# Uniqueness of the translating bowl in $\mathbb{H}^2 \times \mathbb{R}$

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**Abstract**

In this paper we extend the proof of Theorem 5.2 in [1], which characterizes the translating bowl $\mathcal{B}$ in $\mathbb{H}^2 \times \mathbb{R}$ as the only properly immersed translating soliton with finite topology and one end that is smoothly asymptotic to $\mathcal{B}$. We exploit the asymptotic behavior at infinity of $\mathcal{B}$ and the possibility of applying Alexandrov's reflection technique with respect to vertical planes.

# 1 Introduction

An immersed surface in the product space $\mathbb{H}^2 \times \mathbb{R}$ is a *translating soliton* if its mean curvature $H_M$ satisfies at each $p \in M$

$$H_M(p) = \langle \eta_p, \partial_z \rangle, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the usual product metric in $\mathbb{H}^2 \times \mathbb{R}$, $\partial_z$ is the vertical Killing vector field and $\eta_p$ is a unit normal of $M$.

This class of immersed surfaces was described independently in [1, 4], where the authors took as starting point the fruitful theory of translating solitons in $\mathbb{R}^3$ to obtain existence and uniqueness results. We remark that in [4] the authors studied translating solitons in a general Riemannian product. One of the examples arising is the *bowl soliton*, denoted by $\mathcal{B}$: the only rotational translating soliton intersecting the axis of rotation[2], and that is an entire vertical graph. It was used in [1] as barrier in order to obtain a height estimate for compact translating solitons with planar boundary, while the asymptotic behavior of $\mathcal{B}$ was computed in [1, 4].

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[2] If a rotational translating soliton intersects the axis of rotation, it must do it orthogonally.

One of the main results for translating solitons in $\mathbb{R}^3$ is the characterization of the bowl soliton exhibited in Theorem A in [5], for which its asymptotic behavior obtained in [2] and Alexandrov's reflection technique were the key tools for the proof. These ideas have motivated further uniqueness theorems for rotational entire graphs, arising for rather prescribed mean curvature problems, see e.g. Theorem 6.5 in [6].

Inspired by this result, in [1] we obtained the following uniqueness result for $\mathcal{B}$:

**Theorem 1** *Let $M$ be a properly immersed translating soliton with finite topology and one end, that is smoothly asymptotic to the bowl soliton $\mathcal{B}$. Then, $M$ is a vertical translation of $\mathcal{B}$.*

In the same fashion as in [5], the proof of this result relies in the possibility of applying Alexandrov's reflection technique with respect to vertical planes and in the asymptotic behavior of $\mathcal{B}$, but some details were missed in the proof.

Here we present an extended proof of Theorem 1, which covers the aforementioned key facts. For that, in Section 2 we recall the asymptotic behavior at infinity of a rotational translating soliton that is a vertical graph, which was already obtained in Section 4 in [1]. In Definition 2 we give the notion of a translating soliton that is *smoothly asymptotic* to $\mathcal{B}$, which is one of the main hypothesis in Theorem 1. Finally, in Section 3 we prove two fundamental lemmas that are the cornerstone to conclude the proof of Theorem 1.

## 2 Translating solitons smoothly asymptotic to $\mathcal{B}$

We begin by introducing some notation. We regard the hyperbolic plane $\mathbb{H}^2$ as the paraboloid $\{(x, y, \sqrt{1 + x^2 + y^2}),\ (x, y) \in \mathbb{R}^2\}$ in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$. Hence, the product space $\mathbb{H}^2 \times \mathbb{R}$ is a submanifold in $\mathbb{L}^4 = (\mathbb{R}^4, +, +, -, +)$, which we endow with coordinates $(x, y, t, z)$. The $z$-coordinate is the so called *height function* in $\mathbb{H}^2 \times \mathbb{R}$. Note that we can write $x = \sinh r \cos\theta$, $y = \sinh r \sin\theta$ and so $t = \cosh r$. With these coordinates, $r$ is the (hyperbolic) distance from $(x, y, t)$ to $(0, 0, 1) \in \mathbb{H}^2$. The origin of $\mathbb{H}^2 \times \mathbb{R}$ is defined as the point $\mathbf{o} = (0, 0, 1, 0)$.

One of the key features of translating solitons in $\mathbb{H}^2 \times \mathbb{R}$ is that they can be realized as minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ endowed with the conformal metric $e^z\langle\cdot,\cdot\rangle$. This also occurs for translating solitons in $(\mathbb{R}^3, g_0)$ (here $g_0$ denotes the usual flat metric), and the conformal space $(\mathbb{R}^3, e^z g_0)$ is known in the literature as the *Ilmanen space* [3].

As a consequence, translating solitons satisfy the *tangency principle*, see Theorem 2.1 in [1], and so two translating solitons that are tangent in some interior point with one lying locally at one side of the other, must agree.

Next, we focus on rotational translating solitons. Let $M$ be a translating soliton that is rotational around the vertical axis passing through $\mathbf{o}$, and suppose moreover that $M$

is a vertical graph. Such a soliton can be parametrized by

$$\psi(r,\theta) = \big(\sinh r\cos\theta, \sinh r\sin\theta, \cosh r, f(r)\big),\ \ r\in I,\ \ \theta\in(0,2\pi), \tag{2.1}$$

In Lemma 4.1 in [1] we derived that for $r$ big enough, $f(r)$ has the following expansion

$$f(r) = 2r + \frac{1}{4}\log\frac{\cosh^2 r}{5\cosh(2r)-3} + O(e^{-2r}). \tag{2.2}$$

When $r\to\infty$, the function $\frac{1}{4}\log\frac{\cosh^2 r}{5\cosh(2r)-3}$ has as limit the value $-\frac{1}{4}\log 10$. Hence, is easy to check that the function $\xi(r) := \frac{1}{4}\log\frac{\cosh^2 r}{5\cosh(2r)-3} + \frac{1}{4}\log 10$ tends to zero as fast as the function $e^{-2r}$. In fact,

$$\lim_{r\to\infty}\xi(r)e^{2r} = 4/5.$$

Moreover, it holds $\lim_{r\to\infty}\xi'(r)e^{2r} = -8/5$, i.e. $|\xi(r)| + |\xi'(r)| \le Ce^{-2r}$ for some $C>0$ and $r$ big enough.

Thus, by rotational symmetry and up to vertical translations, for $R_0>0$ big enough we can express $\mathcal{B}$ as the vertical graph of a function $f_{\mathcal{B}}(x,y) := 2\operatorname{arcsinh}\sqrt{x^2+y^2} + \varphi(\sqrt{x^2+y^2})$ with $|\varphi(\sqrt{x^2+y^2})| + |\varphi'(\sqrt{x^2+y^2})| \le Ke^{-2\operatorname{arcsinh}\sqrt{x^2+y^2}}$, $K>0$, for every $(x,y)\in\mathbb{R}^2$ such that $x^2+y^2>R_0^2$.

Finally, we give the following definition concerning a translating soliton *close enough* to $\mathcal{B}$. The terminology is motivated by Theorem A in [4], see also Theorem 5.2 in [1] and Theorem 6.5 in [6], and the asymptotic behavior of the bowl soliton.

**Definition 2** *A translating soliton $M$ is smoothly asymptotic to $\mathcal{B}$ if $M$ can be expressed outside a ball of radius $R_0>0$ as the vertical graph*

$$(x, y, \sqrt{1+x^2+y^2}, G(x,y))$$

*of a function $G:\mathbb{R}^2\to\mathbb{R}$ such that for every $(x,y)\in\mathbb{R}^2$ with $x^2+y^2>R_0^2$,*

$$G(x,y) = 2\operatorname{arcsinh}\sqrt{x^2+y^2} + \phi(x,y), \tag{2.3}$$

*where $\phi\in C^1$ satisfies*

$$|\phi(x,y)| + |d\phi_{(x,y)}(v)| \le Ce^{-2\operatorname{arcsinh}\sqrt{x^2+y^2}},$$

*for some $C>0$.*

# 3 Proof of Theorem 1

The proof of Theorem 1 relies on Alexandrov's reflection technique with respect to vertical planes and the asymptotic expansion of $\mathcal{B}$ given by (2.2). First, we introduce some notation.

Let $\sigma(r)$ be the horizontal[3] geodesic in $\mathbb{H}^2 \times \mathbb{R}$ such that $\sigma(0) = \mathbf{o}$ and $\sigma'(0) = e_1$. Consider the 1-parameter family of hyperbolic translations $\{T_r\}$ along $\sigma(r)$ such that $T_r(\sigma(0)) = \sigma(r)$ for all $r$. Let $\Pi$ be the vertical plane in $\mathbb{H}^2 \times \mathbb{R}$ passing through $\mathbf{o}$ and orthogonal to the $e_1$-direction. We define $\{\Pi(r) := T_r(\Pi)\}_r$ as the family of vertical planes in $\mathbb{H}^2 \times \mathbb{R}$ at distance $r$ to $\Pi$ and orthogonal to $e_1$ at $\sigma(r)$. We denote by $\Pi(r)^+$ (resp. $\Pi(r)^-$) to the closed half-space $\bigcup_{\lambda \geq r} \Pi(\lambda)$ (resp. $\bigcup_{\lambda \leq r} \Pi(\lambda)$), and by $M_+(r)$ (resp. $M_-(r)$) to the intersection $M \cap \Pi(r)^+$ (resp. $M \cap \Pi(r)^-$). The reflection of $M_+(r)$ with respect to the plane $\Pi(r)$ will be denoted by $M_+^*(r)$.

Denote by $\mathfrak{p}$ to the projection onto the plane $\Pi$ defined as follows: Let be $x \in \mathbb{H}^2 \times \mathbb{R}$ and consider the curve $\alpha_x(r) = T_r(x)$ given as the flow of $T_r$ passing through $x$. Then, we define $\mathfrak{p}(x)$ as the intersection of $\alpha_x(r)$ with the plane $\Pi$. This intersection is unique, and thus $\mathfrak{p}(x)$ is well defined. Moreover, after a translation in the arc-length parameter of $\alpha_x(r)$, we will suppose henceforth that $\alpha_x(0) = \mathfrak{p}(x)$.

For a point $x \in \mathbb{H}^2 \times \mathbb{R}$, let us denote by $I(x)$ to the (unique) instant of time $r_0$ such that $\alpha_x(r_0) = x$. We say that $A$ is on the *right hand side* of $B$ if for every $x \in \Pi$ such that

$$\mathfrak{p}^{-1}(x) \cap A \neq \varnothing, \quad \text{and} \quad \mathfrak{p}^{-1}(x) \cap B \neq \varnothing,$$

we have

$$\inf\left\{I\left(\mathfrak{p}^{-1}(x) \cap A\right)\right\} \geq \sup\left\{I\left(\mathfrak{p}^{-1}(x) \cap B\right)\right\}.$$

The condition *$A$ is on the right hand side of $B$* is denoted by $A \geq B$, see Figure 1. Although two arbitrary sets in $\mathbb{H}^2 \times \mathbb{R}$ may not be related by the relation $'\geq'$, it is a well-order for horizontal graphs over the plane $\Pi$.

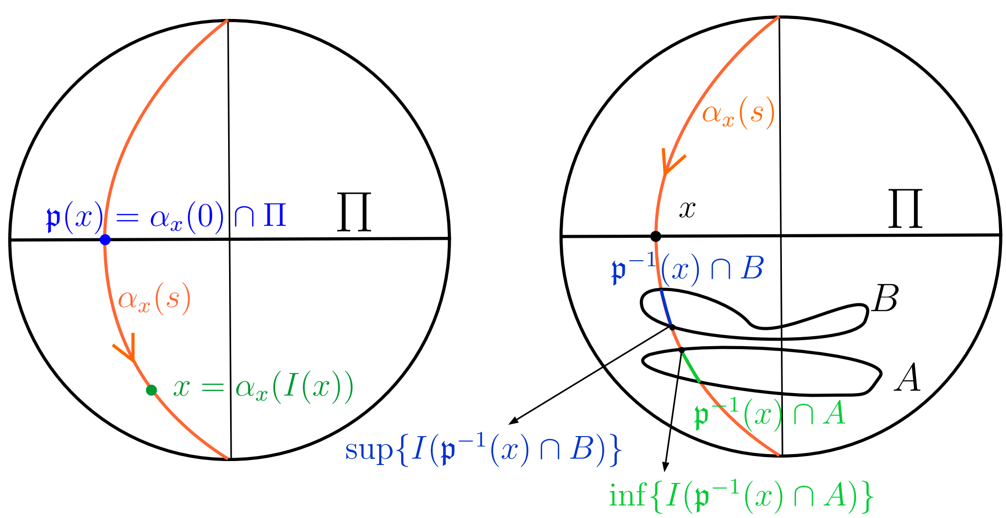


Figure 1: Left: the definition of the projection $\mathfrak{p}$. Right: an example of one set $A$ on the right hand side of other set $B$.

[3] A geodesic is horizontal if its height function is constant.

We restate the main result for the reader's convenience.

**Theorem 1** *Let $M$ be a properly embedded translating soliton with a single end, that is smoothly asymptotic to the bowl soliton $\mathcal{B}$. Then, $M$ is a vertical translation of $\mathcal{B}$.*

*Proof*: We follow the previously introduced notation, and thus $\Pi$ denotes the vertical plane passing through $\mathbf{o} \in \mathbb{H}^2 \times \mathbb{R}$ and $\{\Pi(r)\}_{r\geq 0}$ the family of vertical planes at distance $r$ to $\Pi$ and orthogonal to the $e_1$-direction.

We define the set

$$\mathcal{A} = \{r \geq 0;\ M_+(r) \text{ is a graph over } \Pi(r), \text{ and } M_+^*(r) \geq M_-(r)\}.$$

Our objective is to prove that $0 \in \mathcal{A}$. As discussed in Theorem 5.2 in [1] where we followed the ideas developed in Theorem A in [5], this is done by applying Alexandrov's reflection technique for proving that $\mathcal{A}$ is a non-empty, open and closed subset of the interval $[0, \infty)$ and such that $0 = \min \mathcal{A}$.

The main issue here is to prove that $\mathcal{A} \neq \varnothing$, i.e. we can *start* the reflection procedure with respect to some vertical plane. Once here, it is easy to prove that $\mathcal{A}$ is open and closed in $[0, \infty)$, and that its minimum is 0.

As emphasized in the proof of Theorem 6.5 in [6], the following two claims are the fundamental facts in order to start Alexandrov's reflection technique with respect to a vertical plane far away from the origin.

**Claim 1.** There exists $R_0 > 0$ big enough such that if $r > R_0$ then $M_+(r)$ is a graph over $\Pi(r)$.

*Proof*: Since $M$ is smoothly asymptotic to $\mathcal{B}$ by hypothesis, then, outside a big enough ball, $M$ can be expressed as a vertical graph $(x, y, \sqrt{1 + x^2 + y^2}, G(x, y))$ such that

$$G(x, y) = 2 \operatorname{arcsinh} \sqrt{x^2 + y^2} + \phi(x, y), \qquad \phi \in C^1, |\phi(x, y)| \leq C e^{-2 \operatorname{arcsinh} \sqrt{x^2+y^2}}, \qquad C > 0.$$

So, we have

$$\begin{aligned} dG_{(x,y)}(e_1) &\geq 2\frac{1}{\sqrt{1+x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} - Ce^{-2\operatorname{arcsinh}\sqrt{x^2+y^2}}2\frac{1}{\sqrt{1+x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} = \\ &= 2\frac{1}{\sqrt{1+x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}}\left(1 - Ce^{-2\operatorname{arcsinh}\sqrt{x^2+y^2}}\right) \end{aligned}$$

Note that $x = \sinh r \cos\theta$ and $r$ is the distance to the origin. Hence, for $r$ big enough (and so for $x$ big enough) the previous inequality is positive.

Finally, denote by $T_r(\mathfrak{p}(M_+(r)))$ to the projection $\mathfrak{p}(M_+(r))$ translated to the plane $\Pi(r)$. Now the claim holds since $M_+(r)$ is embedded and $M_+(r) \cup T_r(\mathfrak{p}(M_+(r)))$ bounds a domain in $\mathbb{H}^2 \times \mathbb{R}$. □

**Claim 2.** For $R_1 > 0$ big enough and $r > R_1$, $M_+^*(r)$ lies on the right hand side of $M_-(r)$.

*Proof*: In order to save notation, let us write $X = (x, y) \in \mathbb{R}^2$, and so $(X, \sqrt{1+|X|^2}) \in \mathbb{H}^2$. For $z \in \mathbb{R}$, the reflection $(\widetilde{X}, \sqrt{1+|\widetilde{X}|^2}, z)$ of $(X, \sqrt{1+|X|^2}, z)$ in $\mathbb{H}^2 \times \mathbb{R}$ with respect to $\Pi(r)$ has coordinates

$$\widetilde{X} = \left(A\sqrt{1+|X|^2} + Bx, y\right), \qquad A, B \in \mathbb{R}. \tag{3.1}$$

Moreover, suppose that $\Pi(r) = \gamma \times \mathbb{R}$ where $\gamma \subset \mathbb{H}^2$ is a geodesic orthogonal to the $e_1$-direction at distance $r$ to $\mathbf{o}$, and denote by $\gamma^*$ to the image of $\gamma$ under the projection of $\mathbb{H}^2$ onto the Poincare disk $\mathbb{D}^2$. Since $\Pi(r)$ is orthogonal to the $e_1$-direction, $\gamma^*$ is an arc in $\mathbb{D}^2$ symmetric with respect to the axis $y = 0$. Define $\alpha > 0$ by $(\alpha, 0) = \gamma^* \cap \{y = 0\}$. It easy to check that $\alpha = \tanh(r/2)$. With this notation, the constants $A, B$ appearing in Equation (3.1) are

$$A = \frac{4\alpha(1+\alpha^2)}{(1-\alpha^2)^2}, \qquad B = \frac{-(1+6\alpha^2+\alpha^4)}{(1-\alpha^2)^2}.$$

Also, if $(X, \sqrt{1+|X|^2}, z) \in \mathbb{H}^2 \times \mathbb{R}$ is at the left-hand side of $\Pi(r)$, then the distance from $(X, \sqrt{1+|X|^2})$ to the origin of $\mathbb{H}^2$ is smaller than the distance from $(\widetilde{X}, \sqrt{1+|\widetilde{X}|^2})$ to the origin, i.e. $\operatorname{arcsinh}|\widetilde{X}| > \operatorname{arcsinh}|X|$.

By hypothesis, outside a ball of radius $R_0 > 0$ big enough, $M$ can be written as a vertical graph $(X, \sqrt{1+|X^2|}, G(X))$, where $G(X) = 2\operatorname{arcsinh}|X| + g(X)$, $g(X) \leq Ce^{-2|X|}$ and $C > 0$.

Let be $r_0 > R_0$ and consider $M_-(r_0)$ and the reflection $\widetilde{M_+}(r_0)$. Then, $\widetilde{M_+}(r_0)$ is defined by the graph $\widetilde{G}$ such that $\widetilde{G}(X) := G(\widetilde{X})$. Hence, if we compare the graphs $\widetilde{G}$ and $G$ of $\widetilde{M_+}(r_0)$ and $M_-(r_0)$ respectively around some $X \in \mathbb{R}^2$, $|X| > r_0$, we obtain

$$2(\operatorname{arcsinh}|\widetilde{X}| - \operatorname{arcsinh}|X|) + g(\widetilde{X}) - g(X) \geq 2(\operatorname{arcsinh}|\widetilde{X}| - \operatorname{arcsinh}|X|) + C(e^{-2|\widetilde{X}|} - e^{-2|X|}).$$

Finally, note that for $|X|$ big enough and even if $\widetilde{X}$ and $X$ are close enough, the difference $e^{-2|\widetilde{X}|} - e^{-2|X|}$ tends to zero faster than the difference $\operatorname{arcsinh}\widetilde{X} - \operatorname{arcsinh}X$. Hence, for $r_0$ big enough (and so for $|X|$ enough) the above inequality is positive, which yields that $\widetilde{M_+}(r_0)$ is at the right-hand side of $M_-(r_0)$. □

Thus, there exists $R_0 > 0$ big enough such that Claims 1 and 2 hold, hence $R_0 \in \mathcal{A}$ and so $\mathcal{A} \neq \varnothing$. Once here, we conclude as in the final steps of the proof of Theorem 5.2 in [1] (see pg. 21) to ensure that $0 = \min \mathcal{A}$, which implies $M_+^*(0) \geq M_-(0)$.

By defining the set

$$\mathcal{A}_- = \{r \le 0;\ M_-(r) \text{ is a graph over } \Pi(r) \text{ and } M_-^*(r) \le M_+(r)\},$$

and after a similar argument we have $M_-^*(0) \le M_+(0)$. After symmetrization we conclude $M_-(0) \ge M_+^*(0)$, which finally implies $M_+^*(0) = M_-(0)$, i.e. $M$ is symmetric with respect to the plane $\Pi$.

By rotational symmetry of the problem, $\Pi$ can be chosen to be orthogonal to any horizontal vector of $\mathbb{H}^2 \times \mathbb{R}$ and so $M$ is symmetric with respect to any vertical plane passing through the origin of $\mathbb{H}^2 \times \mathbb{R}$, which yields that $M$ is rotational. Since $M$ has only one end and it is rotational, it touches the axis of rotation and must do it orthogonally. By uniqueness, $M$ is a vertical translation of $\mathcal{B}$. □

# References


[1] A. Bueno, Translating solitons of the mean curvature flow in the space $\mathbb{H}^2 \times \mathbb{R}$, *J. Geom.* **109** (2018). https://doi.org/10.1007/s00022-018-0447-x.

[2] J. Clutterbuck, O. Schnurer, and F. Schulze, Stability of translating solutions to mean curvature flow, *Calc. Var. Partial Differential Equations* **29** (2007), no. 3, 281–293.

[3] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature, *Mem. Amer. Math. Soc.* **108** (1994), no. 520.

[4] F. Martin, J. H. S. de Lira, Translating solitons in Riemannian products, *J. Differential Equations* **266** (2019), 7780–7812.

[5] F. Martín, A. Savas-Halilaj, K. Smoczyk, On the topology of translating solitons of the mean curvature flow, *Calc. Var. Partial Differential Equations* **54** (2015), no. 3, 2853-2882.

[6] A. Martínez, A. Martínez-Triviño, Equilibrium of surfaces in a vertical force field, preprint arXiv:1910.07795.



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