ON THE ROOTS OF THE WILLS FUNCTIONAL

MARÍA A. HERNÁNDEZ CIFRE AND JESÚS YEPES NICOLÁS

Abstract. We investigate the roots of Wills polynomials of convex bodies. We study the structure, showing that the set of roots in the upper half plane is a convex cone, monotonous with respect to the dimension. In particular, we give its precise description for dimensions $n = 2, 3$. We also show that for $n \leq 7$ this cone is completely contained in the (open) left half plane, which is not true in dimensions ≥ 14. Moreover, we study the size of the roots of the Wills polynomial, bounding them in terms of functionals like the in- and circumradius of the set. We also relate the roots of the Steiner and the Wills polynomials.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$, and let B_n and C_n be the n-dimensional unit ball and cube of edge-length 1, respectively. The volume of a set $M \subsetneq \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$, and with $\mathrm{cl}\, M$, $\mathrm{conv}\, M$ we represent its closure and convex hull, respectively. For $K \in \mathcal{K}^n$ and a non-negative real number λ, the volume of the Minkowski sum $K + \lambda B_n$, is expressed as a polynomial of degree n in λ,

$$\mathrm{vol}(K + \lambda B_n) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K)\lambda^i. \tag{1.1}$$

This expression is known as the *Steiner formula* of K. The coefficients $\mathrm{W}_i(K)$ are the *quermassintegrals* of K, and they are a special case of the more general defined *mixed volumes* for which we refer to [21, s. 5.1]. In particular $\mathrm{W}_0(K) = \mathrm{vol}(K)$, $\mathrm{W}_n(K) = \mathrm{vol}(B_n) = \kappa_n$, $n\mathrm{W}_1(K) = \mathrm{S}(K)$ is the *surface area* of K and $(2/\kappa_n)\mathrm{W}_{n-1}(K) = \mathrm{b}(K)$ is the *mean width* of K ([21, p. 42]). The volume of the n-dimensional unit ball B_n takes the value

$$\kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}, \tag{1.2}$$

where Γ denotes the gamma function.

2000 *Mathematics Subject Classification.* Primary 52A20, 52A39; Secondary 30C15.

Key words and phrases. Wills functional, location of roots, successive minima, inradius, circumradius, Steiner polynomial.

Supported by MCI and DAAD, Ref. AIB2010DE-00082, by Subdirección General de Proyectos de Investigación (MCI) MTM2009-10418 and by "Programa de Ayudas a Grupos de Excelencia de la Región de Murcia", Fundación Séneca, 04540/GERM/06.

In [18], McMullen considered the normalized quermassintegrals

$$\mathrm{V}_i(K) = \binom{n}{i} \frac{\mathrm{W}_{n-i}(K)}{\kappa_{n-i}}, \tag{1.3}$$

and proposed to call these measures the *intrinsic volumes* of K, since, if K is k-dimensional, then $\mathrm{V}_k(K)$ is the usual k-dimensional volume of K. The intrinsic volumes depend only on the convex body K but not on the dimension of the embedding space (see e.g. [2, s. 6.4]). Thus the Steiner formula (1.1) can be represented via (1.3) as

$$\mathrm{vol}(K + \lambda B_n) = \sum_{i=0}^{n} \kappa_i \mathrm{V}_{n-i}(K) \lambda^i.$$

In [25] Wills introduced and studied the functional

$$W(\lambda K) = \sum_{i=0}^{n} \mathrm{V}_i(K) \lambda^i$$

because of its possible relation with the so-called lattice-point enumerator $G(K) = \#(K \cap \mathbb{Z}^n)$. Many nice properties of this functional, as well as relations with other measures, have been studied in the last years, see, for instance, [3, 4, 19, 25, 26, 27]. More recently, the Wills functional has been also considered from a more general point of view or in a probabilistic context (see [13] and [22, 23], respectively).

In the following we will write, for $K \in \mathcal{K}^n$,

$$g_K(z) = \sum_{i=0}^{n} \mathrm{V}_i(K) z^i = \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_{n-i}(K)}{\kappa_{n-i}} z^i \tag{1.4}$$

to denote the *Wills polynomial* of K, regarded as a formal polynomial in a complex variable $z \in \mathbb{C}$. Similarly, we will represent the classical Steiner polynomial (cf. (1.1)) in a variable $z \in \mathbb{C}$ by $f_{K;B_n}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K) z^i$.

Notice that $g_K(z)$ (and hence its roots) does not depend on the dimension of the space $\mathbb{R}^n$ where K is embedded, because the intrinsic volumes of K have this property. Thus, from now on and unless we explicitly say the opposite, we will always assume that for $K \in \mathcal{K}^n$, its dimension $\dim K = n$.

In [6, 7, 8, 9, 10, 11, 12, 16], geometric properties of the roots of (relative and classical) Steiner polynomials have been studied: their location, size, relation with other geometric magnitudes (in- and circumradius) and characterization of (families of) convex bodies.

Here we are interested in studying properties of the roots of the Wills polynomial $g_K(z)$, as, for instance, its location or relation with other functionals. To this end, we fix the notation which will be used along the paper. Denoting by $\mathrm{Re}\, z$, $\mathrm{Im}\, z$ and $\arg z$, the real part, imaginary part and the principal argument of a complex number z, respectively, let $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}\, z \geq 0\}$ be the set of complex numbers with non-negative imaginary part, and let

$$\mathcal{R}_W(n) = \{z \in \mathbb{C}^+ : g_K(z) = 0 \ \text{ for } K \in \mathcal{K}^n\} \cup \{0\} \tag{1.5}$$

be the set of all roots of $g_K(z)$, $K \in \mathcal{K}^n$, in the upper half plane, plus the origin; notice that $g_K(0) \neq 0$ for any convex body K, since the constant term of $g_K(z)$ is always 1 for all $K \in \mathcal{K}^n$.

Theorem 1.1. *$\mathcal{R}_W(n)$ is a convex cone, containing the non-positive real axis $\mathbb{R}_{\leq 0}$ and monotonous in the dimension, i.e., $\mathcal{R}_W(n) \subset \mathcal{R}_W(n+1)$.*

Now, for a fixed convex body $E \in \mathcal{K}^n$, let

$$\theta_E = \min\bigl\{\arg z : z \in \mathbb{C}^+,\, g_E(z) = 0\bigr\}$$

and then we denote by

$$\mathcal{R}_W(E) = \bigl\{z \in \mathbb{C}^+ : \arg z \geq \theta_E\bigr\} \cup \{0\}$$

the convex cone, in the upper half plane, generated as the positive hull of the roots of the polynomial $g_E(z)$ and $\mathbb{R}_{\leq 0}$. Using this notation, we can precisely describe the cones $\mathcal{R}_W(2)$ and $\mathcal{R}_W(3)$, which are given by the roots of the Wills polynomial of the 2 and 3-dimensional unit balls, respectively. More precisely, we have the following result.

Theorem 1.2. *$\mathcal{R}_W(2) = \mathcal{R}_W(B_2)$ and $\mathcal{R}_W(3) = \mathcal{R}_W(B_3)$.*

We observe that, in particular, $\mathcal{R}_W(2)$ and $\mathcal{R}_W(3)$ are closed convex cones, but we do not know whether this holds in general.

Regarding the stability of the Wills polynomial, i.e., the fact that all its roots lie in the left half plane, we study the inclusion

$$\mathcal{R}_W(n) \subset \bigl\{z \in \mathbb{C}^+ : \operatorname{Re} z < 0\bigr\} \cup \{0\}, \tag{1.6}$$

property that we call "weak" stability.

Proposition 1.1. *Wills polynomials are weakly stable if $n \leq 7$. For $n \geq 14$ we have $\bigl\{z \in \mathbb{C}^+ : \operatorname{Re} z \leq 0\bigr\} \subsetneq \mathcal{R}_W(n)$.*

We also show that not all roots of $g_K(z)$ can be pure imaginary complex numbers (see Proposition 2.1); Wills polynomials share this property with Steiner polynomials.

The above results will be proved in Section 2, as well as some additional properties and consequences.

Observe that several of the above properties present restrictions in the dimension, in contrast with the known results for the roots of the general relative Steiner polynomial ([9]). It is due to the fact that in higher dimension we do not have enough information about the so called "full system" of inequalities among the quermassintegrals (cf. e.g., [2, Problem 6.1]).

Here we show an asymptotic relation between the roots of the Steiner and the Wills polynomials. It will be shown in Section 3.

Theorem 1.3. *For $s \in \mathbb{N}$ fixed, let $K \in \mathcal{K}^s$ and let $\mu_1, \dots, \mu_s$ be the roots of $g_K(z)$. Embedding $K \subsetneq \mathbb{R}^n$, $n \geq s$, let $\gamma_{1,n}, \dots, \gamma_{s,n}$ be the non-zero roots of $f_{K;B_n}(z)$. Then, reordering if necessary, it holds*

$$\lim_{n\to\infty} \frac{\kappa_n}{\kappa_{n-1}} \gamma_{i,n} = \frac{\mu_i}{|\mu_i|^2}, \quad i = 1, \dots, s.$$

In a sense, this theorem is saying that for high dimensions n, the Steiner polynomial $f_{K;B_n}(z)$ of a convex body K with fixed dimension $\dim K = s$ "behaves as" its Wills polynomial $g_K(z)$.

We also consider the problem to relate the roots of the Wills polynomial of a convex body K with other functionals, namely, the in- and circumradius of K and the so called successive minima of K with respect to the integer lattice. Section 4 is devoted to this topic; there we will state the precise definitions and results.

Finally, in Section 5 we study a very particular Wills polynomial, the one of the unit ball, which has interesting and nice properties.

2. The cone of the roots of the Wills functional

We start this section stating some preliminary lemmas which will be needed for the proof of Theorem 1.1.

In [15, Theorem 5.2] the following result is proved.

Theorem 2.1. [15, Theorem 5.2] *Let $\xi(t)$ be an unordered n-tuple of complex numbers, depending continuously on a real variable t in a (closed or open) interval I. Then there exist n continuous functions $\mu_i(t)$, $i = 1, \ldots, n$, the values of which constitute the n-tuple $\xi(t)$ for each $t \in I$.*

As a consequence of it, we get the following lemma.

Lemma 2.1. *Let $K(t) \in \mathcal{K}^n$, $t \in [a, b]$, be a one-parameter family of convex bodies with $\dim K(t) = n$ for all $t \in (a, b]$, $\dim K(a) = n - 1$ and so that $K(t)$ varies continuously on $t \in [a, b]$, and let $g_{K(t)}(z)$ be the corresponding one-parameter family of Wills polynomials, $t \in [a, b]$. Then:*

i) *There exist $n-1$ continuous functions $\mu_1, \ldots, \mu_{n-1} : [a, b] \longrightarrow \mathbb{C}$ joining the $n-1$ roots of $g_{K(a)}(z)$ and $n-1$ roots of $g_{K(b)}(z)$, such that $\mu_1(t), \ldots, \mu_{n-1}(t)$ are $n-1$ of the n roots of $g_{K(t)}(z)$ for all $t \in [a, b]$.*
ii) *Moreover, there exists another continuous function $\mu_n : (a, b] \longrightarrow \mathbb{C}$ such that $\mu_n(t)$ is the remaining root of $g_{K(t)}(z)$ for all $t \in (a, b]$, verifying that $\lim_{t \to a^+} \mu_n(t) = \infty$.*

Proof. We take the polynomials $\tilde{g}_{K(t)}(z) = \sum_{i=0}^{n} \mathrm{V}_{n-i}\big(K(t)\big) z^i$, $t \in [a, b]$, whose (non-zero) roots are the inverses of the roots of $g_{K(t)}(z)$ and have leading coefficients 1 for all $t \in [a, b]$. Then the result is a direct consequence of Theorem 2.1 and the fact that the roots of a polynomial are continuous functions of the coefficients of the polynomial (see e.g. [17, p. 3]). $\square$

Remark 2.1. *It is also well-known (see e.g.* [27, Proposition 3]*) that if P is an orthogonal box with edge lengths $a_1, \ldots, a_n > 0$, then the roots of $g_P(z)$ are $\mu_i = -1/a_i$, $i = 1, \ldots, n$. In particular, the Wills polynomial of the n-dimensional cube of edge length a, $g_{aC_n}(z)$, has an n-fold root $\mu = -1/a$.*

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. The inclusion $\mathcal{R}_W(n) \subset \mathcal{R}_W(n+1)$ is a direct consequence of the fact that intrinsic volumes remain unchanged if a convex body K is embedded in any Euclidean space of bigger dimension.

By the homogeneity of the intrinsic volumes (see e.g. [2, p. 105]) we have that for any $K \in \mathcal{K}^n$ and all $\lambda > 0$, $g_{\lambda K}(z) = g_K(\lambda z)$. Hence, if $\mu \in \mathcal{R}_W(n)$, $\mu \neq 0$, there exists $K \in \mathcal{K}^n$ such that $g_K(\mu) = 0$ and so, for each $\lambda > 0$,

$$0 = g_K(\mu) = g_{(1/\lambda)K}(\lambda\mu).$$

It implies that $\lambda\mu \in \mathcal{R}_W(n)$. This, together with the fact that for the cube $g_{C_n}(z) = (z+1)^n$ (see Remark 2.1), shows that $\mathcal{R}_W(n)$ is a cone containing the non-positive real axis.

In order to prove the convexity of $\mathcal{R}_W(n)$, we will proceed in two steps:

Step 1: First we show that if we consider the convex cone $\mathcal{R}_W(B_n)$, determined by the roots of B_n and the non-positive real axis, then all its points are roots of some Wills polynomial, i.e., $\mathcal{R}_W(B_n) \subset \mathcal{R}_W(n)$. We proceed by induction on n.

If $n = 1$, the result is obviously true, so, we suppose $n > 1$ and that the cone $\mathcal{R}_W(B_{n-1}) \subset \mathcal{R}_W(n-1)$. Notice that we can assume the strict inclusion $\mathcal{R}_W(B_{n-1}) \subsetneq \mathcal{R}_W(B_n)$, otherwise we directly have the required result.

For each $t \in [0,1]$, we consider the convex body $K(t) = tB_{n-1} + (1-t)B_n$ and its Wills polynomial $g_{K(t)}(z) = \sum_{i=0}^n \mathrm{V}_i\big(K(t)\big)z^i$, and let μ_n be a root of $g_{B_n}(z)$ such that $\arg\mu_n = \theta_{B_n}$. Thus, we have a one-parameter family of polynomials satisfying the conditions of Lemma 2.1, and hence there exists a continuous map $\mu : [0,1] \longrightarrow \mathbb{C}$ with $\mu(0) = \mu_n$ and $\mu(1) = \mu_{n-1}$ a root of $g_{B_{n-1}}(z)$, such that $\mu(t)$ is a root of $g_{K(t)}(z)$ for all $t \in [0,1]$. We notice that without loss of generality we may assume that μ_n is not the root which "gets lost" because, otherwise, we can work with its conjugate $\overline{\mu}_n$.

Therefore $f : [0,1] \longrightarrow (0, 2\pi)$ given by $f(t) = \arg\mu(t)$ is a continuous function with $f(1) = \arg\mu_{n-1} \geq \theta_{B_{n-1}}$ and $f(0) = \theta_{B_n}$, and so, using the intermediate value theorem, together with the fact that $\mathcal{R}_W(n)$ (and $\mathcal{R}_W(B_n)$) is a cone and the induction hypothesis, we can conclude that $\mathcal{R}_W(B_n) \subset \mathcal{R}_W(n)$.

Step 2: Finally we show that $\mathcal{R}_W(n)$ is convex, for which it suffices to prove that, fixed $\mu_0 \in \mathcal{R}_W(n)$, $\mu_0 \neq 0$, the cone

$$\mathcal{R}_W(n) \cap \big\{z \in \mathbb{C}^+ : \arg z \geq \arg\mu_0\big\} \tag{2.1}$$

is convex. To this end, let $K \in \mathcal{K}^n$, $\dim K = s$, be such that $g_K(\mu_0) = 0$, and let $K(t) = tB_s + (1-t)K$, $t \in [0,1]$. Since $\dim K(t) = s$ for all $t \in [0,1]$, $g_{K(t)}$ is always a polynomial of degree s, and thus (cf. Lemma 2.1) there exists a continuous map $\mu : [0,1] \longrightarrow \mathbb{C}$ with $\mu(0) = \mu_0$ and $\mu(1) = \mu_1$ a root of $g_{B_s}(z)$, such that $\mu(t)$ is a root of $g_{K(t)}(z)$ for all $t \in [0,1]$. Using an analogous argument as before and since $\mu_1 \in \mathcal{R}_W(B_s) \subset \mathcal{R}_W(s) \subset \mathcal{R}_W(n)$ by Step 1, we obtain that the cone given in (2.1) is convex. $\square$

Before giving the precise characterization of the cones $\mathcal{R}_W(2)$ and $\mathcal{R}_W(3)$, we study the stability of the Wills polynomial, since it will be needed in the proof of Theorem 1.2. The main ingredient in order to do it are the well-known inequalities

$$\mathrm{W}_i(K)^2 \geq \mathrm{W}_{i-1}(K)\mathrm{W}_{i+1}(K), \quad 1 \leq i \leq n-1, \tag{2.2}$$

and

$$\mathrm{W}_i(K)\mathrm{W}_j(K) \geq \mathrm{W}_k(K)\mathrm{W}_l(K), \;\; 0 \leq k < i < j < l \leq n, \tag{2.3}$$

particular cases of the Aleksandrov-Fenchel inequality (see e.g. [21, s. 6.3]).

Proof of Proposition 1.1. We use the following stability criterion (see [20, Theorem 3] and [14, Theorem 1]): a real polynomial $f(z) = \sum_{i=0}^{n} a_i\, z^i$, with $a_i > 0$ for $i = 0, \dots, n$, is stable if $a_{i-1}a_{i+2} \leq \beta\, a_i a_{i+1}$, $i = 1, \dots, n-2$, where $\beta \approx 0.4655$ is the only real solution of $z(z+1)^2 = 1$. It is easy to check that (2.3) ensures that this criterion is fulfilled for $n = 7$. The weak stability property for all $n \leq 6$ follows from the monotonicity of the cone of the roots (see Theorem 1.1).

Finally, it can be checked with a computer or by applying the Routh-Hurwitz criterion (see e.g. [17, p. 181]) that the polynomial

$$g_{B_{14}}(z) = \kappa_{14} \sum_{i=0}^{14} \binom{14}{i} \frac{1}{\kappa_{14-i}} z^i$$

has a root with positive real part ($\mu \approx 0.04562{+}1.81036\mathrm{i}$). The non-stability property for all $n \geq 14$ is deduced again from the monotonicity of the cones (see Theorem 1.1). $\square$

So only in dimensions $8 \leq n \leq 13$ we do not know whether Wills polynomials may have roots with positive real parts. Obviously, by the convexity of the cone $\mathcal{R}_W(n)$, the existence of a root with positive real part implies the existence of a pure imaginary complex root. However, not all roots can be of that type. More precisely:

Proposition 2.1. *There exists no convex body $K \in \mathcal{K}^n$ such that all roots of $g_K(z)$ are imaginary pure complex numbers (excluding the real root existing in odd dimension).*

The proof of this result is similar to the one of the corresponding result for the Steiner polynomial in [8, Proposition 2.1]. We sketch it here for completeness.

Proof. By Proposition 1.1 all roots of $g_K(z)$ are contained in the (open) left half plane if $n \leq 7$, and so we may assume that $n = \dim K \geq 8$.

Let $K \in \mathcal{K}^n$ be a convex body, n even, such that all roots of $g_K(z)$ are $\{\pm b_j\,\mathrm{i}, j = 1, \dots, n/2\}$, with all $b_j > 0$. Then we get

$$g_K(z) = \sum_{i=0}^{n} \mathrm{V}_i(K) z^i = \mathrm{vol}(K) \prod_{j=1}^{n/2} (z^2 + b_j^2),$$

which implies $\mathrm{V}_{2i+1}(K) = 0$ for all $i = 0, \dots, (n-2)/2$. In particular, $\mathrm{V}_1(K) = 0$, i.e., $\dim K = 0$, a contradiction.

For n odd, let $K \in \mathcal{K}^n$ be a convex body such that the roots of $g_K(z)$ are $\{-a, \pm b_j\,\mathrm{i}, j = 1, \dots, (n-1)/2\}$, with all $a, b_j > 0$. Then

$$g_K(z) = \sum_{i=0}^{n} \mathrm{V}_i(K) z^i = \mathrm{vol}(K)(z+a) \prod_{j=1}^{(n-1)/2} (z^2 + b_j^2)$$

and, in particular, we have

$$\mathrm{vol}(K)\, a \prod_{j=1}^{(n-1)/2} b_j^2 = 1, \quad \mathrm{vol}(K) \prod_{j=1}^{(n-1)/2} b_j^2 = \mathrm{V}_1(K), \quad \mathrm{vol}(K)\, a = \mathrm{V}_{n-1}(K).$$

Thus we get the relation $\mathrm{V}_{n-1}(K)\mathrm{V}_1(K) = \mathrm{vol}(K)$, which implies, by (1.3) and inequality (2.3), that $\kappa_{n-1}/\kappa_n \geq n^2/2$. It contradicts the well-known inequality

$$\sqrt{\frac{2\pi}{n+1}} < \frac{\kappa_n}{\kappa_{n-1}} < \sqrt{\frac{2\pi}{n}} \tag{2.4}$$

(see e.g. [24, Theorem 5.3.2 and p. 216]) since $n > 1$. □

Next we come to the proof of Theorem 1.2 in which we characterize the cones $\mathcal{R}_W(2)$ and $\mathcal{R}_W(3)$.

Proof of Theorem 1.2. We start determining the 2-dimensional cone $\mathcal{R}_W(2)$.

Let $-a + b\mathrm{i} \in \mathbb{C}^+$ be a root of a Wills polynomial $g_K(z)$ for some planar convex body $K \in \mathcal{K}^2$. By Proposition 1.1 and Theorem 1.1 we may assume that both $a, b > 0$. Thus $g_K(z) = \mathrm{vol}(K)(z^2 + 2az + a^2 + b^2)$, and we have the identities $2\mathrm{vol}(K)a = \mathrm{V}_1(K)$, $\mathrm{vol}(K)(a^2 + b^2) = 1$, from which we get

$$\mathrm{vol}(K) = \frac{1}{a^2 + b^2}, \qquad \mathrm{V}_1(K) = \frac{2a}{a^2 + b^2}.$$

Then, the isoperimetric inequality (cf. (2.2), $i = 1$) in terms of the intrinsic volumes, namely, $\mathrm{V}_1(K)^2 \geq \pi \mathrm{vol}(K)$, yields

$$b \leq \sqrt{\frac{4-\pi}{\pi}}\, a. \tag{2.5}$$

If we have equality in (2.5) then equality in the isoperimetric inequality holds, which implies that K is the Euclidean ball. Conversely, if $K = B_2$ then $g_{B_2}(z) = \pi z^2 + \pi z + 1$, whose (complex) roots give equality in (2.5). Therefore, equality holds in (2.5) if and only if $K = B_2$.

This together with the fact that $\mathcal{R}_W(2)$ is a cone (Theorem 1.1) shows that $\mathcal{R}_W(2) = \mathcal{R}_W(B_2) = \left\{x + y\mathrm{i} \in \mathbb{C}^+ : \sqrt{(4-\pi)/\pi}\, x + y \leq 0\right\}$.

Now we consider the 3-dimensional case.

Since $g_{B_3}(z) = (4\pi/3)z^3 + 2\pi z^2 + 4z + 1$, it can be checked that

$$m_0 = |\tan \theta_{B_3}| = \frac{\sqrt{3}(t_- + t_+)}{t_- - t_+ + 2\sqrt{\pi}} \approx 0.9624,$$

where $t_\pm=\left(\sqrt{6\pi^2-39\pi+64}\pm\sqrt{\pi}(\pi-3)\right)^{1/3}$.

Let $-a+b\mathrm{i}\in\mathbb{C}^+$ be a root of a Wills polynomial $g_K(z)$ for some $K\in\mathcal{K}^3$. By Proposition 1.1 and Theorem 1.1 we may assume that both $a,b>0$ and taking $m=b/a$, $m>0$, we have to show that $m\le m_0$. Let $-c$ be the real root of $g_K(z)$, $c>0$. Then we have the identities

$$(2a+c)=\frac{\mathrm{V}_2(K)}{\mathrm{vol}(K)},\quad (a^2+b^2+2ac)=\frac{\mathrm{V}_1(K)}{\mathrm{vol}(K)},\quad c(a^2+b^2)=\frac{1}{\mathrm{vol}(K)}, \tag{2.6}$$

and using (1.3), inequalities (2.2) for $i=1,2$ yield, in terms of a,c,m,

$$\begin{aligned} &i) \quad \frac{4}{3}c^2+\left(\frac{16}{3}-2\pi\right)ac+\left[\frac{16}{3}-\pi(1+m^2)\right]a^2\ge 0,\\ &ii) \quad \left[4\pi-8(1+m^2)\right]c^2+\left[4a(1+m^2)(\pi-4)\right]c+\pi a^2(1+m^2)^2\ge 0, \end{aligned} \tag{2.7}$$

respectively.

We assume $m>m_0$. On the one hand it can be seen that, since $c>0$, inequality (2.7) i) is equivalent to

$$c\ge\widetilde{c}=\frac{a\left(\sqrt{3\pi(4m^2+3\pi-12)}+3\pi-8\right)}{4}.$$

On the other hand, a direct computation shows that the above condition on m also implies that inequality (2.7) ii) holds if and only if

$$0<c\le\bar{c}=\frac{a(m^2+1)\left(\sqrt{2(\pi m^2-3\pi+8)}+\pi-4\right)}{2(2m^2-\pi+2)}.$$

Hence, $\widetilde{c}\le c\le\bar{c}$, which is a contradiction because it can be checked that condition $m>m_0$ gives $\bar{c}<\widetilde{c}$. Therefore $m\le m_0$, and using the convexity of the cone $\mathcal{R}_W(3)$ we get the result. Moreover, since equality in (2.2), $i=2$, holds only for the ball, an analogous argument to the one of the case $n=2$ shows that equality $m=m_0$ holds if and only if $K=B_3$. □

Remark 2.2. *From the above proof, it is also obtained that the ball B_n is the only convex body such that one of the roots of $g_{B_n}(z)$ lies on (determines) the boundary* $\mathrm{bd}\,\mathcal{R}_W(n)\backslash\mathbb{R}_{\le 0}$, $n=2,3$.

3. Relating the roots of the Wills and Steiner polynomials

Let $\mathcal{R}(n)=\left\{z\in\mathbb{C}^+ : f_{K;B_n}(z)=0 \text{ for some } K\in\mathcal{K}^n\right\}$ denote the set of all roots of the Steiner polynomial $f_{K;B_n}(z)$ in the upper half plane. In [8, Theorem 1.2] and [9, Proposition 1.2] it is proved that

$$\begin{aligned} \mathcal{R}(2)&=\mathbb{R}_{\le 0},\\ \mathcal{R}(3)&=\left\{x+y\mathrm{i}\in\mathbb{C}^+ : x+\sqrt{3}\,y<0\right\}\cup\{0\},\\ \mathcal{R}(4)&=\left\{x+y\mathrm{i}\in\mathbb{C}^+ : x+y<0\right\}\cup\{0\}. \end{aligned}$$

A first direct observation from Theorem 1.2 is that $\mathrm{cl}\,\mathcal{R}(n)\subsetneq\mathcal{R}_W(n)$ for $n=2,3$. Moreover, it is easy to check that in dimension 4, the cone $\mathcal{R}_W(B_4)=$

$\{x+y\mathrm{i}\in\mathbb{C}^+ : \alpha x+y<0\}$, $\alpha=1.42224\ldots$, and hence we also have the strict inclusion $\operatorname{cl}\mathcal{R}(4)\subsetneq\mathcal{R}_W(B_4)\subset\mathcal{R}_W(4)$. We cannot expect, however, that $\operatorname{cl}\mathcal{R}(n)\subsetneq\mathcal{R}_W(B_n)$ for any dimension; indeed, it can be checked with a computer or by applying the Routh-Hurwitz criterion that $g_{B_{12}}(z)$ is weakly stable, whereas the (weak) stability of the Steiner polynomial fails for $n=12$ (see [6, Remark 3.2]).

For complex numbers $z_1,\ldots,z_r\in\mathbb{C}$ let

$$\mathrm{s}_i\left(z_1,\ldots,z_r\right)=\sum_{\substack{J\subset\{1,\ldots,r\}\\ \#J=i}}\prod_{j\in J}z_j$$

denote the i-th elementary symmetric function of $z_1,\ldots,z_r$, $1\leq i\leq r$. In addition we set $\mathrm{s}_0\left(z_1,\ldots,z_r\right)=1$. Moreover, let γ_i, $i=1,\ldots,n$, be the roots of the Steiner polynomial $f_{K;B_n}(z)=\sum_{i=0}^n\kappa_i\mathrm{V}_{n-i}(K)z^i$ of $K\in\mathcal{K}^n$. From the identity $\sum_{i=0}^n\kappa_i\mathrm{V}_{n-i}(K)z^i=\kappa_n\prod_{i=1}^n(z-\gamma_i)$ we get

$$(-1)^i\frac{\kappa_{n-i}}{\kappa_n}\mathrm{V}_i(K)=\mathrm{s}_i\left(\gamma_1,\ldots,\gamma_n\right). \tag{3.1}$$

Similarly, taking the Wills polynomial $g_K(z)$ with roots μ_i, $i=1,\ldots,n$, from the relation $\sum_{i=0}^n\mathrm{V}_i(K)z^i=\mathrm{vol}(K)\prod_{i=1}^n(z-\mu_i)$ we get

$$(-1)^i\frac{\mathrm{V}_{n-i}(K)}{\mathrm{vol}(K)}=\mathrm{s}_i\left(\mu_1,\ldots,\mu_n\right). \tag{3.2}$$

Then from (3.1) and (3.2) we easily obtain the following relations between the roots of the Wills and the Steiner polynomials:

$$\begin{aligned}&\mathrm{s}_i\left(\gamma_1^{-1},\ldots,\gamma_n^{-1}\right)=\kappa_i\,\mathrm{s}_i\left(\mu_1,\ldots,\mu_n\right)\qquad\text{and}\\ &\mathrm{s}_i\left(\mu_1^{-1},\ldots,\mu_n^{-1}\right)=\frac{\kappa_n}{\kappa_{n-i}}\,\mathrm{s}_i\left(\gamma_1,\ldots,\gamma_n\right).\end{aligned}$$

However, just checking some easy examples, it can be seen that it is not possible to get relations of the type $\gamma_i=c(n)\mu_i$, for an n-dependent constant $c(n)$. Theorem 1.3 states a kind of asymptotic relation between them. For the proof of this theorem we need the following lemma.

Lemma 3.1. *Let $k\geq 0$. Then*

$$\lim_{n\to\infty}\frac{\kappa_{n-k}/\kappa_n}{\left(\kappa_{n-1}/\kappa_n\right)^k}=1.$$

Proof. Stirling's formula (see e.g. [24, Theorem 5.3.12] and [1, p. 24]) for the gamma function together with (1.2) yield the asymptotic formula

$$\lim_{n\to\infty}\frac{\kappa_n}{\left(\frac{2\pi e}{n}\right)^{n/2}\frac{1}{\sqrt{n\pi}}}=1.$$

Therefore we get

$$\lim_{n\to\infty}\frac{\kappa_{n-k}/\kappa_n}{(\kappa_{n-1}/\kappa_n)^k}=\lim_{n\to\infty}\frac{\left(\frac{2\pi e}{n-k}\right)^{\frac{n-k}{2}}\frac{1}{\sqrt{(n-k)\pi}}}{\left(\frac{2\pi e}{n}\right)^{n/2}\frac{1}{\sqrt{n\pi}}}\left(\frac{\left(\frac{2\pi e}{n}\right)^{n/2}\frac{1}{\sqrt{n\pi}}}{\left(\frac{2\pi e}{n-1}\right)^{\frac{n-1}{2}}\frac{1}{\sqrt{(n-1)\pi}}}\right)^k$$

$$=\lim_{n\to\infty}\frac{(n-1)^{k/2}\sqrt{n}}{n^{k/2}\sqrt{n-k}}\,\frac{(n-1)^{(n-1)k/2}\,n^{n/2}}{(n-k)^{(n-k)/2}\,n^{nk/2}}=1. \qquad \square$$

Proof of Theorem 1.3. We observe that for any $i=1,\dots,s$, μ_i is a root of $g_K(z)$ if and only if $\mu_i/|\mu_i|^2=1/\overline{\mu_i}$ is a root of $\tilde{g}_K(z)=\sum_{i=0}^{s}\mathrm{V}_{n-i}(K)z^i$. Thus it suffices to show that (reordering if necessary)

$$\lim_{n\to\infty}\frac{\kappa_n}{\kappa_{n-1}}\gamma_{i,n}=\tilde{\mu}_i,\qquad i=1,\dots,s, \tag{3.3}$$

where $\tilde{\mu}_i$, $i=1,\dots,s$, are the roots of $\tilde{g}_K(z)$.

Since $\dim K=s$, the Steiner polynomial takes the form

$$f_{K;B_n}(z)=\sum_{i=n-s}^{n}\kappa_i\mathrm{V}_{n-i}(K)z^i=z^{n-s}\sum_{j=0}^{s}\kappa_{n-s+j}\mathrm{V}_{s-j}(K)z^j,$$

and then, for any $i=1,\dots,s$, $\gamma_{i,n}$ is a (non-zero) root of $f_{K;B_n}(z)$ if and only if the complex number $\tilde{\gamma}_{i,n}=(\kappa_n/\kappa_{n-1})\gamma_{i,n}$ satisfies the relation

$$\sum_{j=0}^{s}\frac{\kappa_{n-s+j}}{\kappa_n}\left(\frac{\kappa_{n-1}}{\kappa_n}\right)^j\mathrm{V}_{s-j}(K)\,\tilde{\gamma}_{i,n}^j=0,$$

or equivalently, dividing by $(\kappa_{n-1}/\kappa_n)^s$, if and only if $\tilde{\gamma}_{i,n}$ is a root of the polynomial

$$\sum_{k=0}^{s}\frac{\kappa_{n-k}/\kappa_n}{(\kappa_{n-1}/\kappa_n)^k}\mathrm{V}_k(K)z^{s-k}=z^s+\mathrm{V}_1(K)z+\sum_{k=2}^{s}\beta_{k,n}\mathrm{V}_k(K)z^{s-k},$$

where for short we write $\beta_{k,n}=(\kappa_{n-k}/\kappa_n)/(\kappa_{n-1}/\kappa_n)^k$, $k=2,\dots,s$. By Lemma 3.1, the $s-2$ sequences $(\beta_{k,n})_n$ tends to 1 when n goes to infinity, which shows that the pointwise limit

$$\lim_{n\to\infty}\left(z^s+\mathrm{V}_1(K)z+\sum_{k=2}^{s}\beta_{k,n}\mathrm{V}_k(K)z^{s-k}\right)=\tilde{g}_K(z).$$

This, together with the fact that the roots of a polynomial are continuous functions of the coefficients, proves (3.3) and concludes the proof. $\square$

From Theorem 1.3 we immediately get the following corollary, which shows the asymptotic behavior of the (modulus and the argument of the) roots of the Steiner polynomial with respect to the ones of $g_K(z)$.

Corollary 3.1. *Let $K\in\mathcal{K}^s$ and let $\mu_1,\dots,\mu_s$ be the roots of $g_K(z)$. Embedding $K\subsetneq\mathbb{R}^n$, $n\geq s$, let $\gamma_{1,n},\dots,\gamma_{s,n}$ be the non-zero roots of $f_{K;B_n}(z)$. Then the following properties hold:*

i) $\lim_{n\to\infty} |\gamma_{i,n}| = \infty$, $i = 1, \ldots, s$.
ii) *Reordering if necessary,* $\lim_{n\to\infty} \arg \gamma_{i,n} = \arg \mu_i$, $i = 1, \ldots, s$.

Proof. Using (2.4),

$$\lim_{n\to\infty} |\gamma_{i,n}| = \lim_{n\to\infty} \frac{\kappa_{n-1}}{\kappa_n} \frac{1}{|\mu_i|} \geq \lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} \frac{1}{|\mu_i|} = \infty.$$

Property (ii) is straightforward. $\square$

4. The roots of the Wills polynomial and other functionals

For $K \in \mathcal{K}^n$, we denote by

$$\mathrm{r}(K) = \max\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\, B_n \subset K\},$$
$$\mathrm{R}(K) = \min\{R : \exists\, x \in \mathbb{R}^n \text{ with } K \subset x + R\, B_n\},$$

the usual in- and circumradius of K. Since, up to translations, $\mathrm{r}(K)B_n \subset K$ and $K \subset \mathrm{R}(K)B_n$ the following inequalities are a direct consequence of the monotonicity of the mixed volumes (cf. e.g. [21, p. 277]):

$$\mathrm{r}(K)\mathrm{W}_{i+1}(K) \leq \mathrm{W}_i(K) \leq \mathrm{R}(K)\mathrm{W}_{i+1}(K), \tag{4.1}$$

for $i \in \{0, \ldots, n-1\}$.

We start this section bounding the roots of the Wills functional in terms of the in- and circumradius.

Proposition 4.1. *Let $K \in \mathcal{K}^n$. Then the roots μ_i, $i = 1, \ldots, n$, of the Wills polynomial $g_K(z)$ are bounded by*

$$\frac{1}{\mathrm{V}_1(K)} \leq |\mu_i| \leq \frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)}. \tag{4.2}$$

Both inequalities are sharp. In particular, we have

$$\frac{1}{2n}\frac{1}{\mathrm{R}(K)} \leq |\mu_i| \leq \frac{n}{2}\frac{1}{\mathrm{r}(K)}.$$

Proof. It is known (see e.g. [17, p. 137]) that the roots of a polynomial $\sum_{j=0}^n a_j z^j$ with real coefficients $a_j > 0$ lie in the ring $\min\{a_j/a_{j+1}\} \leq |z| \leq \max\{a_j/a_{j+1}\}$, for $j = 0, 1, \ldots, n-1$. Hence in order to bound the roots of $g_K(z)$ we have to find the minimum and maximum of $\mathrm{V}_j(K)/\mathrm{V}_{j+1}(K)$, $j = 0, \ldots, n-1$. Writing this quotient via (1.3) in terms of the quermassintegrals, we get

$$\frac{\mathrm{V}_j(K)}{\mathrm{V}_{j+1}(K)} = \frac{j+1}{n-j}\frac{\kappa_{n-j-1}}{\kappa_{n-j}}\frac{\mathrm{W}_{n-j}(K)}{\mathrm{W}_{n-j-1}(K)}.$$

Aleksandrov-Fenchel inequalities (2.2) ensure that $\mathrm{W}_{n-j}(K)/\mathrm{W}_{n-j-1}(K)$ is increasing in j, and clearly $j+1$ is so. So we have to study the monotonicity of $\kappa_{n-j-1}/\big((n-j)\kappa_{n-j}\big)$ in j.

In order to do it, we consider the sequence $y_m = \kappa_{m-1}/(m\kappa_m)$. By (1.2) and properties of the gamma function (see e.g. [24, Section 5.3]), it is an easy computation to check that $\kappa_m/\kappa_{m-2} = 2\pi/m$. Then

$$\frac{1}{m+1}\frac{\kappa_m}{\kappa_{m-2}} = \frac{1}{m+1}\frac{2\pi}{m} = \frac{1}{m}\frac{\kappa_{m+1}}{\kappa_{m-1}},$$

and using Aleksandrov-Fenchel inequalities (2.2) for $\kappa_m = \mathrm{W}_m(C_n)$, we get

$$y_{m+1} = \frac{1}{m+1}\frac{\kappa_m}{\kappa_{m+1}} = \frac{1}{m}\frac{\kappa_{m-2}}{\kappa_{m-1}} \leq \frac{1}{m}\frac{\kappa_{m-1}}{\kappa_m} = y_m.$$

Therefore, y_m is a decreasing sequence in m, i.e., $\kappa_{n-j-1}/\bigl((n-j)\kappa_{n-j}\bigr)$ is an increasing sequence in j. Thus, altogether we get

$$\frac{1}{\mathrm{V}_1(K)} = \frac{1}{n}\frac{\kappa_{n-1}}{\mathrm{W}_{n-1}(K)} \leq \frac{\mathrm{V}_j(K)}{\mathrm{V}_{j+1}(K)} \leq \frac{n}{2}\frac{\mathrm{W}_1(K)}{\mathrm{W}_0(K)} = \frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)}$$

for $j = 0, \dots, n-1$, which shows (4.2).

We notice that for $n = 1$, any line segment gives equality in both inequalities. Moreover, for any dimension, let $Q(\ell)$ be the n-dimensional orthogonal box with edge-lengths $1, \ell, \dots, \ell$, $\ell \geq 1$, for which $\mathrm{V}_i\bigl(Q(\ell)\bigr) = \mathrm{s}_i\,(1, \ell, \dots, \ell)$ and $\mu_1 = 1$ is one of the roots of $g_{Q(\ell)}(z)$ (see Remark 2.1). Then

$$\lim_{\ell\to\infty} \frac{\mathrm{V}_{n-1}\bigl(Q(\ell)\bigr)}{\mathrm{vol}\bigl(Q(\ell)\bigr)} = \lim_{\ell\to\infty} \frac{\ell^{n-1} + (n-1)\ell^{n-2}}{\ell^{n-1}} = 1 = |\mu_1|,$$

which shows that the upper bound is sharp. Analogously, taking $\bar{Q}(\ell)$ the n-dimensional orthogonal box with edge-lengths $1, \ell, \dots, \ell$, $\ell \leq 1$, then

$$\lim_{\ell\to 0} \frac{1}{\mathrm{V}_1\bigl(\bar{Q}(\ell)\bigr)} = \lim_{\ell\to 0} \frac{1}{(n-1)\ell + 1} = 1 = |\mu_1|,$$

which shows that the lower bound is sharp.

The bounds in terms of the in- and circumradius follow immediately from (4.1) (via (1.3)), taking into account that $\kappa_{n-1}/\kappa_n \geq 1/2$ for all $n \geq 1$. □

For the next proposition, we need to deal with a special kind of sets. Tangential bodies can be defined in several equivalent ways; here we will use the following one: a convex body $K \in \mathcal{K}^n$ containing a ball rB_n is called a *tangential body*, if it holds the equality $\mathrm{W}_0(K) = \mathrm{r}(K)\mathrm{W}_1(K)$ (cf. (4.1), $i = 0$). The n-dimensional cube is an example of this type of bodies. For an exhaustive study of the more general defined p-tangential bodies we refer to [21, Section 2.2] and [21, Theorem 6.6.16].

Proposition 4.2. *Let $K \in \mathcal{K}^n$ and let μ_i, $i = 1, \dots, n$, be the roots of the Wills polynomial $g_K(z)$. If $\operatorname{Re}\mu_i = -a$, $a > 0$, for all $i = 1, \dots, n$, then*

$$\frac{1}{2\mathrm{R}(K)} \leq a \leq \frac{1}{2\mathrm{r}(K)}.$$

Equality holds in the right inequality if and only if K is a tangential body.

Proof. Using (3.2) for $i=1$ and (1.3), we have

$$-na=\sum_{i=1}^{n}\operatorname{Re}\mu_i=\sum_{i=1}^{n}\mu_i=-\frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)}=-\frac{n}{2}\frac{\mathrm{W}_1(K)}{\mathrm{W}_0(K)},$$

and thus, with by (4.1),

$$\frac{1}{2\mathrm{R}(K)}\leq a\leq\frac{1}{2\mathrm{r}(K)}.$$

Finally, equality $a=1/\bigl(2\mathrm{r}(K)\bigr)$ holds if and only if we have equality in $\mathrm{W}_0(K)\geq\mathrm{r}(K)\mathrm{W}_1(K)$, i.e., when K is a tangential body. $\square$

Proposition 4.2 contrasts with the case of the Steiner polynomial, where only the one of the ball can have all its roots with equal real part (in fact, it has an n-fold real root).

Remark 4.1. *From the above argument we also notice that*

$$\frac{n}{2\mathrm{R}(K)}\leq\bigl|\operatorname{Re}\mu_1+\cdots+\operatorname{Re}\mu_n\bigr|\leq|\operatorname{Re}\mu_1|+\cdots+|\operatorname{Re}\mu_n|.$$

In [27] Wills studied relations between the roots of the Wills polynomial of a 0-symmetric convex body, i.e., such that $K=-K$, and its successive minima, which we introduce next. Here we slightly improve some of those relations.

We denote by $\mathbb{Z}^n$ the integer lattice, i.e., the lattice of all points with integral coordinates in $\mathbb{R}^n$. For $K\in\mathcal{K}^n$ 0-symmetric, the i-th successive minimum $\lambda_i(K)$ of K, $i=1,\dots,n$, is defined as

$$\lambda_i(K)=\min\bigl\{\lambda\in\mathbb{R}:\lambda>0,\dim(\lambda K\cap\mathbb{Z}^n)\geq i\bigr\}.$$

Clearly $0<\lambda_1(K)\leq\cdots\leq\lambda_n(K)$, and they are homogeneous of degree -1, i.e., $\lambda_i(\alpha K)=(1/\alpha)\lambda_i(K)$. As a general reference for lattices and successive minima we refer to [2]. Here we show the following result.

Proposition 4.3. *Let $K\in\mathcal{K}^n$ be 0-symmetric and let μ_i, $i=1,\dots,n$, be the roots of the Wills polynomial, ordered such that $|\mu_1|\leq\cdots\leq|\mu_n|$. Then:*

i) $\lambda_{i+1}(K)\dots\lambda_n(K)<2^{n-i}\binom{n}{i}|\mu_{i+1}|\dots|\mu_n|$, $i=1,\dots,n-1$.
ii) $\lambda_n(K)+(n-1)\mathrm{r}(K)^{n-1}/\mathrm{R}(K)^n\leq-2(\mu_1+\cdots+\mu_n)$.

Equality holds in (ii) if and only if $K=B_n$.

It improves items (d) and (b) in [27, Theorem 1], respectively.

Proof. In [5] the following sharp inequality was proved for a 0-symmetric convex body $K\in\mathcal{K}^n$:

$$\lambda_{i+1}(K)\dots\lambda_n(K)\mathrm{vol}(K)<2^{n-i}\mathrm{V}_i(K),$$

$i=1,\dots,n-1$. This, together with (3.2), gives

$$\lambda_{i+1}(K)\dots\lambda_n(K)<2^{n-i}(-1)^{n-i}\mathrm{s}_{n-i}(\mu_1,\dots,\mu_n)\leq2^{n-i}\binom{n}{i}|\mu_{i+1}\dots\mu_n|.$$

On the other hand, the known Wills conjecture, proved independently by Bokowski and Diskant, states that $\mathrm{vol}(K)-\mathrm{r}(K)\mathrm{S}(K)+(n-1)\kappa_n\mathrm{r}(K)^n\leq 0$ (see e.g. [21, p. 324] and the references inside). Taking into account that $\lambda_n(K)\leq 1/\mathrm{r}(K)$, because $K\supset\mathrm{r}(K)B_n$, and also that $\mathrm{vol}(K)\leq\kappa_n\mathrm{R}(K)^n$ (cf. (4.1)), then using (3.2) we get the required inequality:

$$\begin{aligned}-2\sum_{i=1}^{n}\mu_i=2\frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)}=\frac{\mathrm{S}(K)}{\mathrm{vol}(K)}&\geq\frac{1}{\mathrm{r}(K)}+(n-1)\frac{\kappa_n}{\mathrm{vol}(K)}\mathrm{r}(K)^{n-1}\\&\geq\lambda_n(K)+(n-1)\frac{\mathrm{r}(K)^{n-1}}{\mathrm{R}(K)^n}.\end{aligned}$$

Since equality in Wills' conjecture holds if and only if K is the Euclidean ball, we obtain the same characterization for the equality case in (ii). $\square$

5. A brief note on the Wills polynomial of the ball

The Wills polynomial of the ball verifies the nice property (see [26, (4.4)])

$$i!\,\kappa_i\,g_{B_n}^{(n-i)}(z)=n!\,\kappa_n\,g_{B_i}(z), \tag{5.1}$$

where $g^{(k)}(z)$ denotes the k-th derivative of a polynomial $g(z)$.

We have also proved that the Wills polynomial of the ball determines the cone of roots, i.e., $\mathcal{R}_W(n)=\mathcal{R}_W(B_n)$, for dimensions $n=2,3$. In this section we show some additional properties of this particular polynomial $g_{B_n}(z)$ and the cone $\mathcal{R}_W(B_n)$.

Proposition 5.1. *The Wills polynomial $g_{B_n}(z)$ is weakly stable for $n\leq 13$ and it is not for $n\geq 14$. Moreover, $\mathcal{R}_W(B_{n-1})\subsetneq\mathcal{R}_W(B_n)$ if $n\leq 14$.*

Proof. Applying the stability criterion used in Proposition 1.1, it is easy to check that $g_{B_n}(z)$ is weakly stable for $n\leq 13$, whereas the polynomial $g_{B_{14}}(z)$ has a root with positive real part ($\mu\approx 0.04562+1.81036\mathrm{i}$).

Let $n\geq 14$ be any positive integer such that the polynomial $g_{B_n}(z)$ is not weakly stable. If we assume that $g_{B_{n+1}}(z)$ is weakly stable, then we have $\mathrm{conv}\bigl\{\mu: g_{B_{n+1}}(\mu)=0\bigr\}\subsetneq\{z\in\mathbb{C}:\mathrm{Re}\,z<0\}$. The well-known Gauss-Lucas theorem states that all roots of the derivative of a non-constant polynomial lie in the convex hull of the set of zeros of the polynomial (see e.g. [17, Theorem 6.1]). This result together with the fact $g'_{B_n}(z)=(n\kappa_n/\kappa_{n-1})g_{B_{n-1}}(z)$ (cf. (5.1)) shows that $g_{B_n}(z)$ is weakly stable, a contradiction. So, $g_{B_{n+1}}(z)$ is also weakly stable.

On the other hand, let $\bar{A}$ denote the set of conjugates of complex numbers in $A\subset\mathbb{C}^+$. Because of the (weak) stability of $g_{B_n}(z)$, the cone $\mathcal{R}_W(B_n)\cup\overline{\mathcal{R}_W(B_n)}$ is convex for $n<14$, and then it contains the set $\mathrm{conv}\bigl\{\mu: g_{B_n}(\mu)=0\bigr\}$. Again, Gauss-Lucas' theorem together with (5.1) prove that $\mathcal{R}_W(B_{n-1})\subset\mathcal{R}_W(B_n)$, $n<14$. Numerical computations give the strict inclusion. Finally, the non-stability of $g_{B_{14}}(z)$ concludes the proof. $\square$

Acknowledgement. The authors would like to thank the anonymous referee for very valuable comments and suggestions.

References

[1] E. Artin, *The gamma function*, Athena Series: Selected Topics in Mathematics, Holt, Rinehart and Winston, New York-Toronto-London 1964.

[2] P. M. Gruber, *Convex and Discrete Geometry*. Springer, Berlin Heidelberg, 2007.

[3] H. Hadwiger, Das Wills'sche Funktional, *Monatsh. Math.* **79** (1975), 213-221.

[4] H. Hadwiger, Gitterpunktanzahl im Simplex und Wills'sche Vermutung, *Math. Ann.* **239** (3) (1979), 271-288.

[5] M. Henk, Inequalities between successive minima and intrinsic volumes of a convex body, *Monatsh. Math.* **110** (3-4) (1990), 279–282.

[6] M. Henk, M. A. Hernández Cifre, Notes on the roots of Steiner polynomials, *Rev. Mat. Iberoamericana* **24** (2) (2008), 631–644.

[7] M. Henk, M. A. Hernández Cifre, Intrinsic volumes and successive radii, *J. Math. Anal. Appl.* **343** (2) (2008), 733–742.

[8] M. Henk, M. A. Hernández Cifre, On the location of roots of Steiner polynomials, *Bull. Braz. Math. Soc.* **42** (1) (2011), 153–170.

[9] M. Henk, M. A. Hernández Cifre, E. Saorín, Steiner polynomials via ultra-logconcave sequences, *To appear in Commun. Contemp. Math.* (2012).

[10] M. A. Hernández Cifre, E. Saorín, On the roots of the Steiner polynomial of a 3-dimensional convex body, *Adv. Geom.* **7** (2007), 275–294.

[11] M. A. Hernández Cifre, E. Saorín, Differentiability of quermassintegrals: a classification of convex bodies, to appear in *Trans. Amer. Math. Soc.*

[12] M. Jetter, Bounds on the roots of the Steiner polynomial, *Adv. Geom.* **11** (2011), 313–319.

[13] J. Kampf, On weighted parallel volumes, *Beiträge Algebra Geom.* **50** (2) (2009), 495-519.

[14] O. M. Katkova, A. M. Vishnyakova, A sufficient condition for a polynomial to be stable, *J. Math. Anal. Appl.* **347** (1) (2008), 81–89.

[15] T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition, Springer-Verlag, Berlin, 1995.

[16] V. Katsnelson, On H. Weyl and J. Steiner polynomials, *Complex Anal. Oper. Theory* **3** (1) (2009), 147–220.

[17] M. Marden, *Geometry of polynomials*, Mathematical Surveys, No. 3, American Mathematical Society, Providence, R.I., 1966, 2nd ed.

[18] P. McMullen, Non-linear angle-sum relations for polyhedral cones and polytopes. *Math. Proc. Cambridge Philos. Soc.* **78** (2) (1975), 247-261.

[19] P. McMullen, Inequalities between intrinsic volumes, *Monatsh. Math.* **111** (1) (1991), 47-53.

[20] Y. Y. Nie, X. K. Xie, New criteria for polynomial stability, *IMA J. Math. Control Inform.* **4** (1) (1987), 1–12.

[21] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.

[22] R. A. Vitale, The Wills functional and Gaussian processes, *Ann. Probab.* **24** (4) (1996), 2172-2178.

[23] R. A. Vitale, Y. Wang, The Wills functional for Poisson processes, *Statist. Probab. Lett.* **78** (14) (2008), 2181-2187.

[24] R. Webster, *Convexity*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1994.

[25] J. M. Wills, Zur Gitterpunktanzahl konvexer Mengen, *Elem. Math.* **28** (1973), 57-63.

[26] J. M. Wills, Nullstellenverteilung zweier konvexgeometrischer Polynome, *Beiträge Algebra Geom.* **29** (1989), 51-59.

[27] J. M. Wills, Minkowski's successive minima and the zeros of a convexity-function, *Monatsh. Math.* **109** (2) (1990), 157-164.

Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100-Murcia, Spain

E-mail address: mhcifre@um.es

E-mail address: jesus.yepes@um.es