ON DISCRETE BORELL-BRASCAMP-LIEB INEQUALITIES

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Abstract. If $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ are non-negative measurable functions such that $h(x+y)$ is greater than or equal to the p-sum of $f(x)$ and $g(y)$, where $-1/n \leq p \leq \infty$, $p \neq 0$, then the Borell-Brascamp-Lieb inequality asserts that the integral of h is not smaller than the q-sum of the integrals of f and g, for $q = p/(np+1)$.

In this paper we obtain a discrete analog for the sum over finite subsets of the integer lattice $\mathbb{Z}^n$: under the same assumption as before, for $A, B \subset \mathbb{Z}^n$, then $\sum_{A+B} h \geq [(\sum_{\mathrm{r}_f(A)} f)^q + (\sum_B g)^q]^{1/q}$, where $\mathrm{r}_f(A)$ is obtained by removing points from A in a particular way, and depending on f. We also prove that the classical Borell-Brascamp-Lieb inequality for Riemann integrable functions can be obtained as a consequence of this new discrete version.

1. Introduction

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space. The n-dimensional volume of a compact set $K \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(K)$ (when integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$), and as a discrete counterpart, we use $|A|$ to represent the cardinality of a finite subset $A \subset \mathbb{R}^n$.

We write $\pi_{i_1,\dots,i_k}$, $1 \leq i_1, \dots, i_k \leq n$, to denote the orthogonal projection onto the k-dimensional coordinate plane $\mathbb{R}\mathrm{e}_{i_1} + \cdots + \mathbb{R}\mathrm{e}_{i_k}$, and also write $\pi_{(i)} = \pi_{1,\dots,i-1,i+1,\dots,n}$ to represent the corresponding orthogonal projection onto the i-th coordinate hyperplane $\mathbb{R}\mathrm{e}_1 + \cdots + \mathbb{R}\mathrm{e}_{i-1} + \mathbb{R}\mathrm{e}_{i+1} + \cdots + \mathbb{R}\mathrm{e}_n$.

Let $\mathbb{Z}^n$ be the integer lattice, i.e., the lattice of all points with integer coordinates in $\mathbb{R}^n$, and let $\mathbb{Z}^n_{\geq 0} = \{x \in \mathbb{Z}^n : x_i \geq 0\}$.

Relating the volume with the Minkowski addition of compact sets, one is led to the famous *Brunn-Minkowski inequality*. One form of it states that if $K, L \subset \mathbb{R}^n$ are compact and non-empty, then

$$\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{1.1}$$

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with equality, if $\mathrm{vol}(K)\mathrm{vol}(L) > 0$, if and only if K and L are homothetic compact convex sets. Here $+$ is used for the *Minkowski* (vectorial) sum, i.e.,

$$A + B = \{a + b : a \in A, b \in B\}$$

for any $A, B \subset \mathbb{R}^n$. The Brunn-Minkowski inequality is one of the most powerful theorems in Convex Geometry and beyond: it implies, among others, strong results such as the isoperimetric and Urysohn inequalities (see e.g. [14, s. 7.2]) or even the Aleksandrov-Fenchel inequality (see e.g. [14, s. 7.3]). It would not be possible to collect here all references regarding versions, applications and/or generalizations of the Brunn-Minkowski inequality. So, for extensive and beautiful surveys on them, we refer the reader to [1, 5, 10].

Regarding an analytical counterpart for functions of the Brunn-Minkowski inequality, one is naturally led to the so-called *Borell-Brascamp-Lieb inequality*, originally proved in [2] and [3]. In order to introduce it, we first recall the definition of the p-sum of two non-negative numbers, where $p \neq 0$ is a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$ (for a general reference for p-sums of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood, and Pólya [7] and to the handbook [4]). We consider first the case $p \in \mathbb{R}$, with $p \neq 0$: given $a, b > 0$ we set

$$\mathcal{S}_p\,(a, b) = (a^p + b^p)^{1/p}.$$

To complete the picture, for $p = \pm\infty$ we set $\mathcal{S}_\infty\,(a, b) = \max\{a, b\}$ and $\mathcal{S}_{-\infty}\,(x, y) = \min\{a, b\}$. Finally, if $ab = 0$, we define $\mathcal{S}_p\,(a, b) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$, $p \neq 0$. Note that $\mathcal{S}_p\,(a, b) = 0$, if $ab = 0$, is redundant for all $p < 0$, however it is relevant for $p > 0$. The reason to modify in this way the definition of p-sum given in [7] is due to the classical statement of the Borell-Brascamp-Lieb inequality, which is collected below. In fact, without such a modification, the thesis of the latter result would not have mathematical interest.

The following theorem (see also [5] for a detailed presentation), as previously stated, can be regarded as the functional counterpart of the Brunn-Minkowski inequality. In fact, a straightforward proof of (1.1) can be obtained by applying (1.2) to the characteristic functions $f = \chi_K$, $g = \chi_L$ and $h = \chi_{K+L}$ with $p = \infty$.

Theorem A (The Borell-Brascamp-Lieb inequality)**.** *Let $-1/n \leq p \leq \infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that*

$$h(x + y) \geq \mathcal{S}_p\,(f(x), g(y))$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \mathcal{S}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right). \tag{1.2}$$

Next we move to the discrete setting, i.e., we consider finite subsets of $\mathbb{Z}^n$. Note that one cannot expect to obtain a discrete analog of the Borell-Brascamp-Lieb inequality just by replacing integrals by sums since it is not even possible to get a Brunn-Minkowski inequality in its classical form for the cardinality. Indeed, simply taking $A = \{0\}$ to be the origin and any finite set $B \subset \mathbb{Z}^n$, then

$$|A+B|^{1/n} < |A|^{1/n} + |B|^{1/n}.$$

Therefore, discrete counterparts for both the Brunn-Minkowski inequality and the Borell-Brascamp-Lieb inequality should either have a different structure or involve modifications of the sets.

In [6], Gardner and Gronchi obtained a beautiful and powerful discrete Brunn-Minkowski inequality: they proved that if A, B are finite subsets of the integer lattice $\mathbb{Z}^n$, with dimension $\dim B = n$, then

$$|A+B| \geq \left|D^B_{|A|} + D^B_{|B|}\right|. \tag{1.3}$$

Here $D^B_{|A|}, D^B_{|B|}$ are *B-initial segments*: for any $m \in \mathbb{N}$, D^B_m is the set of the first m points of $\mathbb{Z}^n_{\geq 0}$ in the so-called "B-order", which is a particular order defined on $\mathbb{Z}^n_{\geq 0}$ depending only on the cardinality of B. For a proper definition and a deep study of it we refer the reader to [6]. As consequences of (1.3) they also get two additional nice discrete Brunn-Minkowski type inequalities which improve previous results obtained by Ruzsa [12, 13].

2. Background and main results

An alternative to getting a "classical" Brunn-Minkowski type inequality might be to transform (one of) the sets involved in the problem, either by adding or removing some points. In this spirit, two (equivalent) new discrete Brunn-Minkowski type inequalities are obtained in [8]. Similarly, by removing points from the original set $A \subset \mathbb{Z}^n$, $A \neq \emptyset$, we may define a new set $\mathrm{r}_f(A)$ to *reduce* it according to a particular function f.

To this aim, we need the following notation. If $\Lambda \subset \mathbb{Z}^k$ is finite, $k \in \{1, \dots, n\}$, for each $m \in \mathbb{Z}$ we write $\Lambda(m)$ to represent the section of Λ at m orthogonal to the (last) coordinate line $\mathbb{R}\mathrm{e}_k$, i.e.,

$$\Lambda(m) = \left\{p \in \mathbb{Z}^{k-1} : (p, m) \in \Lambda\right\}.$$

Next, given a non-negative function $f : \Lambda \longrightarrow \mathbb{R}_{\geq 0}$ (which will be often referred to as a *weight function*), let $m_0 = m_0(\Lambda, f) \in \pi_k(\Lambda)$ be such that $\sum_{x \in \Lambda(m_0)} f(x, m_0) = \max_m \sum_{x \in \Lambda(m)} f(x, m)$. Certainly the integer m_0 providing the "maximum section" with respect to the weight function f does not necessarily have to be unique. In that case, we define

$$m_0 = \max\left\{m' \in \pi_k(\Lambda) : \sum_{x \in \Lambda(m')} f(x, m') = \max_m \sum_{x \in \Lambda(m)} f(x, m)\right\}. \tag{2.1}$$

Now we define the function

$$\rho_k : \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\} \longrightarrow \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\}$$

given by

$$\rho_k(\Lambda) = \begin{cases} \Lambda \setminus \{m_0\} & \text{if } k = 1, \\ \Lambda \setminus \Big(\Lambda(m_0) \times \{m_0\}\Big) & \text{if } k > 1; \end{cases}$$

i.e., ρ_k acts on Λ by just removing the "maximum section" $\Lambda(m_0)$, with respect to the weight function f, from the set. To complete the picture we set $\rho_k(\emptyset) = \emptyset$.

Then, for $1 \leq k < n$, we write

$$A_k^- = \bigcup_{m \in \pi_{n,\dots,k+1}(A_{k-1}^-)} \Big(\rho_k\big(A_{k-1}^-(m)\big) \times \{m\}\Big),$$

with $A_0^- = A$. Then we define

$$\mathrm{r}_f(A) = \rho_n\big(A_{n-1}^-\big).$$

In other words, $\mathrm{r}_f(A)$ is given by

$$\mathrm{r}_f(A) = \bigcup_{m \in \pi_n(A) \setminus \{m_0(A_{n-1}^-, f)\}} \Big(\mathrm{r}_f\big(A(m)\big) \times \{m\}\Big). \tag{2.2}$$

Using this technique, in [8, Theorem 2.2] the following result is shown, where $\varphi : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$ is the constant weight function given by $\varphi(x) = 1$ for all $x \in \mathbb{Z}^n$.

Theorem B. *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$|A + B|^{1/n} \geq \big|\mathrm{r}_\varphi(A)\big|^{1/n} + |B|^{1/n}. \tag{2.3}$$

The inequality is sharp.

Equality holds in (2.3) when both A and B are *lattice cubes*. By a lattice cube we mean (up to a translation) the intersection of a cube $r[0,1]^n$, $r \in \mathbb{N}$, with the lattice $\mathbb{Z}^n$.

In [8] it is shown that inequalities (1.3) and (2.3) are not comparable.

The main goal of this paper is to obtain a discrete analog of Theorem A, in the spirit of Theorem B. This is the content of the following result.

Theorem 2.1. *Let $A, B \subset \mathbb{Z}^n$ be finite sets. Let $-1/n \leq p \leq \infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h(x + y) \geq \mathcal{S}_p\left(f(x), g(y)\right)$$

for all $x \in A$, $y \in B$. Then

$$\sum_{z \in A+B} h(z) \geq \mathcal{S}_{\frac{p}{np+1}}\left(\sum_{x \in \mathrm{r}_f(A)} f(x), \sum_{y \in B} g(y)\right). \tag{2.4}$$

We note that, as in the case of Theorem B in [8], the above result holds for finite subsets $A, B \subset \mathbb{R}^n$, via a suitable construction of the set $\mathrm{r}_f(A)$. We state and prove Theorem 2.1 in the case of $\mathbb{Z}^n$ for the sake of simplicity.

As in the continuous setting, inequality (2.4) can be seen as a functional extension of the discrete Brunn-Minkowski inequality (2.3), just by considering the characteristic functions $f = \chi_{_A}$, $g = \chi_{_B}$ and $h = \chi_{_{A+B}}$, and taking $p = \infty$.

Moreover, we also show that the classical Borell-Brascamp-Lieb inequality (1.2) can be obtained from the discrete version (2.4) under the mild (but necessary) assumption that the functions f, g are Riemann integrable:

Theorem 2.2. *The discrete Borell-Brascamp-Lieb inequality* (2.4) *implies the classical Borell-Brascamp-Lieb inequality* (1.2)*, provided that the functions f, g are Riemann integrable.*

We finish this section by recalling the simple inequality

$$|A + B| \geq |A| + |B| - 1 \tag{2.5}$$

for finite subsets A, B in $\mathbb{Z}^n$ (see e.g. [15, Chapter 2]), which provides, in particular, a 1-dimensional discrete Brunn-Minkowski inequality.

3. Proofs of the main results

Before the proof of Theorem 2.1 we state some auxiliary results. The following lemma can be regarded as a discrete analog of the well-known *Cavalieri Principle*.

Lemma 3.1. *Let $\Omega \subset \mathbb{Z}^n$ be finite, let $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ and set $f(\Omega) \subset \{k_0, k_1, \dots, k_r\}$ where $0 = k_0 < k_1 < \cdots < k_r$. Then*

$$\sum_{x\in\Omega} f(x) = \sum_{i=1}^{r} (k_i - k_{i-1}) \Big| \{x \in \Omega : f(x) \geq k_i\} \Big| = \int_0^\infty \Big| \{x \in \Omega : f(x) \geq t\} \Big| \,\mathrm{d}t.$$

Proof. The second equality is immediate and, hence, we will show the first one. To this aim, let $x \in \Omega$ and consider $k_s = f(x)$ for some $s \in \{1, \dots, r\}$ (we may assume, without loss of generality, that $f(x) > 0$). Then, with

$$\delta_i(x) = \begin{cases} 1 & \text{if } f(x) \geq k_i, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$f(x) = \sum_{i=1}^{s} (k_i - k_{i-1}) = \sum_{i=1}^{r} (k_i - k_{i-1}) \delta_i(x),$$

and thus we can conclude that

$$\begin{aligned} \sum_{x\in\Omega} f(x) &= \sum_{x\in\Omega} \sum_{i=1}^{r} (k_i - k_{i-1}) \delta_i(x) = \sum_{i=1}^{r} (k_i - k_{i-1}) \sum_{x\in\Omega} \delta_i(x) \\ &= \sum_{i=1}^{r} (k_i - k_{i-1}) \Big| \{x \in \Omega : f(x) \geq k_i\} \Big|. \end{aligned}$$

□

We note that, under the conditions of the above result, on one hand we may ensure that for any $k' \in (k_{i-1}, k_i)$,

$$(k_i - k_{i-1})\Big|\{x \in \Omega : f(x) \geq k_i\}\Big| = (k_i - k')\Big|\{x \in \Omega : f(x) \geq k_i\}\Big| + (k' - k_{i-1})\Big|\{x \in \Omega : f(x) \geq k'\}\Big|.$$

On the other hand, for every $k' > k_m = \max_{x\in\Omega} f(x)$, we clearly have $\big|\{x \in \Omega : f(x) \geq k'\}\big| = 0$. Hence, the set $\{k_0, k_1, \dots, k_r\}$ is not relevant.

The following result essentially yields the case $n = 1$ of Theorem 2.1 and will be used to derive (2.4). Note, however, that it holds not only for 1-dimensional sets but also for n-dimensional sets, in contrast to the case $n = 1$ of the classical Borell-Brascamp-Lieb inequality.

Lemma 3.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{Z}^n$ be finite sets. Let $-1 \leq p \leq \infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h(x + y) \geq \mathcal{S}_p\left(f(x), g(y)\right)$$

for all $x \in \Omega_1$, $y \in \Omega_2$. Then

$$\sum_{z\in\Omega_1+\Omega_2} h(z) \geq \mathcal{S}_{\frac{p}{p+1}}\left(\sum_{x\in\Omega_1\setminus\{x_0\}} f(x), \sum_{y\in\Omega_2} g(y)\right),$$

where $x_0 \in \Omega_1$ is such that $f(x_0) = \max_{x\in\Omega_1} f(x)$.

Proof. Clearly, we may assume that $\sum_{x\in\Omega_1\setminus\{x_0\}} f(x), \sum_{y\in\Omega_2} g(y) > 0$. We consider the non-negative functions $F, G, H : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$F(x) = \frac{f(x)}{A}, \quad G(y) = \frac{g(y)}{B}, \quad H(z) = \frac{h(z)}{C_p},$$

where

$$A = \max_{x\in\Omega_1} f(x) > 0, \quad B = \max_{y\in\Omega_2} g(y) > 0 \quad \text{and} \quad C_p = \mathcal{S}_p\left(A, B\right) > 0.$$

Then

$$\max_{x\in\Omega_1} F(x) = \max_{y\in\Omega_2} G(y) = 1.$$

First, we show that, for any $x \in \Omega_1$, $y \in \Omega_2$, we have

$$H(x + y) \geq \min\{F(x), G(y)\}. \tag{3.1}$$

To this aim, it is enough to consider $x \in \Omega_1$, $y \in \Omega_2$ with $f(x)g(y) > 0$. For $p \neq \infty$, and writing $\theta = B^p/C_p^p \in (0, 1)$, we get

$$\begin{aligned} h(x + y) &\geq \left(f(x)^p + g(y)^p\right)^{1/p} \\ &= C_p\left(\frac{F(x)^p A^p + G(y)^p B^p}{C_p^p}\right)^{1/p} \\ &= C_p\big((1-\theta)F(x)^p + \theta G(y)^p\big)^{1/p} \\ &\geq C_p \min\{F(x), G(y)\}. \end{aligned}$$

For $p=\infty$, $h(x+y)\geq\max\{f(x),g(y)\}\geq C_\infty\min\{F(x),G(y)\}$ clearly holds. Therefore, we have shown (3.1).

The definition of F and G now implies that the level sets

$$\{x\in\Omega_1:F(x)\geq t\},\quad \{y\in\Omega_2:G(y)\geq t\}$$

are non-empty for any $t\in[0,1]$. Moreover, from (3.1) we deduce that

$$\{z\in\Omega_1+\Omega_2:H(z)\geq t\}\supset\{x\in\Omega_1:F(x)\geq t\}+\{y\in\Omega_2:G(y)\geq t\}$$

and thus, by (2.5) together with the fact that $f(x_0)=\max_{x\in\Omega_1}f(x)$, we have

$$\begin{aligned}\Big|\{z\in\Omega_1+\Omega_2:H(z)\geq t\}\Big|&\geq\Big|\{x\in\Omega_1:F(x)\geq t\}\Big|+\Big|\{y\in\Omega_2:G(y)\geq t\}\Big|-1\\&=\Big|\{x\in\Omega_1\backslash\{x_0\}:F(x)\geq t\}\Big|+\Big|\{y\in\Omega_2:G(y)\geq t\}\Big|\end{aligned}$$

for all $t\in[0,1]$.

Finally, set $\{k_0,k_1,\ldots,k_r\}\supset F\big(\Omega_1\big)\cup G\big(\Omega_2\big)\cup H\big(\Omega_1+\Omega_2\big)$, with $0=k_0<k_1<\cdots<k_r$ where, for some $s\in\{1,\ldots,r\}$, $k_s=\max_{y\in\Omega_2}G(y)=1\geq\max_{x\in\Omega_1\backslash\{x_0\}}F(x)$. Then, by the above inequality, and using Lemma 3.1, we get

$$\begin{aligned}\sum_{z\in\Omega_1+\Omega_2}h(z)&=\sum_{z\in\Omega_1+\Omega_2}C_pH(z)=C_p\sum_{i=1}^{r}(k_i-k_{i-1})\Big|\{z\in\Omega_1+\Omega_2:H(z)\geq k_i\}\Big|\\&\geq C_p\sum_{i=1}^{s}(k_i-k_{i-1})\Big|\{z\in\Omega_1+\Omega_2:H(z)\geq k_i\}\Big|\\&\geq C_p\sum_{i=1}^{s}(k_i-k_{i-1})\Big(\Big|\{x\in\Omega_1\backslash\{x_0\}:F(x)\geq k_i\}\Big|\\&\qquad\qquad+\Big|\{y\in\Omega_2:G(y)\geq k_i\}\Big|\Big)\\&=C_p\left(\sum_{x\in\Omega_1\backslash\{x_0\}}F(x)+\sum_{y\in\Omega_2}G(y)\right)\\&=C_p\left(\frac{1}{A}\sum_{x\in\Omega_1\backslash\{x_0\}}f(x)+\frac{1}{B}\sum_{y\in\Omega_2}g(y)\right),\\&\geq\mathcal{S}_{\frac{p}{p+1}}\left(\sum_{x\in\Omega_1\backslash\{x_0\}}f(x),\sum_{y\in\Omega_2}g(y)\right).\end{aligned}$$

For $p\neq\infty$, the last inequality follows from the reverse Hölder inequality [4, Theorem 1, p. 178],

$$a_1b_1+a_2b_2\geq\big(a_1^{-p}+a_2^{-p}\big)^{-1/p}\big(b_1^q+b_2^q\big)^{1/q},$$

where $q=p/(p+1)$ is the Hölder conjugate of $(-p)\leq 1$, just by taking $a_1=1/A$, $a_2=1/B$, $b_1=\sum_{x\in\Omega_1\backslash\{x_0\}}f(x)$ and $b_2=\sum_{y\in\Omega_2}g(y)$.

The case $p=\infty$ is immediate. □

Now we are in a position to prove our main result. The main idea of the proof we present here is exploiting the above result (for $n=1$) via an inductive procedure, and it goes back to the classical proof of the Borell-Brascamp-Lieb inequality (see e.g. [2, 3, 9, 11]). We present it here for the sake of completeness.

Proof of Theorem 2.1. We may assume, without loss of generality, that both $\sum_{x\in\mathrm{r}_f(A)} f(x)>0$ and $\sum_{y\in B} g(y)>0$.

If $n=1$, the result follows immediately from Lemma 3.2 with $\Omega_1=A$, $\Omega_2=B$ and noticing that $\mathrm{r}_f(A)=A\setminus\{m_0\}$; recall that $f(m_0)=\max_{m\in A} f(m)$, cf. (2.1).

Now suppose that $n>1$ and assume that the theorem holds for dimension $n-1$. Let $m_A\in\pi_n(A)$, $m_B\in\pi_n(B)$, let $\Omega_1=A(m_A)\subset\mathbb{Z}^{n-1}$, $\Omega_2=B(m_B)\subset\mathbb{Z}^{n-1}$ and consider the functions $f_1,g_1,h_1:\mathbb{R}^{n-1}\longrightarrow\mathbb{R}_{\geq 0}$ given by

$$f_1(x)=f(x,m_A),\quad g_1(x)=g(x,m_B),\quad h_1(x)=h(x,m_A+m_B),$$

for any $x\in\mathbb{R}^{n-1}$. Since for all $x\in\Omega_1$, $y\in\Omega_2$ we have

$$h_1(x+y)=h(x+y,m_A+m_B)\geq\mathcal{S}_p\left(f(x,m_A),g(y,m_B)\right)=\mathcal{S}_p\left(f_1(x),g_1(y)\right),$$

we may assert that

$$\sum_{z\in A(m_A)+B(m_B)} h_1(z)\geq\mathcal{S}_{\frac{p}{(n-1)p+1}}\left(\sum_{x\in\mathrm{r}_f(A(m_A))} f_1(x),\sum_{y\in B(m_B)} g_1(y)\right).$$

This, together with the fact that

$$(A+B)(m_A+m_B)\supset A(m_A)+B(m_B),$$

yields, in terms of f, g and h,

$$\sum_{z\in(A+B)(m_A+m_B)} h(z,m_A+m_B)\geq\mathcal{S}_{\frac{p}{(n-1)p+1}}\left(\sum_{x\in\mathrm{r}_f(A(m_A))} f(x,m_A),\sum_{y\in B(m_B)} g(y,m_B)\right).\tag{3.2}$$

Now, let $f_2,g_2,h_2:\mathbb{Z}\longrightarrow\mathbb{R}_{\geq 0}$ be the functions defined by

$$f_2(m)=\sum_{x\in\mathrm{r}_f(A(m))} f(x,m),\quad g_2(m)=\sum_{y\in B(m)} g(y,m),\quad\text{and}$$

$$h_2(m)=\sum_{z\in(A+B)(m)} h(z,m).$$

Let $m_0 = m_0(A^-_{n-1}, f) \in \pi_n(A)$ be the value for which

$$\mathrm{r}_f(A) = \bigcup_{m\in\pi_n(A)\setminus\{m_0\}} \Big(\mathrm{r}_f(A(m)) \times \{m\}\Big)$$

holds (see (2.2)). Then we clearly have $f_2(m_0) = \max_{m\in\pi_n(A)} f_2(m)$.

Hence, (3.2) yields, in terms of f_2, g_2 and h_2,

$$h_2(m_A + m_B) \geq \mathcal{S}_{\frac{p}{(n-1)p+1}} \left(f_2(m_A), g_2(m_B)\right)$$

for any $m_A \in \pi_n(A)$, $m_B \in \pi_n(B)$, and thus we may use Lemma 3.2 with $\Omega_1 = \pi_n(A)$, $\Omega_2 = \pi_n(B)$ and the functions f_2, g_2 and h_2 to obtain

$$\sum_{m\in\pi_n(A)+\pi_n(B)} h_2(m) \geq \mathcal{S}_{\frac{p}{np+1}} \left(\sum_{m_A\in\pi_n(A)\setminus\{m_0\}} f_2(m_A), \sum_{m_B\in\pi_n(B)} g_2(m_B) \right).$$

This, together with the relations

$$\sum_{m\in\pi_n(A)+\pi_n(B)} h_2(m) = \sum_{z\in A+B} h(z), \qquad \sum_{m_A\in\pi_n(A)\setminus\{m_0\}} f_2(m_A) = \sum_{x\in\mathrm{r}_f(A)} f(x),$$

$$\sum_{m_B\in\pi_n(B)} g_2(m_B) = \sum_{y\in B} g(y),$$

finishes the proof. □

As a straightforward consequence of the above result we get the following Brunn-Minkowski type inequality for discrete measures associated to *p-additive functions*, in the spirit of (2.3). Indeed, it is enough to apply Theorem 2.1 to the functions $f = \chi_A \phi$, $g = \chi_B \phi$ and $h = \chi_{A+B} \phi$.

Corollary 3.1. *Let $-1/n \leq p \leq \infty$, $p \neq 0$, and let $\phi : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative function such that*

$$\phi(x + y) \geq \mathcal{S}_p \left(\phi(x), \phi(y)\right)$$

for any $x, y \in \mathbb{Z}^n$. Let μ be the discrete measure on $\mathbb{Z}^n$ with mass function ϕ, i.e., such that

$$\mu(M) = \sum_{x\in M} \phi(x)$$

for any finite set $M \subset \mathbb{Z}^n$, and let $A, B \subset \mathbb{Z}^n$ be finite. Then

$$\mu(A + B) \geq \mathcal{S}_{\frac{p}{np+1}} \left(\mu\big(\mathrm{r}_\phi(A)\big), \mu(B)\right).$$

We conclude the paper by proving Theorem 2.2. To this aim, we need the following definition.

Definition 3.1. *Let $K \subset \mathbb{R}^n$ be a compact set. The k-*discretization *of K, $k \in \mathbb{N}$, is defined as*

$$K_k = \left\{ x \in 2^{-k}\mathbb{Z}^n : \left(x + \left[0, 2^{-k}\right)^n\right) \cap \operatorname{int} K \neq \emptyset \right\},$$

where, by $\operatorname{int} K$, we denote the interior of K.

Proof of Theorem 2.2. Let $m \in \mathbb{N}$ and let $K = [-m, m]^n$. For each $k \in \mathbb{N}$, let K_k be the k-discretization of K.

We define the functions $f_k, g_k : K_k \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_k(x) = \inf_{z \in x + [0, 2^{-k})^n} f(z), \quad g_k(x) = \inf_{z \in x + [0, 2^{-k})^n} g(z),$$

and let $h_k : K_k + K_k \longrightarrow \mathbb{R}_{\geq 0}$ be the function defined by

$$h_k(x) = \inf_{z \in x + [0, 2^{-k})^n} h(z).$$

Note that for any $x, y \in K_k$ we have

$$\begin{aligned} h_k(x+y) &= \inf_{z \in x + y + [0,2^{-k})^n} h(z) \\ &\geq \inf_{z \in x + [0,2^{-k})^n + y + [0,2^{-k})^n} h(z) \\ &= \inf_{z_1 \in x + [0,2^{-k})^n,\, z_2 \in y + [0,2^{-k})^n} h(z_1 + z_2) \\ &\geq \inf_{z_1 \in x + [0,2^{-k})^n,\, z_2 \in y + [0,2^{-k})^n} \mathcal{S}_p\left(f(z_1), g(z_2)\right) \\ &\geq \mathcal{S}_p\left(\inf_{z_1 \in x + [0,2^{-k})^n} f(z_1), \inf_{z_2 \in y + [0,2^{-k})^n} g(z_2)\right) \\ &= \mathcal{S}_p\left(f_k(x), g_k(y)\right), \end{aligned}$$

and thus, since K_k is a finite set, we can use Theorem 2.1 to deduce that, for any $k \in \mathbb{N}$, we have

$$2^{-kn} \sum_{z \in K_k + K_k} h_k(z) \geq \mathcal{S}_{\frac{p}{np+1}}\left(2^{-kn} \sum_{x \in \mathrm{r}_f(K_k)} f_k(x), 2^{-kn} \sum_{y \in K_k} g_k(y)\right). \tag{3.3}$$

First, we clearly have

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \sum_{z \in K_k + K_k} 2^{-kn} h_k(z). \tag{3.4}$$

Now, using (3.4), (3.3) and taking into account that g is Riemann integrable we immediately get

(3.5)

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)\,\mathrm{d}x &\geq \lim_{k\to\infty} \mathcal{S}_{\frac{p}{np+1}}\left(2^{-kn} \sum_{x \in \mathrm{r}_f(K_k)} f_k(x), 2^{-kn} \sum_{y \in K_k} g_k(y)\right) \\ &= \mathcal{S}_{\frac{p}{np+1}}\left(\lim_{k\to\infty} 2^{-kn} \sum_{x \in \mathrm{r}_f(K_k)} f_k(x), \lim_{k\to\infty} 2^{-kn} \sum_{y \in K_k} g_k(y)\right) \\ &= \mathcal{S}_{\frac{p}{np+1}}\left(\lim_{k\to\infty} 2^{-kn} \sum_{x \in \mathrm{r}_f(K_k)} f_k(x), \int_K g(x)\,\mathrm{d}x\right), \end{aligned}$$

because $2^{-kn}\sum_{y\in K_k} g_k(y)$ is a *lower sum* of g for the partition $K_k+[0,2^{-k})^n$ of K.

In the following, we show that

$$\lim_{k\to\infty} 2^{-kn}\sum_{x\in \mathrm{r}_f(K_k)} f_k(x)=\int_K f(x)\,\mathrm{d}x. \tag{3.6}$$

Since the function f is Riemann integrable and non-negative, it is bounded and then there exists a constant $c\in\mathbb{R}_{\geq 0}$ such that $f(x)\leq c$ for all $x\in\mathbb{R}^n$, which implies that $f_k(x)\leq c$ for any $x\in K_k$. For the sake of brevity let $K_{k,i}=(K_k)^-_{i-1}\backslash(K_k)^-_i$, $i=1,\dots,n$, i.e., the set of all points removed from K_k in the i-th step of the construction of $\mathrm{r}_f(K_k)$. Then it is clear that

$$\left|K_{k,i}\right|=\left|\pi_{(i)}(K_{k,i})\right|\leq\left|\pi_{(i)}(K_k)\right|,$$

and since K is compact, $\left(\pi_{(i)}(K)\right)_k=\pi_{(i)}(K_k)$. So we have

$$0=\int_{\pi_{(i)}(K)} c\,\mathrm{d}x=\lim_{k\to\infty}2^{-kn}\sum_{x\in\left(\pi_{(i)}(K)\right)_k} c\geq\lim_{k\to\infty}2^{-kn}\sum_{x\in K_{k,i}} c,$$

which implies that

$$\lim_{k\to\infty}2^{-kn}\sum_{x\in K_{k,i}} f_k(x)=0.$$

This shows that

$$\begin{aligned}\lim_{k\to\infty}2^{-kn}\sum_{x\in\mathrm{r}_f(K_k)} f_k(x)&=\lim_{k\to\infty}\left(2^{-kn}\sum_{x\in K_k}f_k(x)-2^{-kn}\sum_{x\in K_k\backslash \mathrm{r}_f(K_k)}f_k(x)\right)\\&=\lim_{k\to\infty}\left(2^{-kn}\sum_{x\in K_k}f_k(x)-2^{-kn}\sum_{i=1}^n\sum_{x\in K_{k,i}}f_k(x)\right)\\&=\lim_{k\to\infty}2^{-kn}\sum_{x\in K_k}f_k(x)-\sum_{i=1}^n\lim_{k\to\infty}2^{-kn}\sum_{x\in K_{k,i}}f_k(x)\\&=\int_K f(x)\,\mathrm{d}x.\end{aligned}$$

This proves (3.6) and then, by (3.5),

$$\int_{\mathbb{R}^n}h(x)\,\mathrm{d}x\geq\mathcal{S}_{\frac{p}{np+1}}\left(\int_K f(x)\,\mathrm{d}x,\int_K g(x)\,\mathrm{d}x\right).$$

Since this is true for $K=[-m,m]^n$, for every $m\in\mathbb{N}$, the proof is now concluded because

$$\lim_{m\to\infty}\int_{[-m,m]^n}\phi(x)\,\mathrm{d}x=\int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x$$

for every non-negative measurable function $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$. □

Note that the assumption that the functions f and g are Riemann integrable in Theorem 2.2 seems to be necessary. Indeed, in order to derive the classical Borell-Brascamp-Lieb inequality (1.2) from the discrete version (2.4), one needs to consider some functions to which one may apply (2.4), and then the corresponding finite sums should approximate in some sense the integrals of f and g. The point is that, while these sums may be seen as Riemann sums over uniform partitions, there seems to be no natural way to involve integrals of arbitrary (measurable) simple functions.

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