ON A BRUNN-MINKOWSKI INEQUALITY FOR MEASURES WITH QUASI-CONVEX DENSITIES

JESÚS YEPES NICOLÁS

Abstract. In this paper we prove that the classical Brunn-Minkowski inequality holds for product measures on the Euclidean space with quasi-convex densities when considering certain classes of sets that contain, among others, the complements (within a centered box) of unconditional sets. As a consequence, we derive an isoperimetric type inequality.

1. Introduction

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space, and we denote by e_i the i-th canonical unit vector. For $i = 1, \dots, n$, we represent by $H_i = \bigl\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0\bigr\}$ the i-th coordinate hyperplane. The n-dimensional volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (when integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$). We write $M(t) = \{x \in \mathbb{R}^{n-1} : (x,t) \in M\}$ for the $(n-1)$-dimensional section at height $t \in \mathbb{R}$ (in the direction of e_n), whereas the orthogonal projection of M onto an i-dimensional linear subspace H is denoted by $M|H$. Moreover, $H^\perp$ represents the orthogonal complement of H and, for any $x \in M|H_i$, we set $M_i(x) = \{t \in \mathbb{R} : x + t\mathrm{e}_i \in M\}$ to denote the one-dimensional section of M through the point x in the direction of e_i. Finally, given $r > 0$, rM stands for the set $\{rm : m \in M\}$.

The Minkowski sum of two non-empty sets $A, B \subset \mathbb{R}^n$ is the classical vector addition of them: $A + B = \{a + b : a \in A,\, b \in B\}$. It is natural to wonder about the possibility of bounding the volume of the Minkowski sum of two sets in terms of their volumes; this is the statement of the *Brunn-Minkowski inequality* (for extensive and beautiful surveys on this inequality we refer the reader to [1, 7]). One form of it asserts that if $\lambda \in (0,1)$ and A and B are non-empty measurable subsets of $\mathbb{R}^n$ such that $(1-\lambda)A + \lambda B$ is also measurable then

$$\mathrm{vol}\bigl((1-\lambda)A + \lambda B\bigr)^{1/n} \geq (1-\lambda)\mathrm{vol}(A)^{1/n} + \lambda \mathrm{vol}(B)^{1/n}. \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 52A40, 28A35; Secondary 26B25.

Key words and phrases. Brunn-Minkowski inequality, quasi-convex density, product measure, isoperimetric inequality.

Supported by MINECO/FEDER project MTM2015-65430-P and "Programa de Ayudas a Grupos de Excelencia de la Región de Murcia", Fundación Séneca, 19901/GERM/15.

The Brunn-Minkowski inequality was generalized to different types of measures, including the cases of log-concave measures [10, 15] and of p-concave measures (see e.g. [3, 4]). It is interesting to note that it was proved by Borell [2, 3] that such generalizations would require a p-concavity assumption on the density of the underlying measure (see (2.1) below for the precise definition). As a consequence of this approach (see also [21]), when dealing with arbitrary measurable sets and a Radon measure on $\mathbb{R}^n$, the $(1/n)$-form of the Brunn-Minkowski inequality (1.1) is only true, in general, for the volume (up to a constant). However, when considering some special families of sets (e.g. that of *unconditional sets*), the $(1/n)$-Brunn-Minkowski inequality holds for some types of measures, such as the standard Gaussian measure, which is given by

$$\mathrm{d}\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{\frac{-|x|^2}{2}} \mathrm{d}x$$

(see e.g. [8, 11, 12, 14, 16]). Furthermore, for the family of *C-coconvex sets* (complements of closed convex sets, of positive and finite volume, within a pointed closed convex cone with non-empty interior C), a "complemented" version of the Brunn-Minkowski inequality (1.1) holds for the volume (see [9, 19]), namely

$$\mathrm{vol}\big(C \setminus ((1-\lambda)K + \lambda L)\big)^{1/n} \leq (1-\lambda)\mathrm{vol}(C \setminus K)^{1/n} + \lambda \mathrm{vol}(C \setminus L)^{1/n}$$

for all $\lambda \in (0,1)$. And again, this (complemented) Brunn-Minkowski inequality can be also generalized for certain general measures (see [13]).

To complete the picture, one may ask about possible p-convexity conditions on the density of the underlying measure. Among others, what can be said about the measure ν_n on $\mathbb{R}^n$ given by

$$\mathrm{d}\nu_n(x) = e^{|x|^2} \mathrm{d}x,$$

whose density is log-convex? In [13], when dealing with measures involving certain log-convex functions as part of their densities, the authors showed another type of complemented Brunn-Minkowski inequality. Nevertheless, not much more seems to be known regarding Brunn-Minkowski inequalities for log-convex densities or, more generally, quasi-convex densities (see (2.2) below for the precise definition).

To this regard, and inspired by the above-mentioned (complemented) Brunn-Minkowski inequalities, it is natural to wonder whether one may find certain classes of sets for which a measure on $\mathbb{R}^n$ of the kind of ν_n satisfies the $(1/n)$-form of the Brunn-Minkowski inequality. Here we give a positive answer to this question, by showing that it is enough to consider *congruous sets* (see Definition 2.1): a family that contains, among others, the complements of unconditional sets within a centered box (cf. Example 2.1). This is the content of the following result, in the more general setting of product measures with quasi-convex densities (with minimum at the origin).

Theorem 1.1. *Let $\mu=\mu_1\otimes\cdots\otimes\mu_n$ be a product measure on $\mathbb{R}^n$ such that μ_i is the measure given by $\mathrm{d}\mu_i(x)=\phi_i(x)\,\mathrm{d}x$, where $\phi_i:\mathbb{R}\longrightarrow[0,\infty)$ is quasi-convex with $\phi_i(0)=\min_{x\in\mathbb{R}}\phi_i(x)$, for all $i=1,\dots,n$.*

Let $\lambda\in(0,1)$ and let $A,B\subset\mathbb{R}^n$ be non-empty measurable congruous sets such that $(1-\lambda)A+\lambda B$ is also measurable. Then

$$\mu\big((1-\lambda)A+\lambda B\big)^{1/n}\geq(1-\lambda)\mu(A)^{1/n}+\lambda\mu(B)^{1/n}. \tag{1.2}$$

Section 2 is mainly devoted to showing this result. Finally, in Section 3, we derive an isoperimetric type inequality as a consequence of (1.2).

2. Proof of the main result

2.1. **Background.** We recall that a function $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is p-concave, for $p\in\mathbb{R}\cup\{\pm\infty\}$, if

$$\phi\big((1-\lambda)x+\lambda y\big)\geq M_p\big(\phi(x),\phi(y),\lambda\big) \tag{2.1}$$

for all $x,y\in\mathbb{R}^n$ such that $\phi(x)\phi(y)>0$ and any $\lambda\in(0,1)$. Here M_p denotes the *p-mean* of two non-negative numbers a,b:

$$M_p(a,b,\lambda)=\begin{cases}\big((1-\lambda)a^p+\lambda b^p\big)^{1/p}, & \text{if } p\neq 0,\pm\infty,\\ a^{1-\lambda}b^{\lambda} & \text{if } p=0,\\ \max\{a,b\} & \text{if } p=\infty,\\ \min\{a,b\} & \text{if } p=-\infty.\end{cases}$$

A 0-concave function is usually called *log-concave* whereas a $(-\infty)$-concave function is called *quasi-concave*. Quasi-concavity is equivalent to the fact that the superlevel sets $\{x\in\mathbb{R}^n:\phi(x)\geq t\}$ are convex for all $t\in[0,1]$.

On the other side of the coin, one is led to p-convex functions, where $p\in\mathbb{R}\cup\{\pm\infty\}$, i.e., those functions satisfying

$$\phi\big((1-\lambda)x+\lambda y\big)\leq M_p\big(\phi(x),\phi(y),\lambda\big) \tag{2.2}$$

for all $x,y\in\mathbb{R}^n$ and all $\lambda\in(0,1)$. Again, 0-convex functions are referred to as *log-convex* whereas ∞-convex functions are called *quasi-convex*.

Now we define a new class of (pairs of) sets that will play a relevant role throughout this paper.

Definition 2.1. *Let $A,B\subset\mathbb{R}^n$ be non-empty bounded sets. For $n=1$, we say that A and B are* congruous *if one of the following assertions holds.*

i) $A\cap(-\infty,0),B\cap(-\infty,0)=\emptyset$ *and* $\max(A)=\max(B)$.
ii) $A\cap(0,\infty),B\cap(0,\infty)=\emptyset$ *and* $\min(A)=\min(B)$.
iii) $A\cap(0,\infty),B\cap(0,\infty),A\cap(-\infty,0),B\cap(-\infty,0)\neq\emptyset$, $\min(A)=\min(B)$ *and* $\max(A)=\max(B)$.

For $n\geq 2$, we say that A and B are congruous *if, for any $i=1,\dots,n$, the sets $A_i(x)$ and $B_i(y)$ are congruous for all $x\in A|H_i$ and all $y\in B|H_i$.*

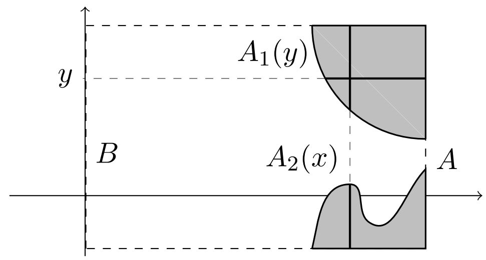

FIGURE 1. The congruous sets A (in gray) and B (the box), with the sections $A_2(x)$, $A_1(y)$ for given $x \in A|H_2$, $y \in A|H_1$.

We notice that the fact that, for any $i = 1, \dots, n$, the sets $A_i(x)$ and $B_i(y)$ are congruous (for all $x \in A|H_i$ and all $y \in B|H_i$) does not mean that the same condition in Definition 2.1 holds for all i (see Figure 1; there $A_2(x), B_2(x')$ satisfy condition iii) of Definition 2.1, for all $x \in A|H_2$ and all $x' \in B|H_2$, whereas $A_1(y), B_1(y')$ fulfil condition i), for any $y \in A|H_1$ and any $y' \in B|H_1$).

Unconditional convex sets are of particular interest in convexity, also regarding Brunn-Minkowski type inequalities (see e.g. [11, 18]). A subset $A \subset \mathbb{R}^n$ is said to be unconditional (not necessarily convex) if for every $(x_1, \dots, x_n) \in A$ and every $(\epsilon_1, \dots, \epsilon_n) \in [-1, 1]^n$ one has $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in A$. As announced before, the family of congruous sets contains certain complements of unconditional sets:

Example 2.1. *Let $P = \prod_{i=1}^n [-\alpha_i, \alpha_i]$, $\alpha_i > 0$ for $i = 1, \dots, n$, be a centered orthogonal compact box and let $A, B \subset P$ be non-empty compact sets such that $P \setminus A, P \setminus B$ are unconditional. Then A and B are congruous. Indeed, from the unconditionality of $P \setminus A$ and $P \setminus B$, we have that $\max\bigl(A_i(x)\bigr) = \max\bigl(B_i(y)\bigr) = \alpha_i$ and $\min\bigl(A_i(x)\bigr) = \min\bigl(B_i(y)\bigr) = -\alpha_i$, for all $x \in A|H_i$ and all $y \in B|H_i$; thus $A_i(x)$ and $B_i(y)$ are congruous for any $i = 1, \dots, n$ since they satisfy condition iii) in Definition 2.1 (see Figure 2).*

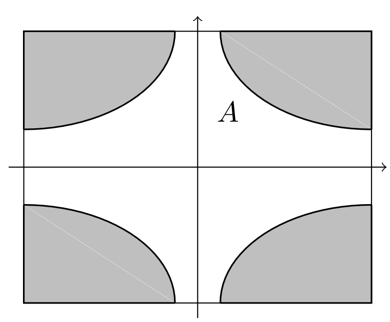

FIGURE 2. A set A (in gray) contained in a centered box P such that $P \setminus A$ is unconditional.

The following result is well-known in the literature (see e.g. the one-dimensional case of [6, Theorem 4.1] and the references therein. Regarding its statement, and following the notation used in [6], we notice that for a quasi-concave function $\phi : \mathbb{R} \longrightarrow [0,\infty)$ we have $(1-\lambda)\phi\chi_A \star_{-\infty} \lambda\phi\chi_B = \phi\chi_{(1-\lambda)A+\lambda B}$, where χ_M denotes the characteristic function of the set $M \subset \mathbb{R}$).

Lemma 2.1. *Let μ be the measure on $\mathbb{R}$ given by $\mathrm{d}\mu(x) = \phi(x)\mathrm{d}x$, where $\phi : \mathbb{R} \longrightarrow [0,\infty)$ is quasi-concave with $\phi(0) = \max_{x\in\mathbb{R}} \phi(x)$. Let $\lambda \in (0,1)$ and let $A, B \subset \mathbb{R}$ be measurable sets with $0 \in A \cap B$. Then*

$$\mu(C) \geq (1-\lambda)\mu(A) + \lambda\mu(B)$$

for any measurable set C such that $C \supset (1-\lambda)A + \lambda B$.

As a consequence of such a Brunn-Minkowski inequality for quasi-concave densities on $\mathbb{R}$, we will obtain the one-dimensional Brunn-Minkowski inequality for measures associated to quasi-convex functions when working with congruous sets. This is the content of Lemma 2.2.

2.2. **Proof.** We start this subsection by showing the one-dimensional case of our main result, Theorem 1.1.

Lemma 2.2. *Let μ be the measure on $\mathbb{R}$ given by $\mathrm{d}\mu(x) = \phi(x)\mathrm{d}x$, where $\phi : \mathbb{R} \longrightarrow [0,\infty)$ is quasi-convex with $\phi(0) = \min_{x\in\mathbb{R}} \phi(x)$. Let $\lambda \in (0,1)$ and let $A, B \subset \mathbb{R}$ be non-empty measurable congruous sets. Then*

$$\mu(C) \geq (1-\lambda)\mu(A) + \lambda\mu(B)$$

for any non-empty measurable set C such that $C \supset (1-\lambda)A + \lambda B$.

Proof. Let A and B satisfy condition iii) in Definition 2.1. Assuming that the result is true if either i) or ii) (of Definition 2.1) holds, it is enough to consider $A^+, A^-, B^+, B^-, C^+, C^-$ where, for any $M \subset \mathbb{R}$, the sets M^+ and M^- stand for $M^+ = M \cap (0,\infty)$ and $M^- = M \cap (-\infty, 0)$. Indeed, applying the result to the sets A^+, B^+, C^+ and A^-, B^-, C^-, respectively, we have

$$\begin{aligned}(1-\lambda)\mu(A) + \lambda\mu(B) &= (1-\lambda)\mu(A^+) + \lambda\mu(B^+) + (1-\lambda)\mu(A^-) + \lambda\mu(B^-)\\ &\leq \mu(C^+) + \mu(C^-) = \mu(C).\end{aligned}$$

Moreover, we note that the function $\bar{\phi} : \mathbb{R} \longrightarrow [0,\infty)$ given by $\bar{\phi}(x) = \phi(-x)$ is quasi-convex (and, clearly, $\bar{\phi}(0) = \min_{x\in\mathbb{R}} \bar{\phi}(x)$). Thus, considering if necessary $\bar{A} = -A$, $\bar{B} = -B$, $\bar{C} = -C$, and the measure $\bar{\mu}$ with density $\bar{\phi}$, it is enough to prove the result for congruous sets satisfying i). Now, the quasi-convexity of ϕ implies that $\phi(x) \leq \max\{\phi(0), \phi(y)\} = \phi(y)$ for any $0 < x < y$. This shows that ϕ is increasing on $(0,\infty)$ and then $\phi \cdot \chi_{(0,\infty)}$ is quasi-concave. Thus, setting $x_0 = \max(A) = \max(B)$, the result follows from applying Lemma 2.1 to the function $\psi : \mathbb{R} \longrightarrow [0,\infty)$ given by $\psi(x) = \phi(x + x_0) \cdot \chi_{(-\infty,0]}(x)$ and the sets $A - x_0, B - x_0, C - x_0$. $\square$

As stated in Theorem 1.1, the above result extends to dimension n. The approach we follow here is based on the underlying idea of [16, Theorem 1.3], and it goes back to some classical proofs of functional versions of the Brunn-Minkowski inequality (such as the *Prékopa-Leindler inequality*) and other related results.

Proof of Theorem 1.1. For the sake of brevity we write $C = (1-\lambda)A + \lambda B$ and, given $t_1, t_2 \in \mathbb{R}$, $t_\lambda = (1-\lambda)t_1 + \lambda t_2$. We also set $\bar{\mu} = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{n-1}$ (i.e., $\mu = \bar{\mu} \otimes \mu_n$).

Since μ is inner regular (i.e., $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\}$ for any measurable set A), we may assume, without loss of generality, that A and B are compact. Indeed, given sequences of compact sets $(K_n)_{n\in\mathbb{N}}$, $(L_n)_{n\in\mathbb{N}}$ that approximate from inside the congruous sets A and B, respectively, one may clearly consider certain sequences of congruous compact sets $(K'_n)_{n\in\mathbb{N}}$, $(L'_n)_{n\in\mathbb{N}}$ such that $\mu(K'_n) = \mu(K_n)$ and $\mu(L'_n) = \mu(L_n)$, for all $n \in \mathbb{N}$. In fact, it is enough to add to K_n and L_n, respectively, the projections $\big(A|H_i\big)$ and $\big(B|H_i\big)$, located at the appropriate height(s) in the direction of e_i, for $i = 1, \dots, n$.

Moreover, we observe that we may assume that $\mu(A)\mu(B) > 0$. Indeed, the case in which one of the sets, say B, has measure zero whereas the other one, A, has positive measure can be obtained (cf. [16, Proposition 2.7]) by applying the positive measures case to A and the following set: let P be an orthogonal compact box congruous with B (and so, with A) and let C_m be a decreasing sequence of (unions of) boxes, which are congruous with B, that shrinks (as $m \to \infty$) to the subset of vertices of P that belong to B; then we take $B_m = B \cup C_m$, which is also congruous with A for all $m \in \mathbb{N}$. We note that this congruence ensures that the points in the limit case belong to B, and hence $\bigcap_{m\in\mathbb{N}}\big((1-\lambda)A + \lambda B_m\big) = (1-\lambda)A + \lambda B$. Taking into account that

$$\mu\left(\bigcap_{m\in\mathbb{N}}\big((1-\lambda)A + \lambda B_m\big)\right) = \lim_m \mu\big((1-\lambda)A + \lambda B_m\big),$$

we get (1.2).

We then show the result by (finite) induction on the dimension n. The case $n = 1$ is just Lemma 2.2. So, we suppose that $n \geq 2$ and that the inequality is true for dimension $n-1$. The sets $A(t_1), B(t_2)$, for $t_1, t_2 \in \mathbb{R}$ such that $t_1\mathrm{e}_n \in A|H_n^\perp$, $t_2\mathrm{e}_n \in B|H_n^\perp$, are clearly congruous and thus, applying the induction hypothesis (i.e., (1.2) in $\mathbb{R}^{n-1}$ for $\bar{\mu}$) together with the fact that $C(t_\lambda) \supset (1-\lambda)A(t_1) + \lambda B(t_2)$, we have

$$\bar{\mu}\big(C(t_\lambda)\big) \geq \Big((1-\lambda)\bar{\mu}\big(A(t_1)\big)^{1/(n-1)} + \lambda\bar{\mu}\big(B(t_2)\big)^{1/(n-1)}\Big)^{n-1}. \tag{2.3}$$

Now, we take the non-negative functions $f, g, h : \mathbb{R} \longrightarrow [0, \infty)$ given by

$$f(t) = \frac{\bar{\mu}(A(t))}{|\bar{\mu}(A(\cdot))|_\infty},\ g(t) = \frac{\bar{\mu}(B(t))}{|\bar{\mu}(B(\cdot))|_\infty},\ h(t) = \frac{\bar{\mu}(C(t))}{c},$$

where

$$c=\Big((1-\lambda)\,|\bar{\mu}(A(\cdot))|_{\infty}^{1/(n-1)}+\lambda\,|\bar{\mu}(B(\cdot))|_{\infty}^{1/(n-1)}\Big)^{n-1}.$$

We notice that the above functions are well-defined: denominators are positive since $\mu(A)\mu(B)>0$, and they are finite because $A|H_{n-1}$ and $B|H_{n-1}$ are compact sets and $\bar{\mu}$ is locally finite. Furthermore, $\sup_{t\in\mathbb{R}} f(t)=\sup_{t\in\mathbb{R}} g(t)=1$.

Using (2.3), and setting $\theta=\dfrac{\lambda\,|\bar{\mu}(B(\cdot))|_{\infty}^{1/(n-1)}}{c^{1/(n-1)}}\in(0,1)$, we get

$$\begin{aligned}\bar{\mu}\big(C(t_\lambda)\big)&\geq\Big((1-\lambda)\bar{\mu}\big(A(t_1)\big)^{1/(n-1)}+\lambda\bar{\mu}\big(B(t_2)\big)^{1/(n-1)}\Big)^{n-1}\\&=c\left((1-\theta)f(t_1)^{1/(n-1)}+\theta g(t_2)^{1/(n-1)}\right)^{n-1}\\&\geq c\,\min\big\{f(t_1),g(t_2)\big\}.\end{aligned}$$

This shows that $h((1-\lambda)t_1+\lambda t_2)\geq\min\{f(t_1),g(t_2)\}$ for any $t_1,t_2\in\mathbb{R}$, which clearly implies that

$$\{t\in\mathbb{R}:h(t)\geq s\}\supset(1-\lambda)\{t\in\mathbb{R}:f(t)\geq s\}+\lambda\{t\in\mathbb{R}:g(t)\geq s\}\tag{2.4}$$

for all $s\in[0,1)$. Moreover, since $A_n(x)$ and $B_n(y)$ are congruous for all $x\in A|H_n$ and all $y\in B|H_n$ then the superlevel sets $\{t\in\mathbb{R}:f(t)\geq s\}$ and $\{t\in\mathbb{R}:g(t)\geq s\}$ are also congruous for any $s\in[0,1)$. Indeed, assuming without loss of generality that $A_n(x),B_n(y)$ satisfy condition i) of Definition 2.1, for all $x\in A|H_n$ and all $y\in B|H_n$, then there exists $s_0>0$ such that $\big(A|H_n\big)+s_0\mathrm{e}_n\subset A$, $\big(B|H_n\big)+s_0\mathrm{e}_n\subset B$ and $A,B\subset[0,s_0\mathrm{e}_n]+H_n$. Hence, both f and g attain their maximum at s_0 and vanish on $(-\infty,0)\cup(s_0,\infty)$, which implies that their superlevel sets satisfy condition i) of Definition 2.1 and thus are congruous.

Therefore, we may apply Lemma 2.2 to get

$$\mu_n\big(\{t\in\mathbb{R}:h(t)\geq s\}\big)\geq(1-\lambda)\mu_n\big(\{t\in\mathbb{R}:f(t)\geq s\}\big)+\lambda\mu_n\big(\{t\in\mathbb{R}:g(t)\geq s\}\big)$$

for any $s\in[0,1)$. This, together with Fubini's theorem and the Cavalieri Principle

$$\int_{\mathbb{R}}\psi(x)\,\mathrm{d}\mu_n(x)=\int_0^{|\psi|_\infty}\mu_n\big(\{t\in\mathbb{R}:\psi(t)\geq s\}\big)\,\mathrm{d}s$$

for $\psi=f,g,h$, jointly with the fact that $|h|_\infty\geq1=|f|_\infty=|g|_\infty$ (cf. (2.4)), allows us to obtain

$$\begin{aligned}\mu\big((1-\lambda)A+\lambda B\big)&=c\int_{\mathbb{R}}h(x)\,\mathrm{d}\mu_n(x)\\&\geq c\left((1-\lambda)\int_{\mathbb{R}}f(x)\,\mathrm{d}\mu_n(x)+\lambda\int_{\mathbb{R}}g(x)\,\mathrm{d}\mu_n(x)\right)\\&=c\left((1-\lambda)\frac{\mu(A)}{|\bar{\mu}(A(\cdot))|_\infty}+\lambda\frac{\mu(B)}{|\bar{\mu}(B(\cdot))|_\infty}\right).\end{aligned}$$

And then, applying the (reverse) Hölder inequality (see e.g. [5, Theorem 1, page 178]),

$$a_1b_1 + a_2b_2 \geq \left(a_1^p + a_2^p\right)^{1/p}\left(b_1^q + b_2^q\right)^{1/q},$$

with parameters $p = 1/n$ and $q = -1/(n-1)$, and taking $a_1 = (1-\lambda)^{1/p}\mu(A)$, $a_2 = \lambda^{1/p}\mu(B)$, $b_1 = (1-\lambda)^{1/q}\,|\bar{\mu}(A(\cdot))|_\infty^{-1}$ and $b_2 = \lambda^{1/q}\,|\bar{\mu}(B(\cdot))|_\infty^{-1}$, we conclude that

$$\mu\big((1-\lambda)A + \lambda B\big) \geq \Big((1-\lambda)\mu(A)^{1/n} + \lambda\mu(B)^{1/n}\Big)^n,$$

as desired. □

3. A remark on an isoperimetric inequality

Given a set $M \subset \mathbb{R}^n$, let $\operatorname{pos} M$ and $\operatorname{int} M$ denote, respectively, the positive hull and interior of M. Moreover, let $\varepsilon_1, \dots, \varepsilon_{2^n}$ denote the elements of $\{-1,1\}^n$. Then, setting $\varepsilon_j = (\varepsilon_1^j, \dots, \varepsilon_n^j)$ for any $j = 1, \dots, 2^n$, we write

$$O_j = \operatorname{pos}\{\varepsilon_1^j \mathrm{e}_1, \dots, \varepsilon_n^j \mathrm{e}_n\}$$

for the corresponding orthant of $\mathbb{R}^n$.

Along this section, we deal with certain sets contained in an orthogonal compact box (which, for the sake of simplicity, will be assumed to be centered): fixing a box $P = \prod_{i=1}^n [-\alpha_i, \alpha_i]$, with $\alpha_i > 0$ for all i, we consider unions of orthants of unconditional compact convex sets 'embedded' in the corners of P. More precisely, such a set A satisfies that, for all $j = 1, \dots, 2^n$,

$$A \cap O_j = x_j + (K_j \cap (-O_j)) \tag{3.1}$$

for some unconditional compact convex set $K_j \subset \operatorname{int} P$ (cf. Figure 3), where $x_j = (\varepsilon_1^j \alpha_1, \dots, \varepsilon_n^j \alpha_n)$ is the corresponding vertex of P. In the following, for the sake of brevity, we will write $A_j = A \cap O_j$.

As in the Euclidean setting, we will obtain an isoperimetric type inequality as a consequence of (1.2). To this aim, we introduce some notation. Let

$$\mathrm{W}_1^\mu(A; B) = \frac{1}{n} \liminf_{t \to 0^+} \frac{\mu(A + tB) - \mu(A)}{t}$$

be the first *quermassintegral* of A with respect to the set B associated to the measure μ. Here we assume that A and B are measurable sets such that $A + tB$ is also measurable for all $t \geq 0$.

In a similar way, and denoting by B_n the n-dimensional Euclidean (closed) unit ball, we may define

$$\mu^+(A) = \liminf_{t \to 0^+} \frac{\mu(A + tB_n) - \mu(A)}{t},$$

the surface area measure associated to μ, i.e., its (lower) Minkowski content. Clearly, $\mu^+(A) = n\mathrm{W}_1^\mu(A; B_n)$. The relative Minkowski content of a set

$A \subset \mathbb{R}^n$ with respect to a second set $\Omega \subset \mathbb{R}^n$ is defined by

$$\mu^+(A, \Omega) = \liminf_{t \to 0^+} \frac{\mu\big((A + tB_n) \cap \Omega\big) - \mu(A \cap \Omega)}{t}.$$

Moreover, given $x \in \mathbb{R}^n$, we set

$$M^\mu(x, A) = n\mu(x + A) - \frac{\mathrm{d}^-}{\mathrm{d}t}\Big|_{t=1} \mu(x + tA),$$

provided that $((x, A), \mu)$ is so that the above (left) derivative exists. When dealing with a set $A \subset \mathbb{R}^n$ satisfying (3.1) for all $j = 1, \ldots, 2^n$, we also write $M^\mu(A) = \sum_{j=1}^{2^n} M_j^\mu(A_j)$, where $M_j^\mu(A_j) = M^\mu(x_j, K_j \cap (-O_j))$. We notice that, from the convexity of $K_j \cap (-O_j)$ and using Theorem 1.1, the function $t \mapsto \mu(x_j + t(K_j \cap (-O_j)))^{1/n}$ is (increasing and) concave on $(0, 1]$ for any product measure μ in the conditions of the latter result. This implies that the left derivative of $\mu(x_j + t(K_j \cap (-O_j)))$ at $t = 1$ (possibly infinite) exists (cf. [17, Theorem 23.1]) and hence, for all $j = 1, \ldots, 2^n$, $M_j^\mu(A_j)$ (and so $M^\mu(A)$) is well-defined. Clearly, $M^{\mathrm{vol}}(A) = 0$ for such a set A and thus this functional does not appear in the classical isoperimetric inequality. For more information about this functional, we refer the reader to [11, 16] and the references therein.

Now we show an isoperimetric type inequality for unions of orthants of unconditional compact convex sets embedded in the corners of a fixed orthogonal box, in the setting of product measures with quasi-convex densities. This a straightforward consequence of the following result for (such) a sole orthant.

Theorem 3.1. *Let $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ be a product measure on $\mathbb{R}^n$ such that μ_i is the measure given by $\mathrm{d}\mu_i(x) = \phi_i(x)\,\mathrm{d}x$, where $\phi_i : \mathbb{R} \longrightarrow [0, \infty)$ is quasi-convex with $\phi_i(0) = \min_{x \in \mathbb{R}} \phi_i(x)$, for all $i = 1, \ldots, n$.*

Let $P = \prod_{i=1}^n [-\alpha_i, \alpha_i]$, with $\alpha_i > 0$ for all i and let $K \subset \operatorname{int} P$ be a non-empty unconditional compact convex set. Let $A = x_1 + (K \cap (-O_1))$, where $x_1 = (\alpha_1, \ldots, \alpha_n)$ and $O_1 = \operatorname{pos}\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$. Then, for any $r > 0$ such that $rB_n \subset \operatorname{int} P$,

$$r\mu^+(A, P) + M^\mu(x_1, K_1 \cap (-O_1)) \geq n\mu(A)^{1-1/n} \mu\big(x_1 + (rB_n \cap (-O_1))\big)^{1/n},$$

with equality if $A = x_1 + (rB_n \cap (-O_1))$.

Following the same argument for any orthant A_j of a non-empty set $A \subset P$ satisfying (3.1) for all $j = 1, \ldots, 2^n$, we get that, for any $r_1, \ldots, r_{2^n} > 0$ such that $r_j B_n \subset \operatorname{int} P$ for all j, we have

$$\sum_{j=1}^{2^n} \Big(r_j \mu^+(A_j, P) + M_j^\mu(A_j)\Big) \geq n \sum_{j=1}^{2^n} \mu(A_j)^{1-1/n} \mu\big(x_j + (r_j B_n \cap (-O_j))\big)^{1/n},$$

with equality if $A_j = x_j + (r_j B_n \cap (-O_j))$ for all $j = 1, \ldots, 2^n$.

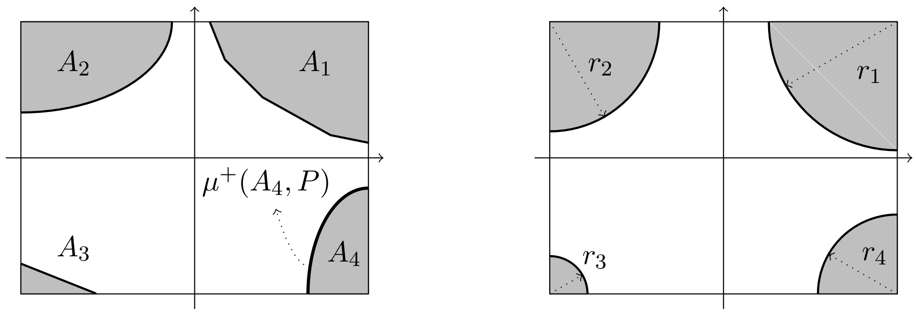

FIGURE 3. Union of orthants A_j of unconditional compact convex sets (left) and the corresponding orthants of balls $x_j + r_j(B_n \cap (-O_j))$ of the same measure (right).

The particular case $r_1 = \cdots = r_{2^n} (=: r)$ of the latter inequality shows that

$$r\mu^+(A, P) + M^\mu(A) \geq n \sum_{j=1}^{2^n} \mu(A_j)^{1-1/n} \mu\big(x_j + (rB_n \cap (-O_j))\big)^{1/n}.$$

In other words: among all unions A of orthants of unconditional compact convex sets embedded in the corners of a fixed centered orthogonal box P (i.e., satisfying (3.1) for all $j = 1, \ldots, 2^n$) *with predetermined measure* $\mu(A_j) = \mu(x_j + (rB_n \cap (-O_j)))$, (union of orthants embedded in the corners of P of) *Euclidean balls rB_n minimize the functional $r\mu^+(A, P) + M^\mu(A)$*.

The main idea of the proof we present here goes back to the classical proof of the Minkowski first inequality that can be found in [20, Theorem 7.2.1]. We refer also the reader to [16, Section 4] and the references therein.

Proof. We consider $L = rB_n$ and we denote by $B = x_1 + L^-$, where $L^- = L \cap (-O_1)$. In the same way, we will write $K^- = K \cap (-O_1)$.

Notice that, for any $\epsilon > 0$ such that $K^- + \epsilon L^- \subset P$, we have that $x_1 + K^- + t_1 L^-$ and $x_1 + K^- + t_2 L^-$ are congruous for all $t_1, t_2 \in [0, \epsilon]$ (since each one-dimensional section of them in the direction of e_i, $i = 1, \ldots, n$, satisfies condition i) in Definition 2.1, with maximum equal to α_i). Then, from the convexity of L^- (and K^-) and using Theorem 1.1, the function $t \mapsto \mu(A + tL^-)^{1/n}$ is concave on $[0, \epsilon]$. This implies that the right derivative of $\mu(A + tL^-)$ at $t = 0$ (possibly infinite) exists (cf. [17, Theorem 23.1]). Similarly, the left derivative of $\mu(x_1 + tK^-)$ at $t = 1$ exists.

Now, we consider the function $f : [0, 1] \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f(t) = \mu\big((1-t)A + tB\big)^{1/n} - \big((1-t)\mu(A)^{1/n} + t\mu(B)^{1/n}\big).$$

By Theorem 1.1 (and from the convexity of both K^- and L^-) f is concave (we notice that the fact of being an unconditional set is closed under convex combinations) and, moreover, $f(0) = f(1) = 0$. Thus, the right derivative of f at $t=0$ exists and furthermore

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_{t=0} f(t) \geq 0 \tag{3.2}$$

with equality if and only if $f(t)=0$ for all $t\in[0,1]$, i.e., if and only if (1.2) holds with equality for all $t\in[0,1]$.

Now, since

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_{t=0} f(t) = \frac{1}{n}\mu(A)^{(1/n)-1}\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_{t=0}\mu\big((1-t)A+tB\big)+\mu(A)^{1/n}-\mu(B)^{1/n},$$

we just must compute the right derivative at 0 of $\mu\big((1-t)A+tB\big)$. Writing $g(r,s)=\mu\big(x_1+r(K^-+sL^-)\big)$, we have

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_{t=0}\mu\big((1-t)A+tB\big) &= \frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_{t=0} g\left(1-t,\frac{t}{1-t}\right)\\ &= -\frac{\mathrm{d}^-}{\mathrm{d}t}\Big|_{t=1}\mu(x_1+tK^-)+\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_{t=0}\mu(A+tL^-)\\ &= M^\mu(x_1,K^-)-n\mu(A)+n\mathrm{W}_1^\mu(A;L^-),\end{aligned}$$

and thus

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_{t=0} f(t) = \frac{1}{n}\mu(A)^{(1/n)-1}\big(M^\mu(x_1,K^-)+n\mathrm{W}_1^\mu(A;L^-)\big)-\mu(B)^{1/n}.$$

Hence, the latter identity, together with (3.2), gives

$$\mathrm{W}_1^\mu(A;L^-)+\frac{1}{n}M^\mu(x_1,K^-)\geq\mu(A)^{1-1/n}\mu(B)^{1/n},$$

with equality if $A=B$.

Finally, from the unconditionality of K^- we clearly have that $\big((A+tL)\cap P\big)=A+tL^-$, which yields $n\mathrm{W}_1^\mu(A;L^-)=r\mu^+(A,P)$. Then, we have

$$r\mu^+(A,P)+M^\mu(x_1,K_1\cap(-O_1))\geq n\mu(A)^{1-1/n}\mu\big(x_1+(rB_n)^-\big)^{1/n},$$

with equality if $A=x_1+(rB_n\cap(-O_1))$. This concludes the proof. □

Acknowledgements. I would like to thank the anonymous referees for their very valuable comments and remarks which have allowed me to improve the presentation of this work. I also thank Prof. M. A. Hernández Cifre for carefully reading the manuscript and her very helpful suggestions during the preparation of it.

References

[1] Barthe, F. "Autour de l'inégalité de Brunn-Minkowski." *Ann. Fac. Sci. Toulouse Math. (6)* 12, no. 2 (2003): 127–178.

[2] Borell, C. "Convex measures on locally convex spaces." *Ark. Mat.* 12 (1974): 239–252.

[3] Borell, C. "Convex set functions in *d*-space." *Period. Math. Hungar.* 6, no. 2 (1975): 111–136.

[4] Brascamp, H. J. and Lieb, E. H. "On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions and with an application to the diffusion equation." *J. Func. Anal.* 22, no. 4 (1976): 366–389.

[5] Bullen, P. S. *Handbook of means and their inequalities*, Mathematics and its Applications vol. 560, Revised from the 1988 original, Kluwer Academic Publishers Group, Dordrecht, 2003.
[6] Colesanti, A., Saorín Gómez, E. and Yepes Nicolás, J. "On a linear refinement of the Prékopa-Leindler Inequality." *Canad. J. Math.* 68, no. 4 (2016): 762–783.
[7] Gardner, R. J. "The Brunn-Minkowski inequality." *Bull. Amer. Math. Soc.* 39, no. 3 (2002): 355–405.
[8] Gardner, R. J. and Zvavitch, A. "Gaussian Brunn-Minkowski inequalities." *Trans. Amer. Math. Soc.* 362, no. 10 (2010): 5333–5353.
[9] Khovanskii, A. and Timorin, V. "On the theory of coconvex bodies." *Discrete Comput. Geom.* 52, no. 4 (2014) 806–823.
[10] Leindler, L. "On a certain converse of Hölder's inequality II." *Acta Sci. Math. (Szeged)* 33, no. 3-4 (1972): 217–223.
[11] Livshyts, G., Marsiglietti, A., Nayar, P. and Zvavitch, A. "On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities." *Trans. Amer. Math. Soc.* 369, no. 12 (2017): 8725–8742.
[12] Marsiglietti, A. "On the improvement of concavity of convex measures." *Proc. Amer. Math. Soc.* 144, no. 2 (2016): 775–786.
[13] Milman, E. and Rotem, L. "Complemented Brunn-Minkowski inequalities for homogeneous and non-homogeneous measures." *Adv. Math.* 262 (2014): 867-908.
[14] Nayar, P. and Tkocz, T. "A note on a Brunn-Minkowski inequality for the Gaussian measure." *Proc. Amer. Math. Soc.* 141 (2013): 4027–4030.
[15] Prékopa, A. "Logarithmic concave measures with application to stochastic programming." *Acta Sci. Math. (Szeged)* 32 (1971): 301–315.
[16] Ritoré, M. and Yepes Nicolás, J. "Brunn-Minkowski inequalities in product metric measure spaces." *Adv. Math.* 325 (2018): 824–863.
[17] Rockafellar, R. T. *Convex analysis*, Princeton University Press, Princeton, New Jersey, 1970.
[18] Saroglou, C. "Remarks on the conjectured log-Brunn-Minkowski inequality." *Geom. Dedicata* 177, no. 1 (2015): 353–365.
[19] Schneider, R. "A Brunn-Minkowski theory for coconvex sets of finite volume." *Adv. Math.* 332 (2018), 199–234.
[20] Schneider, R. *Convex bodies: The Brunn-Minkowski theory*, 2nd expanded ed. Encyclopedia of Mathematics and its Applications, 151. Cambridge: Cambridge University Press, 2014.
[21] Yepes Nicolás, J. "Characterizing the volume via a Brunn-Minkowski type inequality." *Analytic aspects of convexity*, 103–120. Springer INdAM Ser., 25. Cham: Springer, 2018.

Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100-Murcia, Spain
E-mail address: jesus.yepes@um.es