BRUNN-MINKOWSKI TYPE INEQUALITIES FOR THE LATTICE POINT ENUMERATOR

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ABSTRACT. Geometric and functional Brunn-Minkowski type inequalities for the lattice point enumerator $G_n(\cdot)$ are provided. In particular, we show that

$$G_n((1-\lambda)K + \lambda L + (-1,1)^n)^{1/n} \geq (1-\lambda)G_n(K)^{1/n} + \lambda G_n(L)^{1/n}$$

for any non-empty bounded sets $K, L \subset \mathbb{R}^n$ and all $\lambda \in (0,1)$.

We also show that these new discrete versions imply the classical results, and discuss some links with other related inequalities.

1. INTRODUCTION AND NOTATION

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space, and we denote by e_i the i-th canonical unit vector. We set (x, y) for the open segment with endpoints $x, y \in \mathbb{R}^n$. The n-dimensional volume of a compact set $K \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(K)$, and as a discrete counterpart, we use $|A|$ to represent the cardinality of a finite subset $A \subset \mathbb{R}^n$.

Let $\mathbb{Z}^n$ be the integer lattice, i.e., the lattice of all points with integral coordinates in $\mathbb{R}^n$. We will denote by $G_n(\cdot)$ the lattice point enumerator for the integer lattice $\mathbb{Z}^n$, i.e., $G_n(M) = |M \cap \mathbb{Z}^n|$. By $\lfloor x \rfloor$ we denote the floor function of the real number x, i.e., the greatest integer less than or equal to x. Similarly, $\lceil x \rceil$ represents the ceiling function of x, namely, the least integer greater than or equal to x.

Finally, given a set $M \subset \mathbb{R}^n$, χ_M represents the characteristic function of M and, moreover, we denote by $\mathrm{int}\, M$, $\mathrm{bd}\, M$ and $\mathrm{cl}\, M$ its interior, boundary and closure, respectively. Furthermore, given $r > 0$, rM stands for the set $\{rm : m \in M\}$.

Relating the volume of the Minkowski addition of two sets in terms of their volumes, one is led to the famous *Brunn-Minkowski inequality* (for extensive survey articles on this and related inequalities we refer the reader to [1, 5]). One form of it asserts that if $\lambda \in (0, 1)$ and K and L are non-empty compact subsets of $\mathbb{R}^n$ then

$$(1.1) \qquad \mathrm{vol}\big((1 - \lambda)K + \lambda L\big)^{1/n} \geq (1 - \lambda)\mathrm{vol}(K)^{1/n} + \lambda\mathrm{vol}(L)^{1/n}.$$

Here $+$ is used for the Minkowski sum, i.e., $A + B = \{a + b : a \in A, b \in B\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$. Moreover, from the homogeneity of the volume, (1.1) is equivalent to

$$(1.2) \qquad \mathrm{vol}\big(K + L\big)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}.$$

Next we move to the discrete setting, i.e., we consider finite sets of (integer) points which are not necessarily full-dimensional unless indicated otherwise. It can easily be seen that one cannot expect to obtain a Brunn-Minkowski type inequality for the cardinality in the classical form. Indeed, simply taking $A = \{0\}$ to be the origin and any finite set $B \subset \mathbb{Z}^n$, we get

$$|A + B|^{1/n} < |A|^{1/n} + |B|^{1/n}.$$

Therefore, a discrete Brunn-Minkowski type inequality should either have a different structure or involve modifications of the sets. A first example is the simple inequality

$$(1.3) \qquad |A + B| \geq |A| + |B| - 1,$$

for finite $A, B \subset \mathbb{Z}^n$ (see e.g. [17, Chapter 2]).

In [6], Gardner and Gronchi obtained an engaging discrete Brunn-Minkowski inequality: they proved that if A, B are finite subsets of the integer lattice $\mathbb{Z}^n$, with dimension $\dim B = n$, then

$$(1.4) \qquad |A + B| \geq \big|D_{|A|}^B + D_{|B|}^B\big|.$$

Here $D_{|A|}^B, D_{|B|}^B$ are B-*initial segments*: for $m \in \mathbb{N}$, D_m^B is the set of the first m points of $\mathbb{Z}_+^n$ in the so-called "B-order", which is a particular order defined on $\mathbb{Z}_+^n$ depending only on the cardinality of B. For a proper definition and a deep study of it we refer the reader to [6]. As consequences of (1.4) they also get two additional nice discrete Brunn-Minkowski type inequalities:

$$|A + B|^{1/n} \geq |A|^{1/n} + \frac{1}{(n!)^{1/n}}\big(|B| - n\big)^{1/n}$$

and, if $|B| \leq |A|$, then

$$|A + B| \geq |A| + (n - 1)|B| + \big(|A| - n\big)^{(n-1)/n}\big(|B| - n\big)^{1/n} - \frac{n(n - 1)}{2}.$$

These inequalities improve previous results obtained by Ruzsa in [14, 15].

An alternative to getting a "classical" Brunn-Minkowski type inequality might be to transform (one of) the sets involved in the problem. In this regard, in [10] an extension $\bar{A}$ of A, is defined for any non-empty and finite

set $A \subset \mathbb{Z}^n$ (for $n = 1$ then $\bar{A} = A \cup \{\max(A) + 1\}$, whereas for $n > 1$ the set $\bar{A}$ is obtained by first adding the maximal cardinality section of A, and then applying the corresponding extension to every section of the latter new set; we refer to Section 2.1 in [10] for its precise definition and properties). Using this technique the following discrete counterpart for (1.2) was shown:

Theorem A ([10]). *Let* $A, B \subset \mathbb{Z}^n$ *be finite,* $A, B \neq \emptyset$. *Then*

$$(1.5) \qquad \left|\bar{A} + B\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

The inequality is sharp.

Aiming to get a discrete version of (1.1), it is worth noting the following: for any pair of non-empty finite sets $A, B \subset \mathbb{R}^n$, by (1.3) (and using that $|A|, |B| \geq 1$), one has

$$\left|(1-\lambda)A + \lambda B\right| \geq \left|(1-\lambda)A\right| + \left|\lambda B\right| - 1 = |A| + |B| - 1$$
$$\geq (1-\lambda)|A| + \lambda|B| \geq \left((1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}\right)^n,$$

where the last inequality follows from the convexity of the function $t \mapsto t^n$, for $t \geq 0$. Nevertheless this inequality is meaningless from a geometric point of view: the point is that while the quantities $|A|, |B|$ on the right-hand side are reduced by the factors $(1 - \lambda)$ and λ, the sets $(1 - \lambda)A$ and λB on the left-hand side have the same cardinality as A and B, respectively. A possible solution would be to involve a more natural way to "count points" according to dilatations, namely, using the lattice point enumerator G_n (for compact subsets of $\mathbb{R}^n$) instead of the cardinality $|\cdot|$ (for finite subsets of $\mathbb{R}^n$).

Again, one cannot expect to obtain a Brunn-Minkowski type inequality for G_n in the classical form (1.1) (which, as we have mentioned before, would be a similar situation to what happens for $|\cdot|$ regarding a discrete version of (1.2)). Indeed, just by taking $\lambda = 1/2$, $K = [0, m-\epsilon]^n$ and $L = [0, m+\epsilon/2]^n$, with $m \in \mathbb{N}$ and $0 < \epsilon < 1$, one gets

$$(1.6) \qquad \mathrm{G}_n\left(\frac{K+L}{2}\right)^{1/n} = m < m + \frac{1}{2} = \frac{\mathrm{G}_n(K)^{1/n} + \mathrm{G}_n(L)^{1/n}}{2}.$$

Thus, as in (1.5), an alternative to get such an inequality for the lattice point enumerator would be to consider a certain extension of $(1-\lambda)K + \lambda L$. So, we pose the following question:

Question 1.1. *Given compact sets* $K, L \subset \mathbb{R}^n$ *containing points from* $\mathbb{Z}^n$, *what is the "best" way to define a set* M_λ *such that* $(1 - \lambda)K + \lambda L \subset M_\lambda$ *and*

$$\mathrm{G}_n(M_\lambda)^{1/n} \geq (1 - \lambda)\mathrm{G}_n(K)^{1/n} + \lambda\mathrm{G}_n(L)^{1/n}$$

holds for a given $\lambda \in (0, 1)$?

2. Main results

Here we give an answer to Question 1.1, which we view as a discrete counterpart of (1.1):

Theorem 2.1. *Let $\lambda \in (0,1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Then*

$$(2.1) \quad \mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}.$$

The inequality is sharp.

We point out the following: when dealing with arbitrary non-empty subsets $K, L \subset \mathbb{R}^n$ (i.e., not necessarily bounded), from (2.1) we immediately get that

$$\mathrm{G}_n\big(M_\lambda + (-1,1)^n\big)^{1/n} \geq \sup_{m \in \mathbb{N}} \mathrm{G}_n\big((1-\lambda)K_m + \lambda L_m + (-1,1)^n\big)^{1/n}$$

$$\geq \sup_{m \in \mathbb{N}} \left((1-\lambda)\mathrm{G}_n(K_m)^{1/n} + \lambda \mathrm{G}_n(L_m)^{1/n}\right)$$

$$= (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}$$

for any $\lambda \in (0,1)$, where $M_\lambda = (1-\lambda)K + \lambda L$, $K_m = K \cap [-m,m]^n$ and $L_m = L \cap [-m,m]^n$. So, for the sake of simplicity, we will present here the results in the setting of bounded sets, although they also hold in full generality.

The core ingredient in the proof of Theorem 2.1 is its functional analogue. To introduce such a result, we present an analytical counterpart for functions of the Brunn-Minkowski inequality, the so-called *Borell-Brascamp-Lieb inequality*, originally proved in [2] and [3]. To this end, we first recall the definition of the p-mean $\mathcal{M}_p^\lambda(\cdot, \cdot)$ of two non-negative numbers, where p is a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$ (for a general reference for p-means of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood, and Pólya [9] and to the handbook [4]): we consider first the case $p \in \mathbb{R}$, with $p \neq 0$: given $a, b > 0$ we set

$$\mathcal{M}_p^\lambda(a,b) = \big((1-\lambda)a^p + \lambda b^p\big)^{1/p}.$$

For $p = 0$, we write $\mathcal{M}_0^\lambda(a,b) = a^{1-\lambda}b^\lambda$. To complete the picture, for $p = \pm\infty$ we set $\mathcal{M}_\infty^\lambda(a,b) = \max\{a,b\}$ and $\mathcal{M}_{-\infty}^\lambda(x,y) = \min\{a,b\}$. Finally, if $ab = 0$, we define $\mathcal{M}_p^\lambda(a,b) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$. Note that $\mathcal{M}_p^\lambda(a,b) = 0$, if $ab = 0$, is redundant for all $p \leq 0$, however it is relevant for $p > 0$. The reason to modify in this way the definition of p-mean given in [9] is due to the classical statement of the Borell-Brascamp-Lieb inequality, which is collected below. In fact, without such a modification, the thesis of the latter result would not have mathematical interest.

The following theorem (see also [5] for a detailed presentation), as previously stated, can be regarded as the functional counterpart of the Brunn-Minkowski inequality. In fact, a straightforward proof of (1.1) for compact

sets $K, L \subset \mathbb{R}^n$ of positive volume can be obtained by applying (2.2) to the characteristic functions $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)K+\lambda L}$ with $p = \infty$. Moreover, in the literature, the case $p = 0$ is referred to as the *Prékopa-Leindler inequality*, which is a powerful tool, in particular when dealing with *log-concave functions*.

Theorem B (The Borell-Brascamp-Lieb inequality). *Let $\lambda \in (0,1)$. Let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be measurable functions such that*

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda (f(x), g(y))$$

for all $x, y \in \mathbb{R}^n$. Then

$$(2.2) \qquad \int_{\mathbb{R}^n} h(x)\, \mathrm{d}x \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x, \int_{\mathbb{R}^n} g(x)\, \mathrm{d}x \right).$$

To state our main result and henceforth, we will need the following notation: for a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, we denote by $\phi^\diamond : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ the function given by

$$\phi^\diamond(z) = \sup_{u \in (-1,1)^n} \phi(z + u) \quad \text{for all } z \in \mathbb{R}^n.$$

Such an extension of the function ϕ is nothing but the *Asplund product* $\star$ (also referred to as the *sup-convolution*, which can be regarded as the functional analogue of the Minkowski sum in the setting of log-concave functions, for which we refer the reader to [16, Section 9.5] and the references therein) of the functions ϕ and $\chi_{(-1,1)^n}$. Indeed,

$$\phi^\diamond(z) = \sup_{u \in (-1,1)^n} \phi(z+u) = \sup_{u \in \mathbb{R}^n} \phi(z+u)\chi_{(-1,1)^n}(-u)$$

$$= \sup_{u_1 + u_2 = z} \phi(u_1)\chi_{(-1,1)^n}(u_2) = \big(\phi \star \chi_{(-1,1)^n}\big)(z).$$

Our main result reads as follows:

Theorem 2.2. *Let $\lambda \in (0,1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be functions such that*

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda (f(x), g(y))$$

for all $x \in K$, $y \in L$. Then

$$(2.3) \qquad \sum_{z \in M \cap \mathbb{Z}^n} h^\diamond(z) \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\sum_{x \in K \cap \mathbb{Z}^n} f(x), \sum_{y \in L \cap \mathbb{Z}^n} g(y) \right),$$

where $M = (1-\lambda)K + \lambda L + (-1,1)^n$.

Let $\mathcal{B} = \{v_1 \ldots, v_n\}$ be a basis of an n-dimensional lattice $\Lambda \subset \mathbb{R}^n$. Under the same assumptions of the above result, we may consider the auxiliary functions $f_\mathcal{B}, g_\mathcal{B}, h_\mathcal{B} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by

$$f_\mathcal{B}(x) = f(\varphi(x)), \quad g_\mathcal{B}(x) = g(\varphi(x)) \text{ and } \quad h_\mathcal{B}(x) = h(\varphi(x)),$$

for all $x \in \mathbb{R}^n$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is the linear (bijective) map given by $\varphi(x) = \sum_{i=1}^n x_i v_i$ for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Thus, as an immediate consequence of the previous result (applied to the functions $f_\mathcal{B}, g_\mathcal{B}$ and $h_\mathcal{B}$), we get that

$$\sum_{z \in M \cap \Lambda} h^{\diamond_\mathcal{B}}(z) \geq \mathcal{M}^{\lambda}_{\frac{p}{np+1}} \left(\sum_{x \in K \cap \Lambda} f(x), \sum_{y \in L \cap \Lambda} g(y) \right),$$

where $M = (1-\lambda)K + \lambda L + \varphi((-1,1)^n)$ and $h^{\diamond_\mathcal{B}}(z) = \sup_{u \in \varphi((-1,1)^n)} h(z+u)$ for all $z \in \mathbb{R}^n$. So, Theorem 2.2 also holds in the setting of an n-dimensional lattice $\Lambda \subset \mathbb{R}^n$.

We would like to point out that Theorem B admits an equivalent version for p-sums (instead of p-means). In this regard, in [11] a discrete version of the Borell-Brascamp-Lieb inequality for p-sums was provided. However, in contrast to the continuous setting, where one may directly obtain Theorem B from the corresponding version for p-sums (and vice versa) because of the homogeneity of the volume, one cannot expect to derive in a similar way a discrete version of Theorem B (like Theorem 2.2) from [11, Theorem 2.1].

In order to prove Theorem 2.2, we state an auxiliary result that will allow us to get the one-dimensional case of the above-mentioned Brunn-Minkowski type inequality for the lattice point enumerator (cf. Question 1.1).

Lemma 2.1. *Let $\lambda \in (0,1)$ and let $K, L, M \subset \mathbb{R}$ be non-empty sets such that $(1-\lambda)K + \lambda L \subset M$. If $M = \bigcup_{i=1}^s [a_i, b_i]$ is a finite union of pairwise disjoint compact intervals then*

(2.4) $$\mathrm{G}_1(M) + \Delta(M) \geq (1-\lambda)\mathrm{G}_1(K) + \lambda \mathrm{G}_1(L),$$

where $\Delta(M)$ denotes the number of non-integer extreme points of M, namely

$$\Delta(M) = |\{a_i \notin \mathbb{Z} : 1 \leq i \leq s\}| + |\{b_i \notin \mathbb{Z} : 1 \leq i \leq s\}|.$$

Proof. We show the result by induction on the number of intervals s of M. For the case $s = 1$, i.e., when $M = [a_1, b_1]$ is a (non-empty) compact interval, we have on the one hand that $\mathrm{G}_1(M) = \lfloor b_1 \rfloor - \lceil a_1 \rceil + 1$. Moreover, denoting by $a = \inf K$, $b = \sup K$, $c = \inf L$ and $d = \sup L$, we clearly get $\mathrm{G}_1(K) \leq \mathrm{G}_1([a,b]) = \lfloor b \rfloor - \lceil a \rceil + 1$ and $\mathrm{G}_1(L) \leq \mathrm{G}_1([c,d]) = \lfloor d \rfloor - \lceil c \rceil + 1$. On the other hand, the inclusion $(1-\lambda)K + \lambda L \subset M$ implies that

$$\lfloor b_1 \rfloor \geq b_1 - \chi_{\mathbb{R} \setminus \mathbb{Z}}(b_1) \geq (1-\lambda) \lfloor b \rfloor + \lambda \lfloor d \rfloor - \chi_{\mathbb{R} \setminus \mathbb{Z}}(b_1)$$

and

$$-\lceil a_1 \rceil \geq -a_1 - \chi_{\mathbb{R} \setminus \mathbb{Z}}(a_1) \geq -(1-\lambda) \lceil a \rceil - \lambda \lceil c \rceil - \chi_{\mathbb{R} \setminus \mathbb{Z}}(a_1),$$

and thus

$$\lfloor b_1 \rfloor - \lceil a_1 \rceil + 1 \geq (1-\lambda)(\lfloor b \rfloor - \lceil a \rceil + 1) + \lambda(\lfloor d \rfloor - \lceil c \rceil + 1) - \Delta(M).$$

This, together with the above upper bounds for the lattice point enumerator G_1 of K and L, yields $\mathrm{G}_1(M) \geq (1-\lambda)\mathrm{G}_1(K) + \lambda \mathrm{G}_1(L) - \Delta(M)$, which shows the case $s = 1$.

So, we suppose that the inequality is true for $s \geq 1$ and assume that $M = \bigcup_{i=1}^{s+1}[a_i, b_i]$, where $b_i < a_{i+1}$ for all $1 \leq i \leq s$.

Denoting by $M_1 = [a_1, b_1]$ and $M_2 = \bigcup_{i=2}^{s+1}[a_i, b_i]$, we may assume, without loss of generality, that $M_1 \cap ((1-\lambda)K + \lambda L) \neq \emptyset$ (otherwise, the result follows directly from the induction hypothesis applied to the sets K, L and M_2). Hence, we may define $m = \sup(M_1 \cap ((1-\lambda)K + \lambda L))$ and then, since K and L are bounded (because $(1-\lambda)K + \lambda L \subset M$), there exist $k \in \operatorname{cl} K$ and $l \in \operatorname{cl} L$ such that $(1-\lambda)k + \lambda l = m$. Thus, considering the sets $K_1 = \{x \in K : x \leq k\}$, $K_2 = K \setminus K_1$, $L_1 = \{x \in L : x \leq l\}$ and $L_2 = L \setminus L_1$, we clearly have that $(1-\lambda)K_1 + \lambda L_1 \subset M_1$ and $(1-\lambda)K_2 + \lambda L_2 \subset M_2$. Therefore, applying the induction hypothesis (and taking into account that M_1 are M_2 are disjoint), we get

$$
\begin{aligned}
G_1(M) + \Delta(M) &= G_1(M_1) + G_1(M_2) + \Delta(M_1) + \Delta(M_2) \\
&\geq (1-\lambda)\big(G_1(K_1) + G_1(K_2)\big) + \lambda\big(G_1(L_1) + G_1(L_2)\big) \\
&= (1-\lambda)G_1(K) + \lambda G_1(L),
\end{aligned}
$$

as desired. $\qquad\square$

Remark 2.1. *One might think that if $(1-\lambda)K + \lambda L$ is itself a finite union of (pairwise disjoint) compact intervals, the set $M = (1-\lambda)K + \lambda L$ would yield a tighter inequality in (2.4) than other greater sets (with respect to set inclusion). Nevertheless, this is not true in general: if we consider $K = [-2m, -1] \cup [1, 2m]$ with $m \in \mathbb{N}$, $L = \{0\}$ and $\lambda = 1/2$, then for $M = (1-\lambda)K + \lambda L = [-m, -1/2] \cup [1/2, m]$ we get $G_1(M) + \Delta(M) = 2m + 2$ whereas for $M' = [-m, m]$ we have $G_1(M') + \Delta(M') = 2m + 1$.*

We notice that, as shown in (1.6), the quantity $\Delta(M)$ cannot be (in general) omitted. However, we can rewrite (2.4) to provide an answer to Question 1.1 for $n = 1$, which is the one-dimensional case of Theorem 2.1:

Lemma 2.2. *Let $\lambda \in (0,1)$ and let $K, L \subset \mathbb{R}$ be non-empty bounded sets. Then*

$$(2.5) \qquad G_1\big((1-\lambda)K + \lambda L + (-1, 1)\big) \geq (1-\lambda)G_1(K) + \lambda G_1(L).$$

The inequality is sharp.

Proof. Let $M = \bigcup_{x \in (1-\lambda)K + \lambda L}[\lfloor x \rfloor, \lceil x \rceil]$. Clearly, M is a finite union of compact intervals (since K and L are bounded) containing $(1-\lambda)K + \lambda L$. From Lemma 2.1 we then obtain $G_1(M) + \Delta(M) \geq (1-\lambda)G_1(K) + \lambda G_1(L)$, which, together with the facts that $\Delta(M) = 0$ and $M \cap \mathbb{Z} = ((1-\lambda)K + \lambda L + (-1, 1)) \cap \mathbb{Z}$, yields (2.5).

Finally, in order to show that equality may be attained, we consider $K = L = [0, m]$ with $m \in \mathbb{N}$, for which $G_1\big((1-\lambda)K + \lambda L + (-1, 1)\big) = m + 1 = (1-\lambda)G_1(K) + \lambda G_1(L)$ for all $\lambda \in (0,1)$. $\qquad\square$

Remark 2.2. *Since both sides of (2.5) are invariant under translations by integers of the sets K and L, we may replace $(-1, 1)$ (in (2.5)) by any other interval $(m, m+2)$, with $m \in \mathbb{Z}$.*

We note however that the solution to Question 1.1 provided by Lemma 2.2 (for $n = 1$), via $M = (1 - \lambda)K + \lambda L + (-1, 1)$, for all $\lambda \in (0, 1)$, cannot be in general improved by means of any other interval strictly contained in $(-1, 1)$. Indeed, by considering $I = [-a, 1)$, with $-1 < -a < 0$, and taking $K = [-1, 0]$, $L = [-2, 0]$ and $\lambda \in (0, 1)$ such that $\lambda + a < 1$, we get that

$$\mathrm{G}_1\big((1 - \lambda)K + \lambda L + I\big) = \mathrm{G}_1\big([-1 - \lambda - a, 1)\big) = 2$$
$$< 2 + \lambda = (1 - \lambda)\mathrm{G}_1(K) + \lambda\mathrm{G}_1(L).$$

The case $I = (-1, a]$, for $0 < a < 1$, is completely analogous and thus, no interval smaller than $(-1, 1)$ (with respect to set inclusion) can be taken into account.

Now we state some auxiliary results. The following lemma can be regarded as a discrete counterpart of the well-known *Cavalieri Principle* (see [11, Lemma 3.1]).

Lemma 2.3 ([11]). *Let $\Omega \subset \mathbb{R}^n$ be a finite set, let $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ and suppose $f(\Omega) \subset \{k_0, k_1, \ldots, k_r\}$ where $0 = k_0 < k_1 < \cdots < k_r$. Then*

$$\sum_{x \in \Omega} f(x) = \sum_{i=1}^{r}(k_i - k_{i-1})\Big|\{x \in \Omega : f(x) \geq k_i\}\Big| = \int_0^\infty \Big|\{x \in \Omega : f(x) \geq t\}\Big|\,\mathrm{d}t.$$

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ and suppose $f(\Omega \cap \mathbb{Z}^n) \subset \{k_0, k_1, \ldots, k_r\}$ where $0 = k_0 < k_1 < \cdots < k_r$. Then*

$$\sum_{x \in \Omega \cap \mathbb{Z}^n} f(x) = \sum_{i=1}^{r}(k_i - k_{i-1})\mathrm{G}_n\big(\{x \in \Omega : f(x) \geq k_i\}\big).$$

The following result yields the case $n = 1$ of Theorem 2.2 and will be used to derive (2.3).

Lemma 2.4. *Let $\lambda \in (0, 1)$ and let $\Omega_1, \Omega_2 \subset \mathbb{R}$ be non-empty bounded sets. Let $-1 \leq p \leq \infty$ and let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be functions such that*

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda\left(f(x), g(y)\right)$$

for all $x \in \Omega_1$, $y \in \Omega_2$. Then

$$\sum_{z \in \Omega \cap \mathbb{Z}} h^\diamond(z) \geq \mathcal{M}_{\frac{p}{p+1}}^\lambda\left(\sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x), \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y)\right),$$

where $\Omega = (1 - \lambda)\Omega_1 + \lambda\Omega_2 + (-1, 1)$.

Proof. Clearly, we may assume that $\sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x), \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y) > 0$. We consider the functions $F, G, H, H^\diamond : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$F(x) = \frac{f(x)}{a}, \quad G(y) = \frac{g(y)}{b}, \quad H(z) = \frac{h(z)}{c_p}, \quad H^\diamond(z) = \frac{h^\diamond(z)}{c_p},$$

where

$$a = \max_{x \in \Omega_1 \cap \mathbb{Z}} f(x) > 0, \quad b = \max_{y \in \Omega_2 \cap \mathbb{Z}} g(y) > 0 \quad \text{and} \quad c_p = \mathcal{M}_p^\lambda(a,b) > 0.$$

Then

$$\max_{x \in \Omega_1 \cap \mathbb{Z}} F(x) = \max_{y \in \Omega_2 \cap \mathbb{Z}} G(y) = 1.$$

First, we show that, for any $x \in \Omega_1$, $y \in \Omega_2$, we have

$$(2.6) \qquad H((1-\lambda)x + \lambda y) \geq \min(F(x), G(y)).$$

To this aim, it is enough to consider $x \in \Omega_1$, $y \in \Omega_2$ with $f(x)g(y) > 0$. If $p \neq 0$ and $p \neq \infty$, writing $\theta = \lambda b^p / c_p^p \in (0,1)$, we get

$$\begin{aligned} h((1-\lambda)x + \lambda y) &\geq ((1-\lambda)f(x)^p + \lambda g(y)^p)^{1/p} \\ &= c_p \left(\frac{(1-\lambda)a^p F(x)^p + \lambda b^p G(y)^p}{c_p^p} \right)^{1/p} \\ &= c_p \left((1-\theta)F(x)^p + \theta G(y)^p \right)^{1/p} \\ &\geq c_p \min\{F(x), G(y)\}. \end{aligned}$$

For $p = 0$, we have

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda = c_0 F(x)^{1-\lambda} G(y)^\lambda \geq c_0 \min\{F(x), G(y)\}.$$

For $p = \infty$, $h((1-\lambda)x + \lambda y) \geq \max\{f(x), g(y)\} \geq c_\infty \min\{F(x), G(y)\}$ clearly holds. Therefore, we have shown (2.6).

Next, note that the definition of F and G implies that the level sets

$$\{x \in \Omega_1 : F(x) \geq t\}, \quad \{y \in \Omega_2 : G(y) \geq t\}$$

are non-empty for any $t \in [0,1]$. Moreover, writing $\Omega_\lambda = (1-\lambda)\Omega_1 + \lambda \Omega_2$, we deduce from (2.6) that

$$\{z \in \Omega_\lambda : H(z) \geq t\} \supset (1-\lambda)\{x \in \Omega_1 : F(x) \geq t\} + \lambda\{y \in \Omega_2 : G(y) \geq t\}$$

and thus, by Lemma 2.2, we have

$$(2.7) \qquad \begin{aligned} &\mathrm{G}_1\big(\{z \in \Omega_\lambda : H(z) \geq t\} + (-1,1)\big) \\ &\geq (1-\lambda)\mathrm{G}_1\big(\{x \in \Omega_1 : F(x) \geq t\}\big) + \lambda \mathrm{G}_1\big(\{y \in \Omega_2 : G(y) \geq t\}\big) \end{aligned}$$

for all $t \in [0,1]$.

Note that, since $H^\diamond(z+u) \geq H\big((z+u) - u\big) = H(z)$ for all $u \in (-1,1)$, we also have

$$(2.8) \qquad \{z \in \Omega : H^\diamond(z) \geq t\} \supset \{z \in \Omega_\lambda : H(z) \geq t\} + (-1,1).$$

Finally, set $\{k_0, k_1, \ldots, k_r\} \supset F\big(\Omega_1 \cap \mathbb{Z}\big) \cup G\big(\Omega_2 \cap \mathbb{Z}\big) \cup H^\diamond\big(\Omega \cap \mathbb{Z}\big)$, with $0 = k_0 < k_1 < \cdots < k_r$ where, for some $s \in \{1, \ldots, r\}$,

$$k_s = \max_{x \in \Omega_1 \cap \mathbb{Z}} F(x) = \max_{y \in \Omega_2 \cap \mathbb{Z}} G(y) = 1.$$

Then, by Corollary 2.1, (2.7) and (2.8), we get

$$\sum_{z\in\Omega\cap\mathbb{Z}} h^\diamond(z) = \sum_{z\in\Omega\cap\mathbb{Z}} c_p H^\diamond(z) = c_p \sum_{i=1}^{r}(k_i - k_{i-1})\mathrm{G}_1\big(\{z\in\Omega : H^\diamond(z) \geq k_i\}\big)$$

$$\geq c_p \sum_{i=1}^{s}(k_i - k_{i-1})\mathrm{G}_1\big(\{z\in\Omega : H^\diamond(z) \geq k_i\}\big)$$

$$\geq c_p \sum_{i=1}^{s}(k_i - k_{i-1})\Big((1-\lambda)\mathrm{G}_1\big(\{x\in\Omega_1 : F(x) \geq k_i\}\big)$$

$$+ \lambda\mathrm{G}_1\big(\{y\in\Omega_2 : G(y) \geq k_i\}\big)\Big)$$

$$= c_p\left((1-\lambda)\sum_{x\in\Omega_1\cap\mathbb{Z}} F(x) + \lambda\sum_{y\in\Omega_2\cap\mathbb{Z}} G(y)\right)$$

$$= c_p\left(\frac{1-\lambda}{a}\sum_{x\in\Omega_1\cap\mathbb{Z}} f(x) + \frac{\lambda}{b}\sum_{y\in\Omega_2\cap\mathbb{Z}} g(y)\right)$$

$$\geq \mathcal{M}^\lambda_{\frac{p}{p+1}}\left(\sum_{x\in\Omega_1\cap\mathbb{Z}} f(x), \sum_{y\in\Omega_2\cap\mathbb{Z}} g(y)\right).$$

If $p \neq 0$ and $p \neq \infty$, the last inequality follows from the reverse Hölder inequality (see e.g. [4, Theorem 1, page 178]),

$$a_1 b_1 + a_2 b_2 \geq \left(a_1^{-p} + a_2^{-p}\right)^{-1/p}(b_1^q + b_2^q)^{1/q},$$

where $q = p/(p+1)$ is the Hölder conjugate of $(-p) \leq 1$, just by taking $a_1 = ((1-\lambda)^{1/p}a)^{-1}$, $a_2 = (\lambda^{1/p}b)^{-1}$, $b_1 = (1-\lambda)^{1/q}\sum_{x\in\Omega_1\cap\mathbb{Z}} f(x)$ and $b_2 = \lambda^{1/q}\sum_{y\in\Omega_2\cap\mathbb{Z}} g(y)$.

The case $p = 0$ follows from the Arithmetic-Geometric mean inequality, whereas the case $p = \infty$ is immediate, since there the $(p/(p+1))$-mean coincides with the 1-mean. $\qquad\square$

We would like to introduce a few additional notations which will be used to prove Theorem 2.2: we write $M(t) = \{x \in \mathbb{R}^{n-1} : (x,t) \in M\}$ for the $(n-1)$-dimensional section at height $t \in \mathbb{R}$ (in the direction of e_n) whereas $\pi_n(M)$ denotes the orthogonal projection of M onto $\mathbb{R}\mathrm{e}_n$ (regarded as a subset of $\mathbb{R}$), namely $\pi_n(M) = \{t \in \mathbb{R} : M(t) \neq \emptyset\}$.

Proof of Theorem 2.2. If $n = 1$, the result follows immediately from Lemma 2.4. Now suppose that $n > 1$ and assume that the theorem holds for dimension $n - 1$. Let $t_K \in \pi_n(K)$, $t_L \in \pi_n(L)$ and set, for the sake of brevity, $t_\lambda = (1-\lambda)t_K + \lambda t_L$. Moreover, we denote by $C_n = (-1,1)^n$, $C_{n-1} = (-1,1)^{n-1} \times \{0\}$, $M_{n-1} = (1-\lambda)K(t_K) + \lambda L(t_L) + (-1,1)^{n-1}$ and

$M_\lambda = (1-\lambda)K + \lambda L$ (thus, $M = M_\lambda + (-1,1)^n$). Consider the functions $f_1, g_1, h_1 : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_1(x) = f(x, t_K), \quad g_1(x) = g(x, t_L),$$

$$h_1(x) = h(x, t_\lambda)$$

for any $x \in \mathbb{R}^{n-1}$. Since for all $x \in K(t_K)$, $y \in L(t_L)$ we have

$$h_1((1-\lambda)x + \lambda y) = h((1-\lambda)x + \lambda y, (1-\lambda)t_K + \lambda t_L)$$
$$\geq \mathcal{M}_p^\lambda \left(f(x, t_K), g(y, t_L) \right) = \mathcal{M}_p^\lambda \left(f_1(x), g_1(y) \right),$$

we may assert that

$$\sum_{z \in M_{n-1} \cap \mathbb{Z}^{n-1}} h_1^\diamond(z) \geq \mathcal{M}_{\frac{p}{(n-1)p+1}}^\lambda \left(\sum_{x \in K(t_K) \cap \mathbb{Z}^{n-1}} f_1(x), \sum_{y \in L(t_L) \cap \mathbb{Z}^{n-1}} g_1(y) \right).$$

This, together with the fact that

$$((1-\lambda)K + \lambda L)((1-\lambda)t_K + \lambda t_L) \supset (1-\lambda)K(t_K) + \lambda L(t_L),$$

and hence $(M_\lambda + C_{n-1})(t_\lambda) \supset M_{n-1}$, yields, in terms of f, g and h,

$$\sum_{z \in ((M_\lambda + C_{n-1})(t_\lambda)) \cap \mathbb{Z}^{n-1}} h^{\infty\infty}(z, t_\lambda)$$

(2.9)
$$\geq \mathcal{M}_{\frac{p}{(n-1)p+1}}^\lambda \left(\sum_{x \in K(t_K) \cap \mathbb{Z}^{n-1}} f(x, t_K), \sum_{y \in L(t_L) \cap \mathbb{Z}^{n-1}} g(y, t_L) \right),$$

where $h^{\infty\infty} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is the function given by $h^{\infty\infty}(z) = \sup_{v \in C_{n-1}} h(z+v)$, for which we have $h^{\infty\infty}(x, t_\lambda) = h_1^\diamond(x)$ for all $x \in \mathbb{R}^{n-1}$.

Now, let $f_2, g_2, h_2 : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be the functions defined by

$$f_2(t) = \sum_{x \in K(t) \cap \mathbb{Z}^{n-1}} f(x, t), \quad g_2(t) = \sum_{y \in L(t) \cap \mathbb{Z}^{n-1}} g(y, t), \text{ and}$$

$$h_2(t) = \sum_{z \in ((M_\lambda + C_{n-1})(t)) \cap \mathbb{Z}^{n-1}} h^{\infty\infty}(z, t).$$

Hence, (2.9) yields, in terms of f_2, g_2 and h_2,

$$h_2((1-\lambda)t_K + \lambda t_L) \geq \mathcal{M}_{\frac{p}{(n-1)p+1}}^\lambda \left(f_2(t_K), g_2(t_L) \right)$$

for any $t_K \in \pi_n(K)$, $t_L \in \pi_n(L)$, and thus we may use Lemma 2.4 with the sets $\pi_n(K)$, $\pi_n(L)$ and the functions f_2, g_2 and h_2 to obtain

$$\sum_{t \in \Omega \cap \mathbb{Z}} h_2^\diamond(t) \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\sum_{t_K \in \pi_n(K) \cap \mathbb{Z}} f_2(t_K), \sum_{t_L \in \pi_n(L) \cap \mathbb{Z}} g(t_L) \right),$$

where $\Omega = (1 - \lambda)\pi_n(K) + \lambda\pi_n(L) + (-1, 1)$. In the following we prove that $\sum_{t \in \Omega \cap \mathbb{Z}} h_2^\diamond(t) \leq \sum_{z \in M \cap \mathbb{Z}^n} h^\diamond(z)$, and hence the above inequality together with the relations

$$\sum_{t_K \in \pi_n(K) \cap \mathbb{Z}} f_2(t_K) = \sum_{x \in K \cap \mathbb{Z}^n} f(x),$$

$$\sum_{t_L \in \pi_n(L) \cap \mathbb{Z}} g_2(t_L) = \sum_{y \in L \cap \mathbb{Z}^n} g(y),$$

shows the result. Indeed, from the fact that $(u, -w) \in C_n$ for any $(u, 0) \in C_{n-1}$ and $w \in (-1, 1)$, we have $(M_\lambda + C_{n-1})(t+w) \subset M(t)$ for all $w \in (-1, 1)$ and thus we get

$$
\begin{aligned}
\sum_{t \in \Omega \cap \mathbb{Z}} h_2^\diamond(t) &= \sum_{t \in \Omega \cap \mathbb{Z}} \sup_{w \in (-1,1)} h_2(t + w) \\
&= \sum_{t \in \Omega \cap \mathbb{Z}} \sup_{w \in (-1,1)} \sum_{x \in ((M_\lambda + C_{n-1})(t+w)) \cap \mathbb{Z}^{n-1}} h^{\infty\infty}(x, t + w) \\
&\leq \sum_{t \in \Omega \cap \mathbb{Z}} \sum_{x \in M(t) \cap \mathbb{Z}^{n-1}} \sup_{w \in (-1,1)} h^{\diamond\diamond}(x, t + w) \\
&= \sum_{t \in \Omega \cap \mathbb{Z}} \sum_{x \in M(t) \cap \mathbb{Z}^{n-1}} \sup_{w \in (-1,1)} \sup_{v \in (-1,1)^{n-1}} h(x + v, t + w) \\
&= \sum_{z \in M \cap \mathbb{Z}^n} \sup_{u \in (-1,1)^n} h(z + u) = \sum_{z \in M \cap \mathbb{Z}^n} h^\diamond(z),
\end{aligned}
$$

as claimed. This finishes the proof. $\qquad\square$

Corollary 2.2. *Let $\lambda \in (0, 1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq p \leq \infty$. Then*

$$(2.10) \qquad \mathrm{G}_n\big((1 - \lambda)K + \lambda L + (-1, 1)^n\big) \geq \mathcal{M}^\lambda_{\frac{p}{np+1}}\big(\mathrm{G}_n(K), \mathrm{G}_n(L)\big).$$

The inequality is sharp.

Proof. The result is an immediate consequence of Theorem 2.2, just by taking $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)K + \lambda L}$, for which we clearly have that $h^\diamond = \chi_{(1-\lambda)K + \lambda L + (-1,1)^n}$.

Now, in order to show that the equality may be attained, we consider $K = L = [0, m]^n$ with $m \in \mathbb{N}$, for which $\mathrm{G}_n\big((1 - \lambda)K + \lambda L + (-1, 1)^n\big) = \mathrm{G}_n(K) = \mathrm{G}_n(L) = (m + 1)^n$. $\qquad\square$

We notice that (2.10) for $p = \infty$ yields (2.1) for bounded sets $K, L \subset \mathbb{R}^n$ with $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$. So, to prove Theorem 2.1, it is enough to deal with the case in which (only) one of the sets, say L, has no integer points. To this aim, first we show the following auxiliary result:

Theorem 2.3. *Let $K, L \subset \mathbb{R}^n$ be bounded sets such that $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$ and let $\alpha = m/q$ and $\beta = p/q$ with $m, p, q \in \mathbb{N}$ so that $\alpha + \beta \leq 1$. Then*

$$(2.11) \quad \mathrm{G}_n\left(\alpha K + \beta L + \left[-\frac{q-1}{q}, \frac{q-1}{q}\right]^n\right)^{1/n} \geq \alpha \mathrm{G}_n(K)^{1/n} + \beta \mathrm{G}_n(L)^{1/n}.$$

The proof of Theorem 2.3 is ideologically similar to the proof we have provided for Corollary 2.2. So only the sketch of the proof is presented here. The main step is to prove an analogue of Theorem 2.2 and apply it to $f = \chi_K$, $g = \chi_L$ and $h = \chi_{\alpha K + \beta L}$. The main new ingredient in the proof of such an analogue is the one dimensional case, for which we require an adaptation of Lemma 2.2, which is done below in Lemma 2.5.

Lemma 2.5. *Let $K, L \subset \mathbb{R}$ be non-empty bounded sets and let $\alpha = m/q$ and $\beta = p/q$ with $m, p, q \in \mathbb{N}$ so that $\alpha + \beta \leq 1$. Then*

$$(2.12) \qquad \mathrm{G}_1\left(\alpha K + \beta L + \left[-\frac{q-1}{q}, \frac{q-1}{q}\right]\right) \geq \alpha \mathrm{G}_1(K) + \beta \mathrm{G}_1(L).$$

Proof. First we notice that, for any $x, y \in \mathbb{R}$, we have

$$(2.13) \qquad \left\lfloor \frac{mx}{q} + \frac{py}{q} + \frac{q-1}{q} \right\rfloor \geq \frac{m\lfloor x \rfloor + p\lfloor y \rfloor}{q}.$$

Indeed, given $z \in \mathbb{R}$, from the fact that

$$\frac{z}{q} + \frac{q-1}{q} \geq \frac{\lfloor z \rfloor}{q} + \frac{q-1}{q} = c + \frac{r}{q} + \frac{q-1}{q}$$

for some $c, r \in \mathbb{Z}$ with $0 \leq r \leq q-1$, we get that

$$\left\lfloor \frac{z}{q} + \frac{q-1}{q} \right\rfloor \geq \frac{\lfloor z \rfloor}{q}$$

for any $z \in \mathbb{R}$. This now implies that

$$\left\lfloor \frac{mx}{q} + \frac{py}{q} + \frac{q-1}{q} \right\rfloor \geq \frac{\lfloor mx + py \rfloor}{q} \geq \frac{m\lfloor x \rfloor + p\lfloor y \rfloor}{q},$$

which yields (2.13).

Next we show the following: if $K, L, M \subset \mathbb{R}$ are non-empty sets with $\alpha K + \beta L \subset M$ and such that $M + [-(q-1)/q, (q-1)/q] = \bigcup_{i=1}^{s}[a_i, b_i]$ is a finite union of (pairwise disjoint) compact intervals then

$$(2.14) \qquad \mathrm{G}_1\big(M + [-(q-1)/q, (q-1)/q]\big) \geq \alpha \mathrm{G}_1(K) + \beta \mathrm{G}_1(L).$$

We prove it by induction on the number of intervals s of $M + [-(q-1)/q, (q-1)/q]$. For $s = 1$, i.e., when $M + [-(q-1)/q, (q-1)/q] = [a_1, b_1]$ is a (non-empty) compact interval, we have on the one hand that

$$\mathrm{G}_1\big(M + [-(q-1)/q, (q-1)/q]\big) = \lfloor b_1 \rfloor - \lceil a_1 \rceil + 1.$$

Moreover, denoting by $a = \inf K$, $b = \sup K$, $c = \inf L$ and $d = \sup L$, we clearly get $\mathrm{G}_1(K) \leq \mathrm{G}_1([a,b]) = \lfloor b \rfloor - \lceil a \rceil + 1$ and $\mathrm{G}_1(L) \leq \mathrm{G}_1([c,d]) = \lfloor d \rfloor - \lceil c \rceil + 1$. On the other hand, the inclusion $\alpha K + \beta L \subset M$ implies that

$$a_1 \leq \alpha a + \beta c - \frac{q-1}{q} \leq \alpha b + \beta d + \frac{q-1}{q} \leq b_1.$$

Altogether, and using (2.13) jointly with the facts that $\lceil x \rceil = -\lfloor -x \rfloor$ for any $x \in \mathbb{R}$, and $\alpha + \beta \leq 1$, we obtain

$$\begin{aligned}
\mathrm{G}_1\left(M + \left[-\frac{q-1}{q}, \frac{q-1}{q}\right]\right) &= \lfloor b_1 \rfloor - \lceil a_1 \rceil + 1 \\
&\geq \left\lfloor \alpha b + \beta d + \frac{q-1}{q} \right\rfloor - \left\lceil \alpha a + \beta c - \frac{q-1}{q} \right\rceil + 1 \\
&\geq \alpha \lfloor b \rfloor + \beta \lfloor d \rfloor - \alpha \lceil a \rceil + \beta \lceil c \rceil + 1 \\
&\geq \alpha \mathrm{G}_1([a,b]) + \beta \mathrm{G}_1([c,d]) \\
&\geq \alpha \mathrm{G}_1(K) + \beta \mathrm{G}_1(L).
\end{aligned}$$

Thus, we suppose that (2.14) is true for $s \geq 1$ and assume that $M + [-(q-1)/q, (q-1)/q] = \bigcup_{i=1}^{s+1}[a_i, b_i]$, where $b_i < a_{i+1}$ for all $1 \leq i \leq s$.

Denoting by M_1 and M_2 the subsets of M such that $M_1 + [-(q-1)/q, (q-1)/q] = [a_1, b_1]$ and $M_2 + [-(q-1)/q, (q-1)/q] = \bigcup_{i=2}^{s+1}[a_i, b_i]$, we may assume, without loss of generality, that $M_1 \cap (\alpha K + \beta L) \neq \emptyset$ (otherwise, the result follows from the induction hypothesis applied to the sets K, L and M_2). Hence, we may define $m = \sup(M_1 \cap (\alpha K + \beta L))$ and then, since K and L are bounded, there exist $k \in \mathrm{cl}\, K$ and $l \in \mathrm{cl}\, L$ such that $\alpha k + \beta l = m$. Thus, considering the sets $K_1 = \{x \in K : x \leq k\}$, $K_2 = K \backslash K_1$, $L_1 = \{x \in L : x \leq l\}$ and $L_2 = L \setminus L_1$, and taking into account that $m + [-(q-1)/q, (q-1)/q] \subset [a_1, b_1]$, we clearly have that $\alpha K_1 + \beta L_1 \subset M_1$ and $\alpha K_2 + \beta L_2 \subset M_2$. Therefore, applying the induction hypothesis (and taking into account that $M_1 + [-(q-1)/q, (q-1)/q]$ and $M_2 + [-(q-1)/q, (q-1)/q]$ are disjoint), we get

$$\begin{aligned}
&\mathrm{G}_1\big(M + [-(q-1)/q, (q-1)/q]\big) \\
&= \mathrm{G}_1\big(M_1 + [-(q-1)/q, (q-1)/q]\big) + \mathrm{G}_1\big(M_2 + [-(q-1)/q, (q-1)/q]\big) \\
&\geq \alpha \mathrm{G}_1(K_1) + \beta \mathrm{G}_1(L_1) + \alpha \mathrm{G}_1(K_2) + \beta \mathrm{G}_1(L_2) = \alpha \mathrm{G}_1(K) + \beta \mathrm{G}_1(L),
\end{aligned}$$

which shows (2.14).

Next we prove (2.12). We observe that we may assume, without loss of generality, that K and L are compact. Indeed, otherwise, considering the compact sets $K' = K \cap \mathbb{Z}$ and $L' = L \cap \mathbb{Z}$, for which we have $\mathrm{G}_1(K') = \mathrm{G}_1(K)$, $\mathrm{G}_1(L') = \mathrm{G}_1(L)$ and, from the monotonicity of $\mathrm{G}_1(\cdot)$, $\mathrm{G}_1\big(\alpha K + \beta L + [-(q-1)/q, (q-1)/q]\big) \geq \mathrm{G}_1\big(\alpha K' + \beta L' + [-(q-1)/q, (q-1)/q]\big)$, we would get the result. So, (2.12) follows from applying (2.14) with $M = \alpha K + \beta L$, because the fact that M is compact implies that there exists a finite sequence

$m_1, \ldots, m_r$ such that $\{m_1, \ldots, m_r\} + [-(q-1)/q, (q-1)/q] = M + [-(q-1)/q, (q-1)/q]$. This concludes the proof. $\square$

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. For the sake of brevity, we will again denote by $M_\lambda = (1-\lambda)K + \lambda L$. By Corollary 2.2 for $p = \infty$, and the monotonicity of $\mathrm{G}_n(\cdot)$, it is enough to show the result in the case in which $K = K \cap \mathbb{Z}^n$ (for which, clearly, $\mathrm{G}_n(K) > 0$) and $L = \{x\}$ with $x = (x_1, \ldots, x_n) \notin \mathbb{Z}^n$.

First we show the case in which $\lambda = p/q$, $p, q \in \mathbb{N}$, is a rational number. Then, writing $px = z + y$, with $z \in \mathbb{Z}^n$ and $y = (y_1, \ldots, y_n) \in [0,1)^n$, and using Theorem 2.3, we have

$$
\begin{aligned}
\mathrm{G}_n\big(M_\lambda + (-1,1)^n\big)^{1/n} &= \mathrm{G}_n\left(\frac{q-p}{q}K + \frac{1}{q}z + \prod_{i=1}^n\left(-1+\frac{y_i}{q}, 1+\frac{y_i}{q}\right)\right)^{1/n} \\
&\geq \mathrm{G}_n\left(\frac{q-p}{q}K + \frac{1}{q}z + \left[-\frac{q-1}{q}, \frac{q-1}{q}\right]^n\right)^{1/n} \\
&\geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \frac{1}{q} > (1-\lambda)\mathrm{G}_n(K)^{1/n},
\end{aligned}
$$

as desired.

Next we prove the case of an irrational $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Let I be the (possibly empty) subset of $\{1, \ldots, n\}$ defined in the following way: $i \in I$ if and only if $x_i = a_i + b_i/\lambda$ for some $a_i, b_i \in \mathbb{Z}$. We point out that such $a_i, b_i \in \mathbb{Z}$ are necessarily unique, since $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Hence we may then consider the point $x' = (x'_1, \ldots, x'_n)$ given by $x'_i = b_i/\lambda$ if $i \in I$ and $x'_i = 0$ otherwise, for all $i = 1, \ldots, n$. First we notice that, since $\lambda x'$ is an integer point, we have

$$
\begin{aligned}
\mathrm{G}_n\big(M_\lambda + (-1,1)^n\big) &= \mathrm{G}_n\big((1-\lambda)K + \lambda(x - x') + \lambda x' + (-1,1)^n\big) \\
&= \mathrm{G}_n\big((1-\lambda)K + \lambda(x - x') + (-1,1)^n\big).
\end{aligned}
$$

Next, denoting by $x_0 = x - x'$, we will show that there exists $\delta > 0$ such that

$$(2.15) \quad \mathrm{G}_n\big((1-\lambda)K + \lambda x_0 + (-1,1)^n\big) \geq \mathrm{G}_n\big((1-\mu)K + \mu x_0 + (-1,1)^n\big)$$

for all μ with $|\mu - \lambda| < \delta$. Thus, taking a sequence $(r_m)_m \subset \mathbb{Q} \cap (0,1)$ with $\lim_{m\to\infty} r_m = \lambda$ and $|r_m - \lambda| < \delta$ for all $m \in \mathbb{N}$, we get, from the previous case, that

$$
\begin{aligned}
\mathrm{G}_n\big((1-\lambda)K + \lambda x_0 + (-1,1)^n\big)^{1/n} &\geq \mathrm{G}_n\big((1-r_m)K + r_m x_0 + (-1,1)^n\big)^{1/n} \\
&\geq (1-r_m)\mathrm{G}_n(K)^{1/n} + r_m\mathrm{G}_n(\{x_0\})^{1/n} \\
&\geq (1-r_m)\mathrm{G}_n(K)^{1/n}
\end{aligned}
$$

for all $m \in \mathbb{N}$, which yields the result.

To show (2.15) we notice that, since K is finite, $(1-\lambda)K + \lambda x_0 + [-1,1]^n$ is a finite union of closed unit cubes and then, for any μ, $(1-\mu)K + \mu x_0 +$

$[-1,1]^n$ is the union of the corresponding translates of the cubes that consti-
tute $(1-\lambda)K + \lambda x_0 + [-1,1]^n$. Thus, there exists $\delta_1 > 0$ such that if $z \in \mathbb{Z}^n$
satisfies that $z \notin (1-\lambda)K + \lambda x_0 + [-1,1]^n$ then $z \notin (1-\mu)K + \mu x_0 + (-1,1)^n$
for all $|\mu - \lambda| < \delta_1$. Moreover, if $(1-\lambda)K + \lambda x_0 + (-1,1)^n$ contains no integer
boundary points, we may take $\delta = \delta_1$ and we are done.

So, we may assume that $\mathrm{bd}\big(((1-\lambda)K + \lambda x_0 + (-1,1)^n) \cap \mathbb{Z}^n\big) \neq \emptyset$. Let $z \in \mathbb{Z}^n$ be a boundary point of $(1-\lambda)K + \lambda x_0 + (-1,1)^n$ and let $k \in K$. On the
one hand, if z is in the boundary of the cube $(1-\lambda)k + \lambda x_0 + (-1,1)^n$, there
exist $i \in \{1, \ldots, n\}$ and $\varepsilon \in \{-1,1\}$ such that $(1-\lambda)k_i + \lambda(x_i - x_i') + \varepsilon = z_i$,
where $k_i, z_i \in \mathbb{Z}$ are the i-th components of k, z, respectively. This implies
that $x_i - x_i' = k_i + (z_i - \varepsilon - k_i)/\lambda$ and then, from the definition of both
I and x', we get that $i \in I$ and so $a_i = k_i + (z_i - \varepsilon - k_i)/\lambda$, which yields
that $z_i - \varepsilon - k_i = 0$ (because λ is irrational) and thus $x_i - x_i' = a_i = k_i$.
But then $(1-\mu)k_i + \mu(x_i - x_i') + \varepsilon = (1-\lambda)k_i + \lambda(x_i - x_i') + \varepsilon = z_i$ for
all $\mu \in \mathbb{R}$, and hence z lies in the affine hull of a facet of the open cube
$(1-\mu)k + \mu x_0 + (-1,1)^n$, which implies that $z \notin (1-\mu)k + \mu x_0 + (-1,1)^n$
for any μ. On the other hand, if z is not in the boundary of the cube
$(1-\lambda)k + \lambda x_0 + (-1,1)^n$ (and so not contained in it either), there exists
$\delta_{z,k} > 0$ so that $z \notin (1-\mu)k + \mu x_0 + (-1,1)^n$ for all μ with $|\mu - \lambda| < \delta_{z,k}$.

Since the number of integer points z in the boundary of $(1-\lambda)K + \lambda x_0 + (-1,1)^n$ is finite (and so is K), we may define $\delta_2 = \min_{z,k} \delta_{z,k}$. Altogether,
(2.15) holds for $\delta = \min\{\delta_1, \delta_2\}$. $\square$

We conclude this section by showing that the classical Borell-Brascamp-
Lieb inequality (2.2) can be obtained from the discrete version (2.3) under
some mild assumptions on the functions involved:

Theorem 2.4. *The discrete Borell-Brascamp-Lieb type inequality* (2.3) *im-
plies the classical Borell-Brascamp-Lieb inequality* (2.2), *provided that the
functions f, g are Riemann integrable and h is upper semicontinuous.*

Before proving this result, we notice that it is not possible to directly
obtain any of the discrete Brunn-Minkowski type inequalities stated so far
in the paper from the classical one (1.2), by using the method of replacing
the points by suitable compact sets. As pointed out by Gardner and Gronchi
in [6, pages 3996–3997]:

> "*it is worth remarking that the obvious idea of replacing the points in
> the two finite sets by small congruent balls and applying the classical
> Brunn-Minkowski inequality to the resulting compact sets is doomed
> to failure. The fact that the sum of two congruent balls is a ball of
> twice the radius introduces an extra factor of $1/2$ that renders the
> resulting bound weaker than even the trivial bound (11) below*".

We clarify that (11) in [6] coincides with (1.3) of the present paper.

In the following, for a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, we write $\phi^{\diamond_k} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ to denote the function given by $\phi^{\diamond_k}(z) = \sup_{u \in (-2^{-k}, 2^{-k})^n} \phi(z + u)$ for all $z \in \mathbb{R}^n$.

Proof. Consider $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ which satisfy the conditions of Theorem B, i.e., f, g, h are measurable such that

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda (f(x), g(y))$$

holds for all $x, y \in \mathbb{R}^n$ and for some fixed $\lambda \in (0, 1)$ and $-1/n \leq p \leq \infty$. Let $m \in \mathbb{N}$ and let $K = [-m, m]^n$. We will first show that

$$(2.16) \qquad \int_K h(x)\mathrm{d}x \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\int_K f(x)\mathrm{d}x, \int_K g(x)\mathrm{d}x \right),$$

for which we may assume (multiplying by χ_K if necessary) that f, g and h vanish outside K.

For each $k \in \mathbb{N}$, we define the functions $f_k, g_k, h_k : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$f_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} f(z), \quad g_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} g(z)$$

and

$$h_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} h(z).$$

Writing for short $K_0 = \mathrm{int}\, K$, note that for any $x, y \in K_0$ we have

$$\begin{aligned}
h_k((1 - \lambda)x + \lambda y) &= \inf_{z \in (1-\lambda)x + \lambda y + [0, 2^{-k}]^n} h(z) \\
&= \inf_{z \in (1-\lambda)(x + [0, 2^{-k}]^n) + \lambda(y + [0, 2^{-k}]^n)} h(z) \\
&= \inf_{z_1 \in x + [0, 2^{-k}]^n, z_2 \in y + [0, 2^{-k}]^n} h((1 - \lambda)z_1 + \lambda z_2) \\
&\geq \inf_{z_1 \in x + [0, 2^{-k}]^n, z_2 \in y + [0, 2^{-k}]^n} \mathcal{M}_p^\lambda (f(z_1), g(z_2)) \\
&\geq \mathcal{M}_p^\lambda \left(\inf_{z_1 \in x + [0, 2^{-k}]^n} f(z_1), \inf_{z_2 \in y + [0, 2^{-k}]^n} g(z_2) \right) \\
&= \mathcal{M}_p^\lambda (f_k(x), g_k(y)),
\end{aligned}$$

and thus, we can use Theorem 2.2 for $2^{-k}\mathbb{Z}^n$ to deduce that, for any $k \in \mathbb{N}$, we have
(2.17)

$$2^{-kn} \sum_{z \in K \cap 2^{-k}\mathbb{Z}^n} (h_k)^{\diamond_k}(z) \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(2^{-kn} \sum_{x \in K_0 \cap 2^{-k}\mathbb{Z}^n} f_k(x), 2^{-kn} \sum_{y \in K_0 \cap 2^{-k}\mathbb{Z}^n} g_k(y) \right),$$

where, on the left-hand side, we have used that $K = [-m, m]^n$, thus

$$\left(2^k K_0 + (-1, 1)^n \right) \cap \mathbb{Z}^n = 2^k K \cap \mathbb{Z}^n$$

and

$$\left(K_0 + (-2^{-k}, 2^{-k})^n \right) \cap 2^{-k}\mathbb{Z}^n = K \cap 2^{-k}\mathbb{Z}^n.$$

The level sets $\{x \in K : h(x) \geq t\}$ are closed, because h is upper semi-continuous and K is closed (see [13, Theorem 1.6]), and then a standard straightforward computation shows that

$$\{x \in K : h(x) \geq t\} = \bigcap_{k=1}^{\infty} \left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right).$$

Moreover, since h vanishes outside K, we have $\{x \in K : h(x) > t\} + (-2^{-k}, 2^{-k})^n \supset \{x \in K + [0, 2^{-k}]^n : h^{\diamond_k}(x) > t\}$ for all $t > 0$. Thus, by Fubini's theorem and the monotone convergence theorem, we get

(2.18)
$$\begin{aligned}
\int_K h(x)\mathrm{d}x &= \int_0^{\infty} \mathrm{vol}\big(\{x \in K : h(x) \geq t\}\big)\mathrm{d}t \\
&= \int_0^{\infty} \mathrm{vol}\left(\bigcap_{k=1}^{\infty} \left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right) \right) \mathrm{d}t \\
&= \int_0^{\infty} \lim_{k \to \infty} \mathrm{vol}\left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right) \mathrm{d}t \\
&= \lim_{k \to \infty} \int_0^{\infty} \mathrm{vol}\left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right) \mathrm{d}t \\
&\geq \lim_{k \to \infty} \int_0^{\infty} \mathrm{vol}\big(\{x \in K + [0, 2^{-k}]^n : h^{\diamond_k}(x) > t\}\big)\mathrm{d}t \\
&= \lim_{k \to \infty} \int_{K+[0,2^{-k}]^n} h^{\diamond_k}(x)\mathrm{d}x.
\end{aligned}$$

Now we show that, given $z \in \mathbb{R}^n$, $h^{\diamond_k}(x) \geq (h_k)^{\diamond_k}(z)$ for all $x \in z + [0, 2^{-k}]^n$. Indeed, we have

$$\begin{aligned}
h^{\diamond_k}(x) &= \sup_{u \in (-2^{-k}, 2^{-k})^n} h(x + u) \geq \sup_{u \in (-2^{-k}, 2^{-k})^n} \inf_{v \in [0, 2^{-k}]^n} h(z + v + u) \\
&= \sup_{u \in (-2^{-k}, 2^{-k})^n} h_k(z + u) = (h_k)^{\diamond_k}(z).
\end{aligned}$$

This, together with (2.18) and the fact that $K + [0, 2^{-k}]^n = K \cap 2^{-k}\mathbb{Z}^n + [0, 2^{-k}]^n$, implies that

$$\int_K h(x)\mathrm{d}x \geq \lim_{k \to \infty} \int_{K+[0,2^{-k}]^n} h^{\diamond_k}(x)\mathrm{d}x \geq \lim_{k \to \infty} 2^{-kn} \sum_{z \in K \cap 2^{-k}\mathbb{Z}^n} (h_k)^{\diamond_k}(z).$$

Furthermore, since f is Riemann integrable and $2^{-kn} \sum_{x \in K_0 \cap 2^{-k}\mathbb{Z}^n} f_k(x)$ is a *lower sum* of $f \cdot \chi_{(-m,m]^n}$ for the partition $\{x + [0, 2^{-k}]^n \subset K : x \in 2^{-k}\mathbb{Z}^n\}$ of K, it is clear that

$$\lim_{k \to \infty} 2^{-kn} \sum_{x \in K_0 \cap 2^{-k}\mathbb{Z}^n} f_k(x) = \int_K f(x)\mathrm{d}x.$$

Here we observe that it was crucial to work with K_0 in order to get a lower sum of $f \cdot \chi_{(-m,m]^n}$ for the above partition. We also point out the necessity

of considering the characteristic function $\chi_{(-m,m]^n}$ instead of $\chi_{[-m,m]^n}$, which has no influence when computing the above integral: in this way, the function $f \cdot \chi_{(-m,m]^n}$ vanishes on the points of the corresponding facets of the cube.

The same holds for the function g and then, taking limits on both sides of (2.17), we get (2.16). Since (2.16) is true for $K = [-m, m]^n$, for every $m \in \mathbb{N}$, the proof is now concluded because

$$\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = \lim_{m \to \infty} \int_{[-m,m]^n} \phi(x)\mathrm{d}x,$$

for every measurable function $\phi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. $\qquad\square$

It is well-known that a function is Riemann integrable if and only if it is continuous almost everywhere. Since the boundary of a convex set has null measure (and from the characterization of the upper semicontinuity in terms of the level sets) we get the following result, as a straightforward consequence of the previous one.

Corollary 2.3. *The discrete Brunn-Minkowski type inequality* (2.1) *implies the classical Brunn-Minkowski inequality* (1.1) *for bounded convex sets K and L.*

We notice the necessity of assuming convexity in the latter result: for any measurable sets $K, L \subset \mathbb{R}^n$ of positive volume, containing no rational point, one cannot expect to recover the Brunn-Minkowski inequality (1.1) with the above method of shrinking the lattice $\mathbb{Z}^n$, since K, L have no point in $2^{-k}\mathbb{Z}^n$, for any $k \in \mathbb{N}$.

3. Relations with other inequalities

The multiplicative version of the Brunn-Minkowski inequality is, among all its equivalent forms, the one that is naturally connected to the Prékopa-Leindler inequality (the case $p = 0$ of Theorem B). It asserts that if $\lambda \in (0, 1)$ and K and L are non-empty compact subsets of $\mathbb{R}^n$ then

$$(3.1) \qquad \mathrm{vol}\big((1 - \lambda)K + \lambda L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}.$$

In the discrete setting, considering now non-empty bounded sets $K, L \subset \mathbb{R}^n$, from (2.10) for $p = 0$ we get

$$(3.2) \qquad \mathrm{G}_n\big((1 - \lambda)K + \lambda L + (-1, 1)^n\big) \geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda}.$$

Regarding other possible discrete versions of (3.1), we have the following engaging and elegant result, shown very recently by Halikias, Klartag and Slomka in [8] (see also [12]):

Theorem C ([8])**.** *Let $\lambda \in (0, 1)$ and let $f, g, h, k : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$ be functions such that*

$$h(\lfloor (1 - \lambda)x + \lambda y \rfloor)k(\lceil \lambda x + (1 - \lambda)y \rceil) \geq f(x)g(y)$$

*for all $x, y \in \mathbb{Z}^n$, where $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$ and $\lceil x \rceil = (\lceil x_1 \rceil, \ldots, \lceil x_n \rceil)$.
Then*

$$\left(\sum_{x \in \mathbb{Z}^n} h(x) \right) \left(\sum_{x \in \mathbb{Z}^n} k(x) \right) \geq \left(\sum_{x \in \mathbb{Z}^n} f(x) \right) \left(\sum_{x \in \mathbb{Z}^n} g(x) \right).$$

As they observed, when applying the above result to the functions $f = \chi_K$, $g = \chi_L$, $h = \chi_{\frac{K+L}{2} + (-1,0]^n}$ and $k = \chi_{\frac{K+L}{2} + [0,1)^n}$, one has

$$\mathrm{G}_n \left(\frac{K+L}{2} + (-1,0]^n \right) \mathrm{G}_n \left(\frac{K+L}{2} + [0,1)^n \right) \geq \mathrm{G}_n(K)\mathrm{G}_n(L),$$

which yields the discrete multiplicative Brunn-Minkowski type inequality

$$(3.3) \qquad \mathrm{G}_n \left(\frac{K+L}{2} + [0,1]^n \right) \geq \sqrt{\mathrm{G}_n(K)\mathrm{G}_n(L)}.$$

We notice that the sole difference between (3.3) and (3.2) (for $\lambda = 1/2$) is the necessity of adding either the closed cube of edge length 1 or the open cube of edge length 2, respectively. However, they are not comparable. Indeed, let $n = 1$ and let $K = L = [-x, x]$ with $x \in \mathbb{R}_{\geq 0}$. On the one hand, for $x \in \mathbb{Z}$, we have $\mathrm{G}_1\big((K+L)/2 + [0,1]\big) = 2x + 2 > 2x + 1 = \mathrm{G}_1\big((K+L)/2 + (-1,1)\big)$. On the other hand, for $x \notin \mathbb{Z}$, we get $\mathrm{G}_1\big((K+L)/2 + [0,1]\big) = 2\lfloor x \rfloor + 2 < 2\lfloor x \rfloor + 3 = \mathrm{G}_1\big((K+L)/2 + (-1,1)\big)$.

As pointed out in Remark 2.2, inequality (2.1) is in general not true (even for $\lambda = 1/2$) by just adding the cube $[0,1)^n$ to the convex combination $(K+L)/2$. However, this can be solved by just considering the closed cube $[0,1]^n$, i.e., we show that inequality (3.3) also admits a $(1/n)$-form:

Theorem 3.1. *Let $K, L \subset \mathbb{R}^n$ be bounded sets such that $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$. Then*

$$(3.4) \qquad \mathrm{G}_n \left(\frac{K+L}{2} + [0,1]^n \right)^{1/n} \geq \frac{\mathrm{G}_n(K)^{1/n} + \mathrm{G}_n(L)^{1/n}}{2}.$$

The inequality is sharp.

As in Theorem 2.3, we just have to prove the corresponding one-dimensional case, collected in Lemma 3.1. Then, the proof of Theorem 3.1 is completed in a way similar to the proof of Theorem 2.2 with the particular functions $f = \chi_K$, $g = \chi_L$ and $h = \chi_{\frac{K+L}{2}}$, and $\lambda = 1/2$, replacing there the use of inequality (2.5) by (3.5).

Finally, to show that equality may be attained, we consider $K = -L = [0, m]^n$ with $m \in \mathbb{N}$ odd, for which we have $\mathrm{G}_n\big((K + L)/2 + [0,1]^n\big) = \mathrm{G}_n(K) = \mathrm{G}_n(L) = (m + 1)^n$.

Lemma 3.1. *Let $K, L \subset \mathbb{R}$ be non-empty bounded sets. Then*

$$(3.5) \qquad \mathrm{G}_1 \left(\frac{K+L}{2} + [0,1] \right) \geq \frac{\mathrm{G}_1(K) + \mathrm{G}_1(L)}{2}.$$

Proof. The proof is completely analogous to that of Lemma 2.5, and thus we include here just the slight differences.

First we notice that, for any $x, y \in \mathbb{R}$, we have

(3.6)
$$\left\lfloor \frac{x+y}{2} \right\rfloor + \frac{1}{2} \geq \frac{\lfloor x \rfloor + \lfloor y \rfloor}{2}.$$

Indeed, if $\lfloor x \rfloor + \lfloor y \rfloor$ is even then $\lfloor (x+y)/2 \rfloor = (\lfloor x \rfloor + \lfloor y \rfloor)/2$ whereas if $\lfloor x \rfloor + \lfloor y \rfloor$ is odd we get $\lfloor (x+y)/2 \rfloor \geq (\lfloor x \rfloor + \lfloor y \rfloor - 1)/2$.

Now, we notice that if $M \supset (K+L)/2$ is such that $M + [0,1] = [a_1, b_1]$ is a (non-empty) compact interval, we have on the one hand that $\mathrm{G}_1(M + [0,1]) = \lfloor b_1 \rfloor - \lceil a_1 \rceil + 1$. Moreover, denoting by $a = \inf K$, $b = \sup K$, $c = \inf L$ and $d = \sup L$, we clearly get $\mathrm{G}_1(K) \leq \mathrm{G}_1([a,b]) = \lfloor b \rfloor - \lceil a \rceil + 1$ and $\mathrm{G}_1(L) \leq \mathrm{G}_1([c,d]) = \lfloor d \rfloor - \lceil c \rceil + 1$. On the other hand, the inclusion $(K+L)/2 \subset M$ implies that

$$a_1 \leq \frac{a+c}{2} \leq \frac{b+d}{2} \leq b_1 - 1.$$

Altogether, and using (3.6) jointly with the fact that $\lceil x \rceil = -\lfloor -x \rfloor$ for any $x \in \mathbb{R}$, we obtain

$$\mathrm{G}_1(M + [0,1]) = \lfloor b_1 \rfloor - \lceil a_1 \rceil + 1 \geq \left\lfloor \frac{b+d}{2} \right\rfloor - \left\lceil \frac{a+c}{2} \right\rceil + 2$$
$$\geq \frac{\lfloor b \rfloor + \lfloor d \rfloor}{2} - \frac{\lceil a \rceil + \lceil c \rceil}{2} + 1 = \frac{\mathrm{G}_1([a,b]) + \mathrm{G}_1([c,d])}{2}$$
$$\geq \frac{\mathrm{G}_1(K) + \mathrm{G}_1(L)}{2}.$$

The proof is then completed in a way analogous to the proof of Lemma 2.5. $\square$

We would like to point out that the corresponding version of Theorem 3.1 for an arbitrary $\lambda \in (0,1)$ is, in general, not true. Indeed, taking $K = [0,1]^n$, $L = [-5,6]^n$ and $\lambda = 1/3$ one has that

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + [0,1]^n\big)^{1/n} = \mathrm{G}_n\left(\left[\frac{-5}{3}, \frac{11}{3}\right]^n\right)^{1/n} = 5$$
$$< \frac{16}{3} = (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}.$$

We include here an open question that arose during our study:

Question 3.1. *Regarding the statement of Theorem 3.1, is the assumption* $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$ *necessary?*

We conclude the section by proving an inequality similar to (1.5) but in the spirit of (2.1), namely that one may add another (fixed) set to the Minkowski sum $A + B$ instead of considering the extension $\bar{A}$ of A. We show that an appropriate set to be taken into account in this respect is the lattice cube $\{0,1\}^n$, which also fits well with inequality (3.4).

Theorem 3.2. *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$(3.7) \qquad \left|A + B + \{0,1\}^n\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

The inequality is sharp.

The idea of the proof we present here goes back to [7, Lemma 2.4].

Proof. Taking into account the relation

$$A + B + \{0,1\}^n + [0,1]^n = A + [0,1]^n + B + [0,1]^n,$$

jointly with the fact that $\mathrm{vol}(X + [0,1]^n) = |X|$, for any non-empty and finite set $X \subset \mathbb{Z}^n$, the result directly follows from the Brunn-Minkowski inequality (1.2). Finally, to show that the inequality is sharp, we consider the lattice cubes $A = \{0, \ldots, m_1\}^n$ and $B = \{0, \ldots, m_2\}^n$, for $m_1, m_2 \in \mathbb{Z}_{\geq 0}$. $\qquad\square$

We conclude the paper by comparing inequalities (1.5) and (3.7); again, we refer the reader to Section 2.1 in [10] for the precise definition of the extension $\bar{A}$ of a given non-empty finite set $A \subset \mathbb{Z}^n$.

Since $A + (\{0\}^{n-1} \times \{0,1\})$ contains at least the same amount of points that the union of A and its maximal cardinality section, and taking into account that $\{0,1\}^n = \left(\{0,1\}^{n-1} \times \{0\}\right) + \left(\{0\}^{n-1} \times \{0,1\}\right)$, from the definition of $\bar{A}$ is then immediate that $\left|\bar{A}\right| \leq \left|A + \{0,1\}^n\right|$. However, inequalities (1.5) and (3.7) are not comparable. Indeed, if we consider on the one hand $A = \{(0,0), (1,0), (2,0), (1,1)\}$ and $B = \{0,1\}^2$, we have $\left|\bar{A}\right| = 10 = \left|A + \{0,1\}^2\right|$ (with $\bar{A} \neq A + \{0,1\}^2$) and $\left|\bar{A} + B\right| = 20 > 18 = \left|A + \{0,1\}^2 + B\right|$ (see Figure 3). On the other hand, for $A = \{(0,0), (0,1), (1,1), (4,1)\}$ and $B = \{0,1\}^2$ we obtain $\left|\bar{A} + B\right| = 21 < 24 = \left|A + \{0,1\}^2 + B\right|$.
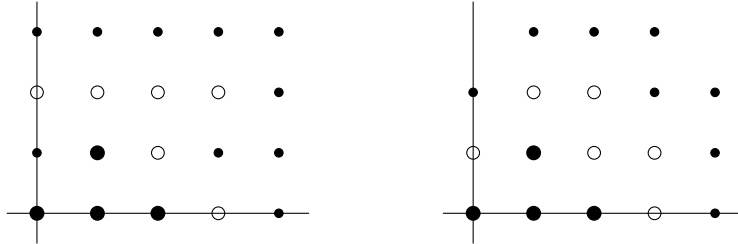

FIGURE 1. Left: the sets A (thick points), $\bar{A}$ (thick and hollow points) and $\bar{A} + B$. Right: the sets A (thick points), $A + \{0,1\}^2$ (thick and hollow points) and $A + \{0,1\}^2 + B$.

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References

[1] Barthe, F. "Autour de l'inégalité de Brunn-Minkowski." *Ann. Fac. Sci. Toulouse Math. (6)* 12, no. 2 (2003): 127–178.

[2] Borell, C. "Convex set functions in d-space." *Period. Math. Hungar.* 6 (1975): 111–136.

[3] Brascamp, H. J. and Lieb, E. H. "On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions and with an application to the diffusion equation." *J. Func. Anal.* 22, no. 4 (1976): 366–389.

[4] Bullen, P. S. *Handbook of means and their inequalities*, Mathematics and its Applications vol. 560, Revised from the 1988 original, Kluwer Academic Publishers Group, Dordrecht, 2003.

[5] Gardner, R. J. "The Brunn-Minkowski inequality." *Bull. Amer. Math. Soc.* 39, no. 3 (2002): 355–405.

[6] Gardner, R. J. and Gronchi, P. "A Brunn-Minkowski inequality for the integer lattice." *Trans. Amer. Math. Soc.* 353, no. 10 (2001): 3995–4024.

[7] Green, B. and Tao, T. "Compressions, convex geometry and the Freiman-Bilu theorem." *Q. J. Math.* 57, no. 4 (2006): 495–504.

[8] Halikias, D., Klartag, B. and Slomka, B. A. "Discrete variants of Brunn-Minkowski type inequalities." *Submitted*, arXiv:1911.04392.

[9] Hardy, G. H., Littlewood, J. E. and Pólya, G. *Inequalities.* Cambridge Mathematical Library, Reprint of the 1952 edition. Cambridge: Cambridge University Press, 1988.

[10] Hernández Cifre, M. A., Iglesias, D. and Yepes Nicolás, J. "On a discrete Brunn-Minkowski type inequality." *SIAM J. Discrete Math.* 32 (2018): 1840–1856.

[11] Iglesias, D. and Yepes Nicolás, J. "On discrete Borell-Brascamp-Lieb inequalities." To appear in *Rev. Matemática Iberoamericana*.

[12] Klartag, B. and Lehec, J. "Poisson processes and a log-concave Bernstein theorem." *Stud. Math.* 247, no. 1, (2019): 85–107.

[13] Rockafellar, R. T. and Wets, R. J.-B. *Variational analysis.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 317. Berlin: Springer-Verlag, 1998.

[14] Ruzsa, I. Z. "Sum of sets in several dimensions." *Combinatorica* 14 (1994): 485–490.

[15] Ruzsa, I. Z. "Sets of sums and commutative graphs." *Studia Sci. Math. Hungar.* 30 (1995): 127–148.

[16] Schneider, R. *Convex bodies: The Brunn-Minkowski theory*, 2nd expanded ed. Encyclopedia of Mathematics and its Applications, 151. Cambridge: Cambridge University Press, 2014.

[17] Tao, T. and Vu, V. *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, 105. Cambridge: Cambridge University Press, 2006.

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