ON BRUNN-MINKOWSKI TYPE INEQUALITIES FOR POLAR BODIES

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Abstract. In this paper we prove a Brunn-Minkowski type inequality for the polar set of the p-sum of convex bodies, which generalizes previous results by Firey, and we show it has an equivalent multiplicative version. We also make some considerations for the polar set of the so-called difference body.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$, and let $\langle\cdot,\cdot\rangle$ be the standard inner product in $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin as an interior point is denoted by $\mathcal{K}^n_0$, and we write B_n for the n-dimensional Euclidean unit ball. The n-dimensional volume of a measurable set $M\subset\mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (or $\mathrm{vol}_n(M)$ if the distinction of the dimension is needed) and with $\mathrm{int}\,M$ and $\mathrm{conv}\,M$ we represent its interior and convex hull, respectively. In particular, we write $\kappa_n=\mathrm{vol}(B_n)$. Finally, the set of all k-dimensional (linear) planes of $\mathbb{R}^n$ is denoted by $\mathcal{L}^n_k$, and for $H\in\mathcal{L}^n_k$, $K\in\mathcal{K}^n$, the orthogonal projection of K onto H is denoted by $K|H$.

Relating the volume with the Minkowski (vectorial) addition of convex bodies, one is led to the famous Brunn-Minkowski inequality. One form of it states that if $K,L\in\mathcal{K}^n$ and $0\leq\lambda\leq 1$, then

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}\geq(1-\lambda)\mathrm{vol}(K)^{1/n}+\lambda\mathrm{vol}(L)^{1/n}.$$

Equality for some $\lambda\in(0,1)$ holds if and only if K and L either lie in parallel hyperplanes or are homothetic.

Brunn-Minkowski inequality has a more general version for the so-called *quermassintegrals* of convex bodies, which are the coefficients (up to a constant) of the polynomial expression which is obtained when computing the volume of the Minkowski addition $K+\lambda B_n$, $\lambda\geq 0$, namely,

$$\mathrm{vol}(K+\lambda B_n)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K)\lambda^i.$$

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It is known as the *Steiner formula* of K (see [13]). Quermassintegrals are a very special case of the more general defined *mixed volumes* for which we refer to [12, s. 5.1]; in particular, $\mathrm{W}_0(K) = \mathrm{vol}(K)$ and $\mathrm{W}_n(K) = \kappa_n$. Thus, Brunn-Minkowski theorem for quermassintegrals states that if $K, L \in \mathcal{K}^n$ and $0 \le \lambda \le 1$, then, for all $i = 0, \dots, n-2$,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda L\big)^{1/(n-i)} \ge (1-\lambda)\mathrm{W}_i(K)^{1/(n-i)} + \lambda \mathrm{W}_i(L)^{1/(n-i)},$$

whereas $\mathrm{W}_{n-1}\big((1-\lambda)K + \lambda L\big) = (1-\lambda)\mathrm{W}_{n-1}(K) + \lambda \mathrm{W}_{n-1}(L)$; in fact, there exists the most general version of Brunn-Minkowski inequality for mixed volumes (see [12, Theorem 7.4.5]).

Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond. For extensive and beautiful surveys on it we refer e.g. to [1, 5]. Among many others, it has been the key for the development of the so-called L_p-Brunn-Minkowski theory (see e.g. [8, 9]), which had its starting point in several works by Firey (see [2, 3, 4]). More precisely, in [4] Firey introduced the following generalization of the classical Minkowski addition (and scalar multiplication), which is usually called p- or *Firey addition/linear combination*: for $1 \le p \le \infty$ fixed, $K, L \in \mathcal{K}^n_0$ and $\lambda, \mu \ge 0$, there exists a (unique) convex body $\lambda \cdot K +_p \mu \cdot L$ for which the support function

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p. \tag{1.1}$$

We recall that $h(K, u) = \max\big\{\langle x, u\rangle : x \in K\big\}$, $u \in \mathbb{S}^{n-1}$, where, as usual, $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$ (see e.g. [12, s. 1.7]). We observe that $\lambda \cdot K = \lambda^{1/p} K$, i.e., the *Firey scalar multiplication* depends on p, and thus, if the distinction of the parameter p is needed, we will write '$\cdot_p$' instead of '$\cdot$'.

Clearly, when $p = 1$, formula (1.1) defines the classical Minkowski addition and scalar multiplication $\lambda K + \mu L$, whereas the case $p = \infty$ gives

$$\lambda \cdot K +_\infty \mu \cdot L = \mathrm{conv}(K \cup L).$$

Moreover, in [4, Theorem 1] it is shown that, for all $1 \le p \le q$,

$$(1-\lambda) \cdot_p K +_p \lambda \cdot_p L \subset (1-\lambda) \cdot_q K +_q \lambda \cdot_q L, \tag{1.2}$$

$\lambda \in [0, 1]$. Firey also proved the extended Brunn-Minkowski inequality

$$\mathrm{W}_i\big((1-\lambda) \cdot K +_p \lambda \cdot L\big)^{p/(n-i)} \ge (1-\lambda)\mathrm{W}_i(K)^{p/(n-i)} + \lambda \mathrm{W}_i(L)^{p/(n-i)}$$

and, moreover, he obtained several Brunn-Minkowski type inequalities for polar bodies. We recall that the polar K^* of a convex body $K \in \mathcal{K}^n_0$ is defined as

$$K^* = \big\{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } y \in K\big\}.$$

In [3], Firey showed that

$$\mathrm{W}_i\Big(\big[(1-\lambda)K + \lambda L\big]^*\Big)^{-1/(n-i)} \ge (1-\lambda)\mathrm{W}_i(K^*)^{-1/(n-i)} + \lambda \mathrm{W}_i(L^*)^{-1/(n-i)}, \tag{1.3}$$

with equality, for some $\lambda \in (0, 1)$, if and only if K and L are dilatates.

In this work we are mainly interested in studying generalizations of the previous relation. Thus, we extend this Brunn-Minkowski type inequality to the p-sum of convex bodies, i.e., in Section 2 we prove the following result:

Theorem 1.1. *Let $K, L \in \mathcal{K}_0^n$ and $1 \leq p \leq \infty$. Then, for all $i = 0, \dots, n-1$ and $\lambda \in [0,1]$,*

$$\mathrm{W}_i\Bigl(\bigl[(1-\lambda)\cdot K +_p \lambda\cdot L\bigr]^*\Bigr)^{-p/(n-i)} \geq (1-\lambda)\mathrm{W}_i(K^*)^{-p/(n-i)} + \lambda \mathrm{W}_i(L^*)^{-p/(n-i)}. \tag{1.4}$$

If $p \neq \infty$, equality holds, for some $\lambda \in (0,1)$, if and only if K and L are dilatates. For $p = \infty$, equality holds if and only if $K \subset L$ or $L \subset K$.

The volume case ($i = 0$) was already obtained by Firey in [2].

Moreover, using standard arguments, we see that (1.4) has an equivalent multiplicative version:

Theorem 1.2. *Let $K, L \in \mathcal{K}_0^n$ and $1 \leq p \leq \infty$. Then, for all $i = 0, \dots, n-1$ and $\lambda \in [0,1]$,*

$$\mathrm{W}_i\Bigl(\bigl[(1-\lambda)\cdot K +_p \lambda\cdot L\bigr]^*\Bigr) \leq \mathrm{W}_i(K^*)^{1-\lambda}\mathrm{W}_i(L^*)^{\lambda}. \tag{1.5}$$

Equality holds, for some $\lambda \in (0,1)$, if and only if $K = L$.

In Section 3 we show this theorem as well as the mentioned equivalence, which provides an alternative proof to the one given by Firey in [3] for inequality (1.3). This would also allow to show Theorem 1.1 independently of Firey's result.

Finally, in Section 4 some considerations for the polar set of the so-called difference body $K - K = K + (-K)$ are made. Indeed, as an immediate consequence of (1.4), the inequality $\mathrm{vol}\bigl((K-K)^*\bigr) \leq (1/2^n)\mathrm{vol}(K^*)$ is obtained, and so it is a natural question whether there exists a constant $c(n) > 0$, depending only on the dimension, bounding from below the ratio $\mathrm{vol}\bigl((K-K)^*\bigr)/\mathrm{vol}(K^*)$. We prove that such an inequality does not exist, which leads to consider other possibilities in order to get such an inequality.

2. A Brunn-Minkowski inequality for the polar set of the p-sum of convex bodies

In order to show Theorem 1.1 a weaker version of inequality (1.3) will be needed, namely: writing (1.3) as

$$\mathrm{W}_i\Bigl(\bigl[(1-\lambda)K + \lambda L\bigr]^*\Bigr)^{1/(n-i)} \leq \frac{1}{(1-\lambda)\mathrm{W}_i(K^*)^{-1/(n-i)} + \lambda\mathrm{W}_i(L^*)^{-1/(n-i)}}$$

and using the convexity of the function $f(x) = 1/x$ when $x > 0$, we immediately get that for $K, L \in \mathcal{K}_0^n$ and $\lambda \in [0,1]$,

$$\mathrm{W}_i\Bigl(\bigl[(1-\lambda)K + \lambda L\bigr]^*\Bigr)^{1/(n-i)} \leq (1-\lambda)\mathrm{W}_i(K^*)^{1/(n-i)} + \lambda\mathrm{W}_i(L^*)^{1/(n-i)}, \tag{2.1}$$

$i = 0, \dots, n-1$.

Proof of Theorem 1.1. First we assume $p<\infty$, and in the following we also suppose $p\neq 1$, because the case $p=1$ is Firey's result (1.3).

Let $i\in\{0,\dots,n-1\}$ be fixed. We notice that $0\in\operatorname{int}K^*,\operatorname{int}L^*$, and therefore $\mathrm{W}_i(K^*),\mathrm{W}_i(L^*)\neq 0$. Thus, we can take $\lambda_1=\mathrm{W}_i(K^*)^{-1/(n-i)}$ and $\lambda_2=\mathrm{W}_i(L^*)^{-1/(n-i)}$, and consider the convex bodies $\overline{K}=(1/\lambda_1)K,\overline{L}=(1/\lambda_2)L\in\mathcal{K}^n_0$. Finally, let

$$\bar\lambda=\frac{\lambda\lambda_2^p}{(1-\lambda)\lambda_1^p+\lambda\lambda_2^p}=\frac{\lambda\mathrm{W}_i(K^*)^{p/(n-i)}}{(1-\lambda)\mathrm{W}_i(L^*)^{p/(n-i)}+\lambda\mathrm{W}_i(K^*)^{p/(n-i)}}.$$

Then, for all $u\in\mathbb{S}^{n-1}$ we have

$$\begin{aligned}h\bigl((1-\bar\lambda)\cdot\overline{K}+_p\bar\lambda\cdot\overline{L},u\bigr)^p&=(1-\bar\lambda)h(\overline{K},u)^p+\bar\lambda h(\overline{L},u)^p\\&=\frac{(1-\lambda)\lambda_1^p h(\overline{K},u)^p+\lambda\lambda_2^p h(\overline{L},u)^p}{(1-\lambda)\lambda_1^p+\lambda\lambda_2^p}\\&=\frac{(1-\lambda)h(K,u)^p+\lambda h(L,u)^p}{(1-\lambda)\lambda_1^p+\lambda\lambda_2^p}\\&=\frac{1}{(1-\lambda)\lambda_1^p+\lambda\lambda_2^p}\,h\bigl((1-\lambda)\cdot K+_p\lambda\cdot L,u\bigr)^p,\end{aligned}$$

which implies that

$$(1-\bar\lambda)\cdot\overline{K}+_p\bar\lambda\cdot\overline{L}=\frac{1}{\bigl((1-\lambda)\lambda_1^p+\lambda\lambda_2^p\bigr)^{1/p}}\bigl[(1-\lambda)\cdot K+_p\lambda\cdot L\bigr],$$

and since $(1-\bar\lambda)\overline{K}+\bar\lambda\overline{L}\subset(1-\bar\lambda)\cdot\overline{K}+_p\bar\lambda\cdot\overline{L}$ (see (1.2)) and polarity inverts inclusions, we obtain that

$$\bigl((1-\lambda)\lambda_1^p+\lambda\lambda_2^p\bigr)^{1/p}\bigl[(1-\lambda)\cdot K+_p\lambda\cdot L\bigr]^*\subset\bigl[(1-\bar\lambda)\overline{K}+\bar\lambda\overline{L}\bigr]^*.$$

Therefore, applying (2.1), we get

$$\begin{aligned}\bigl((1-\lambda)\lambda_1^p+\lambda\lambda_2^p\bigr)^{1/p}&\mathrm{W}_i\Bigl(\bigl[(1-\lambda)\cdot K+_p\lambda\cdot L\bigr]^*\Bigr)^{1/(n-i)}\\&\leq\mathrm{W}_i\Bigl(\bigl[(1-\bar\lambda)\overline{K}+\bar\lambda\overline{L}\bigr]^*\Bigr)^{1/(n-i)}\\&\leq(1-\bar\lambda)\mathrm{W}_i(\overline{K}^*)^{1/(n-i)}+\bar\lambda\mathrm{W}_i(\overline{L}^*)^{1/(n-i)}.\end{aligned}$$

Moreover, we observe that

$$\mathrm{W}_i\bigl(\overline{K}^*\bigr)^{1/(n-i)}=\mathrm{W}_i(\lambda_1K^*)^{1/(n-i)}=1,$$

and analogously $\mathrm{W}_i(\overline{L}^*)^{1/(n-i)}=1$, and hence we can conclude that

$$\bigl((1-\lambda)\lambda_1^p+\lambda\lambda_2^p\bigr)^{1/p}\mathrm{W}_i\Bigl(\bigl[(1-\lambda)\cdot K+_p\lambda\cdot L\bigr]^*\Bigr)^{1/(n-i)}\leq 1,$$

which yields (1.4).

Next we deal with the equality case. If $K=cL$, $c>0$, then equality trivially holds in (1.4). Conversely, if we have equality in (1.4), then, in particular, $(1-\bar\lambda)\overline{K}+\bar\lambda\overline{L}=(1-\bar\lambda)\cdot\overline{K}+_p\bar\lambda\cdot\overline{L}$ holds, i.e., we get that

$$\big[(1-\bar\lambda)h(\overline{K},u)+\bar\lambda h(\overline{L},u)\big]^p=(1-\bar\lambda)h(\overline{K},u)^p+\bar\lambda h(\overline{L},u)^p$$

for every $u\in\mathbb{S}^{n-1}$. This yields $h(\overline{K},u)=h(\overline{L},u)$ for all $u\in\mathbb{S}^{n-1}$, which implies $\overline{K}=\overline{L}$. Therefore, $K=(\lambda_1/\lambda_2)L$, as required.

The case $p=\infty$ is easy: taking limits when $p\to\infty$, inequality (1.4) takes the form

$$\mathrm{W}_i\Big(\big[\mathrm{conv}(K\cup L)\big]^*\Big)^{-1/(n-i)}\geq\max\Big\{W_i(K^*)^{-1/(n-i)},W_i(L^*)^{-1/(n-i)}\Big\},$$

or, equivalently,

$$\mathrm{W}_i\Big(\big[\mathrm{conv}(K\cup L)\big]^*\Big)\leq\min\big\{W_i(K^*),W_i(L^*)\big\},\tag{2.2}$$

which is trivially true because both $K,L\subset\mathrm{conv}(K\cup L)$ and hence $K^*,L^*\supset\big[\mathrm{conv}(K\cup L)\big]^*$.

Moreover, equality holds if and only if either $K\subset L$ or $L\subset K$: indeed, if $K\subset L$ or vice versa, equality in (2.2) holds obviously; conversely, assuming, for instance, that $\min\big\{\mathrm{W}_i(K^*),\mathrm{W}_i(L^*)\big\}=\mathrm{W}_i(K^*)$, if

$$\mathrm{W}_i\Big(\big[\mathrm{conv}(K\cup L)\big]^*\Big)=\mathrm{W}_i(K^*),$$

then $\mathrm{conv}(K\cup L)=K$ and hence, $L\subset K$. $\square$

3. An equivalent multiplicative Brunn-Minkowski inequality for polar sets

In order to prove Theorem 1.2 we need a formula expressing the volume of the polar set of a convex body in terms of its support function (defined in $\mathbb{R}^n$), see e.g. [12, (1.54)]:

$$\mathrm{vol}(K^*)=\frac{1}{n!}\int_{\mathbb{R}^n}e^{-h(K,x)}\mathrm{d}x\tag{3.1}$$

for any $K\in\mathcal{K}^n_0$.

Proof of Theorem 1.2. First we consider the case $p=1$, and prove (1.5) for $i=0$, i.e., the case of the volume, which is just a consequence of (3.1), the additivity of the support function (see e.g. [7, Proposition 6.2]) and Hölder's inequality (see e.g. [7, Corollary 1.5]) for $p=1/(1-\lambda)$ and $q=1/\lambda$:

$$\begin{aligned}\mathrm{vol}\Big(\big[(1-\lambda)K+\lambda L\big]^*\Big)&=\frac{1}{n!}\int_{\mathbb{R}^n}e^{-h\big((1-\lambda)K+\lambda L,x\big)}\,\mathrm{d}x\\&=\frac{1}{n!}\int_{\mathbb{R}^n}\Big(e^{-h(K,x)}\Big)^{1-\lambda}\Big(e^{-h(L,x)}\Big)^{\lambda}\,\mathrm{d}x\\&\leq\left(\frac{1}{n!}\int_{\mathbb{R}^n}e^{-h(K,x)}\,\mathrm{d}x\right)^{1-\lambda}\left(\frac{1}{n!}\int_{\mathbb{R}^n}e^{-h(L,x)}\,\mathrm{d}x\right)^{\lambda}\\&=\mathrm{vol}(K^*)^{1-\lambda}\mathrm{vol}(L^*)^{\lambda}.\end{aligned}$$

Next we deal with a general i-th quermassintegral. Kubota's integral recursion formula (see e.g. [12, p. 301, (5.72)]) states in particular that, for any convex body $K \in \mathcal{K}^n$ and all $i = 1, \dots, n-1$,

$$\mathrm{W}_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \mathrm{vol}_i(K|H)\,\mathrm{d}\mu(H);$$

here μ is the (rotationally invariant) Haar measure on $\mathcal{L}_i^n$ with $\mu(\mathcal{L}_i^n) = 1$. Using the polarity link between sections and projections (see e.g. [6, p. 22]) and since

$$\big((1-\lambda)K + \lambda L\big) \cap H \supset (1-\lambda)(K \cap H) + \lambda(L \cap H),$$

we get that

$$\big((1-\lambda)K+\lambda L\big)^*|H = \Big[\big((1-\lambda)K+\lambda L\big)\cap H\Big]^* \subset \big[(1-\lambda)(K\cap H)+\lambda(L\cap H)\big]^*,$$

where the polar operation on the middle/right is taken in H. This, together with the volume case and Hölder's inequality, yields

$$\begin{aligned}
\mathrm{W}_{n-i}\Big(\big[(1-\lambda)K + \lambda L\big]^*\Big) &= \frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \mathrm{vol}_i\Big(\big((1-\lambda)K + \lambda L\big)^*|H\Big)\,\mathrm{d}\mu(H)\\
&\leq \frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \mathrm{vol}_i\Big(\big[(1-\lambda)(K \cap H) + \lambda(L \cap H)\big]^*\Big)\,\mathrm{d}\mu(H)\\
&\leq \frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \mathrm{vol}_i\big((K \cap H)^*\big)^{1-\lambda} \mathrm{vol}_i\big((L \cap H)^*\big)^{\lambda}\,\mathrm{d}\mu(H)\\
&= \frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \mathrm{vol}_i(K^*|H)^{1-\lambda} \mathrm{vol}_i(L^*|H)^{\lambda}\,\mathrm{d}\mu(H)\\
&\leq \left[\frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \mathrm{vol}_i(K^*|H)\,\mathrm{d}\mu(H)\right]^{1-\lambda} \left[\frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \mathrm{vol}_i(L^*|H)\,\mathrm{d}\mu(H)\right]^{\lambda}\\
&= \mathrm{W}_{n-i}(K^*)^{1-\lambda} \mathrm{W}_{n-i}(L^*)^{\lambda}.
\end{aligned}$$

Next, we deal with the characterization of the equality in (1.5), and first, we consider the inequality for the volume. By the equality case in Hölder's inequality and the continuity of the support function, we have that there exists a constant $c > 0$ such that $e^{-h(K,x)} = c\,e^{-h(L,x)}$ for all $x \in \mathbb{R}^n$. In particular, if $x = 0$, we obtain that $c = 1$ and hence that $h(K,x) = h(L,x)$ for all $x \in \mathbb{R}^n$. Therefore, $K = L$.

This fact allows to characterize the equality case in (1.5) for $p = 1$, since it implies that equality in the last but one inequality above gives $K \cap H = L \cap H$ μ-almost everywhere, and thus $K = L$. The converse is trivially fulfilled.

Finally, using (1.2), we get the required inequality (1.5) for all $p \geq 1$. □

We conclude the section proving that both versions of the polar L_p-Brunn-Minkowski inequality are equivalent.

Proposition 3.1. *Inequalities* (1.4) *and* (1.5) *are equivalent.*

Proof. First we observe that (1.5) is an easy consequence of (1.4) and the arithmetic-geometric mean inequality (see e.g. [7, Corollary 1.2]). Indeed,

$$\begin{aligned}
\mathrm{W}_i\Big(\big[(1-\lambda)\cdot K{+_p}\lambda\cdot L\big]^*\Big)^{p/(n-i)} &\\
&\leq \frac{1}{(1-\lambda)\mathrm{W}_i(K^*)^{-p/(n-i)}+\lambda\mathrm{W}_i(L^*)^{-p/(n-i)}}\\
&\leq \frac{1}{\big(\mathrm{W}_i(K^*)^{-p/(n-i)}\big)^{1-\lambda}\big(\mathrm{W}_i(L^*)^{-p/(n-i)}\big)^{\lambda}}\\
&= \frac{1}{\big[\mathrm{W}_i(K^*)^{1-\lambda}\mathrm{W}_i(L^*)^{\lambda}\big]^{-p/(n-i)}},
\end{aligned}$$

which yields $\mathrm{W}_i\Big(\big[(1-\lambda)\cdot K+_p\lambda\cdot L\big]^*\Big)\leq \mathrm{W}_i(K^*)^{1-\lambda}\mathrm{W}_i(L^*)^{\lambda}$.

Conversely, we assume (1.5) and consider the sets $\overline{K}=\mathrm{W}_i(K^*)^{1/(n-i)}K$, $\overline{L}=\mathrm{W}_i(L^*)^{1/(n-i)}L$ and the positive number

$$\bar{\lambda}=\frac{\lambda\mathrm{W}_i(L^*)^{-p/(n-i)}}{(1-\lambda)\mathrm{W}_i(K^*)^{-p/(n-i)}+\lambda\mathrm{W}_i(L^*)^{-p/(n-i)}}.$$

Then it is easy to check that (see proof of Theorem 1.1)

$$(1-\bar{\lambda})\cdot\overline{K}+_p\bar{\lambda}\cdot\overline{L}=\frac{(1-\lambda)\cdot K+_p\lambda\cdot L}{\big[(1-\lambda)\mathrm{W}_i(K^*)^{-p/(n-i)}+\lambda\mathrm{W}_i(L^*)^{-p/(n-i)}\big]^{1/p}},$$

and since $\mathrm{W}_i\big(\overline{K}^*\big)=\mathrm{W}_i\big(\overline{L}^*\big)=1$, applying the multiplicative inequality (1.5) to $\overline{K},\overline{L}$ and $\bar{\lambda}$, we get

$$\begin{aligned}
1=\mathrm{W}_i\big(\overline{K}^*\big)^{1-\bar{\lambda}}\mathrm{W}_i\big(\overline{L}^*\big)^{\bar{\lambda}} &\geq \mathrm{W}_i\Big(\big[(1-\bar{\lambda})\cdot\overline{K}+_p\bar{\lambda}\cdot\overline{L}\big]^*\Big)\\
&=\Big((1-\lambda)\mathrm{W}_i(K^*)^{-p/(n-i)}+\lambda\mathrm{W}_i(L^*)^{-p/(n-i)}\Big)^{(n-i)/p}\\
&\quad\mathrm{W}_i\Big(\big[(1-\lambda)\cdot K+_p\lambda\cdot L\big]^*\Big).
\end{aligned}$$

It shows (1.4) and concludes the proof of the equivalence. □

4. On the polar set of the difference body

We observe that the definition of p-sum (1.1) yields

$$\lambda\cdot K+_p\lambda\cdot L=\lambda^{1/p}(K+_p L)$$

for any $\lambda>0$. Therefore, from (1.4) when $i=0$ and $\lambda=1/2$, it is immediately obtained that for any $K\in\mathcal{K}_0^n$,

$$\mathrm{vol}\Big(\big[K+_p(-K)\big]^*\Big)\leq\frac{1}{2^{n/p}}\mathrm{vol}(K^*);$$

in particular, if $p=1$, the following inequality is obtained for the polar set of the difference body $K+(-K)=K-K$:

$$\mathrm{vol}\big([K-K]^*\big)\leq\frac{1}{2^n}\mathrm{vol}(K^*).$$

Taking a look at the famous Rogers-Shephard inequality (see [10]), namely,

$$2^n\mathrm{vol}(K)\leq\mathrm{vol}(K-K)\leq\binom{2n}{n}\mathrm{vol}(K),$$

it is at this point a natural question whether it is possible to bound from below the ratio $\mathrm{vol}\big([K-K]^*\big)/\mathrm{vol}(K^*)$ by a constant $c(n)>0$ depending only on the dimension, not on the convex body. Unfortunately, the answer to this question is negative, as the following counterexample shows. Let

$$K=\mathrm{conv}\left\{\left(-\frac{1}{\varepsilon},0\right),\left(\frac{1}{1-\varepsilon},0\right),(0,-2),(0,2)\right\}\in\mathbb{R}^2,$$

where $0<\varepsilon<1$. Clearly $K^*=[-\varepsilon,1-\varepsilon]\times[-1/2,1/2]$ and thus $\mathrm{vol}(K^*)=1$. On the other hand, since

$$\mathrm{conv}\left\{\left(-\frac{1}{(1-\varepsilon)\varepsilon},0\right),\left(\frac{1}{(1-\varepsilon)\varepsilon},0\right),(0,-4),(0,4)\right\}\subset K-K,$$

we have

$$\begin{aligned}\varepsilon\geq(1-\varepsilon)\varepsilon&=\mathrm{vol}\left(\big[-(1-\varepsilon)\varepsilon,(1-\varepsilon)\varepsilon\big]\times\left[-\frac{1}{4},\frac{1}{4}\right]\right)\\&=\mathrm{vol}\left(\left[\mathrm{conv}\left\{\pm\left(\frac{1}{(1-\varepsilon)\varepsilon},0\right),\pm(0,4)\right\}\right]^*\right)\geq\mathrm{vol}\big([K-K]^*\big)\end{aligned}$$

for all $0<\varepsilon<1$, which shows that there exists no constant $c(n)>0$ such that $\mathrm{vol}\big([K-K]^*\big)\geq c(n)=c(n)\mathrm{vol}(K^*)$.

Furthermore, we would like also to point out that a relation of the type

$$\int_0^1\mathrm{vol}\Big(\big[(1-\lambda)K+\lambda(-K)\big]^*\Big)\,\mathrm{d}\lambda\geq c(n)\mathrm{vol}(K^*),$$

$c(n)>0$, cannot be obtained. This would be the natural counterpart to the inequality

$$\int_0^1\mathrm{vol}\big((1-\lambda)K+\lambda(-K)\big)\,\mathrm{d}\lambda\leq\frac{2^n}{n+1}\,\mathrm{vol}(K)$$

proven by Rogers&Shephard in [11].

Indeed, using the above considered convex body K, now with $0<\varepsilon<1/2$, we have that

$$\begin{aligned}&\big[(1-\lambda)K+\lambda(-K)\big]^*\\&\quad\subset\left[\mathrm{conv}\left\{\left(\frac{\lambda(1-2\varepsilon)-1+\varepsilon}{(1-\varepsilon)\varepsilon},0\right),\left(\frac{\lambda(1-2\varepsilon)+\varepsilon}{(1-\varepsilon)\varepsilon},0\right),\pm(0,2)\right\}\right]^*\\&\quad=\left[\frac{(1-\varepsilon)\varepsilon}{\lambda(1-2\varepsilon)-1+\varepsilon},\frac{(1-\varepsilon)\varepsilon}{\lambda(1-2\varepsilon)+\varepsilon}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right],\end{aligned}$$

and thus

$$\begin{aligned}\int_0^1 &\mathrm{vol}\Big(\big[(1-\lambda)K+\lambda(-K)\big]^*\Big)\,\mathrm{d}\lambda\\ &\leq \int_0^1 \mathrm{vol}\left(\left[\frac{(1-\varepsilon)\varepsilon}{\lambda(1-2\varepsilon)-1+\varepsilon},\frac{(1-\varepsilon)\varepsilon}{\lambda(1-2\varepsilon)+\varepsilon}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]\right)\mathrm{d}\lambda\\ &=\frac{2(1-\varepsilon)\varepsilon}{1-2\varepsilon}\log\left(\frac{1-\varepsilon}{\varepsilon}\right)\end{aligned}$$

for all $0<\varepsilon<1/2$. Thus, since

$$\lim_{\varepsilon\to 0}\frac{2(1-\varepsilon)\varepsilon}{1-2\varepsilon}\log\left(\frac{1-\varepsilon}{\varepsilon}\right)=0,$$

we get that there exists no constant $c(n)>0$ verifying

$$\int_0^1 \mathrm{vol}\Big(\big[(1-\lambda)K+\lambda(-K)\big]^*\Big)\,\mathrm{d}\lambda\geq c(n)=c(n)\mathrm{vol}(K^*).$$

Finally, we would like to observe that analogous (counter)examples can be also constructed for any dimension $n\geq 2$.

4.1. **Possible reverse inequalities.** Since it is not possible to get an inequality of the type $\mathrm{vol}\big([K-K]^*\big)\geq c(n)\mathrm{vol}(K^*)$ with $c(n)>0$ a constant depending only on the dimension, either additional assumptions should be imposed in order to get such a relation or a different operation has to be considered. In this respect, one possibility would be to consider the intersection of sets. Thus, for $K,L\in\mathcal{K}^n_0$ and all $\lambda\in[0,1]$, since $K\cap L\subset K,L$ we immediately get that $\mathrm{vol}\big([K\cap L]^*\big)\geq\mathrm{vol}(K^*),\mathrm{vol}(L^*)$, and hence,

$$\mathrm{vol}\big([K\cap L]^*\big)\geq\mathrm{vol}(K^*)^{1-\lambda}\mathrm{vol}(L^*)^{\lambda}, \tag{4.1}$$

with equality if and only if $K=L$. In particular,

$$\mathrm{vol}\Big(\big[K\cap(-K)\big]^*\Big)\geq\mathrm{vol}(K^*), \tag{4.2}$$

and equality holds if and only if K is 0-symmetric.

At this point we would like to make the following observation. The volume Brunn-Minkowski inequality for the polar set of the sum (see the proof of Theorem 1.2) was obtained via Hölder's inequality. An estimate in the opposite direction is given by the famous Prékopa-Leindler inequality (see e.g. [12, Theorem 7.1.2]), which states that for $\lambda\in(0,1)$ fixed and $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}$ nonnegative measurable functions such that

$$h\big((1-\lambda)x+\lambda y\big)\geq f(x)^{1-\lambda}g(y)^{\lambda} \tag{4.3}$$

for any $x,y\in\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n}h\geq\left(\int_{\mathbb{R}^n}f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n}g\right)^{\lambda}. \tag{4.4}$$

Then it is easy to check that the volume bound (4.1) for the 'intersection operation' is also naturally obtained via the Prékopa-Leindler inequality. Indeed, for $x \in \mathbb{R}^n$, since

$$h(K \cap L, x) = \inf_{(1-\lambda)x_1+\lambda x_2=x} \Big((1-\lambda)h(K,x_1) + \lambda h(L,x_2)\Big) \tag{4.5}$$

(see e.g. [12, p. 59]), then

$$e^{-h(K\cap L,x)} \geq \left(e^{-h(K,x_1)}\right)^{1-\lambda} \left(e^{-h(L,x_2)}\right)^{\lambda},$$

and by (3.1) and (4.4) we get

$$\begin{aligned}\mathrm{vol}(K^*)^{1-\lambda}\mathrm{vol}(L^*)^{\lambda} &= \left(\frac{1}{n!}\int_{\mathbb{R}^n} e^{-h(K,x)}\mathrm{d}x\right)^{1-\lambda} \left(\frac{1}{n!}\int_{\mathbb{R}^n} e^{-h(L,x)}\mathrm{d}x\right)^{\lambda} \\ &\leq \frac{1}{n!}\int_{\mathbb{R}^n} e^{-h(K\cap L,x)}\mathrm{d}x = \mathrm{vol}\big([K\cap L]^*\big).\end{aligned}$$

Moreover, since $e^{-h(K\cap L,\cdot)}$ is the 'smallest' function satisfying condition (4.3) (cf. (4.5)), the above argument shows that inequality (4.1) is the best inequality that can be obtained via a 'Prékopa-Leindler approach'.

Having the definition of p-sum (1.1) in mind, and in view of (4.5), one might think of a kind of generalization of the 'intersection operation': for $K, L \in \mathcal{K}^n_0$ and $p \geq 1$, there is a (unique) convex body $K \cap_p L \in \mathcal{K}^n_0$ whose support function is given by

$$h(K \cap_p L, u) = \inf_{u_1+u_2=u} \Big(h(K,u_1)^p + h(L,u_2)^p\Big)^{1/p}. \tag{4.6}$$

It is easy to check that the right-hand side in the above identity defines a sublinear function, and hence $K \cap_p L$ is well-defined. Clearly, when $p = 1$ the intersection is obtained.

The following result shows that this new operation also allows to bound from below the ratio $\mathrm{vol}\big(\big[K \cap_p (-K)\big]^*\big)/\mathrm{vol}(K^*)$ in contrast to the case of the Minkowski/Firey addition.

Proposition 4.1. *Let $K \in \mathcal{K}^n_0$ and $1 \leq p \leq \infty$. Then*

$$\mathrm{vol}\Big(\big[K \cap_p (-K)\big]^*\Big) \geq \mathrm{vol}(K^*), \tag{4.7}$$

and equality holds if and only if K is 0-symmetric and $p = 1$.

Proof. Clearly,

$$h\big(K \cap_p (-K), u\big) \leq \inf_{u_1+u_2=u}\big(h(K,u_1) + h(-K,u_2)\big) = h\big(K \cap (-K), u\big),$$

i.e., $K \cap_p (-K) \subset K \cap (-K)$ and then, using polarity and applying (4.2) we get $\mathrm{vol}\big([K \cap_p (-K)]^*\big) \geq \mathrm{vol}\big([K \cap (-K)]^*\big) \geq \mathrm{vol}(K^*)$.

If K is 0-symmetric and $p = 1$ then equality holds in (4.7). Conversely, if we have equality in (4.7) then, in particular, $K \cap_p (-K) = K \cap (-K)$ and

$\mathrm{vol}\big([K\cap(-K)]^*\big)=\mathrm{vol}(K^*)$. The latter implies $K=-K$ (cf. (4.2)) and thus we get $K\cap_p K=K$. Therefore, for any $u\in\mathbb{R}^n$,

$$h(K,u)\leq\Big(h(K,u_1)^p+h(K,u_2)^p\Big)^{1/p}$$

for all u_1,u_2 with $u_1+u_2=u$, and then, in particular, we have

$$h(K,u)\leq 2^{1/p\,-1}h(K,u),$$

which is true, for $u\neq 0$, only if $p=1$. □

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