

Compact surfaces with boundary with prescribed mean curvature depending on the Gauss map

Antonio Bueno

Departamento de Ciencias

Centro Universitario de la Defensa de San Javier

E-30729 Santiago de la Ribera, Spain

Rafael López

Departamento de Geometría y Topología

Universidad de Granada, Granada, Spain

Abstract

Given a C^1 function \mathcal{H} defined in the unit sphere \mathbb{S}^2 , an \mathcal{H} -surface M is a surface in the Euclidean space \mathbb{R}^3 whose mean curvature H_M satisfies $H_M(p) = \mathcal{H}(N_p)$, $p \in M$, where N is the Gauss map of M . Given a closed simple curve $\Gamma \subset \mathbb{R}^3$ and a function \mathcal{H} , in this paper we investigate the geometry of compact \mathcal{H} -surfaces spanning Γ in terms of Γ . Under mild assumptions on \mathcal{H} , we prove non-existence of closed \mathcal{H} -surfaces, in contrast with the classical case of constant mean curvature. We give conditions on \mathcal{H} that ensure that if Γ is a circle, then M is a rotational surface. We also establish the existence of estimates of the area of \mathcal{H} -surfaces in terms of the height of the surface.

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1 Introduction

In 1910, Bernstein considered the prescribed mean curvature equation

$$\mathcal{M}(u) := \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = (1 + |Du|^2)^m, \quad (1)$$

where m is an integer, and $u = u(x)$ is a function defined in some domain of \mathbb{R}^2 [4, 5]. The left-hand side $\mathcal{M}(u)$ is the known mean curvature operator. In [4, p. 240], and when $m = -1/2$, he called Eq. (1) “*l’équation des surfaces, dont la courbure en chaque point is proportionnelle (égale) au cosinus de l’angle de la normale en ce point avec l’axe des z*”. The unit normal vector field on a surface in Euclidean space \mathbb{R}^3 is known as the Gauss map. Motivated by Bernstein, in this paper we study compact surfaces in Euclidean space \mathbb{R}^3 whose mean curvature is a function of the Gauss map. In PDE terminology, this equation can be viewed as the prescribed mean curvature equation

$$\mathcal{M}(u) = \mathcal{F}(Du), \quad (2)$$

in subsets of \mathbb{R}^2 , where the function $\mathcal{F}: \mathbb{R}^2 \rightarrow \mathbb{R}$ is given. Although the literature is enormous replacing \mathcal{F} by a function of type $\mathcal{F} = \mathcal{F}(x, u)$, the case considered in Eq. (2) is rather smaller. It was Bernstein himself who proved solvability of the Dirichlet problem of (1) for arbitrary boundary data in convex analytic domains, provided $m \leq -1/2$.

Sixty years later, Serrin revisited this equation in his seminal paper [31]. Although this article is rather known by the solvability of the Dirichlet problem for the constant mean curvature equation, Serrin considers many other types of quasilinear elliptic equations. In [31, pages 477-8], he investigated again the Dirichlet problem of the equation studied by Bernstein, changing analyticity of the domain by smoothness as well as he generalized the results to arbitrary dimensions. More recently, equations of type (2) have been considered, specially for the study of radial solutions and the Dirichlet problem: see, for example, [2, 3, 11, 12, 13, 14, 15, 26, 27].

Returning to Eq. (1), it is surprising, even intriguing, that the case $m = -1/2$, that is

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{1}{\sqrt{1 + |Du|^2}}, \quad (3)$$

has acquired considerable interest in recent times. Solutions of Eq. (3) are called translating solitons of the mean curvature flow which evolve purely by translations. The theory of the mean curvature flow is of high activity in the last two decades due to the

pionnering work of Huisken [18]. Translating solitons are eternal solutions of the mean curvature flow in the sense that their evolution exists for all times $-\infty < t < \infty$. These surfaces are of special interest since they arise as a type of singularity in the mean curvature flow. Notice that the flow can develop singularities, that is, solutions which become non-smooth in finite time. The so-called type II singularities are related with translating solitons because if the initial surface in the flow is mean convex, and it develops a type II singularity, then the limit surface, after rescaling the flow, is a translating soliton [19].

A strategy to address with Eq. (2) is that the Gauss map of the graph M of u , when regarded as a surface in Euclidean space \mathbb{R}^3 , is

$$N_p = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}(x), \quad p = (x, u(x)) \in M.$$

If $e_3 = (0, 0, 1)$, then $\langle N, e_3 \rangle = 1/\sqrt{1 + |Du|^2}$, where \langle, \rangle is the Euclidean metric of \mathbb{R}^3 . This says us that the right-hand side in (2) can be expressed as $\mathcal{F}(Du) = \mathcal{H}(N_p)$, where \mathcal{H} is a certain function defined on the unit sphere S^2 . This motivates the following definition:

Definition 1.1 *Let \mathcal{H} be a C^1 function in S^2 . A surface M in \mathbb{R}^3 is said to be an \mathcal{H} -surface if its mean curvature H_M satisfies*

$$H_M(p) = \mathcal{H}(N_p), \quad \forall p \in M. \quad (4)$$

Notice that Eq. (4) is a prescribed mean curvature equation depending on the Gauss map of the surface. This dependence on the Gauss map makes that surfaces defined by (4) can be viewed as a type of anisotropic mean curvature equation [2]. Studying hypersurfaces in Lorentz-Minkowski space, the authors in [14] name “gradient dependent prescribed mean curvature equation” if \mathcal{F} is of type $\mathcal{F}(x, u, Du)$. A last remark about the above definition is that if we replace in (4) the mean curvature by the Gauss curvature, the solutions of the corresponding equations are just the solutions of the well-known Minkowski problem for ovaloids [28].

For particular choices of the prescribed function, some deeply studied geometric theories arise. We highlight the following examples:

1. Surfaces of constant mean curvature (CMC surfaces for short). Here $\mathcal{H} = H_0 \in \mathbb{R}$ is a constant. If $H_0 = 0$ we have minimal surfaces.
2. λ -translators. The prescribed function is $\mathcal{H}(x) = \langle x, \mathbf{w} \rangle + \lambda$, where $\mathbf{w} \in S^2$ and $\lambda \in \mathbb{R}$. The vector \mathbf{w} is called the density vector. When $\lambda = 0$, we have translating solitons of the mean curvature flow.

3. Solitons of powers of the mean curvature flow. The mean curvature of these surfaces is $H_M(p) = \langle N_p, v \rangle^\alpha + \lambda$, where $\alpha > 0$ and $\lambda \in \mathbb{R}$. When $\alpha > 1$, these surfaces are \mathcal{H} -surfaces for $\mathcal{H}(x) = \langle x, v \rangle^\alpha + \lambda$. Notice that $\mathcal{H} \in C^1(\mathbb{S}^2)$.

The theory of complete, non-compact \mathcal{H} -surfaces has been recently being developed by the first author in joint work with Gálvez and Mira [9, 10], taking as starting point the theories of CMC surfaces and translating solitons. See also [7, 23] for a study of invariant λ -translators and [6, 8] for a more general linear Weingarten prescribed curvature problem in \mathbb{R}^3 .

Although the main properties of complete \mathcal{H} -surfaces have been exhibited in the aforementioned papers, less is known about compact \mathcal{H} -surfaces with boundary. When \mathcal{H} is constant, there is a large literature about this topic; see e.g. [22] for an outline of the development of this theory. This line of inquiry has been also studied for translating solitons [29] and λ -translators [24], but as far as we know, there is not a dedicated research for an arbitrary prescribed function \mathcal{H} . To fix the terminology, let $\Gamma \subset \mathbb{R}^3$ a closed curve and $\psi : M \rightarrow \mathbb{R}^3$ be an immersion of a surface M with boundary ∂M . We say that Γ is the boundary of M (if the immersion ψ is understood) if $\psi|_{\partial M}$ is a diffeomorphism onto Γ . When we regard M immersed in \mathbb{R}^3 , we will not distinguish between Γ and ∂M , provided that both are diffeomorphic, and we will commonly say that M spans Γ . The main problem addressed in this paper is to investigate the geometry of compact \mathcal{H} -surfaces in terms of their boundary. For example, it is desirable that M inherits the symmetries of Γ , or that for a fixed curve Γ , not every \mathcal{H} is admissible for the existence of an \mathcal{H} -surface with boundary Γ .

Based on the above examples, in this paper we study the solutions of (4) using ideas and techniques coming from the theory of surfaces with constant mean curvature. This adds a geometric viewpoint to the problems related with (2). For example, we will use the moving plane method introduced by Alexandrov, who proved that spheres are the only compact constant mean curvature surfaces embedded in \mathbb{R}^3 , [1]. Later, Serrin used the same method in the study of elliptic overdetermined problems [32].

In virtue of Eq. (4), the following are three fundamental properties of \mathcal{H} -surfaces that will allow us to approach these questions: (1) \mathcal{H} -surfaces are invariant under Euclidean translations; (2) \mathcal{H} -surfaces are locally solutions of a quasilinear, elliptic PDE and in particular the maximum principle applies; and (3) any symmetry of \mathcal{H} in \mathbb{S}^2 induces a linear isometry of \mathbb{R}^3 that sends \mathcal{H} -surfaces into \mathcal{H} -surfaces. Nonetheless, the arbitrariness of \mathcal{H} entails further incoming difficulties that have to be taken into account. For example, in general, \mathcal{H} -surfaces are not solutions to a variational problem involving

geometric measures such as the area or volume. On the other hand, we need to take into account the loss of symmetry of Eq. (4). For instance, the reflection of an \mathcal{H} -surface with respect to an arbitrary plane is not necessarily an \mathcal{H} -surface, unless this plane is a symmetry plane of \mathcal{H} . In some of the results that we will obtain, we will compare the analogue situation for CMC surfaces, emphasizing the differences, if any.

The organization of this paper is as follows. In Section 2, we state the maximum and comparison principles, which are the cornerstone to obtain the majority of the forthcoming results. We analyze the existence and non-existence of closed \mathcal{H} -surfaces, proving in Prop. 2.4 that the condition $\mathcal{H} \neq 0$ is necessary for the existence of a closed \mathcal{H} -surface. Nonetheless, in contrast to the CMC case, the condition $\mathcal{H} \neq 0$ is not sufficient for the existence of a closed \mathcal{H} -surface, as revealed in Prop. 2.5. In Section 3, we investigate whether the symmetries of Γ are inherited to a compact \mathcal{H} -surface spanning Γ . The usual method to study this problem is the so-called Alexandrov reflection technique, and one of the major issues that we can find is that the compact surface with planar boundary may have points lying at both sides of the plane where its boundary lies, just as in the CMC case. In Section 4, we give sufficient conditions to guarantee that a compact \mathcal{H} -surface lies at one side of the plane containing its boundary: Ths. 4.1 and 4.3. As a particular but important case of \mathcal{H} -surfaces, these results will be applied to λ -translators when $|\lambda| \leq 1$. Finally, in Section 5, we establish an estimate of the area of a compact \mathcal{H} -surface with planar boundary in terms of the height to the boundary (Th. 5.1).

2 The maximum principle for \mathcal{H} -surfaces and first consequences

In this section, we state the maximum principle for \mathcal{H} -surfaces, extending the well-known situation of CMC surfaces. Let (x_1, x_2, x_3) be the canonical coordinates of \mathbb{R}^3 and let $\{e_1, e_2, e_3\}$ denote the usual basis of \mathbb{R}^3 . Given an \mathcal{H} -surface in \mathbb{R}^3 , we can locally express it as the graph of a function u over each tangent plane, being u a solution to an elliptic, second order quasilinear PDE. Specifically, if $u = u(x_1, x_2)$, then u satisfies the equation

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 2\mathcal{H} \left(\frac{(-Du, 1)}{\sqrt{1 + |Du|^2}} \right), \quad (5)$$

where div, D denote respectively the divergence and gradient operators on \mathbb{R}^2 . In this spirit, Eq. (5) can be expressed as $Q(u) = 0$, where

$$Q(u) = \sum_{i,j=1}^2 a_{ij}(x, u, Du) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(x, u, Du), \quad x = (x_1, x_2).$$

As a consequence, the difference of two solutions of $Q(u) = 0$ satisfies a linear elliptic equation. The term $b(x, u, Du)$ gathers the right-hand side of (5). In order to apply the Hopf maximum principle in its interior and boundary versions, it is required that b is continuously differentiable with respect to the variable Du [16, Th. 92], hence we require the function \mathcal{H} to be C^1 in Definition 1.1. A geometric version of the maximum principle is the following:

Lemma 2.1 (Maximum principle of \mathcal{H} -surfaces) *Let M_1, M_2 be two \mathcal{H} -surfaces, possibly with smooth boundary. Assume that one of the following two conditions holds:*

1. *There exists $p \in \operatorname{int}(M_1) \cap \operatorname{int}(M_2)$ such that $(N_1)_p = (N_2)_p$, where N_i is the Gauss map of M_i .*
2. *There exists $p \in \partial M_1 \cap \partial M_2$ such that $(N_1)_p = (N_2)_p$ and $v_1(p) = v_2(p)$, where v_i denotes the interior unit conormal of ∂M_i .*

If M_1 lies at one side of M_2 around p , then $M_1 = M_2$.

In the theory of CMC surfaces, the maximum principle has a stronger version when applied to minimal surfaces, the so-called *tangency principle*. It states that two minimal surfaces cannot have a common tangent point and one lying at one side of the other, regardless of the orientation chosen. This is because the orientation of a minimal surface can be reversed without changing the underlying PDE. In this spirit, we have the corresponding tangency principle of \mathcal{H} -surfaces for a certain type of functions \mathcal{H} .

Corollary 2.2 (Tangency principle of \mathcal{H} -surfaces) *Let \mathcal{H} such that $\mathcal{H}(-x) = -\mathcal{H}(x)$, $x \in S^2$. If M_1, M_2 are two \mathcal{H} -surfaces and M_1 lies at one side of M_2 , then $M_1 = M_2$.*

Proof: Suppose that M_1 lies at one side of M around some common (interior or boundary) point p_0 . If $(N_1)_{p_0} = (N_2)_{p_0}$, then we apply the maximum principle of \mathcal{H} -surfaces to get the result. Otherwise, that is, when $(N_1)_{p_0} = -(N_2)_{p_0}$, we reverse the orientation

of M_1 . Then M_1 is also an \mathcal{H} -surface with this orientation because its mean curvature H'_{M_1} satisfies that for all $p \in M_1$, we have

$$H'_{M_1}(p) = -H_{M_1}(p) = -\mathcal{H}((N_1)_p) = \mathcal{H}(-(N_1)_p).$$

Then we are again under the conditions to apply the maximum principle of \mathcal{H} -surfaces around p_0 , obtaining the result. \square

Some examples of functions \mathcal{H} satisfying the condition $\mathcal{H}(-x) = -\mathcal{H}(x)$ are $\mathcal{H}(x) = \langle x, v \rangle^n$, for a fixed $v \in \mathbb{S}^2$ and n an odd number ($n = 1$ corresponds with translating solitons) and $\mathcal{H}(x) = \sin(\langle x, v \rangle)$, for a fixed $v \in \mathbb{S}^2$.

The following result is the classical mean curvature comparison principle but adapted to \mathcal{H} -surfaces.

Lemma 2.3 (Comparison principle of \mathcal{H} -surfaces) *Let M_i be \mathcal{H}_i -surfaces, $i = 1, 2$. Assume that there exists $p \in M_1 \cap M_2$ such that $(N_1)_p = (N_2)_p$. If M_1 lies locally above M_2 around p , then, $\mathcal{H}_1((N_1)_p) \geq \mathcal{H}_2((N_2)_p)$.*

A fundamental property that will be applied throughout the paper is the following. Let $v \in \mathbb{S}^2$ and assume that $\mathcal{H}(v) = 0$. Then it is immediate that any plane oriented by v is an \mathcal{H} -surface regardless of whether \mathcal{H} is not identically 0. We show next that in the same class of \mathcal{H} -surfaces, one may have different types of examples that widely differ one from the other. Fix some $\varepsilon \in (0, 1)$ and let us define

$$f(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ g(t), & \text{if } t \in [0, \varepsilon] \\ 0, & \text{if } t \geq \varepsilon, \end{cases}$$

where $g(0) = 1, g(\varepsilon) = 0$, g is decreasing and such that f is of class C^∞ . Define $\mathcal{H} \in C^1(\mathbb{S}^2)$ by $\mathcal{H}(x) = f(\langle x, e_3 \rangle)$. Some \mathcal{H} -surfaces are the following:

1. The upper half-sphere \mathbb{S}_+^2 (or any open subset of it) with the downwards orientation.
2. Any plane oriented by v , being $v \in \mathbb{S}^2$ such that $\langle v, e_3 \rangle \geq \varepsilon$.

Note that the coexistence of both planes and half-spheres in the same class of \mathcal{H} -surfaces does not contradict neither the maximum nor the mean curvature comparison principles.

Indeed, if they are tangent at some point their unit normals are necessarily opposite; otherwise, they are no longer \mathcal{H} -surfaces since they do not fulfill (4).

We now derive some direct consequences of the maximum principle. Given $\mathcal{H} \in C^1(\mathbb{S}^2)$, we are interested in the existence of closed (compact without boundary) \mathcal{H} -surfaces. When \mathcal{H} is a positive constant there are such closed examples, namely, spheres. This makes a huge difference when \mathcal{H} is identically zero, since it is well-known the non-existence of closed minimal surfaces. The next two results were exhibited in [10], but we give their proof for the sake of completeness. The first one gives a very general assumption on \mathcal{H} in order to forbid the existence of a closed \mathcal{H} -surface.

Proposition 2.4 *Let $\mathcal{H} \in C^1(\mathbb{S}^2)$. If \mathcal{H} vanishes at some point, then there are no closed \mathcal{H} -surfaces in \mathbb{R}^3 .*

Proof: By contradiction, assume that M is a closed \mathcal{H} -surface. Without loss of generality, we can assume that $\mathcal{H}(e_3) = 0$. Let Π be the plane of equation $x_3 = 0$, which we orient by e_3 . After a vertical translation, and by compactness of M , we can assume that M is contained in the half-space $\{x \in \mathbb{R}^3 : x_3 \geq 0\}$ and that $M \cap \Pi \neq \emptyset$. Let $p_0 \in M \cap \Pi$. Since p_0 is an interior point of M , then $N_{p_0} = \pm e_3$.

If $N_{p_0} = e_3$, we apply the maximum principle for \mathcal{H} -surfaces between the plane Π and M and we conclude that $M \subset \Pi$. This is a contradiction because M is closed. Therefore, $N_{p_0} = -e_3$. We now consider Π oriented with the vector $-e_3$. The comparison principle of \mathcal{H} -surfaces yields $\mathcal{H}(-e_3) \leq 0$. Moreover, $\mathcal{H}(-e_3) < 0$ for if $\mathcal{H}(-e_3) = 0$ we contradict the maximum principle.

By compactness again, let be $p_1 \in M$ a point of maximum height to Π and P the affine tangent plane to M at p_1 . At this point we have $N_{p_1} = \pm e_3$. If $N_{p_1} = e_3$ we contradict the maximum principle. If $N_{p_1} = -e_3$, then M lies locally above P around p_1 but $\mathcal{H}(-e_3) < 0$, which contradicts the comparison principle. In any case, we arrive to the desired contradiction. \square

From this result, the condition $\mathcal{H} \neq 0$ is necessary for the existence of closed \mathcal{H} -surfaces. In the same fashion as in the case that \mathcal{H} is a non-zero constant, it would be expectable that the condition $\mathcal{H} \neq 0$, say $\mathcal{H} > 0$, is also sufficient for the existence of a closed \mathcal{H} -surface. Nevertheless, the following result exhibit examples of positive functions \mathcal{H} for which there are no closed \mathcal{H} -surfaces.

Proposition 2.5 *Let be \mathcal{H} such that $\mathcal{H}(x) = h_0(x) + \lambda$, where $\lambda \in \mathbb{R}$ and $h_0 \in C^1(\mathbb{S}^2)$ is not identically zero. If there is $v \in \mathbb{S}^2$ such that $h_0(x)\langle x, v \rangle \geq 0$ for all $x \in \mathbb{S}^2$, then there are no closed \mathcal{H} -surfaces.*

Proof: By contradiction, assume that M is a closed \mathcal{H} -surface with $\mathcal{H} = h_0 + \lambda$ defined under the hypothesis of the proposition. The function $f \in C^\infty(M)$ defined by $f(p) = \langle p, v \rangle$ satisfies $\Delta f = 2H\langle N, v \rangle$, where Δ is the Laplacian in M . Since $\partial M = \emptyset$, the divergence theorem gives

$$0 = \int_M \mathcal{H}(N)\langle N, v \rangle dM = \int_M h_0(N)\langle N, v \rangle dM + \lambda \int_M \langle N, v \rangle dM.$$

The second integral vanishes since M is closed. Therefore, the first integral must vanish as well, but $h_0(N)\langle N, v \rangle \geq 0$ and h_0 is not identically zero by hypothesis, arriving to the desired contradiction. \square

Examples of positive functions $\mathcal{H} \in C^1(\mathbb{S}^2)$ under the hypothesis of Prop. 2.5 are the following:

1. Let $v \in \mathbb{S}^2$. For $\lambda > 1$ and n an odd natural number, define $\mathcal{H}(x) = \langle x, v \rangle^n + \lambda$.
2. Let $v \in \mathbb{S}^2$. For $\lambda > 2$, define $\mathcal{H}(x) = \langle x, v \rangle(1 + \sin(\langle x, v \rangle)) + \lambda$.

The last two results exhibit that under mild assumptions on \mathcal{H} , there do not exist closed \mathcal{H} -surfaces. Therefore, for these prescribed functions, any compact \mathcal{H} -surface has non-empty boundary. This makes the study of compact \mathcal{H} -surfaces with boundary not only natural, but necessary.

Given a simple closed curve Γ , our first approach to this study is to derive some restrictions for \mathcal{H} in terms of the geometry of Γ for the existence of a compact \mathcal{H} -surface spanning Γ . In the theory of CMC surfaces, it is known that given a closed curve $\Gamma \subset \mathbb{R}^3$, not all $H \in \mathbb{R}$ are allowed to have H -surfaces spanning Γ . This was proved initially by Heinz [17] and generalized later using a flux formula for CMC surfaces: see details in [22]. For example, if Γ is a closed simple curve, then $|H| \leq L/(2\text{area}(\Omega))$, where L is the length of Γ and Ω is the planar domain bounded by Γ . As far as the authors know, there is no an analogue to the flux formula in the theory of \mathcal{H} -surfaces. Nonetheless, we are able to prove a similar necessary condition in case that the surface is a graph.

Proposition 2.6 Let Γ be a closed simple curve contained in a plane $\Pi = v^\perp$, $v \in \mathbb{S}^2$. If $\mathcal{H} > 0$ is a C^1 function defined in \mathbb{S}^2 , then a necessary condition for the existence of a compact \mathcal{H} -graph spanning Γ is

$$\mathcal{H}_{\min} := \min\{\mathcal{H}(x) : x \in \mathbb{S}^2\} \leq \frac{L}{2 \operatorname{area}(\Omega)},$$

where L is the length of Γ and $\Omega \subset \Pi$ is the domain bounded by Γ .

Proof: By (5) and the divergence theorem, we have

$$\int_{\Omega} 2(\mathcal{H} \circ N) d\Omega = \int_{\Omega} \operatorname{div} \left(\frac{(-Du, 1)}{\sqrt{1 + |Du|^2}} \right) d\Omega = \int_{\partial\Omega} \left\langle \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}, \mathbf{n} \right\rangle ds \leq L,$$

where \mathbf{n} is the unit outward normal vector of $\partial\Omega$ as planar curve of Π . Since $\mathcal{H} > 0$, the first term in the above identities can be estimated from below by $2\mathcal{H}_{\min} \int_{\Omega} d\Omega$, obtaining the result. \square

Going back to Prop. 2.4, one may expect that if $\mathcal{H}(v) = 0$, the class of \mathcal{H} -surfaces behaves as minimal surfaces. The following result emphasizes this relationship by generalizing the *convex hull property of minimal surfaces*, which states that any compact minimal surface must lie inside the convex hull of its boundary.

Proposition 2.7 Let Γ be a closed curve in \mathbb{R}^3 . Let $\mathcal{H} \in C^1(\mathbb{S}^2)$ such that $\mathcal{H}(\pm v) = 0$ for some $v \in \mathbb{S}^2$. If M is a compact \mathcal{H} -surface with boundary Γ , then M is contained in the slab

$$\{x \in \mathbb{R}^3 : \mu_1(v) \leq \langle x, v \rangle \leq \mu_2(v)\},$$

where

$$\mu_1 = \min\{\langle p, v \rangle : p \in \Gamma\}, \quad \mu_2(v) = \max\{\langle p, v \rangle : p \in \Gamma\}.$$

Proof: After a change of coordinates, we assume $v = e_3$. Arguing by contradiction, assume that M contains points whose third coordinate x_3 is bigger than $\mu_2(e_3)$. Consider $p \in M$ the highest point of M , which in particular satisfies $x_3(p) > \mu_2(e_3)$, and hence p is an interior point of M . Moreover, $N_p = e_3$ or $N_p = -e_3$. We orient the (affine) tangent plane $T_p M$ by the vector N_p . Since $\mathcal{H}(N_p) = 0$, then $T_p M$ is an \mathcal{H} -surface and the maximum principle implies that M is included in $T_p M$: this a contradiction because $\Gamma \cap T_p M = \emptyset$. \square

3 The Alexandrov method for \mathcal{H} -surfaces

In this section, we investigate how the geometry of a simple closed curve Γ determines the shape of a compact \mathcal{H} -surface that spans it. A first interesting case is to investigate whether the symmetries of the boundary curve Γ are inherited to a compact \mathcal{H} -surface. The main tool for achieving this purpose is the so-called Alexandrov reflection technique. A major difference with the CMC setting is that in the case of an arbitrary prescribed function \mathcal{H} , we have to take into account the loss of symmetries of the resulting PDE fulfilled by the \mathcal{H} -surfaces. As a matter of fact, the reflection about an arbitrary plane no longer sends an \mathcal{H} -surface into an \mathcal{H} -surface. In this spirit, the third hypothesis of the following result is paramount.

Theorem 3.1 *Let Γ be a closed simple curve contained in a plane Π . Assume that:*

1. Γ is symmetric with respect to the reflection about a plane P orthogonal to Π ,
2. P separates Γ into two graphs over the line $\Pi \cap P$.
3. \mathcal{H} is invariant with respect to the reflection about the vector plane parallel to P .

If M is a compact, embedded \mathcal{H} -surface spanning Γ and M lies at one side of Π , then M is symmetric with respect to P .

Proof: After a change of coordinates, we assume that Π is the plane of equation $x_3 = 0$ and that P is the plane of equation $x_1 = 0$. Let $\Omega \subset \Pi$ the domain bounded by Γ . Since M is embedded, $M \cup \Omega$ defines a closed surface, possibly singular along $M \cap \Omega$, that bounds a domain W in \mathbb{R}^3 . Assume without losing generality that $M \subset \{x_3 \geq 0\}$. Let us orient M so N points towards the interior of W .

For any $t \in \mathbb{R}$, we denote P_t the plane of equation $x_1 = t$. We define $M(t)^- = M \cap \{x_1 \leq t\}$, $M(t)^+ = M \cap \{x_1 \geq t\}$ and $M(t)^*$ the reflection of $M(t)^+$ about P_t . At this point, the third item in the hypotheses of Th. 3.1 allows us to assert that $M(t)^*$ is again a \mathcal{H} -surface where the orientation on $M(t)^*$ is N^* , being $N^* = R_{P_t}(N)$. Here R_{P_t} denotes the reflection about the plane P_t . Indeed, if $q \in M(t)^*$, then $q = R_{P_t}(p)$ for some $p \in M(t)^+$. Then

$$H_{M(t)^*}(q) = H_{M(t)^+}(p) = \mathcal{H}(N_p) = \mathcal{H}(R_{P_t}(N_p)) = \mathcal{H}(N_q^*),$$

where we have used the identity $\mathcal{H} = \mathcal{H} \circ R_{P_t}$ thanks to the third item of the hypothesis.

Although the proof of theorem is standard, by completeness we present an outline of it in order to emphasize where the third hypothesis in the statement of Th. 3.1 is crucial. Arguing by contradiction, assume that M is not symmetric with respect to P . Then, there are two points $q_1, q_2 \in M \setminus \Gamma$ such that the line $\overline{q_1 q_2}$ is orthogonal to P , q_1 and q_2 are in different components of $\mathbb{R}^3 \setminus P$ and $\text{dist}(q_1, P) \neq \text{dist}(q_2, P)$; without losing generality, assume $x_1(q_2) < 0 < x_1(q_1)$. If we name x^* to the reflection of $x \in \mathbb{R}^3$ about P , then $x_1(q_1^*) < x_1(q_2)$.

By compactness of M , for t large enough we have $P_t \cap M = \emptyset$. We let t decrease until we arrive to a first instant $t_0 > 0$ where there is a first contact point between P_{t_0} and M ; we know that $t_0 > 0$ since $x_1(q_1) > 0$. The embeddedness of M and since M lies above Π ensures the existence of $\varepsilon > 0$ such that $\text{int}(M(t)^*) \subset W$ for every $t \in (t_0 - \varepsilon, t_0)$. By compactness of M and by decreasing t , we define

$$t_1 = \inf\{t \in \mathbb{R} : \text{int}(M(t)^*) \subset W \text{ and } M(t)^* \text{ is a graph over } \Pi_t\}.$$

Because Π is a plane of symmetry of Γ , $t_1 \geq 0$. For $t < t_1$, whatever the reason for which t fails to be the infimum, it happens $M(t)^* \not\subset \overline{W}$. In our setting, $t_1 > 0$ because of the existence of q_1, q_2 such that $x_1(q_1^*) < x_1(q_2) < 0$.

Since $M(t)^+$ is a graph over P_t for $t > t_1$ and $\Gamma \cap \{x_1 > 0\}$ is also a graph over the line $P \cap \Pi$, we have $\partial M(t_1)^* \cap \Gamma \subset P_{t_1}$. The latter hypothesis is also crucial since it could happen that the first meeting point p between $M(t_1)^*$ and $M(t_1)^-$ lies in Γ where none of the surfaces $M(t_1)^*$ and $M(t_1)^-$ are tangent. There are two possibilities for p :

1. There exists $p \in \text{int}(M(t_1)^*) \cap \text{int}(M(t_1)^-)$. Since p is an interior point, $M(t_1)^*$ and $M(t_1)^-$ are tangent and the orientation of both surfaces point towards the interior of W and in particular they agree. The maximum principle implies that $M(t_2)^* = M(t_2)^-$ and thus P_{t_1} is a plane of symmetry of M . However, this contradicts that Γ is not invariant by reflections about P_{t_1} .
2. The surface M is orthogonal to P_{t_1} at some $p \in \partial M(t_1)^* \cap \partial M(t_1)^-$. This time we use the boundary version of the maximum principle to conclude that P_{t_1} is a plane of symmetry of M , which is again a contradiction.

This implies that M is symmetric with respect to P , proving the result. \square

Some examples of prescribed functions under the hypothesis of Th. 3.1 are

$$\mathcal{H}(x) = \langle x, v \rangle^\alpha + \lambda, \quad x \in \mathbb{S}^2,$$

where $\alpha \geq 1$ (to ensure that \mathcal{H} is C^1) and $\lambda \in \mathbb{R}$. Indeed, if P is a vector plane containing v and R_P is the reflection about P , then for every $x \in \mathbb{S}^2$ we have $\mathcal{H}(x) = \mathcal{H}(R_P(x))$. In particular, if M is an \mathcal{H} -surface then the reflection about any plane parallel to P is again an \mathcal{H} -surface.

For the particular case that the above symmetry property holds for every vector plane containing v , then \mathcal{H} is also invariant under any rotation of \mathbb{S}^2 that preserves the v -direction pointwise fixed. Our purpose is to conclude that M is a rotational \mathcal{H} -surface when its boundary is a circle. In this fashion, we give the following definition.

Definition 3.2 *Let be $v \in \mathbb{S}^2$. A function $\mathcal{H} \in C^1(\mathbb{S}^2)$ is said to be rotational about the v -direction if there exists $\mathfrak{h} \in C^1([-1, 1])$ such that*

$$\mathcal{H}(x) = \mathfrak{h}(\langle x, v \rangle), \quad \forall x \in \mathbb{S}^2.$$

As a consequence, if \mathcal{H} is rotational about some direction v , then rotations about any line parallel to v send \mathcal{H} -surfaces into \mathcal{H} -surfaces and in particular the notion of rotational \mathcal{H} -surface is well-defined. We refer the reader to [9] for a deep study and classification of rotational \mathcal{H} -surfaces, including the achievement of a Delaunay-type classification theorem. Furthermore, as a particular case of a more general study regarding a linear Weingarten relation, it was exhibited in [6] necessary and sufficient conditions for the existence of a rotational \mathcal{H} -surface M^+ (resp. M^-) intersecting orthogonally the rotation axis with unit normal $N = v$ (resp. $N = -v$). The following result gives a characterization of M^\pm .

Corollary 3.3 *Let \mathcal{H} be a rotational functional about $v \in \mathbb{S}^2$. Let Γ be a circle in a plane $\Pi = v^\perp$ such the plane Π is oriented with v . Assume that M is a compact, embedded \mathcal{H} -surface spanning Γ and lying at one side of Π .*

1. *If M lies above Π , then $\mathcal{H}(-v) > 0$ and $M = M^-$.*
2. *If M lies below Π , then $\mathcal{H}(v) > 0$ and $M = M^+$.*

4 Compact \mathcal{H} -surfaces with planar boundary

As a consequence of Th. 3.1, it is interesting to have conditions that ensure that an embedded compact \mathcal{H} -surface with planar boundary lies contained in one side of the boundary plane. We will prove two results in this direction. The first one extends a similar situation for CMC surfaces [20].

Theorem 4.1 *Let Π be a plane and $\Gamma \subset \Pi$ be a closed simple curve that bounds a domain Ω , and let $\text{ext}(\Omega) = \Pi \setminus \overline{\Omega}$. If M is a compact, embedded \mathcal{H} -surface that spans Γ and $M \cap \text{ext}(\Omega) = \emptyset$, then $\text{int}(M)$ lies contained in one of the two closed halfspaces determined by Π .*

Proof: Without loss of generality, we assume that Π is the plane of equation $x_3 = 0$. Let $S^1(r) \subset \Pi$ a circle of radius r and $S_+^2(r)$ the upper hemisphere in the half-space $\{x_3 \geq 0\}$, with $\partial S_+^2(r) = S^1(r)$, and $D(r)$ the disc bounded by $S^1(r)$. Since M is compact, let r big enough such that $M \cap \{x_3 \geq 0\}$ is strictly contained in the domain bounded by $D(r) \cup S_+^2(r)$. In this setting,

$$K = M \cup (D(r) \setminus \Omega) \cup S_+^2(r)$$

is an embedded, closed surface in \mathbb{R}^3 , possibly not smooth along $\Gamma \cup S^1(r)$, that bounds a domain W in \mathbb{R}^3 . Let N be the unit normal of M and take on K the orientation \tilde{N} such that $\tilde{N}|_M = N$. Then \tilde{N} points towards either the interior or the exterior of W . We will assume that \tilde{N} points towards the interior of W : on the contrary, the argument is similar.

By contradiction, assume that M has interior points in both sides of Π . Let $p, q \in \text{int}(M)$ be the points of minimum and maximum height, respectively, to Π with $x_3(p) < 0$ and $x_3(q) \geq 0$ (if $x_3(p) \leq 0$ and $x_3(q) > 0$ the argument is similar). Since \tilde{N} points towards the interior of W , then $N_p = N_q = e_3$. Let T_pM and T_qM be the affine tangent planes to M at p and q , oriented with unit normal e_3 . Since M lies above T_pM (resp. below T_qM) locally around p (resp. around q), hence the mean curvature comparison principle yields $H_M(p) \geq 0$ and $H_M(q) \leq 0$. Thus

$$0 \leq H(p) = \mathcal{H}(N_p) = \mathcal{H}(e_3) = \mathcal{H}(N_q) = H(q) \leq 0.$$

Therefore $\mathcal{H}(e_3) = 0$. Hence T_pM and T_qM are \mathcal{H} -surfaces. The maximum principle of \mathcal{H} -surfaces applied to M and T_pM implies that M is included in T_pM . This is a contradiction because Γ is not included in the plane T_pM . \square

An immediate application of this theorem is for graphs.

Corollary 4.2 *If M is a compact \mathcal{H} -graph spanning a planar closed simple curve, then lies contained in one of the two closed halfspaces determined by the boundary plane.*

In the second result, we prove that the surface lies contained in one side of the boundary plane $\Pi = v^\perp$ if \mathcal{H} takes a special behaviour at $v \in S^2$. Comparing with the hypothesis of Th. 4.1, we drop the assumption of the embeddedness of the surface.

Theorem 4.3 Let $\mathcal{H} \in C^1(\mathbb{S}^2)$ and $v \in \mathbb{S}^2$ such that $\mathcal{H}(v)\mathcal{H}(-v) < 0$ and $\Pi = v^\perp$. Let M be a compact \mathcal{H} -surface whose boundary ∂M is contained in the plane $\Pi = v^\perp$.

1. If $\mathcal{H}(v) > 0$, then $\text{int}(M)$ lies in the halfspace $\{x \in \mathbb{R}^3 : \langle x, v \rangle < 0\}$.
2. If $\mathcal{H}(v) < 0$, then $\text{int}(M)$ lies in the halfspace $\{x \in \mathbb{R}^3 : \langle x, v \rangle > 0\}$.

Examples of functions \mathcal{H} satisfying the condition of this theorem are $\mathcal{H}(x) = \langle x, v \rangle^n$, where n is an odd natural number.

Proof: Without loss of generality, we assume that $v = e_3$. We prove the first item, since the second one is analogous. Assume $\mathcal{H}(e_3) > 0$ and, by hypothesis, $\mathcal{H}(-e_3) < 0$. Arguing by contradiction, suppose the existence of interior points of M in $\{x \in \mathbb{R}^3 : \langle x, e_3 \rangle \geq 0\}$. Let $q = \max\{\langle p, e_3 \rangle : p \in M\}$, in particular, $\langle q, e_3 \rangle \geq 0$. Since q is an interior point, its tangent plane $T_q M$ is the plane of equation $x_3 = x_3(q)$. Thus $N_q = e_3$ or $N_q = -e_3$.

Suppose that $N_q = e_3$. We orient $T_q M$ by the vector e_3 and in particular, $T_q M$ is an \mathcal{H}_1 -surface where $\mathcal{H}_1(x) = \mathcal{H}(x) - \mathcal{H}(e_3)$. Then $T_q M$ lies above M around q and the comparison principle of \mathcal{H} -surfaces implies $\mathcal{H}_1(N_q) \geq \mathcal{H}(N_q)$. This says $0 \geq \mathcal{H}(e_3)$, which is not possible. Thus necessarily $N_q = -e_3$. Now we orient $T_q M$ by $-e_3$, being $T_q M$ an \mathcal{H}_2 -surface for $\mathcal{H}_2(x) = \mathcal{H}(x) - \mathcal{H}(-e_3)$. Since now M lies above $T_q M$ around q , the comparison principle yields $\mathcal{H}(N_q) \geq \mathcal{H}_2(N_q)$, that is, $\mathcal{H}(-e_3) \geq 0$. This contradiction proves the result. \square

A context where Theorem 4.3 is useful is in the study of λ -translators with boundary. Recall that a surface M in \mathbb{R}^3 is a λ -translator if its mean curvature H_M satisfies $H_M(p) = \langle N_p, \mathbf{w} \rangle + \lambda$, where $\mathbf{w} \in \mathbb{S}^2$ is called the *density vector* and $\lambda \in \mathbb{R}$. After a change of the orientation, we can assume $\lambda \geq 0$ without losing generality. In [24], the second author addressed the study of compact λ -translators with boundary. We now improve these results, by showing that the hypothesis $0 \leq \lambda \leq 1$ has a paramount consequence in Th. 4.3.

Corollary 4.4 Let $0 \leq \lambda \leq 1$ and M be a compact λ -translator with density vector \mathbf{w} and whose boundary is contained in a plane $\Pi = v^\perp$, $v \in \mathbb{S}^2$. If $|\langle v, \mathbf{w} \rangle| \geq \lambda$, then $\text{int}(M) \subset \{x \in \mathbb{R}^3 : \text{sign}(\langle v, \mathbf{w} \rangle)\langle x, v \rangle \leq 0\}$. Moreover, there exists $p_0 \in \text{int}(M) \cap \Pi \neq \emptyset$ if and only if M is planar and $|\langle v, \mathbf{w} \rangle| = \lambda$.

Proof: If M is a λ -translator, we have $\mathcal{H}(x) = \langle x, \mathbf{w} \rangle + \lambda$. First, assume $|\langle v, \mathbf{w} \rangle| > \lambda$. This condition yields $\mathcal{H}(v)\mathcal{H}(-v) = \lambda^2 - \langle v, \mathbf{w} \rangle^2 < 0$, hence we are in the hypothesis of Th. 4.3. Moreover, if $\text{sign}(\langle v, \mathbf{w} \rangle) > 0$ then $\mathcal{H}(v) = \langle v, \mathbf{w} \rangle + \lambda > 0$ and M lies in $\{x \in \mathbb{R}^3 : \langle x, v \rangle < 0\}$. In case that $\text{sign}(\langle v, \mathbf{w} \rangle) < 0$, then M lies in $\{x \in \mathbb{R}^3 : \langle x, v \rangle > 0\}$.

For the second part, assume the existence of some $p_0 \in \text{int}(M) \cap \Pi$. Without losing generality, we suppose $\langle v, \mathbf{w} \rangle > 0$, which in particular yields $\langle v, \mathbf{w} \rangle \geq \lambda$. From the first part, we know that M lies in $\{\langle x, v \rangle \leq 0\}$, hence at p_0 we have $N_{p_0} = \pm v$. If $N_{p_0} = v$, then Π lies locally above M around p_0 and the mean curvature comparison principle yields

$$0 \geq H_M(p_0) = \mathcal{H}(v) = \langle v, \mathbf{w} \rangle + \lambda > 0,$$

a contradiction. Consequently, $N_{p_0} = -v$ and M lies locally above Π around p_0 . The comparison principle with the mean curvature gives

$$H_M(p_0) = \mathcal{H}(N_{p_0}) = \mathcal{H}(-v) = -\langle v, \mathbf{w} \rangle + \lambda \geq 0,$$

and consequently $\langle v, \mathbf{w} \rangle = \lambda$. Then, Π with the orientation $-v$ is an \mathcal{H} -surface and the maximum principle yields that M is planar.

If $\langle v, \mathbf{w} \rangle < 0$, which in particular yields $\langle v, \mathbf{w} \rangle \leq -\lambda$, the proof is similar. This time, M lies above Π around p_0 and $N_{p_0} = v$ necessarily. Consequently, by the comparison principle we have $\langle v, \mathbf{w} \rangle + \lambda \geq 0$, from where we conclude $\langle v, \mathbf{w} \rangle = -\lambda$. This time, the maximum principle applied to M and Π with orientation v allows us to assert that M is contained in Π , proving the result. \square

We can combine this corollary together Cor. 3.3 to conclude the following result of symmetry for λ -translators. We illustrate the result in the case that the boundary is a circle.

Corollary 4.5 *Let M be a compact embedded λ -translator with density vector $v \in \mathbb{S}^2$ and $|\lambda| \leq 1$. If the boundary of M is a circle contained in a plane $\Pi = v^\perp$, then M is an open piece of a rotational λ -translator that intersects orthogonally the rotation axis.*

We finish this section with a generalization of a result due to Pyo. In [29] it was proved that a compact translating soliton with density v spanning a circle contained in a plane Π^\perp is a rotational surface. The techniques to prove this result can now be viewed in a natural way in the context of the theory of \mathcal{H} -surfaces.

Theorem 4.6 *Let $v \in \mathbb{S}^2$ and $\mathcal{H} \in C^1(\mathbb{S}^2)$ such that $\mathcal{H}(x) = 0$ for every $x \in \mathbb{S}^2$ orthogonal to v . Assume that $\mathcal{H}(-x) = -\mathcal{H}(x)$ for all $x \in \mathbb{S}^2$. Let Γ be a closed convex curve contained in $\Pi = v^\perp$ and let $\Omega \subset \Pi$ denote the domain bounded by Γ . If M is an \mathcal{H} -surface spanning Γ , then Γ is contained in the solid cylinder $C_\Omega = \Omega \times \mathbb{R}v$. If, in addition, Γ is a circle and \mathcal{H} is rotational about v , then M is a rotational surface.*

Notice that we do not require in the hypothesis that M is embedded.

Proof: Up to a change of coordinates, we assume $v = e_3$ and that Π is the plane of equation $x_3 = 0$. Hence, \mathcal{H} vanishes on the equator $\langle x, e_3 \rangle = 0$ and in particular any vertical plane is an \mathcal{H} -surface. From the hypothesis $\mathcal{H}(-x) = -\mathcal{H}(x)$, $\forall x \in \mathbb{S}^2$ we know that the tangency principle 2.2 applies. We define the solid cylinder $C_\Omega = \Omega \times \mathbb{R}$.

First, we show that $\text{int}(M)$ lies inside C_Ω . By contradiction, assume that there exists some $p \in \text{int}(M)$ lying outside C_Ω . Let \hat{p} be the orthogonal projection of p onto Π , so $\hat{p} \notin \Omega$. Let $q \in \Omega$ be a point of Ω realizing the distance to \hat{p} . If P is the vertical plane with $q \in P$, then Ω lies contained in one side of P because Γ is convex. Moving P outwards Ω and parallel in the direction of its normal vector, let P_0 be the last position that touches M . Then P_0 lies at one side of M and the tangency principle yields a contradiction.

Once we have proved that $\text{int}(M)$ lies inside C_Ω , next we show that M is a graph over Ω , which in particular yields that M is embedded. Let us denote the vertical translations $M_s = M + se_3$, $s \geq 0$, which are again \mathcal{H} -surfaces. By compactness and for s large enough, M_s and M are disjoint. We decrease s until reaching a time s_0 where M_{s_0} meets M at the first time. It is not possible that $s_0 > 0$ because otherwise, the tangency principle implies that $M_{s_0} = M$ and we arrive to the contradiction $\partial M_{s_0} = \Gamma + s_0 e_3 \neq \Gamma$. Thus necessarily by vertical translations, the first contact point occurs at $s = 0$, that is, M comes to its initial position. Arguing similarly with vertical translation of type $M - se_3$, $s \geq 0$, we conclude that M is a graph.

In case that Γ is a circle and \mathcal{H} is rotational, we use Cor. 3.3. □

As we stated, Th. 4.6 generalizes the main result of [29] for translating solitons, since $\mathcal{H}(x) = \langle x, v \rangle$, and there is a large family of functions \mathcal{H} in the conditions of the theorem. For instance, given a C^1 odd function $f(t)$, $t \in [-1, 1]$, the function $\mathcal{H}(x) = f(\langle x, v \rangle)$, for a fixed $v \in \mathbb{S}^2$, lies in the hypothesis of Th. 4.6.

5 A height estimate in terms of the height

We obtain a relation between the area of a compact \mathcal{H} -surface with planar boundary and the height to the plane where its boundary lies. This result extends to the class of \mathcal{H} -surfaces others concerning the height and area of CMC surfaces in \mathbb{R}^3 [25], product spaces $M^2 \times \mathbb{R}$ [21], and of λ -translators in \mathbb{R}^3 [24]. In our result, we need to assume that the surface is a graph.

Theorem 5.1 *Let $\mathcal{H} > 0$ and M be a compact \mathcal{H} -graph on a plane Π . Suppose that the boundary of M is contained in Π . If h is the height of M with respect to Π , then*

$$h \leq \frac{\mathcal{H}_{\max} \text{area}(M)}{2\pi},$$

where $\mathcal{H}_{\max} = \max\{\mathcal{H}(x) : x \in \mathbb{S}^2\}$.

Proof: Without loss of generality, we assume that Π is the horizontal plane of equation $x_3 = 0$. By Cor. 4.2, we know that M lies at one side of Π , say $\{x \in \mathbb{R}^3 : x_3 > 0\}$. Since $\mathcal{H} > 0$, the Gauss map N of M points downwards, that is, $N_3 = \langle N, e_3 \rangle < 0$. In this setting, the height of M is $h = \max\{x_3(p) : p \in M\}$.

For each $t > 0$, we define $M(t) = M \cap \{x \in \mathbb{R}^3 : x_3 \geq t\}$ and $\Pi(t) = \{x \in \mathbb{R}^3 : x_3 = t\}$. Let $A(t)$ be the area of $M(t)$ and $\Gamma(t) = M(t) \cap \Pi(t)$. From the coarea formula [30, Th. 5.8],

$$A'(t) = - \int_{\Gamma(t)} \frac{1}{|\nabla x_3|} ds_t,$$

where ds_t is the line element of $\Gamma(t)$. Let us denote by $L(t)$ to the length of $\Gamma(t)$. On the one hand, by the Cauchy-Schwarz inequality we get

$$L(t)^2 = \left(\int_{\Gamma(t)} ds_t \right)^2 \leq \int_{\Gamma(t)} \frac{1}{|\nabla x_3|} ds_t \int_{\Gamma(t)} |\nabla x_3| ds_t = -A'(t) \int_{\Gamma(t)} |\nabla x_3| ds_t,$$

where ds_t stands for the line element of $\Gamma(t)$.

Since $\Gamma(t)$ is a level curve of x_3 and $M(t) \subset \{x \in \mathbb{R}^3 : x_3 \geq t\}$, we have $|\nabla x_3|_{|\Gamma(t)} = \langle \nu(t), e_3 \rangle$, where $\nu(t)$ is the unit inner conormal of $M(t)$ along $\Gamma(t)$. Therefore,

$$L(t)^2 \leq -A'(t) \int_{\Gamma(t)} \langle \nu(t), e_3 \rangle ds_t. \quad (6)$$

Because $\Delta x_3 = 2HN_3$, the divergence theorem yields

$$\int_{M(t)} 2HN_3 dM_t = - \int_{\Gamma(t)} \langle \nu(t), e_3 \rangle ds_t.$$

We substitute this integral in (6). Using that $A'(t) \leq 0$ and $N_3 \leq 0$, we have

$$L(t)^2 \leq A'(t) \int_{M(t)} 2HN_3 dM_t \leq 2\mathcal{H}_{\max} A'(t) \int_{M(t)} N_3 dM_t.$$

The curve $\Gamma(t)$ is possibly non-connected, hence the inner region bounded by it, $\Omega(t)$ consists on a finite union of compact, connected domains, $\Omega_i(t)$, $i = 1, \dots, n_t$, in $\Pi(t)$. By applying the divergence theorem to the vector field e_3 in the, possibly disconnected, 3-domain $W(t)$ bounded by $M(t)$ and $\Omega(t)$, we arrive to

$$L(t)^2 \leq -2\mathcal{H}_{\max} A'(t) \text{area}(\Omega(t)). \quad (7)$$

By denoting $L_i(t)$ to the length of $\partial\Omega_i(t)$, the classical isoperimetric inequality gives

$$L(t)^2 \geq \sum_{i=1}^{n_t} L_i(t)^2 \geq 4\pi \sum_{i=1}^{n_t} \text{area}(\Omega_i(t)) = 4\pi \text{area}(\Omega(t)).$$

Plugging this inequality in (7), we deduce

$$2\pi \leq -\mathcal{H}_{\max} A'(t).$$

By integrating this inequality from $t = 0$ to $t = h$ and taking into account that $A(0)$ is the area of M , we conclude

$$h \leq \frac{\mathcal{H}_{\max} \text{area}(M)}{2\pi},$$

which proves the result. □

Comparing the statement of Th. 5.1 with the CMC case [25], the absence of a flux formula for \mathcal{H} -surfaces makes that in (6) and for an \mathcal{H} -surface not necessarily a graph, we cannot express the integral $\int_{\Gamma(t)} \langle v(t), e_3 \rangle ds_t$ in terms of \mathcal{H}_{\max} and $\text{area}(\Omega(t))$. Consequently, we need to consider M a graph in the statement of Th. 5.1.

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