(s) := \nabla_{\beta'(s)}\beta'(s)$ then $B(s) = V(s)/|V(s)|$ is an analytic unit vector field along $\beta(s)$ such that $\langle \beta'(s), B(s)\rangle = 0$. Thus, $B(s)$ determines an oriented distribution of planes $\Pi(s)$ along $\beta(s)$ by defining $\Pi(s) = \beta(s) + B(s)^{\perp}$. In particular, the Björling problem generalizes the problem of finding a surface which contains a given curve as a geodesic.

Although we do not have a Weierstrass representation for $\mathcal{H}$-surfaces, and thus we cannot solve the Björling problem with an *explicit* integral representation formula just as in the case $\mathcal{H} = 0$ (see e.g. [Mir]), we can prove existence and uniqueness of this problem by applying different methods, as other have done in similar situations; see e.g. [CMO, MeOn].

Let Σ be an orientable Riemannian surface and let $\psi : \Sigma \to \mathbb{R}^3$ be an isometric immersion of Σ in $\mathbb{R}^3$. Then, it is well known that the coordinates of ψ satisfy the elliptic PDE

$$\Delta_{\Sigma}\psi = 2H_{\Sigma}\eta, \tag{2.1}$$

where Δ_{Σ} stands for the Laplace-Beltrami operator on Σ, η denotes the Gauss map of Σ and H_{Σ} is the mean curvature of Σ computed with respect to η.

Recall that Σ inherits a Riemann surface structure induced by its first fundamental form, and thus we can consider a conformal coordinate $z = s + it$ defined in a simply connected domain $\Omega \subset \mathbb{C}$, and we define the usual *Wirtinger operators* $\partial_z = 1/2(\partial_s - i\partial_t)$, $\partial_{\bar{z}} = 1/2(\partial_s + i\partial_t)$. Then, the induced metric on Σ is expressed as $\langle \cdot, \cdot \rangle = \lambda^2 |dz|^2$, where $|dz|^2$ is the flat metric on Ω and $\lambda^2 = \langle \partial_s, \partial_s \rangle = \langle \partial_t, \partial_t \rangle > 0$ is the *conformal factor*. For such a conformal coordinate, the operator Δ_Σ writes as

$$\Delta_\Sigma = \frac{1}{\lambda^2}\Delta_0 = \frac{1}{\lambda^2}(\partial_{ss} + \partial_{tt}) = \frac{4}{\lambda^2}\partial_{z\bar{z}}, \tag{2.2}$$

where Δ_0 denotes the usual flat Laplacian, and we used the relation of the Laplace-Beltrami operator between two conformal metrics. On the other hand, the Gauss map η of Σ has the following expression with respect to z:

$$\eta = \frac{2}{i} \frac{\psi_z \wedge \psi_{\bar{z}}}{\sqrt{\langle \psi_z \wedge \psi_{\bar{z}}, \psi_z \wedge \psi_{\bar{z}} \rangle}} = \frac{2}{i\lambda^2} \psi_z \wedge \psi_{\bar{z}}. \tag{2.3}$$

Suppose now that the immersion $\psi : \Sigma \to \mathbb{R}^3$ defines an $\mathcal{H}$-surface for some $\mathcal{H} \in C^\omega(\mathbb{S}^2)$. By Equations (2.2) and (2.3), Equation (2.1) writes as

$$\psi_{z\bar{z}} = -i\mathcal{H}(\eta)\psi_z \wedge \psi_{\bar{z}}. \tag{2.4}$$

Viewing the immersion in coordinates $\psi = (\psi_1, \psi_2, \psi_3)$, then Equation (2.4) can be seen as a system of partial differential equations. In this setting, we can prove existence and uniqueness of the Björling problem for the class of $\mathcal{H}$-surfaces, as stated next:

Theorem 2 (Björling problem for $\mathcal{H}$-surfaces) *Let be $\mathcal{H} \in C^\omega(\mathbb{S}^2)$ and $\beta(s), B(s)$ a pair of Björling data defined on a real interval I. Then, there exists an open domain $\Omega \subset \mathbb{C}$ containing $I \times \{0\}$ and a conformal immersion $\psi : \Omega \to \mathbb{R}^3$ that solves the following system*

$$\begin{cases} \psi_{z\bar{z}} = -i\mathcal{H}(\eta)\psi_z \wedge \psi_{\bar{z}}, \\ \psi(s,0) = \beta(s), \\ \psi_t(s,0) = B(s). \end{cases} \tag{2.5}$$

As a consequence, ψ defines an $\mathcal{H}$-surface Σ that contains the curve $\beta(s)$, and the tangent plane distribution $T_{\beta(s)}\Sigma$ at each point $\beta(s) \in \Sigma$ is spanned by the vectors $\beta'(s)$ and $B(s)$.

Proof: The system (2.5) is elliptic and of *Cauchy-Kovalevskaya* type, and thus it has local existence and uniqueness; see [Pet] for a proof of Cauchy-Kovalevskaya theorem. As the system (2.5) is elliptic without characteristic points, we have that the existence and uniqueness extends to the whole interval I where $\beta(s)$ and $B(s)$ are defined. Thus, there exist $\delta > 0$ and functions (ψ_1, ψ_2, ψ_3) defined in $I \times (-\delta, \delta) \subset \mathbb{C}$ such that $\psi = (\psi_1, \psi_2, \psi_3)$ is a solution of (2.4) satisfying $\psi(s,0) = \beta(s), \ \psi_t(s,0) = B(s), \ \forall s \in I$.

First, observe that equation $\langle \psi_{z\bar{z}}, \psi_z \rangle = 0$ holds by substituting $\psi_{z\bar{z}}$ for its expression (2.4), and thus the function $\langle \psi_z, \psi_z \rangle$ is holomophic. As ψ satisfies the described initial conditions, the function $\langle \psi_z, \psi_z \rangle$ evaluated at $(s, 0)$ is equal to $|\beta'(s)|^2 - |B(s)|^2 -$

$2i\langle\beta'(s), B(s)\rangle$. This expression vanishes identically, as $\beta'(s)$ and $B(s)$ are orthogonal vector fields with the same length. In this conditions, $\langle\psi_z, \psi_z\rangle$ is a holomorphic function vanishing at the real axis and by the identity principle of holomorphic functions, $\langle\psi_z, \psi_z\rangle$ is identically zero in $I \times (-\delta, \delta)$. We conclude that the map $\psi : I \times (-\delta, \delta) \to \mathbb{R}^3$ is conformal.

By Equation (2.5), the regularity of $\beta(s)$, and the orthogonality of $\beta'(s)$ and $B(s)$, it is clear that ψ defines an immersion on an open set $\Omega \subset \mathbb{C}$ containing $I \times \{0\}$. Thus, ψ is a conformal immersion of an $\mathcal{H}$-surface.

This concludes the proof of Theorem 2.

□

Remark 3 *As we pointed out, in general system* (2.5) *cannot be explicitly integrable, as happens when* $\mathcal{H} = 0$. *However, we can numerically solve it for producing images of* $\mathcal{H}$*-surfaces in* $\mathbb{R}^3$. *Indeed, let us denote by* $\psi(s,t) = (\psi_1(s,t), \psi_2(s,t), \psi_3(s,t))$. *Given* $\mathcal{H} \in C^\omega(\mathbb{S}^2)$, $\beta(s), B(s) : [s_0, s_1] \to \mathbb{R}^3$ *a pair of Björling data and* $\delta > 0$, *the solution of the Björling problem can be plotted using standard software with the command*

```
ParametricPlot3D[
Evaluate[
First[{ψ[s,t]}/.
NDSolve[{
D[ψ[s,t],s,s] + D[ψ[s,t],t,t] == 2ℋ[η[s,t]]D[ψ[s,t],s] ∧ D[ψ[s,t],t],
Thread[ψ[s,0] == β[s], D[ψ[s,t],t][s,0] == B[s]]},
{ψ1,ψ2,ψ3},{s,s0,s1},{t,−δ,δ}]]],{s,s0,s1},{t,−δ,δ}].
```

For example, consider the analytic function $\mathcal{H}(x) = \langle x, e_3\rangle$, $\forall x \in \mathbb{S}^2$. The $\mathcal{H}$-surfaces arising for this prescribed function are the *self-translating solitons of the mean curvature flow*. The rotational self-translating solitons of the mean curvature flow are classified as follows: an entire, strictly convex vertical graph that intersects the axis of rotation orthogonally, called the *bowl soliton*; and a one parameter family of properly embedded annuli, with both ends pointing towards the e_3 direction, called the *wing-like solitons* or *translating catenoids*. The family of wing-like solitons are parametrized by the *neck size*, i.e. the minimum distance to the axis of rotation, attained at a circumference of radius equal to the *waist*, see [CSS] for details.

Bearing this in mind, we can recover this family by choosing adequate Björling data. Indeed, consider the one parameter family of Björling data $\beta_\tau(s) = \tau(\cos s, \sin s, 0)$ and $B_\tau(s) = (0, 0, \tau)$, $\forall s \in \mathbb{R}$, $\tau > 0$. Then, for each fixed $\tau > 0$, the translating soliton Σ_τ generated by this Björling data corresponds to the wing-like soliton with neck size equal to τ. When $\tau \to 0$, the sequence Σ_τ converges smoothly to a double cover of the bowl soliton minus the vertex. See Figure 1 for a plot of the wing-like soliton with neck size equal to 1.

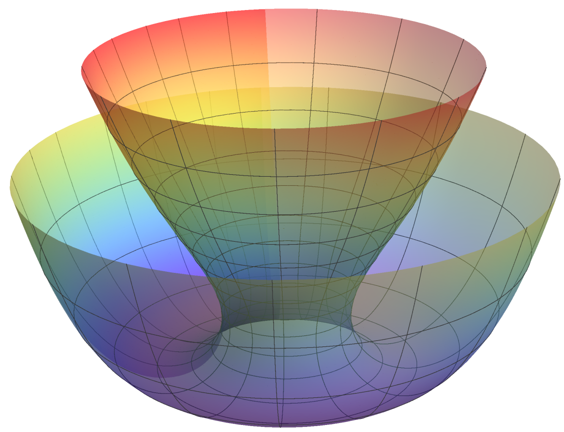

Figure 1: A wing-like soliton for the Björling data $\beta(s) = (\cos s, \sin s, 0)$ and $B(s) = (0, 0, 1)$, $s \in \mathbb{R}$.

3 $\mathcal{H}$-Möbius strips in $\mathbb{R}^3$

The Björling problem adds a large amount of $\mathcal{H}$-surfaces to the ones studied in [BGM], for any given $\mathcal{H} \in C^\omega(\mathbb{S}^2)$. By choosing adequate Björling data, the examples arising may have some prescribed symmetries, as well as some topological properties. In this context, the Björling problem motivates us for the search of *non-orientable* surfaces. However, we must recall that our $\mathcal{H}$-surfaces are supposed to be oriented, in virtue of Definition 1.1, and thus the concept of non-orientable $\mathcal{H}$-surface makes no sense for an arbitrary function $\mathcal{H}$. For instance, if $\mathcal{H}$ is chosen as a positive constant, examples with non-orientable topology do not exist. Thus, some condition on the prescribed function $\mathcal{H} \in C^\omega(\mathbb{S}^2)$ must be imposed in order to make Definition 1.1 independent on the orientation chosen on the $\mathcal{H}$-surface. Bearing this in mind, the mildest hypothesis is that $\mathcal{H}$ has to be *antipodally antisymmetric*, that is $\mathcal{H}(-x) = -\mathcal{H}(x)$ for all $x \in \mathbb{S}^2$.

Going back to the propose of studying non-orientable $\mathcal{H}$-surfaces, the most recurrent examples are the surfaces with the topology of a *Möbius strip*; we advise [Mee, Mir] as two relevant works regarding minimal Möbius strips in $\mathbb{R}^n$ given as the solution of the Björling problem. The following result is inspired in the ideas developed in [Mir].

Proposition 4 *Let $\mathcal{H} \in C^\omega(\mathbb{S}^2)$ such that $\mathcal{H}(-x) = -\mathcal{H}(x)$, $\forall x \in \mathbb{S}^2$, and let $\beta(s), B(s)$ be Björling data such that $\beta(s)$ is T-periodic and $B(s)$ is T-antiperiodic, i.e. $B(s+T) = -B(s)$, for some $T > 0$.*

Then, the $\mathcal{H}$-surface generated by Theorem 2 for this Björling data has the topology of a Möbius strip, with fundamental group generated by $\beta(s)$.

*Conversely, every $\mathcal{H}$-*Möbius strip *is generated in this way.*

Proof: We will give a sketch of the proof, since it is an adaptation of the one used to prove Lemma 3 in [Mir].

First, let $\psi : \mathcal{M} \to \mathbb{R}^3$ be an immersion of an $\mathcal{H}$-Möbius strip, and let $\Gamma(s)$ be a regular, analytic, closed curve in $\mathcal{M}$ that generates its fundamental group. As $\Gamma(s)$ is closed, it is T-periodic for some $T > 0$. Denote by $\widetilde{\mathcal{M}}$ to its *two sheeted cover*, where we have defined an antiholomorphic involution I without fixed points, and let $\pi : \widetilde{\mathcal{M}} \to \mathbb{R}^3$ be the canonical projection. In this setting, there exists a regular, analytic, closed curve $\widetilde{\Gamma}(s)$ that generates the fundamental group of $\widetilde{\mathcal{M}}$, defined by $\Gamma(s) = \pi(\widetilde{\Gamma}(s))$, and in particular it is $2T$-periodic.

If we consider $\widetilde{\psi} : \widetilde{\mathcal{M}} \to \mathbb{R}^3$ given by $\widetilde{\psi} = \psi \circ \pi$, then $\beta(s) = \widetilde{\psi}(\widetilde{\Gamma}(s))$ is a regular, analytic T-periodic curve in $\mathbb{R}^3$. Denoting by $\Pi(s)$ the oriented tangent plane distribution of $\widetilde{\psi}$ along $\beta(s)$ we have that $\Pi(s+T)$ and $\Pi(s)$ agree with opposite orientation. Therefore, if $B(s) := J\beta'(s)$ is the $\pi/2$ rotation of $\beta'(s)$ in $\Pi(s)$, it happens that $B(s+T) = -B(s)$. Thus, we have proved that every $\mathcal{H}$-Möbius strip defines a pair of Björling data with the periodicity properties stated.

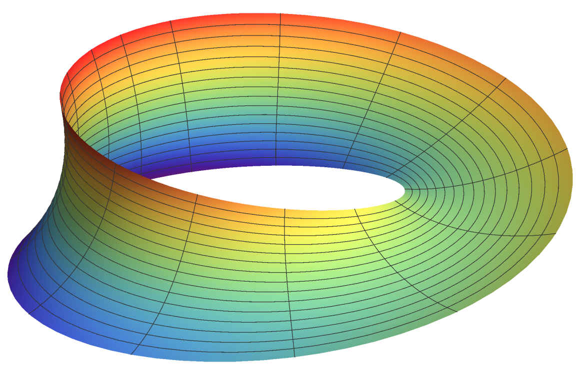

Figure 2: A self-translating soliton of the mean curvature flow with the topology of a Möbius stirp, generated by the Björling data $\beta(s) = 1/2(\cos 2s, \sin 2s, 0)$, $B(s) = \cos s(\cos s, \sin s, 0) + \sin s(0, 0, 1)$, $\forall s \in \mathbb{R}$.

Conversely, let $\beta(s)$ and $B(s)$ be a pair of Björling data such that $\beta(s)$ is T-periodic and $B(s)$ is T-antiperiodic. In particular, they are defined on the entire real line. Let $\psi : \Omega \to \mathbb{R}$ be the solution of this Björling problem given by Theorem 2. The $2T$-periodicity of $\beta(s)$ and $B(s)$ along with the uniqueness of the Björling problem, ensures us that ψ is well defined on the quotient $\Omega/(2T\mathbb{Z})$, which is a topological cylinder.

We can suppose that Ω is symmetric with respect to the conjugation, i.e. $\overline{z} \in \Omega$, $\forall z \in$

Ω, and thus we can define on $\Omega/(2T\mathbb{Z})$ the map

$$\begin{array}{rccc} I & : \Omega/(2T\mathbb{Z}) & \longrightarrow & \Omega/(2T\mathbb{Z}) \\ & z & \longmapsto & \overline{z}+T. \end{array} \tag{3.1}$$

It is also clear that I is an antiholomorphic involution without fixed points, and thus it reverses the orientation of $\Omega/(2T\mathbb{Z})$. Moreover, I defines the following equivalence relation on $\Omega/(2T\mathbb{Z})$: two points $z, w \in \Omega/(2T\mathbb{Z})$ are related if and only if $I(z) = w$. In this situation, the cylinder $\Omega/(2T\mathbb{Z})$ is the orientable two sheeted cover of the space $(\Omega/(2T\mathbb{Z}), I)$, with the canonical projection $\mathfrak{p} : \Omega/(2T\mathbb{Z}) \to (\Omega/(2T\mathbb{Z}), I)$.

Because ψ is an immersion, the unitary vector field $\eta : \Omega \to \mathbb{R}^3$ as defined in (2.3) is a well defined, unitary, normal vector field for ψ. Given $z, w \in \Omega/(2T\mathbb{Z})$ such that $w = I(z)$, the uniqueness of the Björling problem implies that the unit normals $\eta(z)$ and $\eta(w)$ are opposite. Bearing this in mind, we have:

$$H_\psi(\psi(w)) = \mathcal{H}(\eta(w)) = \mathcal{H}(\eta(I(z))) = \mathcal{H}(-\eta(z)) = -\mathcal{H}(\eta(z)) = -H_\psi(\psi(z)), \tag{3.2}$$

and thus the mean curvature at the point $\psi(z)$ has opposite sign to the mean curvature at the point $\psi(I(z))$, for all $z \in \Omega$.

Again, the uniqueness of Theorem 2 allows us to conclude that the quotient map $\widetilde{\psi}(z) = (\psi \circ \mathfrak{p})(z)$ for all $z \in \Omega/(2T\mathbb{Z})$ is a well defined, conformal immersion of an $\mathcal{H}$-surface in $\mathbb{R}^3$ having the topology of a Möbius strip, and in particular is non-orientable. This concludes the proof of Proposition 4.

□

Note that the function $\mathcal{H}(x) = \langle x, e_3\rangle,\ \forall x \in \mathbb{S}^2$ lies in the hypothesis of Proposition 4. Thus, we ensure the existence of self-translating solitons of the mean curvature flow with the topology of a Möbius strip, which we will refer to as *translating Möbius strips*, see Figure 2. After a detailed search in the literature, we can assert that this construction gives the first example of a self-translating soliton of the mean curvature flow with non-orientable topology.

In Figure 3 we show the construction of a non-orientable translating soliton constructed by half-rotating the vector field $B(s)$ 7 times along the curve $\beta(s)$ before it closes. This surface is homeomorphic to the Möbius strip showed in Figure 2. When the mean curvature vanishes, the minimal surfaces as the one appearing in Figure 3 were firstly constructed in [Mir], see also the independent work of [MeWe]. These surfaces are commonly known as *bended helicoids*.

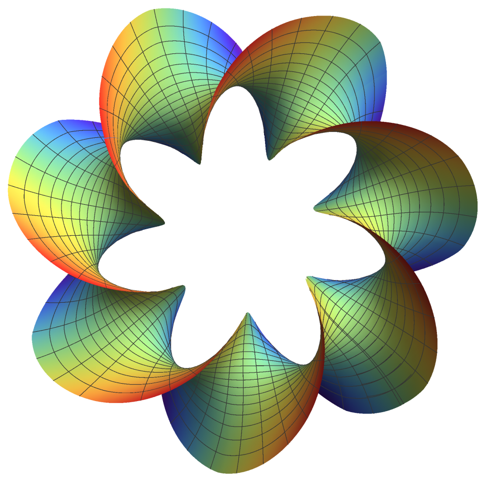

Figure 3: An $\mathcal{H}$-bended helicoid with the topology of a Möbius strip for the analytic function $\mathcal{H}(x) = \langle x, e_3 \rangle,\ x \in \mathbb{S}^2$.

4 Further examples of $\mathcal{H}$-surfaces via the Björling problem

In this last Section we show the existence of some examples of $\mathcal{H}$-surfaces immersed in $\mathbb{R}^3$, motivated by the analogous examples defined in the minimal surface theory. The main difference with the minimal case $\mathcal{H} = 0$ in $\mathbb{R}^3$ is that we fail to have a Weierstrass representation, and thus explicit parametrizations of these surfaces are not expected. Even so, we can prove existence and uniqueness by means of Theorem 2 for adequate Björling data with some prescribed symmetries, and obtain $\mathcal{H}$-surfaces that are the analogous to the famous examples in the minimal surface theory.

$\mathcal{H}$-Helicoids

We choose as Björling data the vertical curve $\beta(s) = (0, 0, s)$ and a T-periodic, analytic, unitary vector field $B(s)$ along $\beta(s)$, for some $T > 0$, and let Σ be $\mathcal{H}$-surface given as the solution of the Björling problem for this Björling data.

The unit normal vector field at the e_3-axis, namely $\eta(s) := \beta'(s) \wedge B(s)$, is a horizontal vector field satisfying $\eta(s) = \eta(s+T)$, $\forall s \in \mathbb{R}$, and the Björling data $\beta(s+T), B(s+T)$ agree with the Björling data $\beta(s) + Te_3, B(s)$. Moreover, as the condition $\eta(s) = \eta(s+T)$, $\forall s \in \mathbb{R}$ holds, the points $\beta(s)$ and $\beta(s+T)$ have the same mean curvature.

Bearing this in mind, if we denote by $\Theta(p) = p + Te_3,\ \forall p \in \mathbb{R}^3$, the uniqueness of the Björling problem ensures us that Σ is invariant by the discrete group of translations $T\mathbb{Z}\Theta$ in the e_3-direction. Moreover, starting at some $s_0 \in \mathbb{R}$, Σ *twists* jointly with $\eta(s)$ around the e_3-axis until reaching the instant $\eta(s_0 + T) = \eta(s_0)$, generating a simply connected *fundamental part* of Σ. Repeating this process, which is just translating this fundamental part of Σ under the action of Θ, we get the whole $\mathcal{H}$-surface Σ.

We will refer to these $\mathcal{H}$-surfaces as $\mathcal{H}$-helicoids, since they generalize the usual minimal helicoids in $\mathbb{R}^3$. See Figure 4 for a plot of an $\mathcal{H}$-helicoid for the particular function $\mathcal{H}(x) = \langle x, e_3 \rangle$, and the Björling data $\beta(s) = (0, 0, s)$, $B(s) = (\cos s, \sin s, 0)$.

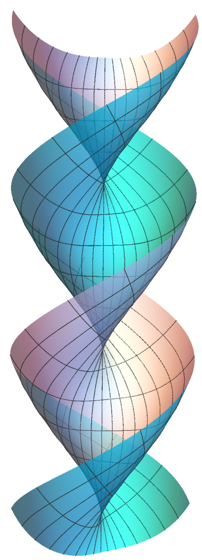

Figure 4: An $\mathcal{H}$-helicoid for the analytic function $\mathcal{H}(x) = \langle x, e_3 \rangle$, $x \in \mathbb{S}^2$.

Enneper-type $\mathcal{H}$-surfaces

Finally, we construct $\mathcal{H}$-surfaces based on some curves contained in the well-known Enneper's surface, one of the most famous examples in the minimal surface theory.

First, consider the curve

$$\beta(s) = (0, -s(1 - s^2/3)/3, -s^2/3)$$

and the vector field

$$B(s) = 1/3(1 + s^2)(1, 0, 0).$$

Then, both $\beta(s)$ and $B(s)$ are analytic and satisfy $|\beta'(s)| - |B(s)| = \langle \beta'(s), B(s) \rangle = 0$, $\forall s \in \mathbb{R}$, i.e. they can be chosen to be Björling data. If $\mathcal{H} = 0$, then the surface given by Theorem 2 is Enneper's minimal surface. In particular, the curve $\beta(s)$ is obtained as intersecting Enneper's minimal surface with the plane $\{x = 0\}$, which is a plane of reflection symmetry of Enneper's minimal surface. For the function $\mathcal{H}(x) = \langle x, e_3 \rangle$, $\forall x \in \mathbb{S}^2$, the $\mathcal{H}$-surface arising is a translating soliton of the mean curvature flow that resembles indeed to Enneper's minimal surface, see Figure 5.

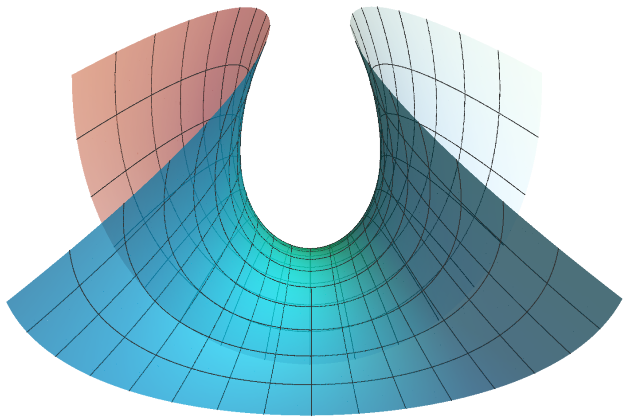

Figure 5: An $\mathcal{H}$-surface constructed over the curve in Enneper's minimal surface invariant under the reflection with respect to the plane $\{x = 0\}$; here $\mathcal{H}(x) = \langle x, e_3 \rangle$, $\forall x \in \mathbb{S}^2$.

We can also choose as Björling data the following

$$\beta(s) = (-\cos s - 1/3\cos(3s), \sin s - 1/3\sin(3s), \sin(2s)),$$
$$B(s) = (\cos s + \cos(3s), 2\cos(2s)\sin s, -2\sin(2s)).$$

Again, for $\mathcal{H} = 0$ the surface given by Theorem 2 is Enneper's minimal surface, and we will call $\beta(s)$ *Enneper's core curve*, see [LoWe]. For the function $\mathcal{H}(x) = \langle x, e_3 \rangle$, $\forall x \in \mathbb{S}^2$, the translating soliton arising also resembles to Enneper's minimal surface, see Figure 6, left. This time we cannot guarantee that the *hole* in the middle will eventually close, as we fail to have an explicit parametrization. If we make the vector $B(s)$ twist along the curve $\beta(s)$ and odd number of times, we get another translating soliton with the topology of a Möbius strip; see Figure 6, right.

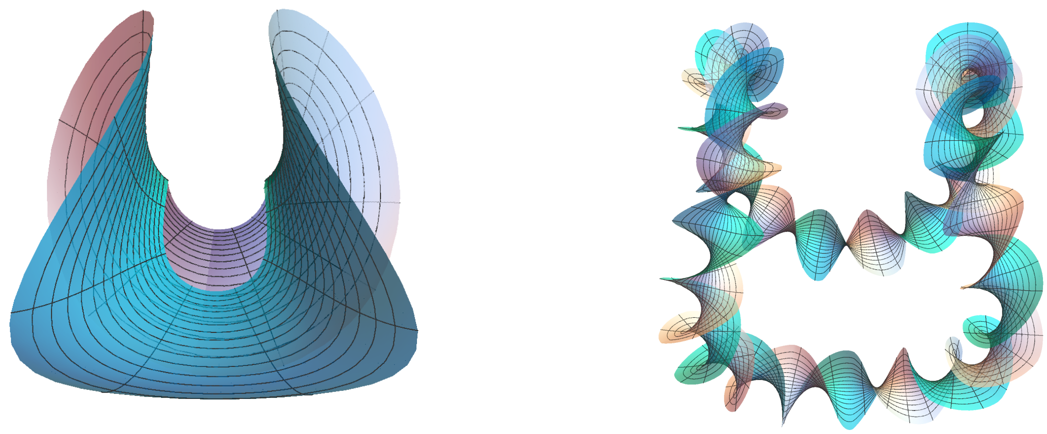

Figure 6: Two self-translating solitons of the mean curvature flow based on Enneper's core curve.

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