ON ROGERS-SHEPHARD TYPE INEQUALITIES FOR GENERAL MEASURES

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Abstract. In this paper we prove a series of Rogers-Shephard type inequalities for convex bodies when dealing with measures on the Euclidean space with either radially decreasing densities, or quasi-concave densities attaining their maximum at the origin. Functional versions of classical Rogers-Shephard inequalities are also derived as consequences of our approach.

1. Introduction and main results

We denote the length of a vector $x \in \mathbb{R}^n$ by $|x|$. We represent by $B_n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ the n-dimensional Euclidean unit ball, by $\mathbb{S}^{n-1}$ its boundary, and σ will denote the standard surface area measure on $\mathbb{S}^{n-1}$. The n-dimensional volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ or $\mathrm{vol}_n(M)$ if the distinction of the dimension is useful (when integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$). With $\mathrm{int}\, M$, $\mathrm{bd}\, M$ and $\mathrm{conv}\, M$ we denote the interior, boundary and convex hull of M, respectively, and we set $[x, y]$ for $\mathrm{conv}\{x, y\}$, $x, y \in \mathbb{R}^n$. The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathrm{G}(n, i)$, and for $H \in \mathrm{G}(n, i)$, the orthogonal projection of M onto H is denoted by $P_H M$. Moreover, $H^\perp \in \mathrm{G}(n, n-i)$ represents the orthogonal complement of H. Finally, let $\mathcal{K}^n$ be the set of all n-dimensional convex bodies, i.e., compact convex sets with non-empty interior, in $\mathbb{R}^n$. We will frequently refer to [3], [17] and [36] for general references for convex bodies and their properties.

The Minkowski sum of two non-empty sets $A, B \subset \mathbb{R}^n$ denotes the classical vector addition of them, $A + B = \{a + b : a \in A, b \in B\}$, and we write $A - B$ for $A + (-B)$.

One of the most famous relations involving the volume and the Minkowski addition is the Brunn-Minkowski inequality (we refer to [16] for an extensive survey of this inequality). One form of it states that if $K, L \in \mathcal{K}^n$, then

$$\mathrm{vol}(K + L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{1.1}$$

and equality holds if and only if K and L are homothetic.

2010 *Mathematics Subject Classification.* Primary 52A40, 28A25; Secondary 52A20.

Key words and phrases. Rogers-Shephard type inequalities, quasi-concave density, radially decreasing density, functional inequalities.

First author is supported by MINECO/FEDER project MTM2016-77710-P. Second and fourth authors are supported by MINECO/FEDER project MTM2015-65430-P and "Programa de Ayudas a Grupos de Excelencia de la Región de Murcia", Fundación Séneca, 19901/GERM/15. Third and fifth authors are supported in part by the U.S. National Science Foundation Grant DMS-1101636. Fifth author is supported in part by la Comue Université Paris-Est.

The Brunn-Minkowski inequality was generalized to different types of measures, including the case of log-concave measures [24, 31], a very powerful generalization to the case of Gaussian measures [9, 10, 13, 14, 37], to p-concave measures and many other extensions (see e.g. [8, 11]). It is interesting to note that it was proved by Borell [7, 8] that most of such generalizations would require a p-concavity assumption on the underlined measure and its density (see (1.6) below for the precise definition). Following those works, recently, many classical results in Convex Geometry were generalized to the case of log-concave (and in some cases p-concave) functions. We mention, among others, the Blaschke-Santaló inequality [4, 5, 15], the Bourgain-Milman and the reverse Brunn-Minkowski inequality [21], the general works on duality and volume [5, 6], as well as the Grünbaum inequality [28, 29] and others [18, 26, 27, 30, 32].

In the particular case when $L = -K$, (1.1) gives

$$\mathrm{vol}(K - K) \geq 2^n \mathrm{vol}(K),$$

with equality if and only if K is centrally symmetric, i.e., there exists a point $x \in \mathbb{R}^n$ such that $K - x = -(K - x)$. An upper bound for the volume of $K - K$ is given by the Rogers-Shephard inequality, originally proven in [34, Theorem 1]. For more details about this inequality, we also refer the reader to [36, Section 10.1] or [3].

Theorem A (The Rogers-Shephard inequality)**.** *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{vol}(K - K) \leq \binom{2n}{n} \mathrm{vol}(K), \tag{1.2}$$

with equality if and only if K is a simplex.

Similarly to the Brunn-Minkowski inequality (1.1), it is natural to wonder about the possibility of extending (1.2) for measures associated to certain densities. The most natural candidates would be the classes of p-concave measures. Nevertheless, it was noticed recently that a number of results in Convex Geometry and Geometric Tomography can be generalized to a class of measures whose densities have no concavity assumption. This includes the solution of the Busemann-Petty problem for general measures [38], the Koldobsky slicing inequality [22, 23, 19, 20], as well as Shephard's problem for general measures [25].

First we observe that one cannot expect to obtain

$$\mu(K - K) \leq \binom{2n}{n} \mu(K) \tag{1.3}$$

without having certain control on the 'position' of the body K. Indeed, it is enough to consider the standard n-dimensional Gaussian measure γ_n given by

$$\mathrm{d}\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{\frac{-|x|^2}{2}} \mathrm{d}x,$$

and $K = x + B_n$ for $|x|$ large enough. In this case it is clear that $\gamma_n(K - K) = \gamma_n(2B_n) > 0$, whereas $\gamma_n(K)$ can be arbitrarily small.

One option to get control, on the right-hand side of (1.3) might be to exchange $\mu(K)$ with a mean of the measures of all the translated copies of K with respect to $-K$. To this end, given a measure μ on $\mathbb{R}^n$, we define its *translated-average* $\overline{\mu}$ as

$$\overline{\mu}(K) = \frac{1}{\mathrm{vol}(K)} \int_K \mu(-y + K) \, \mathrm{d}y,$$

for any $K \in \mathcal{K}^n$. With this notion, our first main result reads as follows.

Theorem 1.1. *Let $K\in\mathcal{K}^n$. Let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x)=\phi(x)\,\mathrm{d}x$, where $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is radially decreasing. Then*

$$\mu(K-K)\leq\binom{2n}{n}\min\bigl\{\overline{\mu}(K),\overline{\mu}(-K)\bigr\}. \tag{1.4}$$

Moreover, if ϕ is continuous at the origin then equality holds in (1.4) if and only if μ is a constant multiple of the Lebesgue measure on $K-K$ and K is a simplex.

A function $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is said to be radially decreasing if $\phi(tx)\geq\phi(x)$ for any $t\in[0,1]$ and any point $x\in\mathbb{R}^n$.

A lower bound for $\mu(K-K)$ when the density function of μ is even and p-concave (see the definition below), $p\geq-1/n$, can be directly obtained from the results by Borell and Brascamp-Lieb [8, 11]:

$$\mu(K-K)\geq\mu(2K). \tag{1.5}$$

Here we extend (1.5) to the case of measures with even and quasi-concave densities (see Theorem 2.2).

We recall that a function $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is p-concave, for $p\in\mathbb{R}\cup\{\pm\infty\}$, if

$$\phi\bigl((1-\lambda)x+\lambda y\bigr)\geq M_p\bigl(\phi(x),\phi(y),\lambda\bigr) \tag{1.6}$$

for all $x,y\in\mathbb{R}^n$ and any $\lambda\in(0,1)$. Here M_p denotes the *p-mean* of two non-negative numbers:

$$M_p(a,b,\lambda)=\begin{cases}\bigl((1-\lambda)a^p+\lambda b^p\bigr)^{1/p}, & \text{if } p\neq0,\pm\infty,\\ a^{1-\lambda}b^{\lambda} & \text{if } p=0,\\ \max\{a,b\} & \text{if } p=\infty,\\ \min\{a,b\} & \text{if } p=-\infty;\end{cases}$$

for $ab>0$; $M_p(a,b,\lambda)=0$, when $ab=0$ and $p\in\mathbb{R}\cup\{\pm\infty\}$. A 0-concave function is usually called *log-concave* whereas a $(-\infty)$-concave function is called *quasi-concave*. Quasi-concavity is equivalent to the fact that the superlevel sets

$$\mathcal{C}_t(\phi)=\bigl\{x\in\operatorname{supp}\phi:\phi(x)\geq t\|\phi\|_\infty\bigr\} \tag{1.7}$$

are convex for $t\in[0,1]$. Here $\operatorname{supp}\phi$ denotes the support of ϕ, i.e., the closure of the set $\bigl\{x\in\mathbb{R}^n:\phi(x)>0\bigr\}$, and with $\|\cdot\|_\infty$ we mean

$$\|\phi\|_\infty=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\phi(x)=\inf\Bigl\{t\in\mathbb{R}:\mathrm{vol}\bigl(\{x\in\mathbb{R}^n:\phi(x)>t\}\bigr)=0\Bigr\}.$$

We notice that if ϕ is p-concave, then $\operatorname{supp}\phi$ is a closed convex set. Furthermore, if a function ϕ is quasi-concave and such that $\max_{x\in\mathbb{R}^n}\phi(x)=\phi(0)$ then it is radially decreasing.

Although the Rogers-Shephard inequality (1.2) has been recently extended to the functional setting (see e.g. [1, 2, 12] and the references therein), there seems to be no direct way to derive inequality (1.4) from the above-mentioned functional versions just by considering the function $\chi_K\,\phi$, where ϕ is the density of the given measure, and χ_K is the characteristic function of a convex body K (see Remark 2.3). More precisely, in [12, Theorems 4.3 and 4.5], Colesanti extended (1.2) to the more general functional inequality

$$\int_{\mathbb{R}^n}\sup_{x=x_1+x_2}\bigl(f(x_1)^p+f(-x_2)^p\bigr)^{1/p}\,\mathrm{d}x\leq\binom{2n}{n}\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x, \tag{1.8}$$

for any p-concave integrable function, with $p \in [-\infty, 0)$. Here, the case $p = -\infty$ has to be understood as $\min\bigl\{f(x_1), f(-x_2)\bigr\}$. In Section 2 we will also generalize (1.8) to general measures (see Theorem 2.3).

In [35], in addition to $K-K$, Rogers and Shephard considered two other centrally symmetric convex bodies associated with K. The first one is

$$CK = \bigl\{(x,\theta) \in \mathbb{R}^{n+1} : x \in (1-\theta)K + \theta(-K),\, \theta \in [0,1]\bigr\},$$

whose volume is given by

$$\mathrm{vol}_{n+1}(CK) = \int_0^1 \mathrm{vol}\bigl((1-\theta)K + \theta(-K)\bigr)\,\mathrm{d}\theta.$$

The second one is just $\mathrm{conv}\bigl(K \cup (-K)\bigr)$. The relation of the volumes of CK and $\mathrm{conv}\bigl(K \cup (-K)\bigr)$ to the volume of K was proved in [35]:

Theorem B. *Let $K \in \mathcal{K}^n$ be a convex body containing the origin. Then*

$$\int_0^1 \mathrm{vol}\bigl((1-\theta)K + \theta(-K)\bigr)\,\mathrm{d}\theta \le \frac{2^n}{n+1}\,\mathrm{vol}(K), \tag{1.9}$$

with equality if and only if K is a simplex. Moreover,

$$\mathrm{vol}\Bigl(\mathrm{conv}\bigl(K \cup (-K)\bigr)\Bigr) \le 2^n\,\mathrm{vol}(K), \tag{1.10}$$

with equality if and only if K is a simplex with the origin as a vertex.

Here we will show an analog of the above result in the setting of measures with radially decreasing density:

Theorem 1.2. *Let $K \in \mathcal{K}^n$ be a convex body containing the origin and let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ is radially decreasing. Then*

$$\int_0^1 \mu\bigl((1-\theta)K + \theta(-K)\bigr)\,\mathrm{d}\theta \le \frac{2^n}{n+1} \sup_{\substack{y\in K\\ \theta\in(0,1]}} \frac{\mu\bigl((1-\theta)y - \theta K\bigr)}{\theta^n} \tag{1.11}$$

and

$$\mu\Bigl(\mathrm{conv}\bigl(K \cup (-K)\bigr)\Bigr) \le 2^n \sup_{\substack{y\in K\\ \theta\in(0,1]}} \frac{\mu\bigl((1-\theta)y - \theta K\bigr)}{\theta^n}. \tag{1.12}$$

Moreover, if ϕ is continuous at the origin then equality holds in (1.11) *if and only if μ is a constant multiple of the Lebesgue measure on* $\mathrm{conv}\bigl(K \cup (-K)\bigr)$ *and K is a simplex, and equality holds in* (1.12) *if and only if μ is a constant multiple of the Lebesgue measure on* $\mathrm{conv}\bigl(K \cup (-K)\bigr)$ *and K is a simplex with the origin as a vertex.*

We note that the upper bounds in Theorem 1.2 are bounded and can be restated using $\|\phi\|_\infty \mathrm{vol}(K)$; indeed, $\mu\bigl((1-\theta)y - \theta K\bigr)/\theta^n$ is bounded from above by $\|\phi\|_\infty \mathrm{vol}(K)$.

In [35, Theorem 1], Rogers and Shephard also gave the following lower bound for the volume of K in terms of the volumes of a projection and a maximal section of K:

Theorem C. *Let $k \in \{1, \dots, n-1\}$, $H \in \mathrm{G}(n, n-k)$ and $K \in \mathcal{K}^n$. Then*

$$\mathrm{vol}_{n-k}\big(P_H K\big) \max_{x_0 \in H} \mathrm{vol}_k\big(K \cap \big(x_0 + H^\perp\big)\big) \leq \binom{n}{k} \mathrm{vol}(K). \tag{1.13}$$

In this paper we will show that the above result remains true for products of measures associated to quasi-concave densities, provided that $P_H K \subset K$, i.e., $P_H K = K \cap H$. The assumption on the projection is necessary, as pointed out in Example 4.1. In particular, this hypothesis does not allow one to prove Theorem 1.2 by directly following the proof of Theorem B (see [35, Theorems 2 and 3]): there, the authors constructed a suitable higher dimensional set to which (1.13) was applied. This will be not possible here.

Before stating the result, we fix the following notation: given a convex body K and $x \in P_H K$, we write $K(x) = (K - x) \cap H^\perp$. We will use the definition of superlevel set $\mathcal{C}_t(\phi)$ given by (1.7).

Theorem 1.3. *Let $k \in \{1, \dots, n-1\}$ and $H \in \mathrm{G}(n, n-k)$. Given a continuous at the origin and quasi-concave function $\phi_k : \mathbb{R}^k \longrightarrow [0, \infty)$ with $\|\phi_k\|_\infty = \phi_k(0)$ and a radially decreasing function $\phi_{n-k} : \mathbb{R}^{n-k} \longrightarrow [0, \infty)$, let $\mu_n = \mu_{n-k} \times \mu_k$ be the product measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu_{n-k}(x) = \phi_{n-k}(x)\,\mathrm{d}x$ and $\mathrm{d}\mu_k(y) = \phi_k(y)\,\mathrm{d}y$. Let $K \in \mathcal{K}^n$ with $P_H K \subset K$ and so that $\mathrm{vol}_k\big(\mathcal{C}_t(\phi_k) \cap K(x)\big)$ attains its maximum at $x = 0$ for every $t \in (0, 1)$. Then*

$$\mu_{n-k}\big(P_H K\big)\mu_k\big(K \cap H^\perp\big) \leq \binom{n}{k} \mu_n(K). \tag{1.14}$$

The above assumption on the maximal section $K(0)$ of K can be omitted when the density of the product measure is also quasi-concave, as shown in Theorem 4.1, which is a straightforward consequence of the following functional version of (1.13).

Theorem 1.4. *Let $k \in \{1, \dots, n-1\}$ and $H \in \mathrm{G}(n, n-k)$. Let $f : \mathbb{R}^n \longrightarrow [0, \infty)$ be a bounded quasi-concave function such that $\mathrm{vol}_k\big(\mathcal{C}_t(f) \cap (x + H^\perp)\big)$, $x \in H$, attains its maximum at $x = 0$ for every $t \in (0, 1)$, and let $g : H \longrightarrow [0, \infty)$ be a radially decreasing function. Then,*

$$\int_H g(x) P_H f(x)\,\mathrm{d}x \int_{H^\perp} f(y)\,\mathrm{d}y \leq \binom{n}{k} \|f\|_\infty \int_{\mathbb{R}^n} g(P_H x) f(x)\,\mathrm{d}x.$$

Here, the projection function $P_H f : H \longrightarrow [0, \infty)$ of f is defined by $P_H f(x) = \sup_{y \in H^\perp} f(x + y)$.

In the particular case of a log-concave integrable function f, this result has been recently obtained in [1, Theorem 1.1].

The paper is organized as follows. Section 2 is mainly devoted to the proofs of Theorems 1.1 and 1.2 as well as the functional analogs of these results. We start Section 3 by deriving a general result for functions with certain concavity conditions, which will play a relevant role along the manuscript. As a consequence of this result we prove, in particular, Theorem 1.4. Next, in Section 4, we study Rogers-Shephard type inequalities for measures with quasi-concave densities, and prove Theorem 1.3. Finally, in Section 5, we present another Rogers-Shephard type inequality when assuming a further concavity for the density of the involved measure.

2. Rogers-Shephard type inequalities for measures with radially decreasing densities

2.1. **The case of convex sets.** As pointed out in the previous section, one cannot expect to obtain (1.3) without having control on the translations of the set K. Moreover, certain requirements on the density of the measure μ must be made (see also the comments after Remark 2.2 and Example 2.1). To this regard, in Section 4 we will show that one may consider quasi-concave densities with maximum at the origin. In this setting, we will also obtain other Rogers-Shephard type inequalities.

Let us now follow a different approach. First we will prove an extension of (1.2) for the more general case of radially decreasing densities, collected in Theorem 1.1. Before showing it, we need the following auxiliary result.

Lemma 2.1. *Let $\phi : [0,\infty) \longrightarrow [0,\infty)$ be a decreasing function and let $n, m \in \mathbb{N}$. Then, for every $x \in (0,\infty)$,*

$$\int_0^x \left(1 - \frac{t}{x}\right)^n t^{m-1}\phi(t)\,\mathrm{d}t \geq \binom{n+m}{n}^{-1} \int_0^x t^{m-1}\phi(t)\,\mathrm{d}t,$$

with equality if and only if ϕ is constant on $(0,x)$.

Proof. Considering the function $F : (0,\infty) \longrightarrow [0,\infty)$ given by

$$F(x) = \binom{n+m}{n}^{-1} \int_0^x t^{m-1}\phi(t)\,\mathrm{d}t - \int_0^x \left(1 - \frac{t}{x}\right)^n t^{m-1}\phi(t)\,\mathrm{d}t,$$

we need to show that it is non-positive.

Expanding the binomial $(1 - t/x)^n$ we may assert on one hand that $F(x) \to 0$ as $x \to 0^+$. On the other hand, and jointly with Lebesgue's differentiation theorem, we get that the derivative of F exists for almost every $x \in (0,\infty)$ and further

$$F'(x) = \binom{n+m}{n}^{-1} x^{m-1}\phi(x) - n\int_0^x \left(1 - \frac{t}{x}\right)^{n-1} \frac{t^m}{x^2}\,\phi(t)\,\mathrm{d}t.$$

Now, applying the change of variable $u = t/x$, we get

$$n\int_0^x \left(1 - \frac{t}{x}\right)^{n-1} t^m\,\mathrm{d}t = \frac{n\,\Gamma(n)\Gamma(m+1)}{\Gamma(n+m+1)} x^{m+1} = \binom{n+m}{n}^{-1} x^{m+1},$$

where Γ represents the Gamma function. This together with the fact that ϕ is decreasing implies that $F'(x) \leq 0$, with equality if and only if ϕ is constant on $(0,x)$.

Since F is absolutely continuous on every interval $[a,b] \subset (0,\infty)$, because it arises as a finite sum of products of absolutely continuous functions,

$$F(x) = F(a) + \int_a^x F'(s)\,\mathrm{d}s \leq F(a)$$

for all $x > 0$ and any $0 < a \leq x$. Taking into account that $\lim_{a\to 0^+} F(a) = 0$ we then have

$$F(x) = \int_0^x F'(s)\,\mathrm{d}s \leq 0,$$

with equality if and only if $F' \equiv 0$ almost everywhere or, equivalently, when ϕ is constant on $(0,x)$. $\square$

Next we prove Theorem 1.1. We follow the idea of the original proof of the Rogers-Shephard inequality ([34]), with the main difference of the application of Lemma 2.1 in (2.4).

Proof of Theorem 1.1. Let $f:\mathbb{R}^n\longrightarrow[0,\infty)$ be the function given by

$$f(x)=\mathrm{vol}\big(K\cap(x+K)\big).$$

Observe that $\operatorname{supp} f=K-K$ and f vanishes on $\mathrm{bd}(K-K)$. Furthermore, using the Brunn-Minkowski inequality (1.1) together with the inclusion

$$K\cap\big[(1-\lambda)x+\lambda y+K\big]\supset(1-\lambda)\big[K\cap(x+K)\big]+\lambda\big[K\cap(y+K)\big], \tag{2.1}$$

which holds for all $\lambda\in[0,1]$ and $x,y\in K-K$, we get that f is $(1/n)$-concave.

On the one hand, by Fubini's theorem, we have

$$\begin{aligned}\int_{K-K}f(x)\,\mathrm{d}\mu(x)&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_K(y)\chi_{y-K}(x)\,\phi(x)\,\mathrm{d}y\,\mathrm{d}x\\&=\int_K\mu(y-K)\,\mathrm{d}y=\mathrm{vol}(K)\,\overline{\mu}(-K).\end{aligned} \tag{2.2}$$

On the other hand, we define the function $g:K-K\longrightarrow[0,\infty)$ given by

$$g(x)=f(0)\left[1-\frac{|x|}{\rho_{K-K}\big(x/|x|\big)}\right]^n,\quad\text{for every } x\neq0,$$

and $g(0)=f(0)$, where

$$\rho_L(u)=\max\{\rho\geq0:\rho u\in L\},\quad u\in\mathbb{S}^{n-1},$$

stands for the radial function of $L\in\mathcal{K}^n$. Notice that $g^{1/n}$ is affine on $\big[0,\rho_{K-K}(u)u\big]$, for all $u\in\mathbb{S}^{n-1}$, and so $g(0)^{1/n}=f(0)^{1/n}$ and

$$g\big(\rho_{K-K}(u)u\big)^{1/n}=0=f\big(\rho_{K-K}(u)u\big)^{1/n}.$$

Hence, since $f^{1/n}$ is concave, it follows that $f^{1/n}\geq g^{1/n}$ on $\big[0,\rho_{K-K}(u)u\big]$. Therefore, using polar coordinates, we have

$$\begin{aligned}\int_{K-K}f(x)\,\mathrm{d}\mu(x)&=\int_{\mathbb{S}^{n-1}}\int_0^{\rho_{K-K}(u)}r^{n-1}f(ru)\phi(ru)\,\mathrm{d}r\,\mathrm{d}\sigma(u)\\&\geq f(0)\int_{\mathbb{S}^{n-1}}\int_0^{\rho_{K-K}(u)}\left(1-\frac{r}{\rho_{K-K}(u)}\right)^n r^{n-1}\phi(ru)\,\mathrm{d}r\,\mathrm{d}\sigma(u).\end{aligned} \tag{2.3}$$

Now, from (2.3) and Lemma 2.1 we obtain

$$\begin{aligned}\int_{K-K}f(x)\,\mathrm{d}\mu(x)&\geq\frac{1}{\binom{2n}{n}}f(0)\int_{\mathbb{S}^{n-1}}\int_0^{\rho_{K-K}(u)}r^{n-1}\phi(ru)\,\mathrm{d}r\,\mathrm{d}\sigma(u)\\&=\frac{1}{\binom{2n}{n}}\mathrm{vol}(K)\mu(K-K),\end{aligned} \tag{2.4}$$

which, together with (2.2), yields

$$\mu(K-K)\leq\binom{2n}{n}\overline{\mu}(-K).$$

By replacing K with $-K$, we obtain the desired inequality.

Finally we notice that equality holds in (1.4) only if there is equality in (2.4). This implies, by Lemma 2.1, that $\phi(ru)$ is constant on $\big(0, \rho_{K-K}(u)\big)$ for σ-almost every $u \in \mathbb{S}^{n-1}$. Since ϕ is continuous at the origin, μ is a constant multiple of the Lebesgue measure on $K - K$ and, by Theorem A, K is a simplex. The converse immediately follows from Theorem A. □

Remark 2.1. *From the proof of the equality case in the above result (and the corresponding one of Lemma 2.1), we notice that the assumption of continuity at the origin for ϕ is necessary in order to 'recover' the Lebesgue measure (up to a constant). Indeed, one could consider a simplex K and a function ϕ that is constant on $\big(0, \rho_{K-K}(u)\big)$ for every $u \in \mathbb{S}^{n-1}$, but not necessarily constant on $K - K$, and thus* (1.4) *would hold with equality.*

The next theorem is obtained just by repeating the same argument given in the proof of Theorem 1.1, but replacing $-K$ with L.

Theorem 2.1. *Let $K, L \in \mathcal{K}^n$ and let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0, \infty)$ is radially decreasing. Then*

$$\mu(K + L)\mathrm{vol}\big(K \cap (-L)\big) \leq \binom{2n}{n} \int_K \mu(x + L)\mathrm{d}x.$$

Remark 2.2. *As a straightforward consequence of Theorem 1.1, we get the following statement. Let $K \in \mathcal{K}^n$ and let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0, \infty)$ is radially decreasing. Then*

$$\mu(K - K) \leq \binom{2n}{n} \min\left\{\sup_{x\in\mathbb{R}^n} \mu(x + K), \sup_{x\in\mathbb{R}^n} \mu(x - K)\right\}. \tag{2.5}$$

The above fact trivially holds in dimension $n = 1$ for an arbitrary measure. Indeed, given $K = [a, b]$, then

$$\begin{aligned}\mu(K - K) = \mu\big([a - b, b - a]\big) &= \mu\big([a, b] - a\big) + \mu\big([a, b] - b\big)\\ &\leq 2 \min\left\{\sup_{x\in\mathbb{R}} \mu(x + K), \sup_{x\in\mathbb{R}} \mu(x - K)\right\}.\end{aligned}$$

However, in dimension $n \geq 2$ the radial decay assumption cannot be omitted, as the following example shows.

Example 2.1. *Fix $0 < \varepsilon < \delta < 2$. Consider the measure μ on $\mathbb{R}^2$ with density*

$$\phi(x) = \begin{cases} 1 & \text{if } x \in \delta B_2 \cup \big(2B_2 \setminus (2 - \varepsilon)B_2\big), \\ 0 & \text{otherwise} \end{cases}$$

(see Figure 1). Then

$$\mu(B_2 - B_2) > 6 \sup_{x\in\mathbb{R}^2} \mu(x + B_2). \tag{2.6}$$

Note that (2.6) *contradicts* (2.5)*. Indeed, on the one hand,*

$$\mu(B_2 - B_2) = \mu(2B_2) = \pi\delta^2 + \big(4 - (2 - \varepsilon)^2\big)\pi = 4\pi\varepsilon + \pi(\delta^2 - \varepsilon^2).$$

On the other hand, we note that we need at least 6 copies of the unit disk in order to cover $\mathrm{bd}(2B_2)$*, which can be seen by considering a regular hexagon inscribed in* $2B_2$ *(see Figure 1). Moreover, if we would cover* $\mathrm{bd}(2B_2)$ *with exactly* 6 *translated*

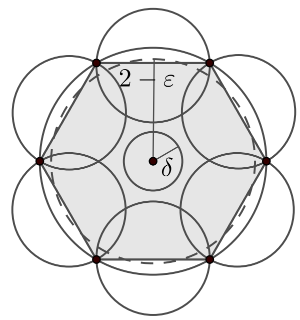

FIGURE 1. Constructing a measure for which (2.5) does not hold.

copies of B_2, then the covering discs would stay away from the origin. Thus, for $\varepsilon > 0$ small enough,

$$\sup_{x\in\mathbb{R}^2} \operatorname{vol}\Big((x+B_2)\cap\big(2B_2\setminus(2-\varepsilon)B_2\big)\Big) = \frac{1}{6}4\pi\varepsilon + o(\varepsilon).$$

Taking, e.g., $\delta = \sqrt{\varepsilon}/100$ we get, for ε small enough, that $\delta > \varepsilon$, and also that $4\pi\varepsilon/6 > \pi\delta^2$ and $o(\varepsilon) < \delta^2$. Thus

$$\begin{aligned} 6\sup_{x\in\mathbb{R}^2} \mu(x+B_2) = 6\sup_{x\in\mathbb{R}^2} \operatorname{vol}\Big((x+B_2)\cap\big(2B_2\setminus(2-\varepsilon)B_2\big)\Big) &= 4\pi\varepsilon + o(\varepsilon) \\ &< 4\pi\varepsilon + \pi(\delta^2-\varepsilon^2). \end{aligned}$$

Moreover, since $\sup_{x\in\mathbb{R}^2}\mu(x+B_2) > \overline{\mu}(B_2)$, this example shows that the radial decay assumption is also needed in Theorem 1.1.

Regarding a reverse inequality for Theorem 1.1 (or (2.5)), we have the following result, which extends (1.5).

Theorem 2.2. *Let $K\in\mathcal{K}^n$. Let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is an even quasi-concave function. Then*

$$\mu(K-K)\geq\mu(2K). \tag{2.7}$$

Equality holds in (2.7) only if $K\cap(\operatorname{supp}\phi)/2$ is centrally symmetric. Moreover, if K is centrally symmetric with respect to the origin, then equality holds in (2.7).

Proof. We write $\overline{K}_t = (2K)\cap\mathcal{C}_t(\phi)$ for every $t\in[0,1]$. On the one hand, by Fubini's theorem, we have

(2.8)
$$\begin{aligned} \mu(2K) = \int_{2K}\phi(x)\,\mathrm{d}x = \|\phi\|_\infty\int_{2K}\int_0^{\frac{\phi(x)}{\|\phi\|_\infty}}\mathrm{d}t\,\mathrm{d}x = \|\phi\|_\infty\int_0^1\int_{2K}\chi_{\mathcal{C}_t(\phi)}(x)\,\mathrm{d}x\,\mathrm{d}t \\ = \|\phi\|_\infty\int_0^1\operatorname{vol}\big(\overline{K}_t\big)\,\mathrm{d}t \leq \|\phi\|_\infty\,2^{-n}\int_0^1\operatorname{vol}\big(\overline{K}_t-\overline{K}_t\big)\,\mathrm{d}t, \end{aligned}$$

where in the last inequality we have used the Brunn-Minkowski inequality (cf. (1.1)).

On the other hand, since ϕ is quasi-concave and even, then $\mathcal{C}_t(\phi)$ is convex and centrally symmetric (with respect to the origin), and hence $\overline{K}_t-\overline{K}_t\subset(2K-2K)\cap$

$2\mathcal{C}_t(\phi) = 2\big((K-K)\cap\mathcal{C}_t(\phi)\big)$. Thus, we get

$$\begin{aligned}\mu(2K) &\le \|\phi\|_\infty\, 2^{-n}\int_0^1 \mathrm{vol}\big(\overline{K}_t - \overline{K}_t\big)\,\mathrm{d}t \le \|\phi\|_\infty \int_0^1 \mathrm{vol}\big((K-K)\cap\mathcal{C}_t(\phi)\big)\,\mathrm{d}t\\ &= \|\phi\|_\infty \int_0^1\int_{\mathbb{R}^n} \chi_{(K-K)\cap\mathcal{C}_t(\phi)}(x)\,\mathrm{d}x\,\mathrm{d}t = \mu(K-K).\end{aligned}$$

For the equality case, we note that the identity $\mu(2K) = \mu(K-K)$ implies that (2.8) holds with equality, and thus $\mathrm{vol}\big(\overline{K}_t\big) = 2^{-n}\mathrm{vol}\big(\overline{K}_t - \overline{K}_t\big)$ for almost every $t\in[0,1]$. Then, there exists a decreasing sequence $(t_m)_m \subset [0,1]$ with $t_m \to 0$ and such that $\mathrm{vol}\big(\overline{K}_{t_m}\big) = 2^{-n}\mathrm{vol}\big(\overline{K}_{t_m} - \overline{K}_{t_m}\big)$ for all $m\in\mathbb{N}$. Therefore, since the boundary of a convex set has null (Lebesgue) measure, we get

(2.9)
$$\begin{aligned}\mathrm{vol}\big((2K)\cap\operatorname{supp}\phi\big) &= \mathrm{vol}\left(\bigcup_{m=1}^{\infty}\overline{K}_{t_m}\right) = \lim_m \mathrm{vol}\big(\overline{K}_{t_m}\big) = \lim_m 2^{-n}\mathrm{vol}\big(\overline{K}_{t_m} - \overline{K}_{t_m}\big)\\ &= 2^{-n}\mathrm{vol}\left(\bigcup_{m=1}^{\infty}\big(\overline{K}_{t_m} - \overline{K}_{t_m}\big)\right)\\ &= 2^{-n}\mathrm{vol}\Big(\big((2K)\cap\operatorname{supp}\phi\big) - \big((2K)\cap\operatorname{supp}\phi\big)\Big).\end{aligned}$$

Since $\operatorname{supp}\phi$ is an n-dimensional convex set containing the origin then $\mu(2K) = \mu(K-K) > 0$, and so $\mathrm{vol}\big((2K)\cap\operatorname{supp}\phi\big) > 0$. Therefore (2.9) implies that $(2K)\cap\operatorname{supp}\phi$ is centrally symmetric. The sufficient condition is evident. □

If we apply (2.7) to the set $K' = K + x/2$ then $\mu(K-K) \ge \sup_{x\in\mathbb{R}^n}\mu(x+2K)$ also holds. We observe, however, that we cannot expect a general reverse inequality for (2.5) in the non-even case, as the following example shows.

Example 2.2. *Let $\theta > 0$ and consider $W_\theta = \big\{r(\cos t, \sin t) : 0 \le t \le \theta,\, r \ge 0\big\} \subset \mathbb{R}^2$. Let μ_θ be the measure on $\mathbb{R}^2$ with density $\phi_\theta(x) = \chi_{W_\theta}(x)$ (see Figure 2).*

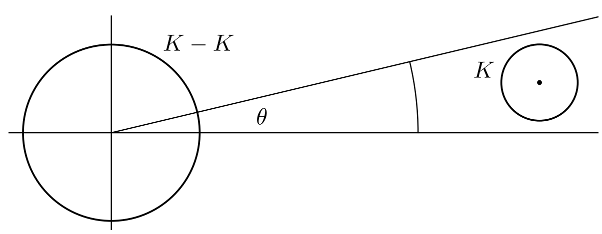

FIGURE 2. A construction for which $\mu(K-K) \to 0$.

By letting $\theta \to 0$, we can move a set K far enough, but keeping the measure of the shifts of K constant, while the measure of $K-K$ will be arbitrarily small. So the left-hand side of (2.5) tends to zero whereas the right-hand side is fixed.

A way to strengthen inequality (2.5) would be to replace $\mu(K-K)$ by the quantity $\sup_{\omega\in\mathbb{R}^n}\mu(K-K+\omega)$:

Question: Given a measure μ on $\mathbb{R}^n$, is it true that for every $K\in\mathcal{K}^n$

$$\sup_{\omega\in\mathbb{R}^n}\mu(K-K+\omega)\leq\binom{2n}{n}\min\left\{\sup_{x\in\mathbb{R}^n}\mu(x+K),\,\sup_{x\in\mathbb{R}^n}\mu(x-K)\right\}?$$

The following result partially solves this question, in the setting of quasi-concave densities, by exploiting the approach carried out in the proof of Theorem 1.1. The idea relies on the possibility of finding a point, for each translated copy of $K-K$, from which the density is radially decreasing over the given translation of $K-K$. The negative counterpart is the apparent necessity of including a factor jointly with the measure of the shift of $K-K$. Nevertheless, we observe that the supremum on the right-hand side can be taken over K. In Section 4, we will provide a different solution to this issue (see Theorem 4.2).

Proposition 2.1. *Let $K\in\mathcal{K}^n$ and let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x)=\phi(x)\,\mathrm{d}x$, where $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is a quasi-concave function whose restriction to its support is continuous. Then, for every $\omega\in\mathbb{R}^n$,*

$$c(\omega)\mu(K-K+\omega)\leq\binom{2n}{n}\sup_{y\in K}\mu(y+\omega-K),\tag{2.10}$$

where $c(\omega)=\mathrm{vol}\big(K\cap(\omega'-\omega+K)\big)\mathrm{vol}(K)^{-1}$, and $\omega'\in K-K+\omega$ is such that $\phi(\omega')=\max_{x\in K-K+\omega}\phi(x)$. Moreover, equality holds for some $\omega_0\in\mathbb{R}^n$ if and only if μ is a constant multiple of the Lebesgue measure on $K-K+\omega_0$, $c(\omega_0)=1$ and K is a simplex.

Proof. Let $f:\mathbb{R}^n\longrightarrow[0,\infty)$ be defined as $f(x)=\mathrm{vol}\big(K\cap(x-\omega+K)\big)$. As before, we get that $\operatorname{supp} f=K-K+\omega$ and f is $(1/n)$-concave (see (1.1) and (2.1)). On the one hand, by Fubini's theorem, we have

$$\int_{K-K+\omega}f(x)\,\mathrm{d}\mu(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_{K}(y)\chi_{y+\omega-K}(x)\,\phi(x)\,\mathrm{d}y\,\mathrm{d}x=\int_K\mu(y+\omega-K)\,\mathrm{d}y.\tag{2.11}$$

On the other hand, from the continuity of ϕ on $\operatorname{supp}\phi$, we know that there exists a point $\omega'\in(K-K+\omega)\cap\operatorname{supp}\phi$, which is a compact set, such that $\phi(\omega')=\max_{x\in K-K+\omega}\phi(x)$. This, together with the quasi-concavity of ϕ, implies that it radially decays from ω' on $K-K+\omega$, i.e., $\phi\big(\omega'+t(x-\omega')\big)\geq\phi(x)$ for any $t\in[0,1]$ and all $x\in K-K+\omega$.

Now we define the function $g:K-K+\omega\longrightarrow[0,\infty)$ given by

$$g(x)=f(\omega')\left[1-\frac{|x-\omega'|}{\rho_{K-K+\omega-\omega'}\big((x-\omega')/|x-\omega'|\big)}\right]^n,\quad\text{for every } x\neq\omega',$$

and $g(\omega')=f(\omega')$. Since $f^{1/n}$ is concave, it follows that $f^{1/n}\geq g^{1/n}$ on $\big[\omega',\omega'+\rho_{K-K+\omega-\omega'}(u)u\big]$, and so, via the polar coordinates $z=x-\omega'=ru$, we get

$$\begin{aligned}\int_{K-K+\omega}f(x)\,\mathrm{d}\mu(x)&=\int_{K-K+\omega-\omega'}f(\omega'+z)\phi(\omega'+z)\,\mathrm{d}z\\&=\int_{\mathbb{S}^{n-1}}\int_0^{\rho_{K-K+\omega-\omega'}(u)}r^{n-1}f(\omega'+ru)\phi(\omega'+ru)\,\mathrm{d}r\,\mathrm{d}\sigma(u)\\&\geq f(\omega')\int_{\mathbb{S}^{n-1}}\int_0^{\rho_{K-K+\omega-\omega'}(u)}\left[1-\frac{r}{\rho_{K-K+\omega-\omega'}(u)}\right]^n r^{n-1}\phi(\omega'+ru)\,\mathrm{d}r\,\mathrm{d}\sigma(u).\end{aligned}$$

Then Lemma 2.1 yields

$$\begin{aligned}\int_{K-K+\omega} f(x)\,\mathrm{d}\mu(x) &\geq \frac{f(\omega')}{\binom{2n}{n}} \int_{\mathbb{S}^{n-1}} \int_0^{\rho_{K-K+\omega-\omega'}(u)} r^{n-1}\phi(\omega'+ru)\,\mathrm{d}r\,\mathrm{d}\sigma(u) \\ &= \frac{1}{\binom{2n}{n}} \mathrm{vol}\big(K\cap(\omega'-\omega+K)\big)\mu(K-K+\omega),\end{aligned} \tag{2.12}$$

which, together with (2.11), gives

$$\begin{aligned}\mu(K-K+\omega)\mathrm{vol}\big(K\cap(\omega'-\omega+K)\big) &\leq \binom{2n}{n}\int_K \mu(y+\omega-K)\,\mathrm{d}y \\ &\leq \binom{2n}{n}\mathrm{vol}(K)\sup_{y\in K}\mu(y+\omega-K).\end{aligned}$$

Finally we notice that equality holds in (2.10) for some $\omega_0\in\mathbb{R}^n$ only if there is equality in (2.12). This implies, by Lemma 2.1, that $\phi(\omega'+ru)$ is constant on $\big(0,\rho_{K-K+\omega_0-\omega'}(u)\big)$ for σ-almost every $u\in\mathbb{S}^{n-1}$. Since ϕ is continuous at $\omega'\in\operatorname{supp}\phi$, μ is a constant multiple of the Lebesgue measure on $K-K+\omega_0$ and, by Theorem A, K is a simplex (in particular, $c(\omega_0)=1$). The converse immediately follows from Theorem A. □

2.2. **The functional case.** In this subsection we draw a consequence of Theorem 1.1 regarding integrals of quasi-concave functions, which extends two results of Colesanti [12, Theorems 4.3 and 4.5] and is collected in Theorem 2.3. To this end, given a quasi-concave function $f:\mathbb{R}^n\longrightarrow[0,\infty)$, we define the $(-\infty)$-difference of f, which remains quasi-concave (cf. [12, Proposition 4.2]), by

$$\Delta_{-\infty}f(z)=\sup_{z=x-y}\min\big\{f(x),f(y)\big\}.$$

Besides $\Delta_{-\infty}f$, we also consider the (difference) functions $\Delta_{-\infty,\theta}f$ (for some $\theta\in[0,1]$) and $\widetilde{\Delta}_{-\infty}f$ given by

$$\begin{aligned}\Delta_{-\infty,\theta}f(z) &= \sup_{z=(1-\theta)x-\theta y}\min\big\{f(x),f(y)\big\},\\ \widetilde{\Delta}_{-\infty}f(z) &= \sup_{\substack{z=(1-\theta)x-\theta y\\ \theta\in[0,1]}}\min\big\{f(x),f(y)\big\}.\end{aligned}$$

These functions can be regarded as the (quasi-concave) functional counterparts of $K-K$, $(1-\theta)K-\theta K$ and $\mathrm{conv}\big(K\cup(-K)\big)$, respectively, as it is shown via their (strict) superlevel sets. For the sake of brevity we will write, for a function $f:\mathbb{R}^n\longrightarrow[0,\infty)$ and $t\in[0,\infty)$,

$$S_{>t}(f)=\big\{x\in\mathbb{R}^n : f(x)>t\big\};$$

analogously, $S_{\geq t}(f)=\big\{x\in\mathbb{R}^n : f(x)\geq t\big\}$. We observe that if $f:\mathbb{R}^n\longrightarrow[0,\infty)$ is a quasi-concave function, then

$$\begin{aligned} &(i) && S_{>t}\big(\Delta_{-\infty}f\big)=S_{>t}(f)-S_{>t}(f),\\ &(ii) && S_{>t}\big(\Delta_{-\infty,\theta}f\big)=(1-\theta)S_{>t}(f)-\theta S_{>t}(f),\\ &(iii) && S_{>t}\Big(\widetilde{\Delta}_{-\infty}f\Big)=\mathrm{conv}\Big(S_{>t}(f)\cup\big(-S_{>t}(f)\big)\Big).\end{aligned} \tag{2.13}$$

Indeed, (i), (ii) and (iii) are completely analogous. To see (i), let $z \in S_{>t}\big(\Delta_{-\infty} f\big)$. Then there exist x, y such that $z = x - y$ and $\min\big\{f(x), f(y)\big\} > t$, which shows the inclusion

$$S_{>t}\big(\Delta_{-\infty} f\big) \subset S_{>t}(f) - S_{>t}(f).$$

For the reverse inclusion, if $z \in S_{>t}(f) - S_{>t}(f)$ then there exist $x, y \in \mathbb{R}^n$, with $z = x - y$, such that $f(x) > t$ and $f(y) > t$. Since $\min\big\{f(x), f(y)\big\} > t$ and $z = x - y$, we get that $\Delta_{-\infty} f(z) > t$, as desired.

Now we collect the above-mentioned consequence of (1.4), which may be seen as its functional version.

Theorem 2.3. *Let $f : \mathbb{R}^n \longrightarrow [0, \infty)$ be an integrable quasi-concave function. Let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0, \infty)$ is radially decreasing. Then*

$$\int_{\mathbb{R}^n} \Delta_{-\infty} f(x)\,\mathrm{d}\mu(x) \leq \binom{2n}{n} \int_0^\infty \min\Big\{\overline{\mu}\big(S_{\geq t}(f)\big), \overline{\mu}\big(-S_{\geq t}(f)\big)\Big\}\,\mathrm{d}t. \tag{2.14}$$

In particular, by choosing $\mathrm{d}\mu(x) = \mathrm{d}x$, the Lebesgue measure, we get

$$\int_{\mathbb{R}^n} \Delta_{-\infty} f(x)\,\mathrm{d}x \leq \binom{2n}{n} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x.$$

Proof. The proof follows the general ideas of those of [12, Theorems 4.3 and 4.5]. Using Fubini's theorem, together with (i) in (2.13), we may write

$$\Delta_{-\infty} f(x) = \int_0^\infty \chi_{S_{>t}(f) - S_{>t}(f)}(x)\,\mathrm{d}t$$

and, consequently,

$$\begin{aligned} \int_{\mathbb{R}^n} \Delta_{-\infty} f(x)\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}^n} \int_0^\infty \chi_{S_{>t}(f) - S_{>t}(f)}(x)\,\mathrm{d}t\,\mathrm{d}\mu(x) \\ &\leq \int_0^\infty \mu\big(S_{\geq t}(f) - S_{\geq t}(f)\big)\,\mathrm{d}t. \end{aligned} \tag{2.15}$$

Since f is quasi-concave and integrable, the closure of the superlevel sets $S_{\geq t}(f)$ are convex bodies for all $0 < t < \|f\|_\infty$. Thus, we may apply (1.4) to $S_{\geq t}(f)$ (since the boundary of a convex set has null measure) which, together with (2.15), allows us to obtain (2.14).

Now we note that, if $\mathrm{d}\mu(x) = \mathrm{d}x$, then we have

$$\min\Big\{\overline{\mathrm{vol}}\big(S_{\geq t}(f)\big), \overline{\mathrm{vol}}\big(-S_{\geq t}(f)\big)\Big\} = \mathrm{vol}\big(S_{\geq t}(f)\big),$$

which completes the proof. □

Given a p-concave function $f : \mathbb{R}^n \longrightarrow [0, \infty)$, for $p \in [-\infty, 0)$, one can define the p-difference of f, which remains p-concave (cf. [12, Proposition 4.2]), by

$$\Delta_p f(z) = \sup_{z=x+y} \big(f(x)^p + f(-y)^p\big)^{1/p} = \sup_{z=x-y} \big(f(x)^p + f(y)^p\big)^{1/p}.$$

where the case $p = -\infty$ is understood as the minimum between both values.

Theorem 2.3 can be established for any $p \in (-\infty, 0)$. It suffices to note that if f is p-concave then it is also quasi-concave, and then, we may apply inequality (2.14) for $p = -\infty$ together with the fact that $(a^p + b^p)^{1/p} \leq \min\{a, b\}$ for each $a, b \geq 0$.

Hence $\Delta_p f \leq \Delta_{-\infty} f$.

Remark 2.3. *As mentioned before, Theorem 2.3 is an application of Theorem 1.1. It is a natural and interesting question whether* (1.4) *could be directly derived from previous functional versions as* (1.8)*. Just considering* $\chi_K\,\phi$ *this is not possible because of item (i) in* (2.13)*: the integral of* $\Delta_{-\infty} f$ *does not provide (in general) the measure of* $K-K$ *with respect to the density* ϕ*.*

2.3. **Rogers-Shephard type inequalities for CK and $\mathrm{conv}\big(K\cup(-K)\big)$ and their functional versions.** Now we prove the corresponding Rogers-Shephard type inequalities for CK and $\mathrm{conv}\big(K\cup(-K)\big)$, as well as their equality cases.

Proof of Theorem 1.2. Let $f:\mathbb{R}^n\times[0,1]\longrightarrow[0,\infty)$ be the function given by

$$f(x,\theta)=\mathrm{vol}\Big(\big((1-\theta)K\big)\cap(x+\theta K)\Big).$$

Note that f is $(1/n)$-concave by (1.1), and $\operatorname{supp} f = CK$. On the one hand, taking the measure μ_{n+1} on $\mathbb{R}^{n+1}$ given by $\mathrm{d}\mu_{n+1}(x,\theta)=\phi(x)\,\mathrm{d}x\,\mathrm{d}\theta$, Fubini's theorem and the change of variable $z=(1-\theta)y$ yield

(2.16)
$$\begin{aligned}
\int_{CK} f(x,\theta)\,\mathrm{d}\mu_{n+1}(x,\theta) &= \int_0^1\int_{\mathbb{R}^n}\mathrm{vol}\Big(\big((1-\theta)K\big)\cap(x+\theta K)\Big)\phi(x)\,\mathrm{d}x\,\mathrm{d}\theta\\
&=\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_{(1-\theta)K}(z)\chi_{x+\theta K}(z)\,\phi(x)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta\\
&=\int_0^1\int_{(1-\theta)K}\int_{\mathbb{R}^n}\chi_{z-\theta K}(x)\,\phi(x)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}\theta\\
&=\int_0^1(1-\theta)^n\int_K\mu\big((1-\theta)y-\theta K\big)\mathrm{d}y\,\mathrm{d}\theta\\
&\leq \mathrm{vol}(K)\int_0^1(1-\theta)^n\theta^n\,\mathrm{d}\theta\sup_{\substack{y\in K\\ \theta\in(0,1]}}\frac{\mu\big((1-\theta)y-\theta K\big)}{\theta^n}\\
&=\frac{1}{\binom{2n+1}{n}}\frac{\mathrm{vol}(K)}{n+1}\sup_{\substack{y\in K\\ \theta\in(0,1]}}\frac{\mu\big((1-\theta)y-\theta K\big)}{\theta^n}.
\end{aligned}$$

Now we define the function $g:CK\longrightarrow[0,\infty)$ given by

$$g(x,\theta)=f\left(0,\frac{1}{2}\right)\left[1-\frac{\big|(x,\theta)-\big(0,\frac{1}{2}\big)\big|}{\rho_{CK-(0,\frac{1}{2})}\Big(\big((x,\theta)-(0,\frac{1}{2})\big)/\big|(x,\theta)-(0,\frac{1}{2})\big|\Big)}\right]^n,$$

for every $(x,\theta)\neq(0,1/2)$ and $g(0,1/2)=f(0,1/2)=\mathrm{vol}(K)/2^n$. Since $f^{1/n}$ is concave, then $f^{1/n}\geq g^{1/n}$ on $\Big[(0,1/2),(0,1/2)+\rho_{CK-(0,\frac{1}{2})}(u)u\Big]$, and so, via the polar coordinates $(x,\theta')=(x,\theta)-(0,1/2)=ru$, we get

$$\begin{aligned}\int_{CK} f(x,\theta)\,\mathrm{d}\mu_{n+1}(x,\theta) &= \int_{CK-(0,\frac{1}{2})} f\left(x,\theta'+\frac{1}{2}\right)\phi(x)\,\mathrm{d}x\,\mathrm{d}\theta' \\ &= \int_{\mathbb{S}^n}\int_0^{\rho_{CK-(0,\frac{1}{2})}(u)} r^n f\left(\left(0,\frac{1}{2}\right)+ru\right)\phi\bigl(rP_H u\bigr)\,\mathrm{d}r\,\mathrm{d}\sigma(u) \\ &\geq f\left(0,\frac{1}{2}\right)\int_{\mathbb{S}^n}\int_0^{\rho_{CK-(0,\frac{1}{2})}(u)}\left(1-\frac{r}{\rho_{CK-(0,\frac{1}{2})}(u)}\right)^n r^n\phi\bigl(rP_H u\bigr)\,\mathrm{d}r\,\mathrm{d}\sigma(u),\end{aligned}$$

where $H=\bigl\{(x,\theta)\in\mathbb{R}^{n+1}:\theta=0\bigr\}$. Then, Lemma 2.1 yields

$$\begin{aligned}\int_{CK} f(x,\theta)\,\mathrm{d}\mu_{n+1}(x,\theta) &\geq \frac{f\left(0,\frac{1}{2}\right)}{\binom{2n+1}{n}}\int_{\mathbb{S}^n}\int_0^{\rho_{CK-(0,\frac{1}{2})}(u)} r^n\phi\bigl(rP_H u\bigr)\,\mathrm{d}r\,\mathrm{d}\sigma(u) \\ &= \frac{1}{\binom{2n+1}{n}}\frac{\mathrm{vol}(K)}{2^n}\mu_{n+1}(CK),\end{aligned}\tag{2.17}$$

which, together with (2.16), gives (1.11).

Finally we notice that equality holds in (1.11) only if there is equality in (2.17). This implies, by Lemma 2.1, that $\phi\bigl(rP_H u\bigr)$ is constant on $\left(0,\rho_{CK-(0,\frac{1}{2})}(u)\right)$ for σ-almost every $u\in\mathbb{S}^n$. Since ϕ is continuous at the origin, μ_{n+1} is a constant multiple of the Lebesgue measure on CK and hence μ is so on $P_H(CK)=\mathrm{conv}\bigl(K\cup(-K)\bigr)$ because μ_{n+1} is a product measure. Since $(1-\theta)y-\theta K\subset CK$ for all $y\in K$ and any $\theta\in[0,1]$, there is equality in (1.9) and therefore, by Theorem B, K is a simplex. The converse is a direct consequence of Theorem B.

Now we prove (1.12). Note that $P_H\Bigl(CK\cap\bigl(\mathcal{C}_t(\phi)\times[0,1]\bigr)\Bigr)=\mathrm{conv}\bigl(K\cup(-K)\bigr)\cap\mathcal{C}_t(\phi)$ and, since $0\in K$, then $CK\cap\bigl(\mathcal{C}_t(\phi)\times[0,1]\bigr)\cap H^\perp=[0,1]$. Hence, Theorem C yields $(n+1)\mathrm{vol}_{n+1}\Bigl(CK\cap\bigl(\mathcal{C}_t(\phi)\times[0,1]\bigr)\Bigr)\geq\mathrm{vol}\Bigl(\mathrm{conv}\bigl(K\cup(-K)\bigr)\cap\mathcal{C}_t(\phi)\Bigr)$, which, together with Fubini's theorem, gives

$$\begin{aligned}\mu_{n+1}(CK) &= \|\phi\|_\infty\int_{CK}\int_0^1\chi_{\mathcal{C}_t(\phi)}(x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \|\phi\|_\infty\int_0^1\int_{CK}\chi_{\mathcal{C}_t(\phi)\times[0,1]}(x,\theta)\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t \\ &= \|\phi\|_\infty\int_0^1\mathrm{vol}_{n+1}\Bigl(CK\cap\bigl(\mathcal{C}_t(\phi)\times[0,1]\bigr)\Bigr)\,\mathrm{d}t \\ &\geq \|\phi\|_\infty\frac{1}{n+1}\int_0^1\mathrm{vol}\Bigl(\mathrm{conv}\bigl(K\cup(-K)\bigr)\cap\mathcal{C}_t(\phi)\Bigr)\,\mathrm{d}t \\ &= \|\phi\|_\infty\frac{1}{n+1}\int_0^1\int_{\mathrm{conv}(K\cup(-K))}\chi_{\mathcal{C}_t(\phi)}(x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \|\phi\|_\infty\frac{1}{n+1}\int_{\mathrm{conv}(K\cup(-K))}\int_0^{\frac{\phi(x)}{\|\phi\|_\infty}}\mathrm{d}t\,\mathrm{d}x \\ &= \frac{1}{n+1}\int_{\mathrm{conv}(K\cup(-K))}\phi(x)\,\mathrm{d}x=\frac{\mu\Bigl(\mathrm{conv}\bigl(K\cup(-K)\bigr)\Bigr)}{n+1}.\end{aligned}$$

This, together with (1.11), shows (1.12). Equality in (1.12) implies, in particular, equality in (1.11) and thus μ is a constant multiple of the Lebesgue measure on $\operatorname{conv}\big(K \cup (-K)\big)$. The proof is now concluded from the equality case of (1.10). $\square$

Remark 2.4. *Taking the function* $f(x,\theta) = \operatorname{vol}\Big(\big((1-\theta)K\big) \cap \big(x + \theta(-L)\big)\Big)$, *and arguing as in the proof of Theorem 1.2, an analogous result can be obtained for two arbitrary convex bodies instead of* K *and* $-K$. *Thus, if* $K, L \in \mathcal{K}^n$ *contain the origin and* μ *is a measure on* $\mathbb{R}^n$ *given by* $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, *where* $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ *is a radially decreasing function, then*

$$\begin{aligned}\frac{\mu\big(\operatorname{conv}(K \cup L)\big)}{n+1} &\leq \int_0^1 \mu\big((1-\theta)K + \theta L\big)\,\mathrm{d}\theta \\ &\leq \frac{2^n}{n+1}\frac{\operatorname{vol}(K)}{\operatorname{vol}\big(K \cap (-L)\big)} \sup_{y\in K \theta\in(0,1]} \frac{\mu\big((1-\theta)y + \theta L\big)}{\theta^n}.\end{aligned}$$

As a consequence of Theorem 1.2, we get in Theorem 2.4 below functional versions of both (1.11) and (1.12). Regarding another functional version of (1.10), in the log-concave setting, we refer the reader to [12, Theorem 1.1]. The advantage of the inequality we present here is that, in contrast to the above-mentioned result, inequality (1.10) may recovered just by taking $f = \chi_K$. We use here the same notation as for Theorem 2.3.

Theorem 2.4. *Let* $f : \mathbb{R}^n \longrightarrow [0,\infty)$ *be an integrable quasi-concave function. Let* μ *be a measure on* $\mathbb{R}^n$ *given by* $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, *where* $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ *is radially decreasing. Then*

(2.18)
$$\int_0^1 \int_{\mathbb{R}^n} \Delta_{-\infty,\theta} f(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\theta \leq \frac{2^n}{n+1}\int_0^\infty \sup_{\substack{y\in S_{\geq t}(f)\\ \theta\in(0,1]}} \frac{\mu\big((1-\theta)y - \theta S_{\geq t}(f)\big)}{\theta^n}\,\mathrm{d}t$$

and

$$\int_{\mathbb{R}^n} \widetilde{\Delta}_{-\infty} f(x)\,\mathrm{d}\mu(x) \leq 2^n \int_0^\infty \sup_{\substack{y\in S_{\geq t}(f)\\ \theta\in(0,1]}} \frac{\mu\big((1-\theta)y - \theta S_{\geq t}(f)\big)}{\theta^n}\,\mathrm{d}t. \tag{2.19}$$

In particular, by choosing $\mathrm{d}\mu(x) = \mathrm{d}x$, *the Lebesgue measure, we get*

$$\int_0^1 \int_{\mathbb{R}^n} \Delta_{-\infty,\theta} f(x)\,\mathrm{d}x\,\mathrm{d}\theta \leq \frac{2^n}{n+1}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^n} \widetilde{\Delta}_{-\infty} f(x)\,\mathrm{d}x \leq 2^n \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x.$$

Proof. Since f is quasi-concave and integrable, the closure of the superlevel sets $S_{\geq t}(f)$ are convex bodies for all $0 < t < \|f\|_\infty$. Thus, we may apply Theorem 1.2 to $S_{\geq t}(f)$ (since the boundary of a convex set has null measure) to obtain

$$\int_0^1 \mu\big((1-\theta)S_{>t}(f) - \theta S_{>t}(f)\big)\,\mathrm{d}\theta \leq \frac{2^n}{n+1}\sup_{\substack{y\in S_{\geq t}(f)\\ \theta\in(0,1]}} \frac{\mu\big((1-\theta)y - \theta S_{\geq t}(f)\big)}{\theta^n}$$

and

$$\mu\Big(\operatorname{conv}\big(S_{>t}(f)\cup(-S_{>t}(f))\big)\Big)\leq 2^n \sup_{\substack{y\in S_{\geq t}(f)\\ \theta\in(0,1]}}\frac{\mu\big((1-\theta)y-\theta S_{\geq t}(f)\big)}{\theta^n}.$$

Integrating on $t\in[0,\infty)$, (2.18) and (2.19) now follow by applying Fubini's theorem together with (ii) and (iii) in (2.13), respectively. Finally, if $\mathrm{d}\mu(x)=\mathrm{d}x$, then we have

$$\sup_{\substack{y\in S_{\geq t}(f)\\ \theta\in(0,1]}}\frac{\operatorname{vol}\big((1-\theta)y-\theta S_{\geq t}(f)\big)}{\theta^n}=\operatorname{vol}\big(S_{\geq t}(f)\big).$$

This concludes the proof. □

3. A projection-section inequality for quasi-concave functions

We start this section by showing a general result for functions that will be exploited throughout the rest of the paper.

Proposition 3.1. *Let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x)=\phi(x)\,\mathrm{d}x$, where $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is quasi-concave and such that $\|\phi\|_\infty=\phi(0)$. Let $f:\mathbb{R}^n\longrightarrow[0,\infty)$ be a p-concave function, $p>0$, with $\|f\|_\infty=f(0)$, and let $g:\mathbb{R}^n\longrightarrow[0,\infty)$ be a measurable function. Then*

$$\int_{\operatorname{supp}f}\int_0^1(1-\theta^p)^n g\big((1-\theta^p)x\big)\,\mathrm{d}\theta\,\mathrm{d}\mu(x)\leq\frac{1}{\|f\|_\infty}\int_{\operatorname{supp}f}g(x)f(x)\,\mathrm{d}\mu(x). \tag{3.1}$$

Moreover, if $\operatorname{supp}f$ is bounded, g is non-zero on $\operatorname{supp}f$ and ϕ is continuous at the origin, equality in (3.1) implies that μ is a constant multiple of the Lebesgue measure on $\operatorname{supp}f$.

Proof. Since f is p-concave, then $\mathcal{C}_\theta(f)$ is a convex set for every $\theta\in[0,1]$. We notice that

$$\frac{\mathcal{C}_{\theta_1}(f)}{1-\theta_1^p}\subset\frac{\mathcal{C}_{\theta_2}(f)}{1-\theta_2^p}$$

for $0\leq\theta_1\leq\theta_2<1$. In particular, taking $\theta_1=0$, we have

$$\operatorname{supp}f\subset\frac{1}{1-\theta^p}\mathcal{C}_\theta(f)\quad\text{for any}\ \ \theta\in[0,1), \tag{3.2}$$

and hence

$$(\operatorname{supp}f)\cap\mathcal{C}_t(\phi)\subset\left(\frac{1}{1-\theta^p}\mathcal{C}_\theta(f)\right)\cap\mathcal{C}_t(\phi)\subset\frac{\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi)}{1-\theta^p} \tag{3.3}$$

for all $\theta\in[0,1)$ and every $t\in[0,1]$. Therefore

$$\big(1-\theta^p\big)\big[(\operatorname{supp}f)\cap\mathcal{C}_t(\phi)\big]\subset\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi),$$

which yields

$$\int_0^1\int_0^1\int_{(1-\theta^p)[(\operatorname{supp}f)\cap\mathcal{C}_t(\phi)]}g(x)\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t\leq\int_0^1\int_0^1\int_{\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi)}g(x)\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t. \tag{3.4}$$

Now we compute both sides of inequality (3.4). On the one hand, by Fubini's theorem and the change of variable $x=\big(1-\theta^p\big)y$, we get

$$\begin{aligned}
&\int_0^1\int_0^1\int_{(1-\theta^p)[(\operatorname{supp} f)\cap\mathcal{C}_t(\phi)]} g(x)\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t\\
&\qquad=\int_0^1\int_0^1\int_{(\operatorname{supp} f)\cap\mathcal{C}_t(\phi)} g\big((1-\theta^p)y\big)(1-\theta^p)^n\,\mathrm{d}y\,\mathrm{d}\theta\,\mathrm{d}t\\
&\qquad=\int_{\operatorname{supp} f}\int_0^1(1-\theta^p)^n g\big((1-\theta^p)y\big)\int_0^1\chi_{\mathcal{C}_t(\phi)}(y)\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}y\\
&\qquad=\int_{\operatorname{supp} f}\int_0^1(1-\theta^p)^n g\big((1-\theta^p)y\big)\frac{\phi(y)}{\|\phi\|_\infty}\,\mathrm{d}\theta\,\mathrm{d}y\\
&\qquad=\frac{1}{\|\phi\|_\infty}\int_{\operatorname{supp} f}\int_0^1(1-\theta^p)^n g\big((1-\theta^p)y\big)\,\mathrm{d}\theta\,\mathrm{d}\mu(y).
\end{aligned}$$

On the other hand, using again Fubini's theorem,

$$\begin{aligned}
\int_0^1\int_0^1\int_{\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi)} g(x)\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t &=\int_0^1\int_0^1\int_{\mathbb{R}^n} g(x)\,\chi_{\mathcal{C}_\theta(f)}(x)\chi_{\mathcal{C}_t(\phi)}(x)\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t\\
&=\int_{\mathbb{R}^n} g(x)\int_0^1\chi_{\mathcal{C}_t(\phi)}(x)\int_0^1\chi_{\mathcal{C}_\theta(f)}(x)\,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}x\\
&=\int_{\operatorname{supp} f} g(x)\frac{f(x)}{\|f\|_\infty}\frac{\phi(x)}{\|\phi\|_\infty}\,\mathrm{d}x\\
&=\frac{1}{\|f\|_\infty\|\phi\|_\infty}\int_{\operatorname{supp} f} g(x)\,f(x)\,\mathrm{d}\mu(x).
\end{aligned}$$

Thus, (3.1) follows from inequality (3.4).

Now we deal with the equality case. First we observe that since $\operatorname{supp} f$ is a bounded set and f is p-concave, then $\mathcal{C}_\theta(f)$ is a bounded convex set for all $\theta\in[0,1)$.

Without loss of generality we may assume that ϕ is upper semicontinuous. Indeed, otherwise we would work with its upper closure, which is determined via the closure of the superlevel sets of ϕ (see [33, page 14 and Theorem 1.6]) and thus defines the same measure because of Fubini's theorem together with the facts that all the superlevel sets of ϕ are convex (since it is quasi-concave) and the boundary of a convex set has null (Lebesgue) measure. Then its superlevel sets $\mathcal{C}_t(\phi)$ are closed (cf. [33, Theorem 1.6]) for every $t\in[0,1]$. In the same way, f may be assumed to be upper semicontinuous (in fact, it is already continuous in the interior of its support, because of the p-concavity). Moreover, since the definitions of both $\mathcal{C}_\theta(f)$ and $\mathcal{C}_t(\phi)$ involve the essential supremum, these superlevel sets have positive volume for all $\theta<1$ and $t<1$, and therefore both $\mathcal{C}_\theta(f)$ and $\mathcal{C}_t(\phi)$ are closed convex sets with non-empty interior, for any $\theta,t\in[0,1)$. From the continuity of ϕ at the origin, we know that $0\in\operatorname{int}\mathcal{C}_t(\phi)$ for all $t<1$ and then $0\in\mathcal{C}_\theta(f)\cap\operatorname{int}\mathcal{C}_t(\phi)$ because $f(0)=\|f\|_\infty$. Hence, and taking into account that $\operatorname{supp} f$ (and thus $\mathcal{C}_\theta(f)$ for any $\theta\in[0,1]$) is bounded, both $\mathcal{C}_\theta(f)\cap(1-\theta^p)\mathcal{C}_t(\phi)$ and $\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi)$ are convex bodies for all $\theta,t\in[0,1)$.

Thus, if equality holds in (3.1) then, in particular, there is equality in the right-hand inclusion of (3.3) for almost all $\theta\in[0,1]$ and almost all $t\in[0,1]$, because $g>0$ on $\operatorname{supp} f$.

Let us assume that there exists $x_0 \in \operatorname{supp} f$ such that $\phi(x_0) < \|\phi\|_\infty$. Taking $t \in \big(\phi(x_0)/\|\phi\|_\infty, 1\big]$, since $x_0 \notin \mathcal{C}_t(\phi)$ then we have that

$$(\operatorname{supp} f) \cap \mathcal{C}_t(\phi) \subsetneq \operatorname{supp} f.$$

Let $x_t \in \operatorname{bd}\big((\operatorname{supp} f) \cap \mathcal{C}_t(\phi)\big) \setminus \operatorname{bd}(\operatorname{supp} f)$. Since both sets are convex bodies, we can always take $x_t \neq 0$. Then for all $t \in \big(\phi(x_0)/\|\phi\|_\infty, 1\big]$, the continuity of f on $\operatorname{int}(\operatorname{supp} f)$ yields the existence of $\theta_t \in (0,1)$ such that

$$x_t \in \mathcal{C}_\theta(f) \cap \mathcal{C}_t(\phi) \quad \text{ for all } \ \theta \in [0, \theta_t).$$

However, since $x_t \in \operatorname{bd} \mathcal{C}_t(\phi)$ and $0 \in \operatorname{int} \mathcal{C}_t(\phi)$,

$$x_t \notin \mathcal{C}_\theta(f) \cap \big(1 - \theta^p\big)\mathcal{C}_t(\phi).$$

This contradicts the equality in the right-hand inclusion of (3.3) for almost every $\theta \in [0,1]$ and $t \in [0,1]$.

Therefore we may conclude that $\phi(x) \geq \|\phi\|_\infty$ for all $x \in \operatorname{supp} f$ and thus $\phi \equiv \|\phi\|_\infty$ almost everywhere on $\operatorname{supp} f$. This implies that μ is a constant multiple of the Lebesgue measure on $\operatorname{supp} f$. □

It is an interesting question whether Proposition 3.1 can be adapted to log-concave functions, i.e., when $p = 0$. We notice that the above approach cannot be followed in this case. Indeed, considering e.g. the function $f : \mathbb{R} \longrightarrow [0, \infty)$ given by $f(x) = e^{-x^2}$, we have that $\operatorname{supp} f = \mathbb{R}$ whereas $\mathcal{C}_\theta(f)$ is a convex body for all $t \in (0,1]$. Hence, there is no chance to get an inclusion of the type (3.2), i.e., $\lambda(\theta) \operatorname{supp} f \subset \mathcal{C}_\theta(f)$ for any $\theta \in [0,1]$ and some $\lambda(\theta) > 0$.

In what follows we use Proposition 3.1 to prove several results, including Theorem 1.4. Let us first introduce a helpful family of constants and notice a few facts. We denote by

$$\alpha_{p,q}^n = \int_0^1 (1-\theta^p)^n \, \theta^{p\,q} \, \mathrm{d}\theta = \frac{\Gamma\left(\frac{1}{p} + q\right)\Gamma(1+n)}{p\,\Gamma\left(1 + n + \frac{1}{p} + q\right)},$$

for each $p, q > 0$. Let us assume that g is concave. Then

$$g\big((1-\theta^p)x\big) \geq \theta^p g(0) + (1-\theta^p) g(x),$$

and so, we get from (3.1) that

$$\alpha_{p,1}^n \, g(0) \, \mu(\operatorname{supp} f) + \alpha_{p,0}^{n+1} \int_{\operatorname{supp} f} g(x) \, \mathrm{d}\mu(x) \leq \frac{1}{\|f\|_\infty} \int_{\operatorname{supp} f} g(x) \, f(x) \, \mathrm{d}\mu(x). \tag{3.5}$$

Another possibility is assuming that g is radially decreasing. Then, from (3.1), we get

$$\alpha_{p,0}^n \int_{\operatorname{supp} f} g(x) \, \mathrm{d}\mu(x) \leq \frac{1}{\|f\|_\infty} \int_{\operatorname{supp} f} g(x) \, f(x) \, \mathrm{d}\mu(x). \tag{3.6}$$

We point out that $\alpha_{p,0}^n = \alpha_{p,1}^n + \alpha_{p,0}^{n+1}$, which shows that the expression on the left-hand side of (3.5) and that of (3.6) are in a sense "similar", as shown by considering the constant function $g(x) = 1$. Indeed, when $g \equiv 1$, (3.6) reads

$$\alpha_{p,0}^n \, \mu(\operatorname{supp} f) \leq \frac{1}{\|f\|_\infty} \int_{\operatorname{supp} f} f(x) \, \mathrm{d}\mu(x). \tag{3.7}$$

Moreover, it can be proved that (3.7) remains true even in the more general case when $\|f\|_\infty = f(x_0)$ for an arbitrary $x_0 \in \mathbb{R}^n$, and without the maximality assumption for ϕ.

Corollary 3.1. *Let $f : \mathbb{R}^n \longrightarrow [0,\infty)$ be a p-concave function, $p > 0$, with $\|f\|_\infty = f(x_0)$ for some $x_0 \in \mathbb{R}^n$, and let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ is a bounded quasi-concave function. Then*

$$\alpha_{p,0}^n \frac{\phi(x_0)}{\|\phi\|_\infty}\,\mu(\operatorname{supp} f) \leq \frac{1}{\|f\|_\infty}\int_{\operatorname{supp} f} f(x)\,\mathrm{d}\mu(x). \tag{3.8}$$

Moreover, if $\operatorname{supp} f$ is bounded and ϕ is continuous at x_0, equality in (3.8) implies that μ is a constant multiple of the Lebesgue measure on $\operatorname{supp} f$.

Proof. The proof follows similar steps as those of Proposition 3.1, but with some key variations. We will highlight these differences.

We consider the function $\psi : \mathbb{R}^n \longrightarrow [0,\infty)$ given by $\psi(x) = f(x + x_0)$, which satisfies $\|\psi\|_\infty = \|f\|_\infty$ and $\operatorname{supp}\psi = (\operatorname{supp} f) - x_0$. Then (cf. (3.2))

$$\operatorname{supp}\psi \subset \frac{1}{1-\theta^p}\mathcal{C}_\theta(\psi) \quad \text{ for all } \ \theta \in [0,1). \tag{3.9}$$

We observe that $y \in \mathcal{C}_\theta(\psi)$ if and only if $f(y + x_0) \geq \theta\|f\|_\infty$, or equivalently, when $y + x_0 \in \mathcal{C}_\theta(f)$. Hence, $\mathcal{C}_\theta(\psi) + x_0 = \mathcal{C}_\theta(f)$, and thus (3.9) turns into

$$(\operatorname{supp} f) - x_0 \subset \frac{1}{1-\theta^p}\big(\mathcal{C}_\theta(f) - x_0\big) \quad \text{ for all } \ \theta \in [0,1).$$

Therefore

$$\begin{aligned}\big((\operatorname{supp} f) - x_0\big) \cap \big(\mathcal{C}_t(\phi) - x_0\big) &\subset \left(\frac{1}{1-\theta^p}\big(\mathcal{C}_\theta(f) - x_0\big)\right) \cap \big(\mathcal{C}_t(\phi) - x_0\big)\\ &\subset \frac{1}{1-\theta^p}\Big(\big[\mathcal{C}_\theta(f) \cap \mathcal{C}_t(\phi)\big] - x_0\Big)\end{aligned}$$

for all $\theta \in [0,1)$ and every $t \in \big[0, \phi(x_0)/\|\phi\|_\infty\big]$, where in the last inclusion we have used that $x_0 \in \mathcal{C}_t(\phi)$. Consequently, we obtain

$$(1-\theta^p)\Big(\big[(\operatorname{supp} f) \cap \mathcal{C}_t(\phi)\big] - x_0\Big) \subset \big(\mathcal{C}_\theta(f) \cap \mathcal{C}_t(\phi)\big) - x_0. \tag{3.10}$$

Next, integrating over $x \in \mathbb{R}^n$ the constant function 1, using (3.10) and the change of variable $x = (1-\theta^p)y$, we get

$$(1-\theta^p)^n \int_{[(\operatorname{supp} f)\cap\mathcal{C}_t(\phi)]-x_0} \mathrm{d}y \leq \int_{[\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi)]-x_0} \mathrm{d}y,$$

which yields

$$(1-\theta^p)^n \int_{(\operatorname{supp} f)\cap\mathcal{C}_t(\phi)} \mathrm{d}x \leq \int_{\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi)} \mathrm{d}x. \tag{3.11}$$

Now, computing the left-hand side in (3.8), we get

$$\begin{aligned}\alpha_{p,0}^n\frac{\phi(x_0)}{\|\phi\|_\infty}\,\mu(\operatorname{supp} f) &= \alpha_{p,0}^n\|\phi\|_\infty\int_{\operatorname{supp} f}\frac{\phi(x_0)}{\|\phi\|_\infty}\,\frac{\phi(x)}{\|\phi\|_\infty}\,\mathrm{d}x\\ &\leq \|\phi\|_\infty\int_0^1(1-\theta^p)^n\,\mathrm{d}\theta\int_{\operatorname{supp} f}\min\left\{\frac{\phi(x)}{\|\phi\|_\infty},\frac{\phi(x_0)}{\|\phi\|_\infty}\right\}\mathrm{d}x\\ &= \|\phi\|_\infty\int_0^1\int_0^{\frac{\phi(x_0)}{\|\phi\|_\infty}}(1-\theta^p)^n\int_{(\operatorname{supp} f)\cap\mathcal{C}_t(\phi)}\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta.\end{aligned}$$

Applying (3.11) we obtain the desired inequality. Indeed from the above computation we get

$$\begin{aligned}\alpha_{p,0}^n\frac{\phi(x_0)}{\|\phi\|_\infty}\,\mu(\operatorname{supp} f) &\leq \|\phi\|_\infty\int_0^1\int_0^{\frac{\phi(x_0)}{\|\phi\|_\infty}}\int_{\mathcal{C}_\theta(f)\cap\mathcal{C}_t(\phi)}\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta\\ &= \frac{\|\phi\|_\infty}{\|f\|_\infty}\int_{\operatorname{supp} f}f(x)\int_0^{\frac{\phi(x_0)}{\|\phi\|_\infty}}\chi_{\mathcal{C}_t(\phi)}(x)\,\mathrm{d}t\,\mathrm{d}x\\ &\leq \frac{\|\phi\|_\infty}{\|f\|_\infty}\int_{\operatorname{supp} f}f(x)\int_0^1\chi_{\mathcal{C}_t(\phi)}(x)\,\mathrm{d}t\,\mathrm{d}x\\ &= \frac{1}{\|f\|_\infty}\int_{\operatorname{supp} f}f(x)\,\mathrm{d}\mu(x).\end{aligned}$$

For the proof of the equality case we observe, on the one hand, that if equality holds in (3.8) then, in particular,

$$\int_{\operatorname{supp} f}f(x)\int_{\frac{\phi(x_0)}{\|\phi\|_\infty}}^1\chi_{\mathcal{C}_t(\phi)}(x)\,\mathrm{d}t\,\mathrm{d}x = 0,$$

which yields $\phi(x_0) = \operatorname{ess\,sup}_{x\in\operatorname{supp} f}\phi(x)$.

On the other hand, we may replace $\|\phi\|_\infty$ by $\operatorname{ess\,sup}_{x\in\operatorname{supp} f}\phi(x)$ in the above argument to get also

$$\alpha_{p,0}^n\frac{\phi(x_0)}{\operatorname{ess\,sup}_{x\in\operatorname{supp} f}\phi(x)}\,\mu(\operatorname{supp} f)\leq\frac{1}{\|f\|_\infty}\int_{\operatorname{supp} f}f(x)\,\mathrm{d}\mu(x),$$

and since

$$\alpha_{p,0}^n\frac{\phi(x_0)}{\|\phi\|_\infty}\,\mu(\operatorname{supp} f)\leq\alpha_{p,0}^n\,\frac{\phi(x_0)}{\operatorname{ess\,sup}_{x\in\operatorname{supp} f}\phi(x)}\,\mu(\operatorname{supp} f) = \alpha_{p,0}^n\,\mu(\operatorname{supp} f),$$

equality in (3.8) implies that $\phi(x_0) = \|\phi\|_\infty$.

Finally, due to the fact that $\phi(x_0) = \|\phi\|_\infty$, the rest of the proof of the equality case is entirely analogous to the one in Proposition 3.1, and we do not repeat it here. □

As an application of Proposition 3.1, and the above-mentioned consequences of it, we show Theorem 1.4.

Proof of Theorem 1.4. For all $t\in[0,1]$, the function $\varphi_t : P_H\mathcal{C}_t(f)\longrightarrow[0,\infty)$ given by

$$\varphi_t(x) = \operatorname{vol}_k\bigl(\mathcal{C}_t(f)\cap(x+H^\perp)\bigr)$$

is $(1/k)$-concave, because of the Brunn-Minkowski inequality (1.1), and $\operatorname{supp}\varphi_t = P_H\mathcal{C}_t(f)$. By hypothesis we have $\|\varphi_t\|_\infty = \varphi_t(0)$. Then, by applying (3.6) to φ_t, we get

$$\alpha_{1/k,0}^{n-k}\int_{P_H\mathcal{C}_t(f)} g(x)\,\mathrm{d}x \leq \frac{1}{\|\varphi_t\|_\infty}\int_H g(x)\,\varphi_t(x)\,\mathrm{d}x \tag{3.12}$$

and hence, integrating each side of inequality (3.12) over $t\in[0,1]$ and noticing that $\alpha_{1/k,0}^{n-k} = \binom{n}{k}^{-1}$, it follows that

$$\int_0^1\int_{P_H\mathcal{C}_t(f)} g(x)\,\mathrm{d}x\int_{H^\perp}\chi_{\mathcal{C}_t(f)}(y)\mathrm{d}y\,\mathrm{d}t \leq \binom{n}{k}\int_0^1\int_H g(x)\int_{x+H^\perp}\chi_{\mathcal{C}_t(f)}(y)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t. \tag{3.13}$$

On the one hand, by Fubini's theorem and noticing that

$$P_H\mathcal{C}_t(f)\supset P_H\Big(\big\{x\in\mathbb{R}^n : f(x) > t\|f\|_\infty\big\}\Big) = \big\{x\in H : P_Hf(x) > t\|f\|_\infty\big\},$$

we obtain

$$\begin{aligned}\int_0^1\int_H g(x)\chi_{P_H\mathcal{C}_t(f)}(x)\,\mathrm{d}x&\int_{H^\perp}\chi_{\mathcal{C}_t(f)}(y)\,\mathrm{d}y\,\mathrm{d}t\\ &= \int_H\int_{H^\perp} g(x)\int_0^1\chi_{P_H\mathcal{C}_t(f)}(x)\chi_{\mathcal{C}_t(f)}(y)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\\ &\geq \int_H\int_{H^\perp} g(x)\min\left\{\frac{P_Hf(x)}{\|f\|_\infty},\frac{f(y)}{\|f\|_\infty}\right\}\mathrm{d}y\,\mathrm{d}x\\ &\geq \int_H\int_{H^\perp} g(x)\frac{P_Hf(x)}{\|f\|_\infty}\frac{f(y)}{\|f\|_\infty}\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_H g(x)\frac{P_Hf(x)}{\|f\|_\infty}\,\mathrm{d}x\int_{H^\perp}\frac{f(y)}{\|f\|_\infty}\,\mathrm{d}y.\end{aligned}\tag{3.14}$$

On the other hand, Fubini's theorem yields

$$\begin{aligned}\int_0^1\int_H g(x)\int_{x+H^\perp}\chi_{\mathcal{C}_t(f)}(y)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t &= \int_H g(x)\int_{x+H^\perp}\int_0^1\chi_{\mathcal{C}_t(f)}(y)\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_H\int_{x+H^\perp} g(x)\frac{f(y)}{\|f\|_\infty}\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n} g(P_Hz)\frac{f(z)}{\|f\|_\infty}\,\mathrm{d}z.\end{aligned}\tag{3.15}$$

Therefore, from (3.13), (3.14) and (3.15) we obtain

$$\int_H g(x)P_Hf(x)\,\mathrm{d}x\int_{H^\perp} f(y)\,\mathrm{d}y \leq \binom{n}{k}\|f\|_\infty\int_{\mathbb{R}^n} g(P_Hx)f(x)\,\mathrm{d}x.$$

This conclues the proof. □

With the above approach, but using (3.7) instead of (3.6), we notice that the maximality assumption at the origin can be relaxed to get the following result, which has been recently obtained in the setting of a log-concave integrable function in [1, Theorem 1.1].

Corollary 3.2. *Let $k \in \{1, \dots, n-1\}$ and $H \in \mathrm{G}(n, n-k)$. Let $f : \mathbb{R}^n \longrightarrow [0, \infty)$ be a quasi-concave function such that*

$$\sup_{x \in H} \mathrm{vol}_k\big(\mathcal{C}_t(f) \cap \big(x + H^{\perp}\big)\big)$$

is attained for all $t \in (0, 1)$. Then

$$\int_H P_H f(x)\,\mathrm{d}x \max_{x_0 \in H} \int_{x_0 + H^{\perp}} f(y)\,\mathrm{d}y \leq \binom{n}{k} \|f\|_\infty \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x. \tag{3.16}$$

We point out that, in the case of an integrable function f whose restriction to its support is continuous, the above assumption on the volume of the sections of $\mathcal{C}_t(f)$ trivially holds, since $\mathcal{C}_t(f)$ is compact for every $t \in (0, 1)$. Notice also that, when dealing with certain classes of functions with a more restrictive concavity (such as log-concave ones), continuity on the interior of their support is already guaranteed.

4. Rogers-Shephard type inequalities for measures with quasi-concave densities

As a direct application of Corollary 3.2 we obtain the following result.

Theorem 4.1. *Let $k \in \{1, \dots, n-1\}$ and $H \in \mathrm{G}(n, n-k)$. Let $\phi_i : \mathbb{R}^i \longrightarrow [0, \infty)$, $i = n-k, k$, be functions with $\|\phi_i\|_\infty = \phi_i(0)$, and such that the function $\phi : \mathbb{R}^n \longrightarrow [0, \infty)$ given by $\phi(x, y) = \phi_{n-k}(x)\phi_k(y)$, $x \in \mathbb{R}^{n-k}$, $y \in \mathbb{R}^k$, is quasi-concave. Let $\mu_n = \mu_{n-k} \times \mu_k$ be the product measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu_{n-k}(x) = \phi_{n-k}(x)\,\mathrm{d}x$ and $\mathrm{d}\mu_k(y) = \phi_k(y)\,\mathrm{d}y$. Let $K \in \mathcal{K}^n$ with $P_H K \subset K$ and so that $\mathrm{vol}\big(\mathcal{C}_t(\phi) \cap K \cap (x + H^{\perp})\big)$ attains its maximum for all $t \in (0, 1)$. Then*

$$\mu_{n-k}\big(P_H K\big) \max_{x_0 \in H} \left[\frac{\phi_{n-k}(x_0)}{\|\phi_{n-k}\|_\infty} \mu_k\big(K \cap (x_0 + H^{\perp})\big) \right] \leq \binom{n}{k} \mu_n(K). \tag{4.1}$$

Proof. It is a straightforward consequence of (3.16) applied to the function $f : \mathbb{R}^n \longrightarrow [0, \infty)$ given by $f(x, y) = \phi_{n-k}(x)\phi_k(y)\chi_{_K}(x, y)$. Indeed, since $P_H K \subset K$ then

$$P_H f(x) = \sup_{y \in H^{\perp}} \phi_{n-k}(x)\phi_k(y)\chi_{_K}(x, y) = \phi_{n-k}(x)\phi_k(0)\chi_{_{P_H K}}(x)$$

and $\|f\|_\infty = \phi_{n-k}(0)\phi_k(0)$. □

We point out that the assumption $P_H K \subset K$ is needed in order to conclude the above Rogers-Shephard type inequality (as well as Theorem 1.3):

Example 4.1. *Let μ_1 be the measure on $\mathbb{R}$ given by $\mathrm{d}\mu_1(x) = e^{-x^2}\,\mathrm{d}x$ and let $\mu_2 = \mu_1 \times \mu_1$, i.e., $\mathrm{d}\mu_2(x) = e^{-|x|^2}\,\mathrm{d}x$. Let $H = \big\{(x, y) \in \mathbb{R}^2 : y = 0\big\}$ and, for a given $0 < \alpha < \pi/2$, let K_α be the centrally symmetric parallelogram $K_\alpha = \mathrm{conv}\big\{(1, \tan\alpha \pm 1), (-1, -\tan\alpha \pm 1)\big\}$.*

On the one hand, $K_\alpha(0) = \big[(0, 1), (0, -1)\big]$ is the 'maximal' section of K_α (with respect to μ_1) and $P_H K_\alpha = \big[(-1, 0), (1, 0)\big]$. On the other hand, since K_α is contained in the infinite strip S_α determined by the straight lines $y = (\tan\alpha)x \pm 1$, and μ_2 is rotationally invariant, we have that

$$\mu_2(K_\alpha) \leq \mu_2(S_\alpha) = \sqrt{2\pi}\,\mu_1(I_\alpha),$$

where I_α denotes the line segment centered at the origin and with length the width of S_α.

Hence, $\mu_1(I_\alpha)$, and so $\mu_2(K_\alpha)$, can be made arbitrarily small when $\alpha \to \pi/2$. However, the term $\mu_1\bigl(P_H K_\alpha\bigr)\mu_1\bigl(K_\alpha(0)\bigr) = \mu_1\bigl([(-1,0),(1,0)]\bigr)^2$ is a fixed positive constant. This shows the necessity of assuming $P_H K \subset K$ in order to derive both (4.1) *and* (1.14).

In order to avoid the assumption $P_H K \subset K$, one may exchange the orthogonal projection by the corresponding maximal section. To this end, first we fix some notation: given a measure μ in $\mathbb{R}^n$ with density ϕ, we will denote by μ_i, $i = 1,\dots,n-1$, the *marginal* of μ in the corresponding i-dimensional affine subspace, i.e., for given $M \subset z + H$ with $H \in \mathrm{G}(n,i)$ and $z \in H^\perp$,

$$\mu_i(M) = \int_H \chi_{_M}(x,z)\phi(x,z)\,\mathrm{d}x.$$

Taking the function $f : \mathbb{R}^n \longrightarrow [0,\infty)$ given by $f(x,y) = \phi(x,y)\chi_{_K}(x,y)$, $x \in H$, $y \in H^\perp$, since

$$P_H f(x) = \sup_{y\in H^\perp} \phi(x,y)\chi_{_K}(x,y) \geq \phi(x,y)\chi_{_K}(x,y) = f(x,y),$$

we get the following result, as direct consequence of (3.16).

Corollary 4.1. *Let $k \in \{1,\dots,n-1\}$ and $H \in \mathrm{G}(n,n-k)$. Let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ is a quasi-concave function with $\|\phi\|_\infty = \phi(0)$. Let $K \in \mathcal{K}^n$ be such that there exists the maximum of $\mathrm{vol}\bigl(\mathcal{C}_t(\phi) \cap K \cap (x + H^\perp)\bigr)$ for all $t \in (0,1)$. Then*

$$\max_{y\in H} \mu_{n-k}\bigl(K \cap (y+H)\bigr) \max_{x_0\in H} \mu_k\bigl(K \cap (x_0 + H^\perp)\bigr) \leq \binom{n}{k}\|\phi\|_\infty \mu(K). \tag{4.2}$$

We notice that, from (4.1),

$$\mu_{n-k}\bigl(P_H K\bigr)\mu_k\bigl(K \cap H^\perp\bigr) \leq \binom{n}{k}\mu_n(K) \tag{4.3}$$

holds provided that the density of μ_n, $\phi(x,y) = \phi_{n-k}(x)\phi_k(y)$, is quasi-concave. Although the latter implies that both ϕ_{n-k}, ϕ_k are quasi-concave, the converse is, in general, not true. In the following we exploit the approach followed in the previous section in order to derive (4.3) for the more general case of measures μ_{n-k}, μ_k, with radially decreasing and quasi-concave densities, respectively, and their product $\mu_n = \mu_{n-k} \times \mu_k$, provided that the maximality assumption

$$\max_{x\in P_H K} \mathrm{vol}_k\bigl(\mathcal{C}_t(\phi_k) \cap K(x)\bigr) = \mathrm{vol}_k\bigl(\mathcal{C}_t(\phi_k) \cap K(0)\bigr)$$

holds. Again, we need to assume the condition $P_H K \subset K$.

Proof of Theorem 1.3. By an appropriate choice of the coordinate axes, we may assume that $H = \{x_{n-k+1} = \cdots = x_n = 0\}$. For every $t \in [0,1]$, and $x \in P_H K$, we consider the set

$$\mathcal{C}_{x,t} = \Bigl(\{0\} \times \mathcal{C}_t(\phi_k)\Bigr) \cap K(x)$$

and the function $\varphi_t : P_H K \longrightarrow [0,\infty)$ given by

$$\varphi_t(x) = \mathrm{vol}_k\bigl(\mathcal{C}_{x,t}\bigr).$$

Since $P_H K \subset K$ and ϕ_k is continuous at the origin (which implies that $0 \in \mathrm{int}\,\mathcal{C}_t(\phi_k)$ for all $t < 1$), we may assure that, for every $t < 1$, $\varphi_t(x) > 0$ for any x in the (relative) interior of $P_H K$ and hence $\operatorname{supp}\varphi_t = P_H K$. Moreover, φ_t is $(1/k)$-concave by (1.1) and, by hypothesis, we have $\|\varphi_t\|_\infty = \varphi_t(0)$.

Then, applying (3.6), with $p = 1/k$, to the function $g : P_H K \longrightarrow [0,\infty)$ given by $g(x,0) = \phi_{n-k}(x)$, $x \in \mathbb{R}^{n-k}$, we get

$$\int_{P_H K} \phi_{n-k}(x)\,\mathrm{d}x \le \binom{n}{k}\frac{1}{\|\varphi_t\|_\infty}\int_{P_H K} \phi_{n-k}(x)\,\varphi_t(x)\,\mathrm{d}x, \tag{4.4}$$

and hence, integrating (4.4) over $t \in [0,1]$, we obtain

$$\int_0^1 \int_{P_H K} \phi_{n-k}(x)\,\mathrm{d}x \int_{\mathbb{R}^k} \chi_{\mathcal{C}_{0,t}}(y)\,\mathrm{d}y\,\mathrm{d}t \le \binom{n}{k}\int_0^1 \int_{P_H K} \phi_{n-k}(x)\int_{\mathbb{R}^k} \chi_{\mathcal{C}_{x,t}}(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore, by Fubini's theorem we have

$$\begin{aligned}
\mu_{n-k}\big(P_H K\big)\mu_k\big(K\cap H^\perp\big) &= \|\phi_k\|_\infty \int_{P_H K} \phi_{n-k}(x)\,\mathrm{d}x \int_{K(0)}\int_0^1 \chi_{\mathcal{C}_t(\phi_k)}(y)\,\mathrm{d}t\,\mathrm{d}y\\
&= \|\phi_k\|_\infty \int_0^1 \int_{P_H K} \phi_{n-k}(x)\,\mathrm{d}x \int_{\mathbb{R}^k} \chi_{\mathcal{C}_{0,t}}(y)\,\mathrm{d}y\,\mathrm{d}t\\
&\le \binom{n}{k}\|\phi_k\|_\infty \int_0^1 \int_{P_H K} \phi_{n-k}(x) \int_{\mathbb{R}^k} \chi_{\mathcal{C}_{x,t}}(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&= \binom{n}{k}\|\phi_k\|_\infty \int_{P_H K} \phi_{n-k}(x) \int_{K(x)}\int_0^1 \chi_{\mathcal{C}_t(\phi_k)}(y)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\\
&= \binom{n}{k}\int_{P_H K} \phi_{n-k}(x)\mu_k\big(K(x)\big)\mathrm{d}x = \binom{n}{k}\mu_n(K).
\end{aligned}$$

This conclude the proof. □

Next we show an extension of the above Rogers-Shephard type inequalities involving maximal sections of convex bodies (cf. (4.2)) in the spirit of [1, Lemma 4.1].

Corollary 4.2. *Let $i,j \in \{2,\dots,n-1\}$, $i+j \ge n+1$, and let $E \in \mathrm{G}(n,i)$, $H \in \mathrm{G}(n,j)$ be such that $E^\perp \subset H$. Let $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ be a $(-1/n)$-concave function and let μ be the measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$. Then, for every $K \in \mathcal{K}^n$,*

$$\sup_{x\in E^\perp} \mu_i\big(K\cap(x+E)\big) \sup_{y\in H^\perp} \mu_j\big(K\cap(y+H)\big) \le \binom{n-k}{n-i}\sup_{x\in\mathbb{R}^n} \mu_k\big(K\cap(x+F)\big)\mu(K), \tag{4.5}$$

where $F = E\cap H$.

Proof. Let $f : F^\perp \longrightarrow [0,\infty)$ be the function given by

$$f(x,y) = \int_{\mathbb{R}^k} \phi(x,y,z)\chi_K(x,y,z)\,\mathrm{d}z.$$

The Borell-Brascamp-Lieb inequality (see e.g. [16, Theorem 10.1]) implies that f is quasi-concave and, in particular, $\mathcal{C}_t(f)$ is a convex body. Then, we may apply Corollary 3.2 to obtain

$$\begin{aligned}
&\int_{E^\perp} \sup_{y\in H^\perp}\int_{\mathbb{R}^k} \phi(x,y,z)\chi_K(x,y,z)\,\mathrm{d}z\,\mathrm{d}x \sup_{x\in E^\perp}\int_{H^\perp}\int_{\mathbb{R}^k} \phi(x,y,z)\chi_K(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\\
&\le \binom{n-k}{n-i}\sup_{(x,y)\in F^\perp}\int_{\mathbb{R}^k} \phi(x,y,z)\chi_K(x,y,z)\,\mathrm{d}z \int_{F^\perp}\int_{\mathbb{R}^k} \phi(x,y,z)\chi_K(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

and thus, in particular, for every $y_0\in H^\perp$ we have

$$\int_{E^\perp}\int_{\mathbb{R}^k}\phi(x,y_0,z)\chi_{_K}(x,y_0,z)\,\mathrm{d}z\,\mathrm{d}x\sup_{x\in E^\perp}\int_{H^\perp}\int_{\mathbb{R}^k}\phi(x,y,z)\chi_{_K}(x,y,z)\,\mathrm{d}z\,\mathrm{d}y \leq\binom{n-k}{n-i}\sup_{(x,y)\in F^\perp}\mu_k\Big(K\cap\big((x,y)+F\big)\Big)\mu(K).$$

Hence, for every $y_0\in H^\perp$, we get

$$\mu_j\big(K\cap(y_0+H)\big)\sup_{x\in E^\perp}\mu_i\big(K\cap(x+E)\big)\leq\binom{n-k}{n-i}\sup_{x\in\mathbb{R}^n}\mu_k\big(K\cap(x+F)\big)\mu(K),$$

which implies (4.5). □

Next we show how one may exploit the approach we are following in this section to obtain an analogous result to Proposition 2.1, in the setting of quasi-concave densities which are not necessarily continuous. Notice that whereas the right-hand side in (4.6) is smaller than the right-hand side in (2.10), the constants $c(\omega)$ and $\phi(\omega)/\|\phi\|_\infty$ are not comparable in general.

Theorem 4.2. *Let $K\in\mathcal{K}^n$ and let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x)=\phi(x)\,\mathrm{d}x$, where $\phi:\mathbb{R}^n\longrightarrow[0,\infty)$ is a bounded quasi-concave function. Then, for every $\omega\in\mathbb{R}^n$,*

$$\frac{\phi(\omega)}{\|\phi\|_\infty}\mu(K-K+\omega)\leq\binom{2n}{n}\min\left\{\sup_{y\in K}\mu(y+\omega-K),\sup_{y\in K}\mu(-y+\omega+K)\right\}. \tag{4.6}$$

Moreover, if ϕ is continuous at ω_0, for some $\omega_0\in\mathbb{R}^n$, then equality holds in (4.6) (for such ω_0) if and only if μ is a constant multiple of the Lebesgue measure on $K-K+\omega_0$, $\phi(\omega_0)=\|\phi\|_\infty$ and K is a simplex.

Proof. Let $\omega\in\mathbb{R}^n$ and consider the function $f_\omega:K-K+\omega\longrightarrow[0,\infty)$ given by

$$f_\omega(x)=\mathrm{vol}\big(K\cap(x-\omega+K)\big).$$

Notice that, f_ω is $(1/n)$-concave by (1.1), $\operatorname{supp}f_\omega=K-K+\omega$ and, moreover, that $\|f_\omega\|_\infty=f_\omega(\omega)=\mathrm{vol}(K)$. Then, using (3.8), we get

$$\begin{aligned}\frac{\phi(\omega)}{\|\phi\|_\infty}\mu(K-K+\omega)&\leq\binom{2n}{n}\frac{1}{\mathrm{vol}(K)}\int_{\mathbb{R}^n}\mathrm{vol}\big(K\cap(x-\omega+K)\big)\,\mathrm{d}\mu(x)\\&=\binom{2n}{n}\frac{1}{\mathrm{vol}(K)}\int_{\mathbb{R}^n}\phi(x)\int_{\mathbb{R}^n}\chi_{_K}(y)\chi_{_{y+\omega-K}}(x)\,\mathrm{d}y\,\mathrm{d}x\\&=\binom{2n}{n}\frac{1}{\mathrm{vol}(K)}\int_K\mu(y+\omega-K)\,\mathrm{d}y\leq\binom{2n}{n}\sup_{y\in K}\mu(y+\omega-K).\end{aligned}$$

Therefore, exchanging the roles of K and $-K$, (4.6) infers.

Finally, if equality holds in (4.6) for some $\omega_0\in\mathbb{R}^n$ then, by Corollary 3.1, μ is a constant multiple of the Lebesgue measure on $K-K+\omega_0$ and $\phi(\omega_0)=\|\phi\|_\infty$. Now, from the equality case of Theorem A, K must be a simplex. The converse is immediate from Theorem A. □

We conclude this section by noticing that, from the proof of the previous result, one may also obtain (1.4) in the slightly less general setting of quasi-concave densities with maximum at the origin. We include it here for the sake of completeness.

Corollary 4.3. *Let $K \in \mathcal{K}^n$ and let μ be the measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ is a quasi-concave function with $\|\phi\|_\infty = \phi(0)$. Then*

$$\mu(K-K) \le \binom{2n}{n} \min\bigl\{\overline{\mu}(K), \overline{\mu}(-K)\bigr\}.$$

Moreover, if ϕ is continuous at the origin then equality holds if and only if μ is a constant multiple of the Lebesgue measure on $K-K$ and K is a simplex.

5. A remark for measures with p-concave densities, $p > 0$

As we have shown in Example 4.1, the assumption $P_H K \subset K$ on Theorems 1.3 and 4.1 is necessary. However, when dealing with measures associated to p-concave densities, $p > 0$, an inequality in the spirit of (1.13) can be obtained for an arbitrary $K \in \mathcal{K}^n$, by setting a binomial coefficient according to the concavity nature of the density. This is the content of the following result.

Theorem 5.1. *Let $k \in \{1, \dots, n-1\}$, $r \in \mathbb{N}$ and $H \in \mathrm{G}(n, n-k)$. Given a $(1/r)$-concave function $\phi_k : \mathbb{R}^k \longrightarrow [0,\infty)$, and a radially decreasing function $\phi_{n-k} : \mathbb{R}^{n-k} \longrightarrow [0,\infty)$, let $\mu_n = \mu_{n-k} \times \mu_k$ be the product measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu_{n-k}(x) = \phi_{n-k}(x)\,\mathrm{d}x$ and $\mathrm{d}\mu_k(y) = \phi_k(y)\,\mathrm{d}y$. Let $K \in \mathcal{K}^n$ be such that $\max_{x\in H} \mu_k\left(K \cap \left(x + H^\perp\right)\right) = \mu_k\left(K \cap H^\perp\right)$. Then*

$$\mu_{n-k}\bigl(P_H K\bigr)\mu_k\bigl(K \cap H^\perp\bigr) \le \binom{n+r}{n-k}\mu_n(K).$$

Proof. Consider the function $f : H \longrightarrow \mathbb{R}$ given by

$$f(x) = \mu_k\left(K \cap \left(x + H^\perp\right)\right),$$

which satisfies $\operatorname{supp} f = P_H K$.

Now, the Borell-Brascamp-Lieb inequality (see [16, Theorem 10.1]) implies that μ_k is $(1/(k+r))$-concave which, together with the convexity of K, yields that f is $(1/(k+r))$-concave. Furthermore, by assumption we have that $\|f\|_\infty = f(0)$. Thus, using (3.6) for $g = \phi_{n-k}$, we obtain

$$\alpha^{n-k}_{1/(k+r),0} \int_{P_H K} \phi_{n-k}(x)\,\mathrm{d}x \le \frac{1}{\mu_k\bigl(K \cap H^\perp\bigr)} \int_{P_H K} \mu_k\left(K \cap \left(x + H^\perp\right)\right)\,\phi_{n-k}(x)\mathrm{d}x$$

and hence

$$\mu_{n-k}\bigl(P_H K\bigr)\mu_k\bigl(K \cap H^\perp\bigr) \le \binom{n+r}{n-k}\mu_n(K),$$

as desired. □

The latter result can be stated for any positive real number r, just replacing $\binom{n+r}{n-k}$ by the suitable constant.

We notice that the above inequality includes (1.13) as a special case, since the constant density (of the Lebesgue measure) is ∞-concave, and thus $r = 0$.

Acknowledgements. We thank the referees for many valuable suggestions and remarks which have allowed us to considerably improve the manuscript.

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