CHARACTERIZING THE VOLUME VIA A BRUNN-MINKOWSKI TYPE INEQUALITY

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ABSTRACT. The Brunn-Minkowski inequality asserts that the n-th root of the functional volume is concave, namely, $\mathrm{vol}\big((1-\lambda)A+\lambda B\big)^{1/n}$ is greater than $(1-\lambda)\mathrm{vol}(A)^{1/n}+\lambda\mathrm{vol}(B)^{1/n}$ for compact sets A, B and $\lambda\in[0,1]$. Here we will show that if a given measure satisfies an inequality like this, with a certain positive power, for the family of all Euclidean balls then it must be a constant multiple of the volume.

1. INTRODUCTION

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., nonempty compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$, and let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. We write B_n for the n-dimensional Euclidean (closed) unit ball whereas $\mathcal{B}^n$ will denote the set of all closed balls in $\mathbb{R}^n$, i.e.,

$$\mathcal{B}^n=\{x+rB_n:\,x\in\mathbb{R}^n,r>0\}.$$

The volume of a measurable set $M\subset\mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ or $\mathrm{vol}_n(M)$ if the distinction of the dimension is useful (when integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$). With $\mathrm{int}\,M$, $\mathrm{bd}\,M$, $\mathrm{aff}\,M$ and $\dim M$ we represent its interior, boundary, affine hull and dimension (namely, the dimension of its affine hull), respectively.

Relating the volume with the Minkowski (vectorial) addition of convex bodies, one is led to the famous *Brunn-Minkowski inequality.* One form of it states that if $K,L\in\mathcal{K}^n$ and $\lambda\in(0,1)$, then

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}\geq(1-\lambda)\mathrm{vol}(K)^{1/n}+\lambda\mathrm{vol}(L)^{1/n}, \tag{1.1}$$

i.e., the n-th root of the volume is a concave function. Equality for some $\lambda\in(0,1)$ holds if and only if K and L either lie in parallel hyperplanes or are homothetic.

The Brunn-Minkowski inequality is one of the most powerful theorems in Convex Geometry and beyond: it implies, among others, strong results such

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as the isoperimetric and Urysohn inequalities (see e.g. [19, s. 7.2]) or even the Aleksandrov-Fenchel inequality (see e.g. [19, s. 7.3]). It would not be possible to collect here all references regarding versions, applications and/or generalizations on the Brunn-Minkowski inequality. So, for extensive and beautiful surveys on them we refer the reader to [1, 7].

The Brunn-Minkowski inequality in its simplest form states that

$$\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{1.2}$$

from where one can verify its equivalence with (1.1) just by using the homogeneity of the volume. Yet some other equivalent forms of (1.1) are that

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}, \tag{1.3}$$

which is often referred to in the literature as its multiplicative or dimension free form, and also that

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big) \geq \min\{\mathrm{vol}(K), \mathrm{vol}(L)\}. \tag{1.4}$$

The main goal of this paper is to know whether the dimensional forms of the Brunn-Minkowski inequality, (1.1) and (1.2), are also satisfied for other measures on $\mathbb{R}^n$ or whether they constitute an inherent property of the volume. To this aim, we need first to overview some results closely related to this inequality.

Regarding an analytical counterpart for functions of the Brunn-Minkowski inequality, one is naturally led to the *Prékopa-Leindler inequality*, originally proved in [15] and [12].

Theorem A (The Prékopa-Leindler inequality)**.** *Let $\lambda \in (0,1)$ and let $f,g,h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that*

$$h\big((1-\lambda)x+\lambda y\big) \geq f(x)^{1-\lambda}g(y)^{\lambda}$$

for all $x,y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{\lambda}. \tag{1.5}$$

In fact, a straightforward proof of (1.3) can be obtained by applying (1.5) to the characteristic functions $f = \chi_{_K}$, $g = \chi_{_L}$ and $h = \chi_{_{(1-\lambda)K+\lambda L}}$.

To further understand how the Prékopa-Leindler inequality is strongly related to the general Brunn-Minkowski inequality (1.1) one must know the so-called *Borell-Brascamp-Lieb inequality*. In order to introduce it, we first recall the definition of the p-th mean of two non-negative numbers, where p is a parameter varying in $\mathbb{R}\cup\{\pm\infty\}$ (for a general reference for p-means of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood, and Pólya [9] and to the handbook [4]). Consider first the case $p \in \mathbb{R}$ and $p \neq 0$; given $a,b \geq 0$ such that $ab \neq 0$ and $\lambda \in (0,1)$, we set

$$M_p(a,b,\lambda) = ((1-\lambda)a^p + \lambda b^p)^{1/p}.$$

For $p = 0$ we define

$$M_0(a, b, \lambda) = a^{1-\lambda} b^{\lambda}$$

and, to complete the picture, for $p = \pm\infty$ we set $M_{+\infty}(a, b, \lambda) = \max\{a, b\}$ and $M_{-\infty}(a, b, \lambda) = \min\{a, b\}$. Finally, if $ab = 0$, we will define $M_p(a, b, \lambda) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$. Note that $M_p(a, b, \lambda) = 0$, if $ab = 0$, is redundant for all $p \leq 0$, however it is relevant for $p > 0$. Furthermore, for $p \neq 0$, we shall allow that a, b take the value $+\infty$ and in that case, as usual, $M_p(a, b, \lambda)$ will be the value that is obtained "by continuity" with respect to p.

Jensen's inequality for means (see e.g [9, Section 2.9] and [4, Theorem 1 p. 203]) implies that if $-\infty \leq p < q < +\infty$ then

$$M_p(a, b, \lambda) \leq M_q(a, b, \lambda), \tag{1.6}$$

with equality for $ab > 0$ and $\lambda \in (0, 1)$ if and only if $a = b$.

The following theorem contains the Borell-Brascamp-Lieb inequality (see [2], [3] and also [7] for a detailed presentation), which uses the p-th mean to generalize the Prékopa-Leindler inequality (the case $p = 0$).

Theorem B (The Borell-Brascamp-Lieb inequality)**.** *Let $\lambda \in (0, 1)$, $-1/n \leq q \leq +\infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that*

$$h\big((1-\lambda)x + \lambda y\big) \geq M_q\left(f(x), g(y), \lambda\right)$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq M_p\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x,\, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x,\, \lambda\right),$$

where $p = q/(nq + 1)$.

As a direct application of this result we notice that, for $f = \chi_{_K}$, $g = \chi_{_L}$ and $h = \chi_{_{(1-\lambda)K+\lambda L}}$, (1.1) is obtained when taking $q = +\infty$ whereas (1.4) holds if we set $q = -1/n$ (and thus $p = -\infty$).

Regarding the functions which are naturally connected to the above theorem, we get to the following definition (see e.g. [3]).

Definition 1.1. *A non-negative function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is q-concave, for a given $q \in \mathbb{R} \cup \{\pm\infty\}$, if for all $x, y \in \mathbb{R}^n$ and all $\lambda \in (0, 1)$,*

$$f\big((1-\lambda)x + \lambda y\big) \geq M_q\left(f(x), f(y), \lambda\right).$$

A 0-concave function is usually called log-concave *whereas a $(-\infty)$-concave function is referred to as* quasi-concave*.*

Let μ be a measure on $\mathbb{R}^n$ with density function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$. If f is q-concave, with $-1/n \leq q \leq +\infty$, then by the Borell-Brascamp-Lieb inequality for $\overline{f} = f\,\chi_{_A}$, $\overline{g} = f\,\chi_{_B}$ and $\overline{h} = f\,\chi_{_{(1-\lambda)A+\lambda B}}$, we have that

$$\mu((1-\lambda)A + \lambda B) \geq M_p(\mu(A), \mu(B), \lambda) = \big((1-\lambda)\mu(A)^p + \lambda\mu(B)^p\big)^{1/p} \tag{1.7}$$

for any pair of measurable sets A, B with $\mu(A)\mu(B) > 0$ and such that $(1-\lambda)A + \lambda B$ is also measurable, where $p = q/(nq+1)$. From now on, a measure μ satisfying (1.7) will be said to be *p-concave*.

Borell [2, Theorem 3.2] (see also [6, Section 3.3]) gave a sort of converse to this statement:

Theorem C. *Let $-\infty \leq p \leq 1/n$ and let μ be a Radon measure on an open convex set $\Omega \subset \mathbb{R}^n$, which is also its support. If μ is p-concave on Ω then there exists a q-concave function f such that $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$, where $-1/n \leq q \leq +\infty$ is so that $p = q/(nq+1)$.*

In other words, p-concave measures are associated to q-concave functions (under the link $p = q/(nq+1)$) and vice versa. When $0 < p < 1/n$, q is non-negative and then f is q-concave if and only if f^q is concave on the convex set $\Omega = \{x \in \mathbb{R}^n : f(x) > 0\}$. Thus, on the one hand, since there are no further non-negative concave functions defined on the whole Euclidean space $\mathbb{R}^n$ than constants (see e.g. [16, Problem-Remark H, p. 8]), if μ is a Radon measure with support $\mathbb{R}^n$ that is p-concave, for some $p > 0$, then it must be, up to a constant, the volume. On the other hand, (for $p = 1/n$) since the sole $(+\infty)$-concave functions supported on open sets are those that are constants over them (see e.g. [3, p. 373]), if μ is a Radon measure supported on a certain open convex subset of $\mathbb{R}^n$ that is $(1/n)$-concave then the only possibility, once again, is that it is a constant multiple of the volume.

Here we are interested in showing this characterization of the volume, via the Brunn-Minkowski inequality, independently of Borell's result. Moreover, we will prove, on the one hand, that it is enough to assume that the measure satisfies the corresponding Brunn-Minkowski inequality for a subfamily of convex bodies: the set of all balls $\mathcal{B}^n$. More precisely, the main result of the paper reads as follows.

Theorem 1.2. *Let $p > 0$ and let $\Omega \subset \mathbb{R}^n$ be an open convex set. Let μ be a locally finite Borel measure on Ω such that*

$$\mu((1-\lambda)K + \lambda L)^p \geq (1-\lambda)\mu(K)^p + \lambda\mu(L)^p \tag{1.8}$$

holds for any pair of balls $K, L \in \mathcal{B}^n$ with $K, L \subset \Omega$, and all $\lambda \in (0,1)$. Then $\mu = c\,\mathrm{vol}_n$ for some (constant) $c \geq 0$ if either $\Omega = \mathbb{R}^n$ or $p = 1/n$.

When dealing with Brunn-Minkowski type inequalities (cf. (1.7)), it is natural to wonder about the improvement of the concavity of the corresponding measure (see e.g. [8], [10], [13] and [14]), i.e., whether it is possible to 'enhance' the exponent p for such an inequality, in the sense of considering a tighter p-th mean (cf. (1.6)). To this aim, many times one shows that the desired inequality is not true for arbitrary convex bodies and thus, it is necessary to consider the problem only for special subfamilies of sets. In this context, the characterization provided in this paper can be viewed as a useful tool for this type of problems, as well as another step for a better understanding of the extent and diversity of the Brunn-Minkowski inequality and its applications.

We will also explicitly show that the only condition needed for the measure is being locally finite (see Lemma 2.3), as well as that both the assumptions $p > 0$ when the support of the measure is $\mathbb{R}^n$ and $p = 1/n$ when it is an arbitrary open convex set are necessary (see Remark 3.5).

In contrast to Theorem 1.2, the additive version of the Brunn-Minkowski inequality (cf. (1.2)) characterizes the volume even in the case in which neither the measure satisfies this inequality on the whole space $\mathbb{R}^n$ nor the exponent p equals $1/n$. The only assumption needed to this aim is assuming that the origin is an interior point. More precisely, we show the following result.

Theorem 1.3. *Let $p > 0$ and let $\Omega \subset \mathbb{R}^n$ be an open convex set with $0 \in \Omega$. Let μ be a locally finite Borel measure on Ω such that*

$$\mu(K + L)^p \geq \mu(K)^p + \mu(L)^p$$

holds for any pair of balls $K, L \in \mathcal{B}^n$ with $K, L \subset \Omega$, and all $\lambda \in (0, 1)$. Then $\mu = c\,\mathrm{vol}_n$ for some (constant) $c \geq 0$.

The paper is organized as follows. Section 2 is mainly devoted to collecting some definitions and auxiliary well-known results, whereas Sections 3 and 4 are devoted to proving (among other results) Theorems 1.2 and 1.3. In Section 3 we will focus on the simpler case in which the measure is absolutely continuous with a continuous density function, for the purpose of showing the general case of an arbitrary locally finite measure along Section 4.

2. Background material and auxiliary results

For the sake of completeness, we will collect some definitions and well-known facts from measure theory that will be used throughout this work. We refer the reader to [5].

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty set. Let (Ω, μ) be a measure space where μ is a Borel measure on Ω, i.e., we will omit for simplicity the σ-algebra Σ which will be assumed to contain the σ-algebra of Borel sets in Ω, and that will be contained in the σ-algebra of Lebesgue measurable sets.*

We recall the definition of the support of a measure μ, which is the set $\mathrm{supp}(\mu) = \big\{x \in \Omega : \mu\big((x + rB_n) \cap \Omega\big) > 0$ *for all* $r > 0\big\}$. *When $\Omega = \mathbb{R}^n$, the support is closed because its complement is the union of the open sets of measure* 0. *Moreover, by compactness arguments, and writing $\mathbb{R}^n \setminus \mathrm{supp}(\mu)$ as a countable union of compact sets (cf.* [5, Proposition 1.1.6]*), we clearly have $\mu(\mathbb{R}^n \setminus \mathrm{supp}(\mu)) = 0$.*

Definition 2.2. *A measure μ on $\Omega \subset \mathbb{R}^n$ is said to be* locally finite *if for every point $x \in \Omega$ there exists $r = r(x) > 0$ such that $\mu\big(x + rB_n\big) < +\infty$. In particular, a locally finite measure is finite on every compact subset of Ω and hence is σ-*finite*, i.e., Ω is the countable union of measurable sets with finite measure.*

Moreover, a measure μ *is called a Radon measure if it is locally finite and* $\mu(A) = \sup\{\mu(K) : K \subset A$ *compact*$\}$ *for every measurable set* A*.*

In this paper we deal with the characterization of measures on (subsets of) $\mathbb{R}^n$ via the Brunn-Minkowski inequality. To this end, from now on, we will omit the measures that are not locally finite because in that case they are trivially defined on the sets with nonempty interior. This is the content of the following result.

Lemma 2.3. *Let* $p > 0$ *and let* $\Omega \subset \mathbb{R}^n$ *be an open convex set. Let* μ *be a Borel measure on* Ω *that is not locally finite. If*

$$\mu((1-\lambda)K + \lambda L)^p \geq (1-\lambda)\mu(K)^p + \lambda\mu(L)^p \tag{2.1}$$

holds for any pair of balls $K, L \in \mathcal{B}^n$ *with* $K, L \subset \Omega$*, and all* $\lambda \in (0,1)$*, then* $\mu(A) = +\infty$ *for all* $A \subset \Omega$ *with nonempty interior.*

Proof. It is enough to show that $\mu(B) = +\infty$ for all $B \in \mathcal{B}^n$, $B \subset \Omega$. Since μ is not locally finite, there exists $x_0 \in \Omega$ such that

$$\mu(x_0 + rB_n) = +\infty$$

for all $r > 0$ such that $x_0 + rB_n \subset \Omega$. Since Ω is open, for any $x \in \Omega$, $x \neq x_0$, there exist $y \in \Omega$ and $\lambda \in (0,1)$ such that $(1-\lambda)x_0 + \lambda y = x$. Thus, taking $K = x_0 + rB_n$ and $L = y + rB_n$ for $r > 0$ small enough, by (2.1) we have

$$\mu(x + rB_n)^p \geq (1-\lambda)\mu(x_0 + rB_n)^p + \lambda\mu(y + rB_n)^p = +\infty, \tag{2.2}$$

which concludes the proof. □

We would like to point out that the proof of the precedent result does not work for further values of p, because in that case we would also have (cf. (2.2)) that $\mu(x+rB_n) \geq M_p\big(\mu(x_0+rB_n), \mu(y+rB_n), \lambda\big)$, but the right-hand side would not be, in general, infinity.

Definition 2.4. *We recall that* μ *is said to be* concentrated *on a measurable set* $A \subset \Omega$ *if* $\mu(\Omega \setminus A) = 0$*. In this sense,* μ *is* singular *with respect to the Lebesgue measure if there exists a measurable set* $A \subset \Omega$ *such that* μ *is concentrated on* A *and* vol *is concentrated on* $\Omega \setminus A$*.*

Conversely, μ *is said to be* absolutely continuous *(with respect to the Lebesgue measure) if for every measurable set* $A \subset \Omega$*,* $\mu(A) = 0$ *whenever* $\mathrm{vol}(A) = 0$*.*

We would like to stress that, for simplicity, we will sometimes omit the Lebesgue measure in those notions, like absolutely continuous or singular, which refer to a specific measure in a context where more than one are mentioned. In the same way, expressions like for almost every x *will mean for* vol-almost every x*.*

The well-known Radon-Nikodym theorem (see e.g. [5, Theorem 4.2.2]) will play a relevant role throughout this paper.

Theorem D (Radon-Nikodym theorem)**.** *Let μ be a σ-finite measure on $\Omega \subset \mathbb{R}^n$. If μ is absolutely continuous (with respect to the Lebesgue measure) then there exists a measurable function $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ such that*

$$\mu(A) = \int_A f(x)\,\mathrm{d}x$$

for any measurable set $A \subset \Omega$.

Such a function f for a given σ-finite absolutely continuous measure is usually called a density function of μ, which will be denoted, for short, as $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$.

Another useful tool along the paper will be the Lebesgue decomposition theorem (we refer the reader to [5, Theorem 4.3.2]), which asserts that, roughly speaking, every σ-finite measure is the sum of an absolutely continuous measure and a singular one.

Theorem E (Lebesgue's decomposition theorem)**.** *Let μ be a σ-finite Borel measure on $\Omega \subset \mathbb{R}^n$. Then there is a unique measure μ_s and a unique (up to a null set) measurable function $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ such that*

$$\mathrm{d}\mu(x) = f(x)\mathrm{d}x + \mathrm{d}\mu_s(x), \tag{2.3}$$

where μ_s is singular with respect to the Lebesgue measure.

For a σ-finite Borel measure μ, (2.3) will be referred to as the Lebesgue decomposition of μ (with respect to the Lebesgue measure).

The following result (see e.g. [18, Theorem 7.7]) shows that the content of the fundamental theorem of calculus over the real line persists in the setting of the Lebesgue integral over the whole Euclidean space $\mathbb{R}^n$. To this end one must consider the so-called symmetric derivative of the measure μ given by $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$. First we recall the following definition (see e.g. [5]).

Definition 2.5. *A function $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is* locally integrable *if for every point $x \in \Omega$ there exists $r = r(x) > 0$ such that $\int_{x+rB_n} f(t)\,\mathrm{d}t < +\infty$. In particular, the integral of a locally integrable function is finite on every compact subset of Ω.*

Theorem F (Lebesgue's differentiation theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ be a locally integrable function. Then*

$$\lim_{r \to 0^+} \frac{1}{\mathrm{vol}(rB_n)} \int_{x+rB_n} f(t)\,\mathrm{d}t = f(x)$$

for almost every $x \in \Omega$.

Furthermore, if f is a continuous function then the above condition holds everywhere.

The above result admits a stronger version in the setting of the Radon-Nikodym theorem and the Lebesgue decomposition, as the following result shows (see e.g. [18, Theorem 7.14]).

Theorem G. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let μ be a locally finite Borel measure on Ω, and let $\mathrm{d}\mu(x) = f(x)\mathrm{d}x + \mathrm{d}\mu_s(x)$ be the Lebesgue decomposition of μ. Then*

$$\lim_{r\to 0^+} \frac{\mu(x+rB_n)}{\mathrm{vol}(rB_n)} = f(x)$$

for almost every $x \in \Omega$.

When working with singular measures, the symmetric derivative satisfies the following property (see e.g. [18, Theorem 7.15]).

Theorem H. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let μ be a Borel measure on Ω that is singular with respect to the Lebesgue measure. Then*

$$\lim_{r\to 0^+} \frac{\mu(x+rB_n)}{\mathrm{vol}(rB_n)} = +\infty$$

for μ-almost every $x \in \Omega$.

By Definition 2.4, we notice that Theorems G and H suppose to be the two faces of the same coin, in the sense that each of them shows what essentially happens in the sets where the absolutely continuous part and the singular one of a given locally finite measure are, respectively, concentrated.

3. Simple case: absolutely continuous measures with continuous Radon-Nikodym derivative

Here we will show the statement of Theorem 1.2 when working with absolutely continuous measures associated to continuous density functions. In other words, we will prove on the one hand that, for such a measure, assuming the Brunn-Minkowski inequality with exponent $p > 0$ (cf. (1.8)) in the whole Euclidean space $\mathbb{R}^n$, is equivalent to say that the measure is (up to a constant) the volume. On the other hand, the Brunn-Minkowski inequality with exponent $1/n$ in a given open convex set $\Omega \subset \mathbb{R}^n$ yields the same consequence. Moreover, the assumptions that either $\Omega = \mathbb{R}^n$ for a given $p > 0$ or $p = 1/n$ for an arbitrary open convex set $\Omega \subset \mathbb{R}^n$, are necessary (see Remark 3.5).

Furthermore, we would like to point out that along this paper we will not assume, in principle, that the exponent p is not bigger than $1/n$. However, we will get this constraint for p (unless we are dealing with the zero measure). Indeed, if the measure satisfies the Brunn-Minkowski inequality for $p > 1/n$, then it also does for $p = 1/n$ because of (1.6) and thus, from Theorem 1.2, the measure must be a constant multiple of the volume. The equality case of (1.1) (together with (1.6)) yields that $1/n$ is the 'largest' exponent for such an inequality for the volume, and hence this implies that the given measure must be the zero one.

We will start by showing that, if an absolutely continuous measure, with continuous Radon-Nikodym derivative, satisfies the Brunn-Minkowski inequality with exponent p then its density function must be quasi-concave (even when $p \leq 0$).

Lemma 3.1. *Let $p \in \mathbb{R} \cup \{\pm\infty\}$ and let μ be the measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$, where f is a (non-negative) continuous function. If*

$$\mu((1-\lambda)K + \lambda L) \geq \big((1-\lambda)\mu(K)^p + \lambda\mu(L)^p\big)^{1/p}$$

holds for any pair of balls $K, L \in \mathcal{B}^n$, and all $\lambda \in (0,1)$, then f is quasi-concave.

Proof. Suppose, by contradiction, that $f((1-\lambda_0)x+\lambda_0 y) < \min\{f(x), f(y)\}$ for certain $x, y \in \mathbb{R}^n$ and $\lambda_0 \in (0,1)$. Let $z = (1-\lambda_0)x + \lambda_0 y$ and let $\varepsilon > 0$ be such that $f(z) + \varepsilon < \min\{f(x), f(y)\} - \varepsilon$.

Since f is continuous there exists $\delta > 0$ such that $|f(t) - f(t')| < \varepsilon$ for all $t \in t' + \delta B_n$ and $t' \in \{x, y, z\}$. Then, taking $K = x + \delta B_n$ and $L = y + \delta B_n$, we have

$$\begin{aligned}
&\mu((1-\lambda_0)K + \lambda_0 L) = \mu\big(z + \delta B_n\big) = \int_{z+\delta B_n} f(t)\,\mathrm{d}t \\
&\leq \delta^n \kappa_n (f(z) + \varepsilon) < \delta^n \kappa_n \big(\min\{f(x), f(y)\} - \varepsilon\big) \\
&\leq \left[(1-\lambda_0)\left(\int_{x+\delta B_n} f(t)\,\mathrm{d}t\right)^p + \lambda_0\left(\int_{y+\delta B_n} f(t)\,\mathrm{d}t\right)^p\right]^{1/p} \\
&= \big((1-\lambda_0)\mu(K)^p + \lambda_0\mu(L)^p\big)^{1/p},
\end{aligned}$$

a contradiction. □

Next we will show the main result of this section, which is the particular case of Theorem 1.2 for absolutely continuous measures with continuous density function and $\Omega = \mathbb{R}^n$.

Theorem 3.2. *Let $p > 0$ and let μ be the measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$, where f is a (non-negative) continuous function. If*

$$\mu((1-\lambda)K + \lambda L)^p \geq (1-\lambda)\mu(K)^p + \lambda\mu(L)^p$$

holds for any pair of balls $K, L \in \mathcal{B}^n$, and all $\lambda \in (0,1)$, then $\mu = c\,\mathrm{vol}_n$ for some (constant) $c \geq 0$.

Proof. Suppose, by contradiction, that $f(x) \neq f(y)$ for some $x, y \in \mathbb{R}^n$. Without loss of generality, $\mathrm{aff}(\{x, y\}) = \{z \in \mathbb{R}^n : z_2 = \cdots = z_n = 0\}$. Let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be the function given by $f_1(s) = f(s, 0, \ldots, 0)$. Then $x = (x_1, 0, \ldots, 0)$, $y = (y_1, 0 \ldots, 0)$ and $f(x) = f_1(x_1)$, $f(y) = f_1(y_1)$.

Without loss of generality, we may assume that $f_1(x_1) > f_1(y_1)$. By Lemma 3.1, f is quasi-concave and then f_1 is so. Hence, assuming that $x_1 < y_1$, the quasi-concavity of f_1 implies that it must be decreasing in $[y_1, +\infty)$ and thus, the limit $0 \leq L = \lim_{t\to+\infty} f_1(t)$ exists (the case $x_1 > y_1$ is analogous). So, $L \leq f_1(y_1) < f_1(x_1)$ and thus we can find $\varepsilon > 0$ and $\lambda_0 \in (0,1)$ such that

$$L + \varepsilon < (f_1(x_1) - \varepsilon)(1-\lambda_0)^{1/p}. \tag{3.1}$$

Moreover, for such an ε, there exists $z_1 \in \mathbb{R}$ such that

$$f_1(t) \leq L + \frac{\varepsilon}{2} \tag{3.2}$$

for all $t \geq z_1$.

On the other hand, since f is continuous, there exists $\delta > 0$ such that

$$\big|f(t) - f(t')\big| < \frac{\varepsilon}{2} \tag{3.3}$$

for all $t \in t' + \delta B_n$ and $t' \in \{x, z\}$, where $z = (z_1, 0\dots, 0)$.

Let $K = x + \delta B_n$ and let $L = w + \delta B_n$, where $w = (w_1, 0\dots, 0)$ is so that $(1-\lambda_0)x + \lambda_0 w = z$. Then, by (3.1), (3.2) and (3.3), we have

$$\begin{aligned}
&\mu((1-\lambda_0)K + \lambda_0 L) = \mu\big(z + \delta B_n\big) = \int_{z+\delta B_n} f(t)\,\mathrm{d}t \\
&\leq \delta^n\kappa_n\left(f(z) + \frac{\varepsilon}{2}\right) = \delta^n\kappa_n\left(f_1(z_1) + \frac{\varepsilon}{2}\right) \leq \delta^n\kappa_n(L+\varepsilon) \\
&< \delta^n\kappa_n(f_1(x_1) - \varepsilon)(1-\lambda_0)^{1/p} = \delta^n\kappa_n(f(x) - \varepsilon)(1-\lambda_0)^{1/p} \\
&\leq \left[(1-\lambda_0)\left(\int_{x+\delta B_n} f(t)\,\mathrm{d}t\right)^p\right]^{1/p} = \big((1-\lambda_0)\mu(K)^p\big)^{1/p} \\
&\leq \big((1-\lambda_0)\mu(K)^p + \lambda_0\mu(L)^p\big)^{1/p},
\end{aligned}$$

a contradiction. □

Second proof of Theorem 3.2. For any $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$\left(\int_{(1-\lambda)x+\lambda y+rB_n} f(t)\,\mathrm{d}t\right)^p \geq (1-\lambda)\left(\int_{x+rB_n} f(t)\,\mathrm{d}t\right)^p + \lambda\left(\int_{y+rB_n} f(t)\,\mathrm{d}t\right)^p$$

for all $r > 0$ and thus, dividing by $\mathrm{vol}(rB_n)^p$ and using Theorem F, we get

$$f((1-\lambda)x + \lambda y)^p \geq (1-\lambda)f(x)^p + \lambda f(y)^p.$$

Then f^p is a concave function on the whole $\mathbb{R}^n$ and thus, since f is non-negative, f must be constant (see e.g. [16, Problem-Remark H, p. 8]). □

Now, we deal with the particular case $p = 1/n$ of Theorem 1.2, i.e., assuming the Brunn-Minkowski inequality with exponent $1/n$ over an arbitrary open convex set Ω. To this aim, we will start with the one-dimensional case, since the proof we present here will help us to better understand which approach 'should' be carried out for the corresponding n-dimensional case, shown in Theorem 3.4.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}$ be an open convex set and let μ be the measure on Ω given by $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$, where f is a (non-negative) continuous function. If*

$$\mu((1-\lambda)K + \lambda L) \geq (1-\lambda)\mu(K) + \lambda\mu(L) \tag{3.4}$$

holds for any pair of balls $K, L \in \mathcal{B}^1$ with $K, L \subset \Omega$, and all $\lambda \in (0,1)$, then $\mu = c\,\mathrm{vol}_1$ for some (constant) $c \geq 0$.

Proof. Let $x_0 \in \Omega$ and let $F : \Omega \longrightarrow \mathbb{R}$ be the function given by

$$F(x) = \int_{x_0}^{x} f(t)\,\mathrm{d}t.$$

Fix $x, y > x_0$, $x, y \in \Omega$, and take $K = [x_0, x]$ and $L = [x_0, y]$. Then, from (3.4), we get $F\big((1-\lambda)x + \lambda y\big) \geq (1-\lambda)F(x) + \lambda F(y)$. Since it is true for arbitrary $x, y \in (x_0, +\infty) \cap \Omega$ and $\lambda \in [0, 1]$, we may assure that F is concave on $(x_0, +\infty) \cap \Omega$. In the same way, we obtain that F is convex on $(-\infty, x_0) \cap \Omega$.

Moreover, since f is continuous, by the fundamental theorem of calculus we get $F'(x) = f(x)$, for all $x \in \Omega$. Now, the concavity of F on $(x_0, +\infty) \cap \Omega$ (resp. the convexity of F on $(-\infty, x_0) \cap \Omega$) implies that $f(x) = F'(x)$ is decreasing in $(x_0, +\infty) \cap \Omega$ (resp. $f(x) = F'(x)$ is increasing in $(-\infty, x_0) \cap \Omega$). Since $x_0 \in \Omega$ is arbitrary, f must be constant. □

As we have just seen in the above result, the 'local nature' of the Brunn-Minkowski inequality on (a subset of) $\mathbb{R}$ suggests us to employ some tools from differential calculus such as its fundamental theorem. Thus, for the general case, it seems to be natural to use the n-dimensional counterpart of the above-mentioned result in the setting of the Lebesgue integral, i.e., the Lebesgue differentiation theorem (Theorem F).

Moreover, we would like to point out that, although Borell's approach in [2] is quite involved and complicated, the underlying key idea of his proof of Theorem C is exploiting the Lebesgue differentiation theorem for boxes with suitable lengths (depending on the range in which the parameter p lies) in order to get the corresponding desired properties of concavity of the density function. This idea is similar to that of the proof we present here for the following result, which is included for the sake of completeness.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open convex set and let μ be the measure on Ω given by $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$, where f is a (non-negative) continuous function. If*

$$\mu((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda)\mu(K)^{1/n} + \lambda\mu(L)^{1/n} \tag{3.5}$$

holds for any pair of balls $K, L \in \mathcal{B}^n$ with $K, L \subset \Omega$, and all $\lambda \in (0, 1)$, then $\mu = c\,\mathrm{vol}_n$ for some (constant) $c \geq 0$.

Proof. Let $r_1, r_2 > 0$ be fixed. Then, for any $x, y \in \Omega$ and $\lambda \in [0, 1]$, by (3.5) we have

$$\left(\int_{(1-\lambda)x+\lambda y\,+\,r((1-\lambda)r_1+\lambda r_2)B_n} f(t)\,\mathrm{d}t\right)^{1/n}$$

$$\geq (1-\lambda)\left(\int_{x\,+\,rr_1B_n} f(t)\,\mathrm{d}t\right)^{1/n} + \lambda\left(\int_{y\,+\,rr_2B_n} f(t)\,\mathrm{d}t\right)^{1/n}$$

for all $r > 0$ small enough. Thus, dividing by $\mathrm{vol}(rB_n)^{1/n}$ and using Theorem F (together with the fact that the volume is homogeneous of degree n), we get

$$f((1-\lambda)x+\lambda y)^{1/n}((1-\lambda)r_1+\lambda r_2)\geq(1-\lambda)f(x)^{1/n}\,r_1+\lambda f(y)^{1/n}\,r_2.$$

Now, by taking limits in the latter expression as $r_2\to 0$ and $r_1\to 0$, respectively, we may assure that $f((1-\lambda)x+\lambda y)\geq\max\{f(x),f(y)\}$ (for any $x,y\in\Omega$ and all $\lambda\in[0,1]$). Hence, and since Ω is open, f must be constant. □

Remark 3.5. *In relation to Theorem 1.2, we would like to point out that the assumptions $p>0$ when $\Omega=\mathbb{R}^n$ or $p=1/n$ when Ω is an arbitrary open convex set, are necessary.*

Indeed, for $p=0$ (and hence, from the monotonicity of the p-means (1.6)*, also for any $p<0$) it is enough to consider the standard Gaussian measure γ in $\mathbb{R}^n$, which is given by*

$$\mathrm{d}\gamma(x)=\frac{1}{(2\pi)^{n/2}}e^{\frac{-|x|^2}{2}}\,\mathrm{d}x,$$

because it is log-concave and thus, by Theorem A, it satisfies the (multiplicative) Brunn-Minkowski inequality on the whole $\mathbb{R}^n$, i.e., (1.7) *holds for any pair of convex bodies $K,L\in\mathcal{K}^n$ and all $\lambda\in(0,1)$.*

On the other hand, let $q\in\mathbb{R}_{>0}$ and let μ_q be the measure given by

$$\mathrm{d}\mu_q(x)=(1-|x|)^{1/q}\chi_{B_n}(x)\,\mathrm{d}x.$$

Then, by the Borell-Brascamp-Lieb inequality, Theorem B, μ_q satisfies the Brunn-Minkowski inequality (1.7) *for $p=q/(nq+1)$. Thus, taking $\Omega=\operatorname{int}B_n$,* (1.8) *holds for any pair of non-degenerate convex bodies $K,L\in\mathcal{K}^n$ with $K,L\subset\Omega$, and all $\lambda\in(0,1)$.*

4. General case

The main goal of this section is to show Theorems 1.2 and 1.3. To this aim, we will prove that it is enough to work with absolutely continuous measures with continuous density function, and thus we may use the results that were previously obtained in Section 3. More precisely, we will show the following:

Lemma 4.1. *Let $p>0$ and let $\Omega\subset\mathbb{R}^n$ be an open convex set. Let μ be a locally finite Borel measure on Ω such that*

$$\mu((1-\lambda)K+\lambda L)^p\geq(1-\lambda)\mu(K)^p+\lambda\mu(L)^p$$

holds for any pair of balls $K,L\in\mathcal{B}^n$ with $K,L\subset\Omega$, and all $\lambda\in(0,1)$. Then μ is absolutely continuous and $\mathrm{d}\mu(x)=f(x)\mathrm{d}x$ where $f:\Omega\longrightarrow\mathbb{R}_{\geq 0}$ is continuous.

For the sake of simplicity, we will split the above result into another two, namely, Lemmas 4.2 and 4.4. We will start this section by showing that a locally finite measure that satisfies the Brunn-Minkowski inequality (1.8) is absolutely continuous. We would like to stress here the relevance of the assumption of locally finiteness, in contrast to what Lemma 2.3 ensures.

Lemma 4.2. *Let $p > 0$ and let $\Omega \subset \mathbb{R}^n$ be an open convex set. Let μ be a locally finite Borel measure on Ω such that*

$$\mu((1-\lambda)K + \lambda L)^p \geq (1-\lambda)\mu(K)^p + \lambda\mu(L)^p \tag{4.1}$$

holds for any pair of balls $K, L \in \mathcal{B}^n$ with $K, L \subset \Omega$, and all $\lambda \in (0,1)$. Then μ is absolutely continuous.

Proof. Since μ is locally finite, by Theorem E, there exist a singular measure μ_s and a measurable function $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ for which $\mathrm{d}\mu(x) = f(x)\mathrm{d}x + \mathrm{d}\mu_s(x)$. Moreover, by means of Theorem H, the set

$$A = \left\{ x \in \Omega : \lim_{r\to 0^+} \frac{\mu_s(x + rB_n)}{\mathrm{vol}(rB_n)} = +\infty \right\}$$

satisfies that $\mu_s(\Omega \setminus A) = 0$.

If A is nonempty, there exists $x_0 \in \Omega$ for which

$$\lim_{r\to 0^+} \frac{\mu_s(x_0 + rB_n)}{\mathrm{vol}(rB_n)} = +\infty. \tag{4.2}$$

Since Ω is open, for any $x \in \Omega$, $x \neq x_0$, there exist $y \in \Omega$ and $\lambda \in (0,1)$ such that $(1-\lambda)x_0 + \lambda y = x$. Then, by (4.1), for all $r > 0$ small enough we get

$$\begin{aligned} \mu(x + rB_n) &\geq \big((1-\lambda)\mu(x_0 + rB_n)^p + \lambda\mu(y + rB_n)^p\big)^{1/p} \\ &\geq (1-\lambda)^{1/p}\mu(x_0 + rB_n) \geq (1-\lambda)^{1/p}\mu_s(x_0 + rB_n). \end{aligned}$$

Now, the above inequality implies, by (4.2), that

$$\lim_{r\to 0^+} \frac{\mu(x + rB_n)}{\mathrm{vol}(rB_n)} = +\infty$$

for all $x \in \Omega$, a contradiction with the statement of Theorem G (we notice that $f(x) \neq +\infty$ for all $x \in \Omega$). Thus, A is empty and hence μ_s is identically zero. ☐

Remark 4.3. *We would like to stress, for the sake of completeness, the role played by both Theorem G and Theorem H in the proof of the precedent result, Lemma 4.2.*

On the one hand, regarding the singular part of μ, μ_s, we have that the set

$$A = \left\{ x \in \Omega : \lim_{r\to 0^+} \frac{\mu_s(x + rB_n)}{\mathrm{vol}(rB_n)} = +\infty \right\}$$

satisfies that $\mu_s(\Omega \setminus A) = 0$, because of Theorem H (and moreover, $\mathrm{vol}(A) = 0$, by Theorem G).

On the other hand, Theorem G together with the p-concavity of μ imply, using the above consequence of Theorem H, that A must be empty and thus μ_s is identically zero.

In order that the general case can be reduced to the one studied in the previous section, we must show that the Radon-Nikodym derivative can be chosen to be continuous. This is the content of the following result, which is proved with a quite standard argument, and whose proof is included for the sake of completeness.

Lemma 4.4. *Let $p>0$ and let $\Omega\subset\mathbb{R}^n$ be an open convex set. Let μ be a locally finite Borel measure on Ω given by $\mathrm{d}\mu(x)=f(x)\mathrm{d}x$. If*

$$\mu((1-\lambda)K+\lambda L)^p\geq(1-\lambda)\mu(K)^p+\lambda\mu(L)^p$$

holds for any pair of balls $K,L\in\mathcal{B}^n$ with $K,L\subset\Omega$, and all $\lambda\in(0,1)$, then there exists a (non-negative) continuous function $\phi:\Omega\longrightarrow\mathbb{R}_{\geq 0}$ such that $\phi(x)=f(x)$ for almost every $x\in\Omega$.

Proof. Let $\phi:\Omega\longrightarrow\mathbb{R}_{\geq 0}\cup\{+\infty\}$ be the function given by

$$\phi(x)=\liminf_{r\to 0^+}\frac{1}{\mathrm{vol}(rB_n)}\int_{x+rB_n}f(t)\,\mathrm{d}t.$$

On the one hand, we have that $\phi(x)=f(x)$ for almost every $x\in\Omega$, by Theorem F (we notice that f is locally integrable because μ is locally finite).

On the other hand, for any $x,y\in\Omega$ and $\lambda\in[0,1]$,

$$\begin{aligned}
&\phi((1-\lambda)x+\lambda y)=\liminf_{r\to 0^+}\frac{1}{\mathrm{vol}(rB_n)}\int_{(1-\lambda)x+\lambda y+rB_n}f(t)\,\mathrm{d}t\\
&\geq\liminf_{r\to 0^+}\frac{1}{\mathrm{vol}(rB_n)}\left[(1-\lambda)\left(\int_{x+rB_n}f(t)\,\mathrm{d}t\right)^p+\lambda\left(\int_{y+rB_n}f(t)\,\mathrm{d}t\right)^p\right]^{1/p}\\
&\geq\left[(1-\lambda)\left(\liminf_{r\to 0^+}\frac{1}{\mathrm{vol}(rB_n)}\int_{x+rB_n}f(t)\,\mathrm{d}t\right)^p\right.\\
&\left.+\lambda\left(\liminf_{r\to 0^+}\frac{1}{\mathrm{vol}(rB_n)}\int_{y+rB_n}f(t)\,\mathrm{d}t\right)^p\right]^{1/p}=\big((1-\lambda)\phi(x)^p+\lambda\phi(y)^p\big)^{1/p}.
\end{aligned}$$

So we have that ϕ^p is concave and thus, since Ω is open, $\phi(x)\neq+\infty$ for all $x\in\Omega$. Hence, we may assert that $\phi:\Omega\longrightarrow\mathbb{R}_{\geq 0}$ is continuous (see e.g. [17, Theorem 10.1]). □

Proof of Theorem 1.2. If $\Omega=\mathbb{R}^n$ the statement follows from Lemma 4.1 and Theorem 3.2. For $p=1/n$, in the same way, the result comes from Lemma 4.1 and Theorem 3.4 (also from Theorem 3.3 if $n=1$). □

One can easily check that a nonzero measure μ satisfying (1.8), for any pair of balls contained in an open set A, fulfills $A\subset\mathrm{supp}(\mu)$. Indeed, given $x_0\in\mathrm{supp}(\mu)$ (we notice that $\mathrm{supp}(\mu)\neq\emptyset$ because μ is nonzero), for any $x\in A$, $x\neq x_0$, there exist $y\in A$ and $\lambda\in(0,1)$ such that $(1-\lambda)x_0+\lambda y=x$, since A is open. Thus, taking $K=x_0+rB_n$ and $L=y+rB_n$ for $r>0$ small enough, from (1.8) we have that $x\in\mathrm{supp}(\mu)$.

Now, as a consequence of Theorem 1.2, we get the following result for p-concave measures with arbitrary support.

Theorem 4.5. *Let $p>0$ and let μ be a nonzero locally finite Borel measure on $\mathbb{R}^n$. Let $X=\operatorname{supp}(\mu)$, $H=\operatorname{aff} X$ and let $m=\dim X$. Suppose that μ is such that*

$$\mu((1-\lambda)K+\lambda L)\geq M_p\big(\mu(K),\mu(L),\lambda\big) \tag{4.3}$$

holds for any pair of balls $K,L\in\mathcal{B}^n$, and all $\lambda\in(0,1)$. Then $\mu_{|X}=c\operatorname{vol}_m$ for some (constant) $c>0$ if either $X=H$ or $p=1/m$.

Proof. Assume first that $m=n$ and let $\Omega=\operatorname{int}X$. By (4.3) and the definition of support, X is clearly convex. In particular, Ω is an open convex subset of $\mathbb{R}^n$ for which $\mu((1-\lambda)K+\lambda L)^p\geq(1-\lambda)\mu(K)^p+\lambda\mu(L)^p$ for any pair of balls $K,L\in\mathcal{B}^n$ with $K,L\subset\Omega$, and all $\lambda\in(0,1)$. Hence, by Theorem 1.2, $\mu_{|\Omega}=c\operatorname{vol}_n$ for some (constant) $c>0$. Without loss of generality we may assume that $c=1$ (otherwise we would work with the measure μ/c).

Thus, if $X=\mathbb{R}^n$, we are done. We next consider the case $X\neq\mathbb{R}^n$ and $p=1/n$. Since the boundary of a convex set has Lebesgue measure zero (see e.g. [11]), and together with the already proved fact that $\mu_{|\Omega}=\operatorname{vol}_n$, it is enough to show that $\mu(\operatorname{bd}X)=0$ (we notice that X is a closed convex set with nonempty interior Ω and then $X=\Omega\cup\operatorname{bd}X$). To this end, by compactness arguments and by means of the relation

$$\mu(\operatorname{bd}X)=\mu\left(\bigcup_{k=1}^{+\infty}\big((\operatorname{bd}X)\cap kB_n\big)\right)=\lim_k\mu\big((\operatorname{bd}X)\cap kB_n\big),$$

it is enough to show that for each $x\in\operatorname{bd}X$ there exists $r=r_x>0$ such that $\mu\big((x+rB_n)\cap\operatorname{bd}X\big)=0$.

Suppose by contradiction that there exists $x\in\operatorname{bd}X$ such that

$$\mu\big((x+rB_n)\cap\operatorname{bd}X\big)>0 \tag{4.4}$$

for all $r>0$. Let $x_0\in\Omega$ and let $r_0>0$ such that $x_0+r_0B_n\subset\Omega$ and $(x+x_0)/2+r_0B_n\subset\Omega$. Let $K=(x+r_0B_n)\cap X$ and $L=(x_0-x)+K\subset x_0+r_0B_n\subset\Omega$. Then $(K+L)/2\subset(x+x_0)/2+r_0B_n\subset\Omega$ and so, by (4.4) and the (equality case of the) Brunn-Minkowski inequality (1.1), we have

$$\begin{aligned}\mu\left(\frac{K+L}{2}\right)&=\operatorname{vol}\left(\frac{K+L}{2}\right)=M_{1/n}\big(\operatorname{vol}(K),\operatorname{vol}(L),1/2\big)\\&<M_{1/n}\big(\operatorname{vol}(K)+\mu(K\cap\operatorname{bd}X),\operatorname{vol}(L),1/2\big)=M_{1/n}\big(\mu(K),\mu(L),1/2\big),\end{aligned}$$

a contradiction with (4.3). We point out that $(K+L)/2$ is measurable because it is convex, since it is a convex combination of convex sets.

Now the general case $m\leq n$ follows from the n-dimensional one because $\mu(\mathbb{R}^n\setminus X)=0$ (cf. Definition 2.1) and thus (4.3) holds for any pair of balls $x+K$, $y+L$, with $x,y\in H$ and $K,L\in\mathcal{B}^m$, $K,L\subset H$, and all $\lambda\in(0,1)$. □

Proof of Theorem 1.3. Let $x_0\in\Omega$ be fixed. Since Ω is open, and $0\in\Omega$, there exists $r_0>0$ such that $r_0B_n\subset\Omega$ and $x_0+r_0B_n\subset\Omega$.

On the one hand, for all $u \in r_0B_n$ and all $r > 0$ small enough, we get

$$\mu(x_0 + u + 2rB_n) \geq \big(\mu(x_0 + rB_n)^p + \mu(u + rB_n)^p\big)^{1/p}.$$

Thus, dividing by $\mathrm{vol}(rB_n)$, we have

$$\frac{2^n\mu(x + 2rB_n)}{\mathrm{vol}(2rB_n)} \geq \frac{\mu_s(x_0 + rB_n)}{\mathrm{vol}(rB_n)}$$

for all $x \in (x_0 + r_0B_n) \subset \Omega$, where $\mathrm{d}\mu(x) = f(x)\mathrm{d}x + \mathrm{d}\mu_s(x)$ is the Lebesgue decomposition of μ.

So, by the above expression (and following the same steps to the proof of Lemma 4.2), one may conclude that μ is absolutely continuous.

Now, on the other hand, let $\phi : \Omega \longrightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ be the function given by

$$\phi(x) = \liminf_{r \to 0^+} \frac{1}{\mathrm{vol}(rB_n)} \int_{x+rB_n} f(t)\,\mathrm{d}t.$$

Then, following similar steps to the proof of Lemma 4.4 and taking $K = x_0 + (1-\lambda)rB_n$, $L = u + \lambda rB_n$ for $\lambda \in (0,1)$ and $r > 0$ small enough, we obtain

$$\phi(x_0 + u) \geq \Big(\big((1-\lambda)^n\phi(x_0)\big)^p + \big(\lambda^n\phi(u)\big)^p\Big)^{1/p} \geq (1-\lambda)^n\phi(x_0).$$

Taking limits as $\lambda \to 0^+$ we may assert that $\phi(x) \geq \phi(x_0)$ for all $x \in x_0 + r_0B_n$. Exchanging the roles of x and x_0 we have that ϕ is constant on $x_0 + r_0B_n$. Since x_0 is arbitrary, we get that ϕ is constant on every compact subset $C \subset \Omega$ and thus ϕ is so on Ω. The proof is now concluded because $\phi(x) = f(x)$ for almost every $x \in \Omega$ (by Theorem F). □

Remark 4.6. *Let E be a convex body with nonempty interior. The role played by $\mathcal{B}^n$ along this paper can be replaced by $\mathcal{F}^n = \{x + rE : x \in \mathbb{R}^n, r > 0\}$ since all the tools here involved are also true when exchanging the Euclidean unit ball B_n by E (see e.g.* [18, Definition 7.9] *and consequent results).*

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