HORO-SHRINKERS IN THE HYPERBOLIC SPACE

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ABSTRACT. A surface Σ in the hyperbolic space $\mathbb{H}^3$ is called a horo-shrinker if its mean curvature H satisfies $H = \langle N, \partial_z \rangle$, where (x, y, z) are the coordinates of $\mathbb{H}^3$ in the upper half-space model and N is the unit normal of Σ. In this paper we study horo-shrinkers invariant by one-parameter groups of isometries of $\mathbb{H}^3$ depending if these isometries are hyperbolic, parabolic or spherical. We characterize totally geodesic planes as the only horo-shrinkers invariant by a one-parameter group of hyperbolic translations along vertical geodesic tangent to ∂_z. The grim reapers are defined as the horo-shrinkers invariant by a one-parameter group of parabolic translations perpendicular to ∂_z. We describe the geometry of the grim reapers proving that they are periodic surfaces. In the last part of the paper, we give a complete classification of horo-shrinkers invariant by spherical rotations about the z-axis, distinguishing if the surfaces intersect or not the rotation axis.

1. INTRODUCTION

The theory of the mean curvature flow (MCF for short) is an area of great activity in geometric analysis in the last decades: see, for example, the surveys [10, 12, 22] and references therein. In Euclidean space $\mathbb{R}^3$, let Σ be an oriented smooth surface and $\Psi\colon \Sigma \to \mathbb{R}^3$ an isometric immersion. A MCF for Ψ is a smooth family of immersions $\{\Psi_t : \Sigma \to \mathbb{R}^3 : t \in [0, T)\}$ satisfying

$$\begin{cases} \dfrac{\partial \Psi_t}{\partial t} & = H(\Psi_t)N(\Psi_t), \\ \Psi_0 & = \Psi, \end{cases}$$

where $H(\Psi_t)$ and $N(\Psi_t)$ are the mean curvature and the unit normal of Ψ_t respectively. Solutions of the MCF develop singularities at finite time, which may cause a change in the topology of the surface. There are two types of singularities. Translators of the MCF (also called translating solitons) appear as the equation of the limit flow by a blow-up procedure near type II singularities, according to Huisken and Sinestrari [15]. In $\mathbb{R}^3$, a translator is a surface Σ characterized by the equation $H = \langle N, \vec{v} \rangle$, where H and N are the mean curvature and the unit normal of Σ, respectively, and $\vec{v}$ is a direction of the ambient space. This direction $\vec{v}$ indicates that the shape of Σ does not change during the evolution because Σ is translated by the MCF at constant velocity.

2020 *Mathematics Subject Classification.* Primary 53E10; Secondary 53C44, 53A10, 53C21, 53C42.

Key words and phrases. hyperbolic space, mean curvature flow, conformal vector field, grim reaper, spherical rotation.

Huisken initiated the study of the MCF in general Riemannian manifolds [14]. When the ambient space is the hyperbolic space, pioneering research on the MCF is [7, 8]; see also [3, 4, 9, 13]. Singularities of MCF have been less studied [17, 25]. Nevertheless, the same notion the of translator can be defined by replacing $\vec{v} \in \mathbb{R}^3$ by a Killing vector field $X \in \mathfrak{X}(\mathbb{H}^3)$ whose flow of isometries consists of translations of $\mathbb{H}^3$. A surface $\Sigma \subset \mathbb{H}^3$ is a translator with respect to X if $H = \langle N, X\rangle$. In the hyperbolic space $\mathbb{H}^3$ there are two types of translations [11]. Parabolic translations are isometries of $\mathbb{H}^3$ that fix one point of the ideal boundary $\mathbb{H}^3_\infty$. Hyperbolic translations are isometries of $\mathbb{H}^3$ that fix two points of $\mathbb{H}^3_\infty$. The corresponding translators have been recently studied in [6] and [19], respectively.

Besides Killing vector fields, another vector fields of $\mathbb{H}^3$ of special relevance are the conformal vector fields. It was in [1] where the authors proposed the study of self-similar solutions of the mean curvature flow in the presence of a conformal vector field due to its formal similarity with Ricci solitons [21, 22]. The class of conformal vector fields in hyperbolic space is rich and it would be an extremely heavy job to consider solitons for an arbitrary conformal field, as just occurs for translators. That is why in this paper, we fix a particular one which can be expressed in a convenient way. For that reason, we first fix the model for the hyperbolic space, which will be the upper half-space model of $\mathbb{H}^3$.

Let $(\mathbb{R}^3_+, \langle\cdot,\cdot\rangle)$, where $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, $\langle\cdot,\cdot\rangle$ is the hyperbolic metric

$$\langle\cdot,\cdot\rangle = \frac{1}{z^2}\langle\cdot,\cdot\rangle_e,$$

and $\langle\cdot,\cdot\rangle_e = dx^2 + dy^2 + dz^2$ is the Euclidean metric of $\mathbb{R}^3$. In this model, the vector field $\partial_z \in \mathfrak{X}(\mathbb{H}^3)$ is a conformal vector field because its Lie derivative is $\mathcal{L}_{\partial_z}\langle,\rangle = -\frac{2}{z}\langle,\rangle$. In this paper, we will investigate translators in $\mathbb{H}^3$ with respect to this conformal vector field.

Definition 1.1. A horo-shrinker in $\mathbb{H}^3$ is an isometric immersion $\Psi\colon \Sigma \to \mathbb{H}^3$ of an oriented smooth surface Σ whose mean curvature H satisfies

$$H = \langle N, \partial_z\rangle, \tag{1}$$

where N is the unit normal of Σ.

The vector field ∂_z also appears as a remarkable conformal vector field in other models, as for example, when $\mathbb{H}^3$ is viewed as the warped product $\mathbb{R} \times_{e^t} \mathbb{R}^2$. In coordinates $(t, p) \in \mathbb{R} \times \mathbb{R}^2$, the vector field $e^t\partial_t$ is conformal and it corresponds with the vector field ∂_z of Eq. (1): see e.g. Examples 2.3 and 3.3 in [1]. A study similar to the one we carry out in this paper is done in [23], where the authors considered the conformal vector field $-\partial_z$. They name the corresponding solutions of (1) horo-expanders. Let us point out that the change of sign with respect to our definition (1) is relevant and there is no relation between the solitons for the two conformal fields. This situation has its analogy in Euclidean space $\mathbb{R}^3$, where self-shrinkers and self-expanders are surfaces which move by homotheties (contractions or expansions, respectively) when they evolve by the MCF. In contrast to the translators of $\mathbb{H}^3$, the shape of horo-shrinkers and horo-expanders is not preserved along the MCF.

A first observation is that horo-shrinkers and horo-expanders in $\mathbb{H}^3$ are minimal surfaces in the sense of Ilmanen [16]. Specifically, if we define the function $\phi(x, y, z) = -2/z$, then a minimal surface for the conformal metric $e^{\phi}\langle,\rangle$ is characterized by Eq. (1), that is, the surface is a horo-shrinker. In case of horo-expanders the function is $\phi(x, y, z) = 2/z$. Recall that being minimal in a conformal space is a property that also fulfill self-shrinkers and self-expanders of $\mathbb{R}^3$. However, and in contrast to the theory of the MCF in Euclidean space, it is unknown if the translators of $\mathbb{H}^3$ determined by parabolic and hyperbolic translations and described in [6, 19] are minimal surfaces in the sense of Ilmanen.

The purpose of this paper is to begin the theory of horo-shinkers in $\mathbb{H}^3$, and for that matter we focus on investigating horo-shrinkers with some type of invariance by a one-parameter group of isometries of $\mathbb{H}^3$. In the hyperbolic 3-space we find three types of such groups: parabolic and hyperbolic translations and spherical rotations [11].

A full classification in all its generality of invariant horo-shrinkers under *any* translation or rotation seems hopeless. When we turn our attention to an analog situation in $\mathbb{R}^3$, non-trivial translators of the MCF invariant by rotations only appear when the rotation axis is parallel to the translation direction. This also happens in other geometric frameworks such as in the product spaces $\mathbb{M}^2 \times \mathbb{R}$ [5, 20] or the Lorentz-Minkowski space [18]. Regarding translators invariant by translations, the celebrated examples arise when the translation direction is orthogonal to the invariant one. Finally, in the similar framework [23] of considering the conformal field $-\partial_z$ in the half-space model of $\mathbb{H}^3$, the authors also restrict to specific isometries that have a geometric relation with their vector field in order to study invariant examples. Inspired by these situations, in this work we will restrict ourselves to the following translations and rotations that have a geometric relation with the conformal vector field ∂_z:

(1) (Hyperbolic isometries) Group of hyperbolic translations along the vertical geodesic tangent to ∂_z;
(2) (Parabolic isometries) Group of parabolic translations perpendicular to ∂_z;
(3) (Spherical isometries) Group of spherical rotations about the z-axis.

In order to save notation and for clarity reasons in the exposition of the results, hereinafter we simply say parabolic or hyperbolic translations and spherical rotations, and they should be understood as one of the particular aforementioned isometries.

We detail the organization of the paper and highlight some of the main results. In Section 2 we show the first examples of horo-shrinkers, such as vertical planes (totally geodesic planes) and the horosphere H_1 of equation $z = 1$. Taking these examples as comparison surfaces and by the tangency principle, we prove that there are no closed horo-shrinkers. We also classify in Thm. 2.3 all horo-shrinkers invariant by hyperbolic translations. In Section 3 we define the grim reapers as those horo-shrinkers invariant by a one-parameter group of parabolic translations. The classification of the grim reapers is given in Thm. 3.5, being these surfaces vertical planes, the horosphere H_1, and a one-parameter family of periodic surfaces along a horizontal direction orthogonal to the parabolic translations. As a consequence of the properties of the grim reapers, we will prove in Thm. 3.7 that there are no solutions of the Dirichlet problem at infinity associated to the non-parametric equation for (1).

Section 4 is devoted to the study of horo-shrinkers invariant by spherical rotations about the z-axis which we will call horo-shrinker of *spherical type*. We distinguish if the surfaces intersect or not the rotation axis. In the first case, the existence of these surfaces is not a direct consequence of standard theory, since the ODE fulfilled is degenerated when the surface intersects the rotation axis. In Thm. 4.1 we prove such existence using Banach's fixed point theorem. In Thm. 4.3 we prove that they are parametrized by one parameter, namely, the initial height at which they intersect the rotation axis. We also prove that these surfaces oscillate around H_1. Finally, in Thm. 4.4 we describe the spherical horo-shrinkers that do not intersect the rotation axis. These surfaces are parametrized by the initial height and the distance to the rotation axis and they also oscillate around H_1.

2. Preliminaries

In this paper we use the upper half-space model of $\mathbb{H}^3$. We will employ the terminology parallel in the Euclidean sense and by vertical and horizontal we mean to be parallel to the z-axis or parallel to the xy-plane, respectively. The ideal boundary $\mathbb{H}^3_\infty$ of $\mathbb{H}^3$ is the one-point compactification of the plane $z = 0$. We show some explicit examples of horo-shrinkers.

1. Vertical totally geodesic planes. These surfaces are minimal ($H = 0$) and the unit normal N is orthogonal to ∂_z.
2. The horosphere of equation $z = 1$. This horosphere will be denoted by H_1. In general, horospheres in the upper half-space model can be viewed as horizontal planes of equation $z = c$, $c > 0$. The mean curvature is $H = 1$ with the orientation $N = c\partial_z$. Then $\langle N, \partial_z\rangle = 1/c$ and consequently, the only horosphere $z = c$ satisfying (1) is when $c = 1$.

Next, we express the condition of being a horo-shrinker in a non-parametric way. For this, we will use a relation between the hyperbolic mean curvature H of a surface Σ in $\mathbb{H}^3$ and its Euclidean mean curvature H_e when Σ is regarded as a surface in $(\mathbb{R}^3_+, \langle,\rangle_e)$. This relation is given by

$$H(x,y,z) = zH_e(x,y,z) + (N^e)_3(x,y,z), \quad (x,y,z) \in \Sigma, \tag{2}$$

where N^e is the Euclidean unit normal of Σ and the subindex $(\cdot)_3$ denotes the third coordinate of the vector. From the viewpoint of PDE theory, equation (1) is of second order and elliptic. Indeed, if Σ is locally expressed as $z = u(x,y)$, in virtue of (2), equation (1) writes as

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{2}{\sqrt{1+|Du|^2}}\frac{1-u}{u^2}. \tag{3}$$

This elliptic equation is of quasilinear type. As stated in the Introduction, horo-shrinkers are minimal surfaces in the conformal space $(\mathbb{H}^3, e^{-2/z}\langle,\rangle)$. This allows to use the tangency principle similarly as for minimal surfaces of $\mathbb{R}^3$. This implies that if two horo-shrinkers have a common tangent point and one horo-shrinker locally lies at one side of the other around that point, then both horo-shrinkers agree in an open set. A first consequence of the tangency principle is a certain control of the height of the points of a horo-shrinker that are critical points of height function.

Proposition 2.1. *Let Σ be a horo-shrinker, $\Sigma \neq \mathsf{H}_1$. If $p \in \Sigma$ is a local maximum (resp. minimum) of the function $z\colon \Sigma \to \mathbb{R}$, then $z(p) > 1$ (resp. $z(p) < 1$).*

Proof. Let $p = (x_0, y_, z_0)$ be a local maximum of the function z. Since $Du(x_0, y_0) = (0,0)$, Eq. (3) becomes simply

$$\Delta u(x_0, y_0) = 2\frac{1 - z_0}{z_0^2},$$

where Δ is the Euclidean Laplacian of $\mathbb{R}^2$. Thus $\Delta u(x_0, y_0) \leq 0$ implies $z_0 \geq 1$. We prove that, in fact, $z_0 > 1$. On the contrary, if $z_0 = 1$, then the horo-shrinker Σ lies in one side of H_1 in an open set of p. Then the tangency principle would imply $\Sigma \subset \mathsf{H}_1$, which it is a contradiction. In case that p is a local minimum, the arguments are analogous. □

A second consequence of the tangency principle is the following result.

Proposition 2.2. *There are no closed (compact without boundary) horo-shrinkers.*

Proof. Arguing by contradiction, assume that Σ is a closed horo-shrinker. Take Π a vertical plane of $\mathbb{R}^3_+$ that does not intersect Σ. We move Π under parabolic translations towards Σ until we arrive to a first contact point between both surfaces. Since Π is a horo-shrinker, the tangency principle asserts that Σ and Π agree in the largest neighborhood of both surfaces containing the tangency point. This implies that $\Sigma \subset \Pi$, which is a contradiction. □

We finish this section giving the classification of horo-shrinkers invariant by a one-parameter group of hyperbolic translations. According to the program established in the Introduction, we will consider the group of hyperbolic translations along the vertical geodesic tangent to the vector field ∂_z. Then such a hyperbolic translation is an Euclidean homothety from a point of the ideal boundary of $\mathbb{H}^3$, which can be assumed to be the origin O. Therefore the corresponding group is

$$\mathcal{H} = \{(x, y, z) \mapsto t(x, y, z) : t > 0\}.$$

In particular, a surface invariant by $\mathcal{H}$ can be viewed as a radial graph on the hemisphere $\mathbb{S}^2_+ = \{(x, y, z) \in \mathbb{R}^3_+ : x^2 + y^2 + z^2 = 1\}$.

Theorem 2.3. *Vertical planes containing O are the only horo-shrinkers invariant by the group $\mathcal{H}$.*

Proof. Let Σ be a horo-shrinker invariant by $\mathcal{H}$. Since Σ is a radial graph on some domain of $\mathbb{S}^2_+$, a parametrization of Σ is

$$\Psi(s,t)=t\alpha(s),\quad s\in I\subset\mathbb{R},\ t\in\mathbb{R},$$

where $\alpha\colon I\to\mathbb{S}^2_+$ is a curve parametrized by the Euclidean arc-length. Then $|\alpha(s)|_e=|\alpha'(s)|_e=1$ for all $s\in I$. The Euclidean mean curvature H_e and the Euclidean unit vector N^e are

$$H_e=\frac{1}{2}\frac{\langle\alpha'\times\alpha,\alpha''\rangle_e}{t},\quad N^e=\alpha'\times\alpha.$$

Then, (1) is

$$2\alpha_3\langle\alpha'\times\alpha,\alpha''\rangle_e+(\alpha'\times\alpha)_3=\frac{\langle\alpha'\times\alpha,\partial_z\rangle_e}{t\alpha_3}=\frac{(\alpha'\times\alpha)_3}{t\alpha_3},$$

where the last equality is because $\langle\alpha'\times\alpha,\partial_z\rangle_e=(\alpha'\times\alpha)_3$. If we write this equation as

$$t\alpha_3\left(2\alpha_3\langle\alpha'\times\alpha,\alpha''\rangle_e+(\alpha'\times\alpha)_3\right)-(\alpha'\times\alpha)_3=0,$$

we have a polynomial equation on t. Thus, we deduce $\langle\alpha'\times\alpha,\alpha''\rangle_e=0$ and $(\alpha'\times\alpha)_3=0$ because $\alpha_3\neq 0$. Since $\langle\alpha'\times\alpha,\alpha''\rangle_e=0$, and α is a unit speed curve in $\mathbb{S}^2_+$, we have $\alpha''=-\alpha$. Thus α is an Euclidean geodesic of $\mathbb{S}^2_+$, that is, a great (hemi) circle of $\mathbb{S}^2_+$. Using that $(\alpha'\times\alpha)_3=0$, we deduce that α is included in a vertical plane through O. In consequence, Σ is a vertical plane containing O. □

Remark 2.4. Notice that the same proof is valid for horo-expanders, proving that vertical planes are the only horo-expanders that are invariant under the group $\mathcal{H}$ of hyperbolic translations. This completes the classification given in [23] of all horo-expanders invariant by the one-parameter group $\mathcal{H}$.

3. The grim reapers

In this section we classify the horo-shrinkers invariant by a specific one-parameter group of parabolic translations. A parabolic translation of $\mathbb{H}^3$ is an isometry that leaves fixed one double point of the ideal boundary $\mathbb{H}^3_\infty$. As stated in the Introduction and similarly to Thm. 2.3, the parabolic translations considered are those whose integral lines are orthogonal to ∂_z. Thus, the fixed point is $\infty\in\mathbb{H}^3_\infty$ and the parabolic translations are simply horizontal Euclidean translations. After a rotation about the z-axis, which does not affect to the orthogonality between the parabolic translations and the vector field ∂_z, we can assume that the group is determined by the horizontal direction $(0,1,0)\in\mathbb{R}^3$, that is,

$$\mathcal{P}=\{(x,y,z)\mapsto(x,y,z)+t(0,1,0):t\in\mathbb{R}\}.$$

Hence, a surface invariant by $\mathcal{P}$ is a ruled surface of $\mathbb{R}^3_+$ whose all rulings are horizontal straight-lines parallel to $(0,1,0)$. In analogy with the Euclidean context, we give the following definition.

Definition 3.1. A grim reaper is a horo-shrinker invariant by the group $\mathcal{P}$.

Let Σ be a grim reaper. A parametrization of Σ is

$$\Psi(s,t) = (x(s), t, z(s)), \quad t \in \mathbb{R}, s \in I \subset \mathbb{R}, \tag{4}$$

where $\alpha(s) = (x(s), 0, z(s))$ is a planar curve contained in the xz- plane. Suppose that s is the hyperbolic arc-length, that is $\langle \alpha'(s), \alpha'(s)\rangle = 1$. This reads as

$$\frac{1}{z(s)^2}(x'(s)^2 + z'(s)^2) = 1,$$

hence there is a smooth function $\theta = \theta(s)$ such that

$$x'(s) = z(s)\cos\theta(s), \qquad z'(s) = z(s)\sin\theta(s).$$

The unit normal is $N = zN^e = (-z', 0, x')$. The Euclidean mean curvature H_e of Σ is $H_e = \kappa/2$, where κ is the Euclidean curvature of α. Since $\kappa = \theta'/z$, then $H_e = \theta'/(2z)$. Using (2), we have

$$H = \frac{\theta'}{2} + \frac{x'}{z}.$$

Since $\langle N, \partial_z\rangle = x'/z^2$, then (1) is

$$\frac{\theta'}{2} + \cos\theta = \frac{\cos\theta}{z}. \tag{5}$$

Therefore, the coordinate functions $x(s)$, $z(s)$ and $\theta(s)$ satisfy

$$\begin{cases} x'(s) = z(s)\cos\theta(s), \\ z'(s) = z(s)\sin\theta(s), \\ \theta'(s) = 2\cos\theta(s)\dfrac{1-z(s)}{z(s)}. \end{cases} \tag{6}$$

Since the aim of this section is the geometric description of the grim reapers, we study the shape of the solution curves of (6). First, we see that each solution of (6) remains at a bounded distance to the plane $z = 0$.

Proposition 3.2. *Let $(x(s), z(s), \theta(s))$ be a solution to (6). Then there is $\delta > 0$ such that $z(s) \geq \delta$ for all $s \in I$.*

Proof. Multiplying the last equation of (6) by $\cos\theta\sin\theta$ and taking into account that $z'/z = \sin\theta$, we have

$$\frac{\sin\theta(\sin\theta)'}{1-\sin^2\theta} = 2\sin\theta\frac{1-z}{z} = 2\frac{1-z}{z^2}z'.$$

Hence we deduce that there exists $c \in \mathbb{R}$ such that

$$\cos\theta = cz^2 e^{2/z}. \tag{7}$$

Now, arguing by contradiction, assume that there is a sequence $s_n \to s_1$ such that $z(s_n) \to 0$ as $n \to \infty$, being s_1 either finite or infinite. Substituting in (7) the right-hand side diverges to ∞, which it is not possible because the left-hand side is bounded. $\square$

As a consequence, each solution of (6) is defined in $\mathbb{R}$ because the functions on the right-hand side of (6) are bounded.

Recall that the function $x(s)$ does not explicitly appear in (6), but only its derivative. Geometrically this implies that any solution of (6) remains a solution after a parabolic translation $(x,0,z)\mapsto(x+t,0,z)$. Consequently, in order to study the properties of the solutions of (6) it is enough to consider the nonlinear autonomous system

$$\begin{pmatrix} z' \\ \theta' \end{pmatrix} = \begin{pmatrix} z\sin\theta \\ 2\cos\theta\dfrac{1-z}{z} \end{pmatrix}, \tag{8}$$

defined in the domain $\{(z,\theta): z>0, \theta\in\mathbb{R}\}$. Indeed, if we fix (x_0,z_0,θ_0), $z_0>0$, let (z,θ) be the unique solution to (6) for the initial data $z(0)=z_0>0$, $\theta(0)=\theta_0$, and define x as the solution to $x'=z\cos\theta$, $x(0)=x_0$. Then, $\alpha(s)=(x(s),0,z(s))$ is the generating curve of a surface parametrized by (4) that is a solution to (1).

By periodicity of the trigonometric functions, we define the *orbits* as the solutions $\gamma(s)=(z(s),\theta(s))$ of (8), which are defined for $z>0$ and $\theta\in(-\pi,\pi)$. By uniqueness of the Cauchy problem, two different orbits cannot intersect, hence the (z,θ)-domain $(0,\infty)\times(-\pi,\pi)$ is foliated by all the orbits.

The following result exhibits that we can reduce the study of the orbits essentially to $\theta\in(0,\pi/2)$. Its proof follows immediately, hence it is omitted.

Proposition 3.3. *The following properties hold:*

1. *If $\gamma(s)=(z(s),\theta(s))$ is an orbit, then $\overline{\gamma}(s)=(z(-s),-\theta(-s))$ is also an orbit. Consequently, every orbit γ is symmetric with respect to the line $\theta=0$.*
2. *If $\gamma(s)=(z(s),\theta(s))$ is an orbit for $\theta\in(-\pi/2,\pi/2)$, then $\overline{\gamma}(s)=(z(s),-\theta(s)+\pi)$ is an orbit for $\theta\in(\pi/2,3\pi/2)$.*

We define the phase plane of (8) as the set

$$\Theta=\{(z,\theta)\colon z>0, \theta\in(-\pi/2,\pi/2)\}.$$

The coordinates (z,θ) are in one-to-one correspondence to the orbits of (8). The motion of any orbit in Θ is uniquely determined by the sign of the functions z' and θ'. From (8), the signs of z' and θ' are determined by the signs of $\cos\theta$ and $\sin\theta$ as well as of the function $z-1$. For example, any orbit intersecting the line $z=1$ changes the monotonicity of its second coordinate, attaining a local maximum or minimum. In the remaining of the phase plane, the second coordinate of any orbit is strictly monotonous.

The explicit examples of horo-shrinkers given in Section 2 are now viewed as trivial solutions of (6) in the following result.

Proposition 3.4. *Explicit examples of orbits of* (8) *are the following:*

1. *The point $(1,0)$. This orbit corresponds to the horosphere H_1.*
2. *The lines $\theta=\pm\pi/2$. These orbits correspond to vertical planes (totally geodesic planes) of equation $x=x_0$, where $x_0\in\mathbb{R}$. The parameter $\theta=\pi/2$ implies that the vertical plane is parametrized with increasing height, and for $\theta=-\pi/2$ the height is decreasing.*

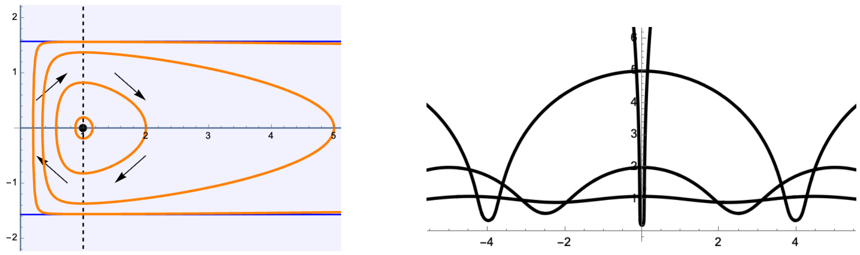

Figure 1. Left: the phase plane of (8) and different orbits portrayed. Right: the generating curves of the corresponding orbits. The initial height are $z_0 = 0.2$, 1.1, 2 and 5.

As usual, for the description of the orbits of the autonomous system (8), we analyze its equilibrium points. It is clear that the point $P_0 = (1, 0)$ is the unique equilibrium. The linearized of the system around P_0 is

$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

Since the eigenvalues are imaginary numbers with zero real part, the equilibrium P_0 has a center structure. Thus the orbits of the linearized system are ellipses enclosing P_0 in their inner regions. Consequently, the orbits that are close enough to P_0 either spiral around P_0 or are closed curves enclosing P_0 in their inner regions. However, by Prop. 3.3 the orbits are symmetric about $\theta = 0$, hence they cannot spiral around P_0. In particular all the orbits stay at a positive distance from P_0: see Fig. 1, left. Note: all figures in this paper have been plotted using the software Mathematica.

We now derive the classification of the grim reapers.

Theorem 3.5. *The classifications of the grim reapers is the following:*

1. *Vertical planes (totally geodesic planes).*
2. *The horosphere H_1.*
3. *A one-parameter family of entire graphs $\mathcal{G}(z_0)$, $z_0 \in (0, 1)$ that are periodic along the x-direction. The value z_0 indicates the Euclidean distance of $\mathcal{G}(z_0)$ at $z = 0$. Moreover:*
   - (a) *If $z_0 \to 0$, then $\mathcal{G}(z_0)$ converges to a double covering of a vertical plane.*
   - (b) *If $z_0 \to 1$ then $\mathcal{G}(z_0)$ converges to the horosphere H_1.*
   - (c) *For each $z_0 \in (0, 1)$ there exists a unique $z_0^* \in (1, \infty)$ that corresponds to the Euclidean height of $\mathcal{G}(z_0)$ at $z = 0$. Hence $\mathcal{G}(z_0)$ can be also parametrized in terms of z_0^*, being equivalent. In fact, if $z_0 \to 0$ (resp. $z_0 \to 1$) then $z_0^* \to \infty$ (resp. $z_0^* \to 1$).*

Proof. The first two types of surfaces were already depicted in Prop. 3.4. Now, fix some $z_0 \in (0, 1)$. We call $\mathcal{G}(z_0)$ the grim reaper which is the graph of the solution of (6) for initial conditions $x(0), z(0), \theta(0)) = (0, z_0, 0)$. Let γ_{z_0} be the orbit with initial data $\gamma_{z_0}(0) = (z_0, 0)$ and let α_{z_0} be the corresponding generating curve of the grim reaper. Then γ_{z_0} is vertical at $(z_0, 0)$, for $s > 0$ and by monotonicity both coordinates functions $z(s)$ and

$\theta(s)$ of γ_{z_0} strictly increase. Since γ_{z_0} cannot intersect the orbit $\theta = \pi/2$, necessarily γ_{z_0} intersects the vertical line $z = 1$ where its θ-coordinate attains a local maximum. Then, θ decreases and since γ_{z_0} cannot converge to P_0 due to its center structure, γ_{z_0} intersects again the line $\theta = 0$ at some $(z_0^*, 0)$, where $z_0^* > 1$. Finally, by symmetry of the phase plane with respect to the line $\theta = 0$, γ_{z_0} closes again at the point $(z_0, 0)$. See Fig. 1, left. Note that if $z_0 \to 1$ then $\gamma_{z_0} \to P_0$, while if $z_0 \to 0$ then γ_{z_0} converges to both $\theta = \pm\pi/2$.

At this point, for each $z_0 \in (0, 1)$ the point $z_0^* \in (1, \infty)$ corresponds with a local maximum of the function $z = z(s)$ by Prop. 2.1. It could happen that $(z_0)_n \to 0$ and $(z_0^*)_n \to (z_0^*)_\infty < \infty$. However, this possibility cannot happen in virtue of Prop. 3.2. Indeed, assume by contradiction that this behavior occurs, take some $z^* > (z_0^*)_\infty$ and let γ_{z^*} be the orbit passing through $(z^*, 0)$ at $s = 0$. Then, when s increases γ_{z^*} cannot intersect $\theta = -\pi/2$, hence γ_{z^*} intersects $z = 1$. Since it cannot intersect again the line $\theta = 0$ (because it would correspond to some $z_* \in (0, 1)$, a contradiction), the only possibility for γ_{z^*} is to converge to $z = 0$. But this contradicts Prop. 3.2. As a consequence, for each $z_0 \in (0, 1)$ there exists exactly one $z_0^* \in (1, \infty)$, being both intervals in a one-to-one correspondence.

Therefore, the z-coordinate of the associated generating curve α_{z_0} is periodic. Since $\theta \in (-\pi/2, \pi/2)$ we conclude that $x' > 0$. This implies that x is strictly increasing, hence the curve α_{z_0} is periodic along the x-axis, in particular, invariant under a discrete group of translations along the x-axis. The curve α_{z_0} is a graph on the x-axis because $x' > 0$. The Euclidean height of α_{z_0} at $z = 0$ is z_0^* and its distance to $z = 0$ is z_0. If $z_0 \to 1$ then the curve α_{z_0} converges to the horizontal line $z = 1$. If $z_0 \to 0$ then α_{z_0} converges to a double covering of a vertical line, since γ_{z_0} converges to both $\theta = \pm\pi/2$. See Fig. 1, right. This concludes the proof. □

With the same notation as in the proof of Thm. 3.5, let us consider initial conditions $(x(0), z(0), \theta(0)) = (0, z_1, 0)$ with $z_1 > 1$ in system (6), and let γ_{z_1} be the corresponding orbit passing through $(z_1, 0)$. Then, γ_{z_1} passes through some $(z_1^0, 0)$, with $z_1^0 < 1$ being the minimum value of the height function z. Definitively, $(z_1^0)^* = z_1$ and therefore up to a parabolic translation orthogonal to the ruling direction $(0, 1, 0)$, the grim reaper $\mathcal{G}(z_1)$ agrees with $\mathcal{G}((z_1^0)^*)$.

Corollary 3.6. *Let $z_0 \in (0, 1)$. Then there is a unique $z_1 \in (1, \infty)$ such that $\mathcal{G}(z_0)$ and $\mathcal{G}(z_1)$ coincide up to a parabolic translation orthogonal to* $(0, 1, 0)$.

To finish this section, we address the Dirichlet problem at infinity for Eq. (2). More precisely, let $\Omega \subset \mathbb{R}^2 = \{z = 0\} \subset \mathbb{H}^3_\infty$ be a bounded domain with smooth boundary. We are asking for functions $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ such that u satisfies (3) in Ω with $u > 0$ in Ω and $u = 0$ along $\partial\Omega$. This is motivated by the pioneering works of the theory of constant mean curvature surfaces in hyperbolic space $\mathbb{H}^3$ due to Anderson ($H = 0$) and to Tonegawa ($0 < H < 1$) [2, 24]. The Dirichlet problem at infinity for horo-expanders was considered in [23] proving existence under general hypothesis on convexity of Ω. However, for Eq. (3) we show that the Dirichlet problem at infinity is not solvable. The key is that the grim reapers of Thm. 3.5 allow us to compare with the possible solutions of the Dirichlet problem at infinity.

Theorem 3.7. *There are no solutions of the Dirichlet problem at infinity for Eq.* (3).

Proof. By contradiction, suppose that u is a solution of (3), $u > 0$ in Ω and with initial condition $u = 0$ along $\partial\Omega$. Let Σ be the graph of u when u is defined in Ω and let $u_M > 0$ be the maximum of u in Ω which exists because $\overline{\Omega}$ is compact and u is continuous in $\overline{\Omega}$. After a parabolic translation along the x-direction, we can assume that Σ is included in the half-space $\{(x,y,z) \in \mathbb{R}^3_+ : x > 0\}$. By Prop. 2.1, let $u_M > 1$ be the maximum value of u in Ω. Using Thm. 3.5, let $z_0 \in (0,1)$ sufficiently close to 0 such that z_0^*, the maximum height of $\mathcal{G}(z_0)$, satisfies $z_0^* > u_M$. Take the piece of $\mathcal{G}(z_0)$ comprised between two consecutive maximum of $\mathcal{G}(z_0)$. To be precise, if we write the generating curve of $\mathcal{G}(z_0)$ as $z = z(r)$, let $r_0 > 0$ be such that $z(0) = z_0$, $z(\pm r_0) = z_0^*$ and $r = \pm r_0$ are the only maximum of $z(r)$ in the interval $[-r_0, r_0]$. Consider $\mathcal{G}(z_0)^F$ the piece of $\mathcal{G}(z_0)$ determined by $z(r)$ in the interval $r \in [-r_0, r_0]$, that is, $\mathcal{G}(z_0)^F = \mathcal{G}(z_0) \cap \{(x,y,z) \in \mathbb{R}^3 : -r_0 \leq x \leq r_0\}$. Notice that the boundary of $\mathcal{G}(z_0)^F$ are two straight-lines parallel to the y-axis and both situated at height z_0^*.

Let us move $\mathcal{G}(z_0)^F$ by translations along the x-direction with $x \searrow -\infty$ until $\mathcal{G}(z_0)^F$ does not intersect Σ. This is possible because Σ is included in the half-space $x > 0$ and the rulings of $\mathcal{G}(z_0)^F$ are parallel to the y-axis. Next, we move $\mathcal{G}(z_0)^F$ by translations along the x-direction with $x \nearrow \infty$ until the first contact point p with Σ. Let $\widetilde{\mathcal{G}(z_0)}^F$ denote the position of $\mathcal{G}(z_0)^F$ when it touches Σ. This point exists because $\overline{\Omega}$ is compact. Since $z_0 > 0$, then $z(p) \geq z_0$, so it is an interior point of Σ (or equivalently, $z(p) \neq 0$). On the other hand, $z(p) \leq u_M < z_0^*$, so p is an interior point of $\widetilde{\mathcal{G}(z_0)}^F$. Definitively, p is a common interior point of Σ and $\widetilde{\mathcal{G}(z_0)}^F$. The tangency principle implies that Σ is included in $\widetilde{\mathcal{G}(z_0)}^F$. This is a contradiction because $\widetilde{\mathcal{G}(z_0)}^F$ is contained in the half-space $\{(x,y,z) \in \mathbb{R}^3_+ : z > z_0\}$ where $z_0 > 0$. □

4. Spherical rotational horo-shinkers

In this section we classify horo-shrinkers invariant by the one-parameter group $\mathcal{S}$ of spherical rotations of $\mathbb{H}^3$ about the z-axis. The elements of $\mathcal{S}$ are simply Euclidean rotations about the z-axis, being

$$\mathcal{S} = \{(x,y,z) \mapsto (x\cos t - y\sin t, x\sin t + y\cos t, z) : t \in \mathbb{R}\}.$$

In the sequel we will indistinctly say rotation axis or z-axis. Thus, a parametrization of a spherical rotational surface Σ is

$$\Psi(s,t) = (x(s)\cos t, x(s)\sin t, z(s)), \quad s \in I \subset \mathbb{R}, t \in \mathbb{R},$$

where $\alpha(s) = (x(s), 0, z(s))$ is the generating curve. Notice that horospheres, viewed as horizontal planes of equation $z = c$, $c > 0$, are also spherical surfaces. Consequently, the horosphere H_1 is a spherical rotational horo-shrinker.

Assume that α is parametrized by the Euclidean arc-length. Then $\alpha'(s) = (\cos\theta(s), 0, \sin\theta(s))$, for some smooth function $\theta = \theta(s)$. The Euclidean mean curvature H_e and the Euclidean unit normal N^e of Σ are, respectively,

$$H_e = \frac{1}{2}\left(\theta' + \frac{\sin\theta}{x}\right), \quad N^e = (-\sin\theta\cos t, -\sin\theta\sin t, \cos\theta).$$

By (2), the equation (1) writes as

$$\frac{z}{2}\left(\theta' + \frac{\sin\theta}{x}\right) + \cos\theta = \frac{\cos\theta}{z}. \tag{9}$$

Thus, Eq. (1) is equivalent to say that the coordinate functions of the curve α satisfy

$$\begin{cases} x'(s) = \cos\theta(s), \\ z'(s) = \sin\theta(s), \\ \theta'(s) = -\dfrac{\sin\theta(s)}{x(s)} + 2\cos\theta(s)\dfrac{1-z(s)}{z(s)^2}. \end{cases} \tag{10}$$

The study of the spherical rotational horo-shrinkers, or equivalently, of the solutions of (10), is separated in two cases depending if the surface, or equivalently the generating curve, intersects or not the rotation axis.

Firstly, we consider the case the surface intersects the rotation axis. We will prove the existence of such surfaces and, in this case, that this intersection must be orthogonal. Since α intersects the z-axis, then at the initial value, say $s = 0$, for (10), $x(0)$ must be 0 and $z'(0) = 0$. However, the existence of solutions of (10) is not assured by the standard ODE theory because (10) is degenerated at $x = 0$. To address the existence, we parametrize the curve α by $r \mapsto (r, 0, z(r))$, $z(r) > 0$, then Eq. (9) writes as

$$\frac{z}{2}\left(\frac{z''}{(1+z'^2)^{3/2}} + \frac{z'}{r(1+z'^2)^{1/2}}\right) + \frac{1}{(1+z'^2)^{1/2}} = \frac{1}{z(1+z'^2)^{1/2}}. \tag{11}$$

Next we prove that there exist solutions of (11) defined at $r = 0$ such that $z'(0) = 0$.

Theorem 4.1. *If $z_0 > 0$, then there exist $R > 0$ and a solution $z \in C^2([0, R])$ of (11) with initial conditions*

$$z(0) = z_0 > 0, \quad z'(0) = 0. \tag{12}$$

Proof. Multiplying (11) by r, we can write (11) as

$$\left(\frac{rz'(r)}{\sqrt{1+z'(r)^2}}\right)' = 2r\frac{1-z(r)}{z(r)^2\sqrt{1+z'(r)^2}}. \tag{13}$$

Define the functions

$$g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \quad g(x,y) = \frac{2(1-x)}{x^2\sqrt{1+y^2}},$$

$$\varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi(y) = \frac{y}{\sqrt{1+y^2}}.$$

From (13), a function $z = z(r) \in C^2([0,R])$ satisfies (11)-(12) if and only if $(r\varphi(z'))' = rg(z,z')$ under initial conditions (12). The inverse function of φ is $\varphi^{-1}(x) = x/\sqrt{1-x^2}$, which is defined in $(-1,1)$. Fix $R > 0$ that will be determined later and define the operator $\mathsf{T} : C^1([0,R]) \to C^1([0,R])$ by

$$(\mathsf{T}z)(r) = z_0 + \int_0^r \varphi^{-1}\left(\frac{1}{s}\int_0^s tg(z,z')dt\right) ds. \tag{14}$$

It is clear that a fixed point of T is a solution of the initial value problem (12)-(13). First, we prove the existence of $\epsilon > 0$ such that T is well defined in a closed ball $\overline{B(z_0,\epsilon)}$ of $C^1([0,R])$. Here we understand that the space $C^1([0,R])$ is endowed the usual sup-norm $\|z\| = \|z\|_\infty + \|z'\|_\infty$. For this, let $\epsilon > 0$ such that $\epsilon < z_0$, and consider g defined in $[z_0-\epsilon, z_0+\epsilon]\times\mathbb{R}$. Let $M > 0$ such that $|\frac{2(1-z)}{z^2}| \le M$ for all $|z - z_0| \le \epsilon$. Let $R \le \min\{\frac{1}{M}, \frac{\sqrt{3}\epsilon}{2}, \frac{\sqrt{3}\epsilon}{2M}\}$. We have

$$\int_0^s \frac{t}{s} g(z,z')\,dt \le \int_0^s \frac{t}{s} M\,dt \le \frac{RM}{2} \le \frac{1}{2},$$

because $R \le 1/M$. This allows to apply φ^{-1} in the parenthesis of (14). In order to use the Banach fixed point theorem, we need the two following steps.

(1) *The map* T *satisfies* $\mathsf{T}(\overline{B(z_0,\epsilon)}) \subset \overline{B(z_0,\epsilon)}$. To prove this inclusion, let $z \in \overline{B(z_0,\epsilon)}$. By using that φ^{-1} is increasing, we have

$$|(\mathsf{T}z)(r) - z_0| \le \int_0^r \varphi^{-1}\left(\int_0^s \frac{t}{s(z_0-\epsilon)}\,dt\right) ds < \varphi^{-1}\left(\frac{1}{2}\right) R = \frac{R}{\sqrt{3}} \le \frac{\epsilon}{2},$$

$$|(\mathsf{T}z)'(r)| \le \varphi^{-1}\left(\int_0^s \frac{t}{s} M\,dt\right) \le \varphi^{-1}\left(\frac{R}{2}M\right) = M\frac{R}{\sqrt{4-RM^2}} \le \frac{RM}{\sqrt{3}} \le \frac{\epsilon}{2}$$

because $R \le \sqrt{3}\epsilon/2$ and $R \le \sqrt{2}\epsilon/(2M)$, respectively. As a conclusion, $\|\mathsf{T}z\| \le \epsilon$.

(2) *The map* T *is a contraction.* The functions g and φ^{-1} are Lipschitz continuous in $[z_0-\epsilon, z_0+\epsilon]\times[-\epsilon,\epsilon]$ and $[-\epsilon,\epsilon]$, respectively provided $0 < \epsilon < \min\{z_0, 1\}$. Let $L = \min\{L_g, L_{\varphi^{-1}}\}$, where L_g and $L_{\varphi^{-1}}$ stand for the Lipschitz constants of g and φ^{-1}, respectively. Given $z, \tilde{z} \in \overline{B(z_0,\epsilon)}$, for all $r \in [0,R]$ we have

$$\begin{aligned}|(\mathsf{T}z)(r) - (\mathsf{T}\tilde{z})(r)| &\le L\int_0^r \frac{1}{s}\int_0^s t\left|g(z,z') - g(\tilde{z},\tilde{z}')\right|\,dt\,ds\\ &\le L^2\int_0^r \frac{1}{s}\int_0^s t\left(\|z-\tilde{z}\|_\infty + \|z'-\tilde{z}'\|_\infty\right)\,dt\,ds\\ &= L^2\|z-\tilde{z}\|\int_0^r \frac{s}{2}\,ds = \frac{r^2L^2}{4}\|z-\tilde{z}\|.\end{aligned}$$

Analogously,

$$|(\mathsf{T}z)'(r) - (\mathsf{T}\tilde{z})'(r)| \le \frac{rL^2}{2}\|z-\tilde{z}\|.$$

Therefore

$$\|\mathsf{T}z - \mathsf{T}\tilde{z}\| \le \min\{\frac{R^2L^2}{4}, \frac{RL^2}{2}\}\|z-\tilde{z}\|.$$

Since L is fixed, by choosing $R > 0$ small enough, we conclude that T is a contraction.

The solution $z = z(r)$ obtained by the Banach fixed point theorem lies in $C^1([0,R]) \cap C^2((0,R])$. We prove that $z(r)$ can be extended up to C^2-regularity at $r = 0$. From (11), the L'Hôpital rule gives

$$\lim_{r\to 0} z''(r) = \frac{1-z_0}{z_0^2}. \tag{15}$$

This completes the proof of the theorem. □

Once we have proved the existence of spherical rotational horo-shrinkers intersecting orthogonally the rotation axis, our next goal is to achieve a full classification of such surfaces. First, we prove the following result which is valid for any solution of (10).

Proposition 4.2. *Let $\alpha(s) = (x(s), 0, z(s))$ be a solution of (10). If α is not a graph, then the x-coordinate has exactly one critical point which is a minimum. In consequence, if α intersects the rotation axis then α is a graph.*

Proof. If α is not a graph, then there is a critical point s_0 of x, $x'(s_0) = 0$ with $x(s_0) > 0$. From (10) we have $\theta'(s_0) = -\sin\theta(s_0)/x(s_0) = \pm\frac{1}{x(s_0)}$ and thus

$$x''(s_0) = \frac{1}{x(s_0)} > 0,$$

which yields that s_0 is a local minimum of $x(s)$. This proves that s_0 must be a local minimum and in such a case, no more critical points of $x(s)$ exist.

If α intersects the rotation axis at $s = 0$, then $x(0) = 0$ and $x'(0) = 1$. If $s_0 > 0$ is the first critical point of $x(s)$, then s_0 would be a local minimum, a contradiction. □

As a consequence of Prop. 4.2, the generating curve α of a spherical rotational horo-shrinker that intersects the rotation axis can be globally parametrized by $z = z(r)$. By Eq. (11), $z(r)$ is a solution of the initial value problem

$$\begin{cases} \dfrac{z''}{1+z'^2} + \dfrac{z'}{r} = 2\dfrac{1-z}{z^2} \\ z(0) = z_0 > 0, \quad z'(0) = 0. \end{cases} \tag{16}$$

Let $J = [0, r_{max})$ stand for the maximal domain of the solutions of (10), where $r_{\max} \in \mathbb{R} \cup \{\infty\}$. We denote by $\mathcal{B}(z_0)$ the spherical rotational horo-shrinker generated by $z(r)$, whose intersection with the rotation axis occurs at $z = z_0$. The following result exhibits the properties of $\mathcal{B}(z_0)$ and its classification. Numerical examples are depicted in Fig. 2.

Theorem 4.3. *The spherical rotational horo-shinkers interesecting the rotation axis are the surfaces $\mathcal{B}(z_0)$, where the parameter $z_0 > 0$ indicates the height of the intersection point of the surface with the rotation axis. Each $\mathcal{B}(z_0)$ is an entire graph that oscillates around the horosphere H_1. Furthermore,*

(1) *If* $z_0 = 1$, *then* $\mathcal{B}(1) = \mathsf{H}_1$.
(2) *If* $z_0 \in (0,1)$, *then* $\mathcal{B}(z_0)$ *is strictly convex at* $r = 0$.
(3) *If* $z_0 \in (1,\infty)$, *then* $\mathcal{B}(z_0)$ *is strictly concave at* $r = 0$.

Proof. The case $z_0 = 1$ follows immediately by just checking that the constant function $z(r) = 1$ fulfills (16). This proves the assertion (1). Suppose now $z_0 \neq 1$. Substituting at (16), we have

$$z''(0) = \frac{1 - {z_0}^2}{z_0}.$$

If $z_0 \in (0,1)$ (resp. $z_0 \in (1,\infty)$) then $z''(0) > 0$, the function $z(r)$ has a local minimum (resp. local maximum) at $r = 0$ and $z(r)$ is strictly convex (resp. concave) for $r > 0$ small enough. This proves (2) and (3).

We now prove that the solutions $z(r)$ of (16) are entire graphs (that is, $r_{max} = \infty$) that oscillate around the horizontal line $z = 1$ in the xz-plane. We assume $z_0 \in (0,1)$, as the arguments when $z_0 \in (1,\infty)$ are analogous. The behavior of $z(r)$ will be deduced by proving a series of claims.

(1) *The function* $z(r)$ *cannot be a convex graph for* $r > 0$. On the contrary, because z', z'' are positive, we have that $z \to \infty$ as $r \to r_{max}$. The left-hand side of (16) remains always positive, but its right-hand side is eventually negative, a contradiction.
(2) *The function* $z(r)$ *cannot fail to be a graph at finite time* $r_0 > 0$. Arguing by contradiction, assume that as $r \to r_0$ it happens $z(r) \to z(r_0)$, $z'(r) \to \infty$, $z''(r) > 0$ for r close to r_0. Recall that $\lim_{r \to r_0} z''(r)$ can be either finite or infinite, but in any case it is positive. Taking limits in (16) as $r \to r_0$ we see that the left-hand side of (16) goes to ∞, while its right-hand side is a finite value, a contradiction.
(3) As a consequence, z must change its convexity, which implies $z''(r_1) = 0$ and $z''(r_1) < 0$ for $r > r_1$ close enough to r_1. In particular, $z'(r_1) > 0$ and from (16), we deduce that $z(r_1) = z_1 < 1$.

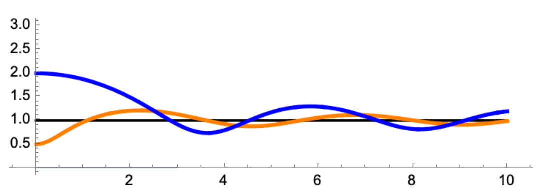

FIGURE 2. Generating curves of spherical rotational horo-shrinkers intersecting the rotational axis, and the horosphere $z = 1$ between them. Here $z_0 = 0,5$, 1 and 2.

(4) *There are* $0 < z_m < z_M < \infty$ *such that* $z_m \leq z(r) \leq z_M$ *for all* $r \in J$. Moreover, $z_m = z_0$ (if $z_0 > 1$, then $z_M = z_0$). In order to prove the claim, let us multiply (16) by z' and integrate from 0 to r. Then we obtain

$$\frac{1}{2}\log(1 + z'^2) + \int_0^r \frac{z'(t)^2}{t} dt = -2\left(\frac{1}{z} + \log z\right) + 2\left(\frac{1}{z_0} + \log z_0\right). \tag{17}$$

If there exists a sequence $r_n \to r_{max}$ such that $z(r_n) \to \infty$, then the right hand-side of (17) goes $-\infty$, a contradiction because the left hand-side is positive. This proves that $z(r)$ is bounded from above. A similar argument shows that $z(r)$ is bounded from below, by taking a sequence $r_n \to r_{max}$ such that $z(r_n) \to 0$.

We now prove that $z_m = z_0$. On the contrary, let $r_* > 0$ be such that $z(r_*) = z_* < z_0$. Letting $r = r_*$ in (17), the right hand-side must be positive. Consider the function $f(t) = -2(\frac{1}{t}+\log t)$, which is negative and increasing in $(0,1)$. The right hand-side of (17) writes as $f(z_*) - f(z_0) > 0$. Hence $z_* > z_0$, a contradiction.

(5) *We have* $r_{max} = \infty$. We write the first equation of (16) as

$$\begin{pmatrix} z \\ z' \end{pmatrix}' = \begin{pmatrix} z' \\ (1+z'^2)\left(2\frac{1-z}{z^2} - \frac{z'}{r}\right) \end{pmatrix}. \tag{18}$$

Then $r_{max} = \infty$ if we show that the right hand-side of (18) is bounded. In fact, by the above claim, it is enough to prove that the function $z'(r)$ is bounded. If there is a sequence $(r_n) \to r_{max}$ such that $|z'(r_n)| \to \infty$, then evaluating (17) at $r = r_n$ and letting $n \to \infty$, we have that the left hand-side of (17) diverges. However, the right hand-side is bounded by Claim 4. This contradiction proves the claim.

(6) *The function* $z(r)$ *attains a local maximum at some* $r_2 > r_1$. By contradiction, assume that such maximum does not exist, which implies that $z''(r) \leq 0$ and $z'(r) > 0$ for every $r > r_1$. By the previous claim, since $z(r)$ is strictly increasing and bounded from above, then $z(r)$ has a limit which, without loss of generality, we can suppose to be z_M. Moreover, $z'(r) \to 0$ as $r \to \infty$.

Next, letting $r \to \infty$ in (16), we deduce that $z''(r)$ has a limit. Since $z''(r) \leq 0$, we conclude that this limit is 0. Thus $\lim_{r\to\infty} z''(r) = 2\frac{1-z_M}{z_M^2} = 0$. This yields $z_M = 1$. In particular, the left hand-side of (16) is positive. Therefore, by dividing in (16) by z' and integrating from r_1 to r for r big enough yields

$$\frac{1}{2}\log\frac{z'^2}{1+z'^2} + \log r + c_1 > 0,.$$

for some integration constant c_1. After some manipulations we arrive to

$$z' > \frac{c_2}{\sqrt{r^2 - c_2^2}}, \quad c_2 = e^{-c_1} > 0.$$

Finally, integrating from r_1 to r we obtain

$$z(r) > c_2 \operatorname{arctanh}\frac{r}{\sqrt{r^2 - c_2^2}} + c_3, \qquad c_3 \in \mathbb{R}.$$

Letting $r \to \infty$, the right hand-side in this inequality diverges and thus $z(r) \to \infty$ as $r \to \infty$. This it is not possible by Claim 4.

After these arguments we ensure the existence of $r_2 > r_1$ such that $z'(r_2) = 0$. Note that $z(r_2) \neq 1$ since otherwise, $\mathcal{B}(z_0) = \mathsf{H}_1$

by uniqueness in (16): this is not possible because $z_0 < 1$. Since $z''(r) \leq 0$ for $r < r_2$ close enough to r_2 we conclude $z_2 > 1$. Then $z''(r_2) < 0$, which implies that $z(r)$ attains a local maximum at r_2.

(7) From the above claims, we deduce that the function $z(r)$ cannot end being a graph at some finite $r_3 > r_2$ because the left-hand side of (16) would be $-\infty$ but the right-hand side is finite. Consequently, z keeps being a graph and for $r > r_2$ small enough we have $z''(r),\ z'(r) < 0$. We show that z cannot keep this behavior. Otherwise, for $r \to r_3 > r_2$ we would have $\lim_{r\to r_3} z(r) = 0$ with $z', z'' < 0$. But this contradicts the Claim 4. Thus $z(r)$ has to change its curvature, i.e. there exists $r_3 > r_2$ such that $z''(r_3) = 0$, for which $z(r_3) > 1$ since $z'(r_3) < 0$. At this point, the only possibilities for z are the following:
 (a) $z(r) \to z_\infty > 0$ as $r \to \infty$;
 (b) $z'(r_4) = 0,\ z''(r_4) > 0$ for some $r_4 > r_3$.

We prove that the latter is the one that holds. By contradiction, if the former holds, the following would occur

$$\lim_{r\to\infty} z'(r) = \lim_{r\to\infty} z''(r) = 0,$$

which yields a contradiction after substituting in (16). We conclude that necessarily $z'(r_4) = 0$ at some $r_4 > r_3$, where by a similar argument as in the case of the maximum we get that $z''(r_4) > 0$, i.e. z attains a local minimum at r_4, and $z(r_4) < 1$ by Prop. 2.1.

At this point, we have a similar structure as when $z(r)$ started at the rotation axis with an orthogonal intersection at a local minimum of $z(r)$. Therefore, this process is repeated and we see that $z = z(r)$ is an entire graph that oscillates around the horosphere H_1.

□

The last result of this section is devoted to show the geometric properties of the spherical rotational horo-shrinkers that do not intersect the rotation axis. See Fig. 3.

Theorem 4.4. *Let Σ be a spherical rotational horo-shrinker about the z-axis such that Σ does not intersect the rotation axis. Then Σ belongs to a two-parameter family of spherical rotational horo-shrinkers, $\mathcal{W}(x_0, z_0)$, where the parameter $x_0 \in (0,\infty)$ indicates the Euclidean distance of $\mathcal{W}(x_0, z_0)$ to the z-axis and z_0 is the Euclidean distance to $z = 0$. Moreover:*

(1) *The surfaces $\mathcal{W}(x_0, z_0)$ are bi-graphs on the xy-plane.*
(2) *Each $\mathcal{W}(x_0, z_0)$ is contained in the closure of the non-bounded domain determined by the Euclidean cylinder about the z-axis and of radius x_0.*
(3) *Each $\mathcal{W}(x_0, z_0)$ has the topology of an annulus, and its ends oscillate around the horosphere H_1.*

Proof. Let us write Eq. (9) considering that the generating curve α is a graph $x = x(r)$ on the z-axis, $z > 0$. Then $x(r)$ satisfies

$$x'' = \frac{(r^2 + 2(r-1)xx')(1 + x'^2)}{r^2x^2}. \tag{19}$$

For $x_0, z_0 > 0$, let $x = x(r)$ be the solution of (19) with initial conditions $x(z_0) = x_0$, $x'(z_0) = 0$. Since $x''(z_0) = 1/x_0$, the function $x(r)$ is strictly convex locally around r_0. This curve generates a spherical rotational horo-shrinker which will be denoted by $\mathcal{W}(x_0, z_0)$. Then $\mathcal{W}(x_0, z_0)$ starts as a bi-graph over the xy-plane around (x_0, z_0). Let $\mathcal{W}_+(x_0, z_0)$ denote the upper graphical component, which is the graph of a function $z = z_+(r)$. Similarly, its lower graphical component is denoted by $\mathcal{W}_-(x_0, z_0)$ and it is the graph of a function $z = z_-(r)$.

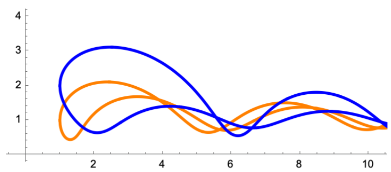

Figure 3. Generating curves of spherical rotational horo-shrinkers which do no intersect the rotational axis. In orange, the initial condition is $(x_0, z_0) = (1, 1)$. In blue, the initial condition is $(x_0, z_0) = (1, 2)$.

Let us analyze the behavior of $\mathcal{W}_+(x_0, z_0)$, that is, of the function $z_+(r)$: for $z_-(r)$ the arguments are analogous. The function z_+ satisfies $z_+(x_0) = z_0$, $z'_+(x_0) = \infty$ and for $r > x_0$ close enough to x_0 we have $z'_+(r) > 0$ and $z''_+(r) < 0$. At this point, we follow similar ideas that the ones developed in the proof of Thm. 4.3, so the graph z_+ must attain a local maximum, decrease, change its curvature and then attain a local minimum. This process is repeated proving that $z_+(r)$ oscillates around $z = 1$ as $r \to \infty$. □

Remark 4.5. The surfaces $\mathcal{B}(z_0)$ and $\mathcal{W}(x_0, z_0)$ of Thms. 4.3 and 4.4 can be thought as the analogous to the bowl soliton and the wing-like examples of translators in the theory of the mean curvature flow in $\mathbb{R}^3$. Up to a translation of $\mathbb{R}^3$, the bowl soliton is the unique translator in $\mathbb{R}^3$ intersecting orthogonally the rotation axis, while the wing-like examples form a one-parameter family of annuli, parametrized in terms of the distance to the rotation axis. In contrast, the situation for horo-shrinkers is a bit different. If the rotation axis is the z-axis (such as it has been considered in this section), the hyperbolic translations of $\mathbb{H}^3$ from the origin O (Euclidean homotheties) do not preserve equation (1). Therefore, two horo-shrinkers $\mathcal{B}(z_0)$ and $\mathcal{B}(z_1)$, $z_0 \neq z_1$, do not coincide by a hyperbolic translation of $\mathbb{H}^3$. For this reason, the family of surfaces $\mathcal{B}(z_0)$ is one-parametric. Similarly, the family $\mathcal{W}(x_0, z_0)$ is two-parametric.

We end this paper with the following observation. Using Mathematica, it is possible to observe that the surfaces $\mathcal{B}(z_0)$ and $\mathcal{W}(x_0, z_0)$ not only oscillate around H_1 but they converge to it at infinity. However, the authors have not been able to prove this convergence. The difficulty is that if we project the system (10) on the (z, θ)-plane, the 2-dimensional system is not autonomous by the presence of x. Or equivalently, the system (18) is non-autonomous. Anyway, it is important to point out that $(z, \theta) = (1, 0)$ (resp.

$(z, z') = (1, 0)$) is an equilibrium point of (10) (resp. (18)) regardless of the value of x (resp. of r). This equilibrium point corresponds to the horosphere H_1.

Acknowledgement

Antonio Bueno has been partially supported by the grant PID2021-124157NB-I00, funded by MCIN/ AEI/10.13039/501100011033/ "ERDF A way of making Europe" and by CARM, Programa Regional de Fomento de la Investigación, Fundación Séneca-Agencia de Ciencia y Tecnología Región de Murcia, reference 21937/PI/22.

Rafael López has been partially supported by MINECO/ MICINN/FEDER grant no. PID2023-150727NB-I00, and by the "María de Maeztu" Excellence Unit IMAG, reference CEX2020-001105- M, funded by MCINN/AEI/ 10.13039/501100011033/ CEX2020-001105-M.

References

[1] L. J. Alías, J. H. de Lira, M. Rigoli, Mean curvature flow solitons in the presence of conformal vector fields. J. Geom. Anal. 30 (2020), 1466–1529.

[2] M. Anderson, Complete minimal varieties in hyperbolic space. Invent Math. 69 (1982), 477–494.

[3] B. Andrews, X. Chen, Curvature flow in hyperbolic spaces. J. Reine Angew. Math. 729 (2017), 29–49.

[4] B. Andrews, Y. Wei, Quermassintegral preserving curvature flow in hyperbolic space. Geom. Funct. Anal. 28 (2018), 1183–1208.

[5] A. Bueno, Translating solitons of the mean curvature flow in the space $\mathbb{H}^2 \times \mathbb{R}$, *J. Geom.* **109** (2018).

[6] A. Bueno, R. López, A new family of translating solitons in hyperbolic space. Preprint (2023).

[7] E. Cabezas-Rivas, V. Miquel, Volume preserving mean curvature flow in the hyperbolic space. Indiana Univ. Math. J. 56 (2007), 2061–2086.

[8] E. Cabezas-Rivas, V. Miquel, Volume-preserving mean curvature flow of revolution hypersurfaces in a rotationally symmetric space. Math. Z. 261 (2009), 489–510.

[9] E. Cabezas-Rivas, V. Miquel, Volume preserving mean curvature flow of revolution hypersurfaces between two equidistants. Calc. Var. 43 (2012), 185–210.

[10] T. Colding, W. P. II. Minicozzi, E. K. Pedersen, Mean curvature flow. Bull. Am. Math. Soc. 52 (2015), 297–333.

[11] M. Do Carmo, M. Dajczer, Rotation hypersurfaces in spaces of constant curvature. Trans. Amer. Math. Soc. 277 (1983), 685–709.

[12] K. Ecker, Regularity theory for mean curvature flow. Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, 2004.

[13] C. Gerhardt, Inverse curvature flows in hyperbolic space. J. Differ. Geom. 89 (2011), 487–527.

[14] G. Huisken, The volume preserving mean curvature flow. J. Reine Angew. Math. 382 (1987), 35–48.

[15] G. Huisken, C. Sinestrari, Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. 183 (1999), 45–70.

[16] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature. Mem. Amer. Math. Soc. (1994) 108.

[17] Z. Ji, Cylindrical estimates for mean curvature flow in hyperbolic spaces. Commun. Pure Appl. Anal. 20 (2021), 1199–1211.

[18] M. A. Lawn, M. Ortega, Translating Solitons in a Lorentzian Setting, Submersions and Cohomogeneity One Actions, *Mediterr. J. Math.* **19** (2022).
[19] R. F. de Lima, A. K. Ramos, J. P. dos Santos, Solitons to mean curvature flow in the hyperbolic 3-space. arXiv:2307.14136 [math.DG] (2023).
[20] R. López, M. Munteanu, Solitons of the mean curvature flow in $\mathbb{S}^2 \times \mathbb{R}$. Differ. Geom. Appl. 99, 102243, (2025).
[21] J. Lott, Mean curvature flow in a Ricci flow background. Commun. Math. Phys. 313 (2012), 517–533..
[22] C. Mantegazza, Lecture Notes on Mean Curvature Flow. Progress in Mathematics, vol. 290, Birkhäuser/Springer Basel AG, Basel, 2011.
[23] L. Mari, J. Rocha de Oliveira, A. Savas-Halilaj, R. Sodré de Sena, Conformal solitons for the mean curvature flow in hyperbolic space. Ann Glob Anal Geom 65, 19 (2024).
[24] Y. Tonegawa, Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space. Math. Z. 221 (1996), 591–615.
[25] K. Wang, Singularities of mean curvature flow and isoperimetric inequalities in H^3. Proc. Amer. Math. Soc. 143 (2015), 2651–2660.

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