Translation surfaces of linear Weingarten type

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Abstract

We give a relatively simple proof that a translation surface in Euclidean space that satisfies a relation of type $aH + bK = c$, for some real numbers a, b, c, where H and K are the mean curvature and the Gauss curvature of the surface, respectively, must have constant H or constant K. Our method of proof extends to the Lorentzian ambient space.

Keywords: translation surface, linear Weingarten surface, mean curvature, Gauss curvature

2000MSC: 53A05, 53A35

1 Introduction and results.

A Weingaten surface in Euclidean space $\mathbb{R}^3$ is a surface S whose mean curvature H and Gauss curvature K satisfies a non-trivial relation $\Psi(H, K) = 0$. This type of surfaces were introduced by the very Weingarten [13] in the context of the problem of finding all surfaces isometric to a given surface of revolution and

*The authors have been partially supported by MEC-FEDER grant no. MTM2011-22547 and Junta de Andalucía grant no. P09-FQM-5088.

have been extensively studied in the literature. For the study of Weingarten surfaces, it is natural to impose some geometric condition, as for example, that the surface is ruled or rotational [1, 3, 4, 7, 12].

Following this strategy, Dillen, Goemans and Van de Woestyne considered Weingarten surfaces that are graphs of type $z = f(x) + g(y)$, where f and g are smooth functions defined in some intervals $I, J \subset \mathbb{R}$, respectively [2]. A surface S in $\mathbb{R}^3$ is called a *translation surface* if it can locally parametrize as $X(x, y) = (x, y, f(x) + g(y))$. In particular, a translation surface S has the property that the translations of a parametric curve $x = ct$ by the parametric curves $y = ct$ remain in S (similarly for the parametric curves $x = ct$). In the cited paper, the authors classify all translation surfaces of Weingarten type:

Theorem ([2])**.** *A translation surface in $\mathbb{R}^3$ of Weingarten type is a plane, a generalized cylinder, a Scherk's minimal surface or an elliptic paraboloid.*

Previously, particular cases were studied in [11, 15]. The proof given in [2] (see also [6]) discusses many cases and it involves the solvability of a big number of ODE systems. In fact, in [2] it is described the procedure and it requieres of calculations which are done with a computer program (as Maple) to manipulate the algebraic operations.

In this paper we provide a significantly simpler proof of this result when the surface satisfies the simplest relation $\Psi(H, K) = aH + bK$, that is, that Ψ is linear in its variables. A *linear Weingarten surface* S is a surface in Euclidean space $\mathbb{R}^3$ such that there exists a relation

$$a\ H + b\ K = c, \tag{1}$$

for some real numbers a, b, c, not all zero. In the set of linear Weingarten surfaces, we mention two families of surfaces that correspond with trivial choices of the constants a and b: surfaces with constant Gauss curvature ($a = 0$) and surfaces with constant mean curvature ($b = 0$). In the above theorem, only the three first surfaces are linear Weingarten surfaces: a plane ($H = K = 0$), a generalized cylinder ($K = 0$) and the Scherk's minimal surface parametrized as $z = 1/\lambda \log(\cos(\lambda y))/\log(\cos(\lambda x))$, $\lambda > 0$ ($H = 0$). Besides these two families of surfaces, the classification of linear Weingarten surfaces in the general case is almost completely open today. See [5, 9, 12].

The result that we prove is:

Theorem 1. *A translation surface in Euclidean surface of linear Weingarten type is a surface with $K = ct$ or $H = ct$. In particular, the surface is congruent with a plane, a Scherk's minimal surface or a generalized cylinder.*

This proves that in the family of translation surfaces, it doesn't exist new linear Weingarten surfaces besides the trivial choices of a, b in (1). We point out that an early work of proved that the only translations surfaces with constant K or constant H are the three first surfaces of Th. 1 ([8]). Finally, and with minor modifications, we extend in Th. 2 our results to the Lorentzian ambient space (see also [2]).

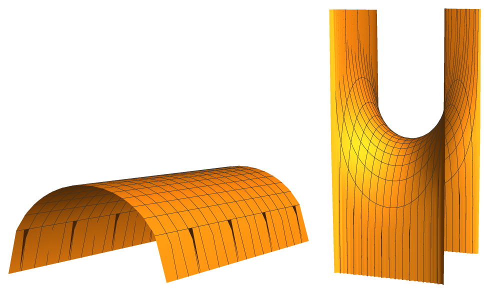

Figure 1: Euclidean translation surfaces
At left, generalized cylinder with H=1. At right, Scherk's minimal surface

2 Proof of Theorem 1

The mean curvature H and the Gauss curvature K are expressed in a local parametrization X as

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}, \quad K = \frac{eg - f^2}{EG - F^2}, \tag{2}$$

where $\{E, F, G\}$ and $\{e, f, g\}$ are the coefficients of the first fundamental form and the second fundamental form, respectively. Assume that S is a translation surface expressed locally as $X(x, y) = (x, y, f(x) + g(y))$ for some smooth functions f and g. Then H and K are

$$H = \frac{f''(1 + g'^2) + g''(1 + f'^2)}{2(1 + f'^2 + g'^2)^{\frac{3}{2}}}, \quad K = \frac{f''g''}{(1 + f'^2 + g'^2)^2}. \tag{3}$$

Assume now that S is also a linear Weingarten surface, where H and K satisfy the linear relation (1). The proof of Theorem 1 is by contradiction and we suppose that $a, b \neq 0$. Let us observe that this implies $f'' \neq 0$ and $g'' \neq 0$ because on the contrary, H is constant. We distinguish two cases according the value of c.

2.1 Case $c = 0$.

Suppose $c = 0$ in (1). With the change $a \to 2a$, Eq. (1) writes using (3) as

$$a \, \frac{f''(1+g'^2) + g''(1+f'^2)}{(1+f'^2+g'^2)^{\frac{3}{2}}} + b \, \frac{f''g''}{(1+f'^2+g'^2)^2} = 0. \tag{4}$$

Let

$$W = EG - F^2 = 1 + f'^2 + g'^2.$$

We multiply (4) by W^2 and divide by $(1+g'^2)(1+f'^2)$ obtaining

$$a \, \left(\frac{f''}{1+f'^2} + \frac{g''}{1+g'^2} \right) \sqrt{W} + b \, \frac{f''}{1+f'^2} \frac{g''}{1+g'^2} = 0. \tag{5}$$

Introduce the next notation:

$$F = \frac{f''}{1+f'^2}, \quad G = \frac{g''}{1+g'^2}. \tag{6}$$

In particular, since $f'' \neq 0$ and $g'' \neq 0$, then $F \neq 0$ and $G \neq 0$. Then (5) writes as

$$a(F+G)\sqrt{W} + bFG = 0. \tag{7}$$

Let us observe that this identity implies $F + G \neq 0$, since on the contrary, $b = 0$. From (7), we have

$$1 + f'^2 + g'^2 = W = \frac{b^2}{a^2} \left(\frac{FG}{F+G} \right)^2.$$

We differentiate this equation with respect to x and next, with respect to y. Because the left hand side is a sum of a function of x and a function y, this calculation yields 0. On the other hand, the right hand side concludes

$$6 \frac{b^2}{a^2} \frac{F^2 G^2 F' G'}{(F+G)^4} = 0. \tag{8}$$

This implies $F' = G' = 0$ and thus, F and G are constants. From (7), we deduce that $W = 1 + f'^2 + g'^2$ is constant, in particular, f' and g' are constant: a contradiction with $f'', g'' \neq 0$.

2.2 Case $c \neq 0$.

Consider $c \neq 0$ in (1). Dividing by c, and after a change of notation, (1) writes as

$$a\ \frac{f''(1+g'^2)+g''(1+f'^2)}{(1+f'^2+g'^2)^{\frac{3}{2}}} + b\ \frac{f''g''}{(1+f'^2+g'^2)^2} = 1, \tag{9}$$

or equivalently

$$a(F+G)\sqrt{W} + bFG = \frac{W^2}{(1+f'^2)(1+g'^2)}, \tag{10}$$

where F and G are given in (6). We differentiate (10) separately with respect to x and with respect to y:

$$a\left(F'\sqrt{W} + (F+G)\frac{f'f''}{\sqrt{W}}\right) + bF'G = \frac{4Wf'f''}{(1+f'^2)(1+g'^2)} - \frac{2f'f''W^2}{(1+f'^2)^2(1+g'^2)}.$$

$$a\left(G'\sqrt{W} + (F+G)\frac{g'g''}{\sqrt{W}}\right) + bF'G = \frac{4Wg'g''}{(1+f'^2)(1+g'^2)} - \frac{2g'g''W^2}{(1+f'^2)(1+g'^2)^2}.$$

Dividing the first equation by $f'f''$ and the second one by $g'g''$, we have

$$a\frac{F'\sqrt{W}}{f'f''} + b\frac{F'G}{f'f''} + \frac{2W^2}{(1+f'^2)^2(1+g'^2)} = a\frac{G'\sqrt{W}}{g'g''} + b\frac{FG'}{g'g''} + \frac{2W^2}{(1+f'^2)(1+g'^2)^2}.$$

From (10), we replace the value of W^2 in the above expression, obtaining

$$a\left(\frac{F'}{f'f''} + \frac{2(F+G)}{1+f'^2} - \frac{G'}{g'g''} - \frac{2(F+G)}{1+g'^2}\right)\sqrt{W} + b\left(\frac{F'G}{f'f''} + \frac{2FG}{1+f'^2} - \frac{FG'}{g'g''} - \frac{2FG}{1+g'^2}\right) = 0. \tag{11}$$

Now we write (9) as

$$a\left(f''(1+g'^2) + g''(1+f'^2)\right)\sqrt{W} + bf''g'' = W^2 \tag{12}$$

and we differentiate this expression with respect to x and with respect to y:

$$a\left(f'''(1+g'^2)+2f'f''g''\right)\sqrt{W}+a\left(f''(1+g'^2)+g''(1+f'^2)\right)\frac{f'f''}{\sqrt{W}}+bf'''g''=4f'f''W.$$

$$a\left(2f''g'g''+g'''(1+f'^2)\right)\sqrt{W}+a\left(f''(1+g'^2)+g''(1+f'^2)\right)\frac{g'g''}{\sqrt{W}}+bf''g'''=4g'g''W.$$

From both equation, we obtain the value of W on the right hand sides and we equal both expressions, obtaining

$$a\left(\frac{f'''}{f'f''}(1+g'^2)+2g''-2f''-(1+f'^2)\frac{g'''}{g'g''}\right)\sqrt{W}=b(f''\frac{g'''}{g'g''}-g''\frac{f'''}{f'f''}). \tag{13}$$

If we write (11) and (13) as $P_1\sqrt{W}=Q_1$ and $P_2\sqrt{W}=Q_2$, respectively, we obtain $P_1Q_2-P_2Q_1=0$. After some manipulations, this identity writes as

$$\left(f'f''^2g'''-f'''g'g''^2\right)\left(2f'f''g'g''(f''-g'')+f'f''(1+f'^2)g'''-f'''g'g''(1+g'^2)\right)=0,$$

that is, $P_2Q_2=0$. We discuss by cases:

1. Case $P_2=0$ and $Q_2\neq 0$. Then (13) implies $a=0$, a contradiction.
2. Case $P_2\neq 0$ and $Q_2=0$ Then (13) implies $b=0$, a contradiction.
3. Case $P_2=Q_2=0$. Both equations write as

$$\frac{f'''}{f'f''^2}=\frac{g'''}{g'g''^2} \tag{14}$$

$$2(f''-g'')+(1+f'^2)\frac{g'''}{g'g''}-\frac{f'''}{f'f''}(1+g'^2)=0. \tag{15}$$

Equation (14) implies the existence of $\lambda\in\mathbb{R}$ such that

$$\frac{f'''}{f'f''^2}=\frac{g'''}{g'g''^2}=2\lambda. \tag{16}$$

and thus

$$\frac{f'''}{f'f''}=2\lambda f'',\quad \frac{g'''}{g'g''}=2\lambda g''.$$

Substituting the above in (15), we get

$$2(f'' - g'') + (1 + f'^2)2\lambda g'' - (1 + g'^2)2\lambda f'' = 0,$$

or

$$f'' - g'' + \lambda g'' - \lambda f'' = \lambda f'' g'^2 - \lambda g'' f'^2. \tag{17}$$

If $\lambda \neq 0$, differentiating this equation with respect to x and then with respect to y, we deduce

$$f' f'' g''' = g' g'' f'''.$$

As we suppose that $f'', g'' \neq 0$, we conclude that

$$\frac{f'''}{f' f''} = \frac{g'''}{g' g''} = \mu$$

for some constant $\mu \in \mathbb{R}$. Substituting in (16) we deduce that f'', g'' are both constant functions, so (16) yields to λ being zero, a contradiction.

Therefore, $\lambda = 0$ in (16). Equation (17) says now that $f'' = g'' = m$, for some real number $m \neq 0$. Then (9) writes as

$$am(2 + f'^2 + g'^2) = W^{\frac{3}{2}} - bm^2 W^{-\frac{1}{2}}.$$

Differentiating with respecto x and simplifying by $f' f''$, we get

$$2am = 3W^{\frac{1}{2}} + bm^2 W^{-\frac{3}{2}},$$

which implies that W is constant and this would say that $f'' = g'' = 0$, a contradiction.

3 The Lorentzian case

We consider the Lorentzian-Minkowski space $\mathbb{L}^3$, that is, $\mathbb{R}^3$ endowed with the metric $(dx)^2 + (dy)^2 - (dz)^2$. A surface immersed in $\mathbb{L}^3$ is said non degenerate if the induced metric on S is non degenerate. The induced metric can only be of two types: positive definite and the surface is called spacelike, or a Lorentzian metric, and the surface is called timelike. For both types of surfaces, it is defined the mean curvature H and the Gauss curvature K, and we consider that there exists a linear relation between H and K as in (1).

Similarly, in Lorentzian setting we can extend the concept of translation surface. A surface S in $\mathbb{L}^3$ is again locally a graph on one of the coordinate planes, since this property is not metric but because S is immersed. Thus a translation surface in $\mathbb{L}^3$ is a surface that writes locally as the graph of a function which is the sum of two real functions. However, in $\mathbb{L}^3$ we can say a bit more. If S is spacelike, then S is a graph on the xy-plane and if S is a timelike surface, then S is a graph on the xz-plane or on the yz-plane [14]. Therefore, if S is a translation surface in $\mathbb{L}^3$, we may suppose that:

1. If S is spacelike, then S writes locally as $z = f(x) + g(y)$.
2. If S is timelike, then S writes locally as $y = f(x) + g(z)$ or as $x = f(y) + g(z)$.

We extend Theorem 1 as follows:

Theorem 2. *A translation non-degenerate surface in Lorentz-Minkowski space $\mathbb{L}^3$ of linear Weingarten type is a surface with $K = ct$ or $H = ct$.*

Translations surfaces in $\mathbb{L}^3$ with constant mean curvature or constant Gauss curvature were classified in [8] and they are a plane, a Scherk's minimal surface or a generalized cylinder.

The proof of Th. 2 is similar as Th. 1 and we only sketch the differences. Again, we suppose by contradiction that $a, b \neq 0$ in (1). The expressions of H and K in local coordinates are

$$H = \epsilon\, \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}, \quad K = \epsilon\, \frac{eg - f^2}{EG - F^2},$$

where $\epsilon = -1$ is S is spacelike and $\epsilon = 1$ if S is timelike ([10, 14]).

3.1 Spacelike case

Suppose that S writes as $z = f(x) + g(y)$. Then

$$H = -\frac{(1 - g'^2)f'' + (1 - f'^2)g''}{2((1 - f'^2 - g'^2))^{3/2}}, \; K = -\frac{f''g''}{(1 - f'^2 - g'^2)^2},$$

with $W = 1 - f'^2 - g'^2 > 0$. Let

$$F = \frac{f''}{1 - f'^2}, \quad G = \frac{g''}{1 - g'^2}.$$

If $c = 0$ in (1), then (7) is the same, obtaining (8). This implies that W is constant, a contradiction.

If $c \neq 0$, then we assume after a change of constants a and b that $c = 1$. Now the Weingarten linearity condition writes as

$$a(F + G)\sqrt{W} + bFG = \frac{W^2}{(f'^2 - 1)(g'^2 - 1)}. \tag{18}$$

Similarly as (11) and (13), we obtain

$$a\left(\frac{F'}{f'f''} + \frac{2(F+G)}{f'^2 - 1} - \frac{G'}{g'g''} - \frac{2(F+G)}{g'^2 - 1}\right)\sqrt{W} + b\left(\frac{F'G}{f'f''} + \frac{2FG}{f'^2 - 1} - \frac{FG'}{g'g''} - \frac{2FG}{g'^2 - 1}\right) = 0 \tag{19}$$

and

$$a\left(\frac{f'''}{f'f''}(g'^2 - 1) + 2g'' - 2f'' - (f'^2 - 1)\frac{g'''}{g'g''}\right)\sqrt{W} = b(f''\frac{g'''}{g'g''} - g''\frac{f'''}{f'f''}). \tag{20}$$

If we write (19) and (20) as $P_1\sqrt{W} = Q_1$ and $P_2\sqrt{W} = Q_2$, respectively, we obtain $P_1Q_2 - P_2Q_1 = 0$. This identity writes as

$$\left(f'f''^2g''' - f'''g'g''^2\right)\left(2f'f''g'g''(f'' - g'') + f'f''(f'^2 - 1)g''' - f'''g'g''(g'^2 - 1)\right) = 0,$$

that is, $P_2Q_2 = 0$. Now the discussion of cases follows the same steps as in the Euclidean case, obtaining a contradiction.

3.2 Timelike case

Without loss of generality, we suppose that S writes as $y = f(x) + g(z)$. Then $W = -1 - f'^2 + g'^2 < 0$ and

$$H = \frac{f''(-1 + g'^2) + g''(1 + f'^2)}{2((1 + f'^2 - g'^2))^{3/2}}, \ K = -\frac{f''g''}{(1 + f'^2 - g'^2)^2}.$$

The linear relation (1) is now

$$a\ \frac{f''(-1+g'^2)+g''(1+f'^2)}{\tilde{W}^{3/2}}+b\ \frac{f''g''}{\tilde{W}^2}=c,$$

with $\tilde{W}=-W>0$. Let

$$F=\frac{f''}{1+f'^2},\quad G=\frac{g''}{-1+g'^2}.$$

If $c=0$, then we get (7) again by replacing W by $\tilde{W}$, obtaining that $\tilde{W}$ (and so W) is constant, a contradiction.

If $c\neq 0$, then we assume that $c=1$. Now the Weingarten linearity condition writes as

$$a(F+G)\sqrt{\tilde{W}}+bFG=\frac{\tilde{W}^2}{(1+f'^2)(-1+g'^2)}.$$

Now (11) and (13) write, respectively, as

$$a\left(\frac{F'}{f'f''}+\frac{2(F+G)}{f'^2+1}+\frac{G'}{g'g''}+\frac{2(F+G)}{g'^2-1}\right)\sqrt{\tilde{W}}+b\left(\frac{F'G}{f'f''}+\frac{2FG}{f'^2+1}+\frac{FG'}{g'g''}+\frac{2FG}{g'^2-1}\right)=0$$

$$a\left(\frac{f'''}{f'f''}(g'^2-1)+2g''+2f''+(f'^2+1)\frac{g'''}{g'g''}\right)\sqrt{\tilde{W}}+b(f''\frac{g'''}{g'g''}+g''\frac{f'''}{f'f''})=0.$$

Now we deduce

$$\left(f'f''^2g'''+f'''g'g''^2\right)\left(2f'f''g'g''(f''+g'')+f'f''(f'^2+1)g'''+f'''g'g''(g'^2-1)\right)=0$$

and the discussion is similar as in the Euclidean case.

3.3 Both cases

Suppose that S writes as $z=f(x)+g(y)$ if S is spacelike or $y=f(x)+g(z)$ if S is timelike. Then

$$H=\epsilon\frac{-\epsilon f''(1-g'^2)+g''(1+\epsilon f'^2)}{2((1+\epsilon f'^2-g'^2))^{3/2}},\ K=-\frac{f''g''}{(1+\epsilon f'^2-g'^2)^2},$$

with $W=1+\epsilon f'^2-g'^2>0$. Let

$$F=\frac{f''}{1+\epsilon f'^2},\quad G=\epsilon\frac{g''}{g'^2-1}.$$

If $c=0$ in (1), then (7) is the same, obtaining (8). This implies that W is constant, a contradiction.

If $c\neq 0$, then we assume after a change of constants a and b that $c=1$. Now the Weingarten linearity condition writes as

$$a(F+G)\sqrt{W}+bFG=\frac{W^2}{-(1+\epsilon f'^2)(g'^2-1)}. \tag{21}$$

Now (11) and (13) write, respectively, as

$$a\left(\frac{F'}{f'f''}+\frac{2(F+G)}{f'^2+\varepsilon}+\varepsilon\frac{G'}{g'g''}+\varepsilon\frac{2(F+G)}{g'^2-1}\right)\sqrt{\tilde{W}}+b\left(\frac{F'G}{f'f''}+\frac{2FG}{f'^2+\varepsilon}+\varepsilon\frac{FG'}{g'g''}+\varepsilon\frac{2FG}{g'^2-1}\right)=0$$

$$a\left(\frac{f'''}{f'f''}(g'^2-1)+2g''+\varepsilon 2f''+\varepsilon(f'^2+\varepsilon)\frac{g'''}{g'g''}\right)\sqrt{\tilde{W}}+\varepsilon b(f''\frac{g'''}{g'g''}+\varepsilon g''\frac{f'''}{f'f''})=0.$$

Now we deduce

$$\left(f'f''^2g'''+\varepsilon f'''g'g''^2\right)\left(2f'f''g'g''(f''+\varepsilon g'')+f'f''(f'^2+\varepsilon)g'''+\varepsilon f'''g'g''(g'^2-1)\right)=0$$

and the discussion is similar as in the Euclidean case.

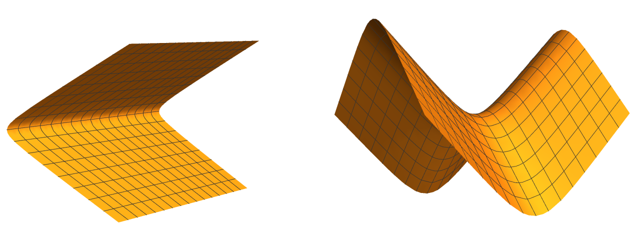

Figure 2: Lorentzian translation surfaces
At left, cylinder with H=1. At right, one of Scherk's minimal surfaces

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