ON A DISCRETE BRUNN-MINKOWSKI TYPE INEQUALITY

MARÍA A. HERNÁNDEZ CIFRE, DAVID IGLESIAS, AND JESÚS YEPES NICOLÁS

Abstract. The Brunn-Minkowski inequality states that the volume of compact sets $K, L \subset \mathbb{R}^n$ satisfies $\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}$. In this paper we obtain two discrete analogs of it for the cardinality of finite subsets of the integer lattice $\mathbb{Z}^n$. On one hand we prove that if $A, B \subset \mathbb{Z}^n$ are finite, then $\left|\bar{A} + B\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$, where $\bar{A}$ is an extension of A which is constructed by adding some new integer points in a particular way; on the other hand, removing points of A, say, an equivalent inequality of the form $|A + B|^{1/n} \geq \left|\mathrm{r}(A)\right|^{1/n} + |B|^{1/n}$ can be obtained, where $r(A)$ is the reduced set of A. Both inequalities are sharp, and it can be seen that the number of additional points in $\bar{A}$ cannot be too large, and depends only on A. Finally we also prove that the classical Brunn-Minkowski inequality for compact sets can be obtained as a consequence of these new discrete versions.

1. Introduction and notation

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space, and we denote by e_i the i-th canonical unit vector. The n-dimensional volume of a compact set $K \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(K)$, and as a discrete counterpart, we use $|A|$ to represent the cardinality of a finite subset $A \subset \mathbb{R}^n$. We write $\pi_{i_1,\dots,i_k}$, $1 \leq i_1, \dots, i_k \leq n$, to denote the orthogonal projection onto the k-dimensional coordinate plane $\mathbb{R}\mathrm{e}_{i_1} + \dots + \mathbb{R}\mathrm{e}_{i_k}$. For the sake of brevity we just write $H_i = \mathbb{R}\mathrm{e}_1 + \dots + \mathbb{R}\mathrm{e}_{i-1} + \mathbb{R}\mathrm{e}_{i+1} + \dots + \mathbb{R}\mathrm{e}_n$ to represent the i-th coordinate hyperplane and $\pi_{(i)} = \pi_{1,\dots,i-1,i+1,\dots,n}$ for the corresponding orthogonal projection onto H_i.

Let $\mathbb{Z}^n$ be the integer lattice, i.e., the lattice of all points with integer coordinates in $\mathbb{R}^n$, and let $\mathbb{Z}^n_+ = \left\{x \in \mathbb{Z}^n : x_i \geq 0\right\}$. Special sets that will appear throughout the paper are the *lattice sets*: a finite set $A \subset \mathbb{Z}^n$ is a (convex) lattice set if $A = (\mathrm{conv}\, A) \cap \mathbb{Z}^n$, where $\mathrm{conv}\, A$ represent the convex hull of A. In particular, we denote by C^n_r, $r \in \mathbb{Z}_{>0}$, the *lattice cube*

$$\mathrm{C}^n_r = r[0,1]^n \cap \mathbb{Z}^n,$$

with $r + 1$ integer points in its edges.

2010 *Mathematics Subject Classification.* Primary 52C07, 11H06; Secondary 52A40.

Key words and phrases. Brunn-Minkowski inequality, integer lattice, sum set, cardinality.

Supported by MINECO/FEDER project MTM2015-65430-P and "Programa de Ayudas a Grupos de Excelencia de la Región de Murcia", Fundación Séneca, 19901/GERM/15.

Relating the volume with the Minkowski addition of compact sets, one is led to the famous Brunn-Minkowski inequality. One form of it states that if $K, L \subset \mathbb{R}^n$ are compact, then

$$\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{1.1}$$

with equality, if $\mathrm{vol}(K)\mathrm{vol}(L) > 0$, if and only if K and L are homothetic compact convex sets. Here $+$ is used for the *Minkowski* (vectorial) sum, i.e.,

$$A + B = \{a + b : a \in A, b \in B\}$$

for any $A, B \subset \mathbb{R}^n$. The Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond: for instance, its equivalent analytic version (the Prékopa-Leindler inequality, see e.g. [8, Theorem 8.14]) and the fact that the convexity/compactness assumption can be weakened to Lebesgue measurability (see [9]), have allowed it to move to much wider fields. It implies very important inequalities such as the isoperimetric and Urysohn inequalities (see e.g. [16, page 382]), and it has been the starting point for new developments like the L_p-Brunn-Minkowski theory (see e.g. [10, 11]), or a reverse Brunn-Minkowski inequality (see e.g. [13]), among many others. It would not be possible to collect here all references regarding versions, applications and/or generalizations of the Brunn-Minkowski inequality. For extensive and beautiful surveys on them we refer to [1, 5].

Next we move to the discrete setting, i.e., we consider finite sets of (integer) points which are not necessarily full-dimensional unless indicated otherwise. It can easily be seen that one cannot expect to obtain a Brunn-Minkowski inequality for the cardinality in the classical form. Indeed, simply taking $A = \{0\}$ to be the origin and any finite set $B \subset \mathbb{Z}^n$, then

$$|A + B|^{1/n} < |A|^{1/n} + |B|^{1/n}.$$

Therefore, a discrete Brunn-Minkowski type inequality should either have a different structure or involve modifications of the sets.

In [6], Gardner and Gronchi obtained a beautiful and powerful discrete Brunn-Minkowski inequality: they proved that if A, B are finite subsets of the integer lattice $\mathbb{Z}^n$, with dimension $\dim B = n$, then

$$|A + B| \geq \left|D^B_{|A|} + D^B_{|B|}\right|. \tag{1.2}$$

Here $D^B_{|A|}, D^B_{|B|}$ are *B-initial segments*: for any $m \in \mathbb{N}$, D^B_m is the set of the first m points of $\mathbb{Z}^n_+$ in the so-called "B-order", which is a particular order defined on $\mathbb{Z}^n_+$ which depends only on the cardinality of B. For a proper definition and a deep study of it we refer the reader to [6]. As consequences of (1.2) they also get two additional nice discrete Brunn-Minkowski type inequalities:

$$|A + B|^{1/n} \geq |A|^{1/n} + \frac{1}{(n!)^{1/n}}\bigl(|B| - n\bigr)^{1/n} \tag{1.3}$$

and, if $|B| \leq |A|$, then

$$|A+B| \geq |A| + (n-1)|B| + \big(|A|-n\big)^{(n-1)/n}\big(|B|-n\big)^{1/n} - \frac{n(n-1)}{2}.$$

These inequalities improve previous results obtained by Ruzsa in [14, 15].

2. How to transform a discrete set. The main results

An alternative way to get a "classical" Brunn-Minkowski type inequality might be to transform (one of) the sets involved in the problem, either by adding or removing some points. Then the question arises as to how many points one should add/remove to ensure the reliability of a Brunn-Minkowski inequality.

2.1. **Transforming one set by adding extra points.** In order to guess how many points one should add, we consider two lattice cubes $\mathrm{C}^n_{r_1}$ and $\mathrm{C}^n_{r_2}$: it is clear that $\mathrm{C}^n_{r_1} + \mathrm{C}^n_{r_2} = \mathrm{C}^n_{r_1+r_2}$, and therefore,

$$|\mathrm{C}^n_{r_1} + \mathrm{C}^n_{r_2}| = (r_1+r_2+1)^n < (r_1+r_2+2)^n = \Big(|\mathrm{C}^n_{r_1}|^{1/n} + |\mathrm{C}^n_{r_2}|^{1/n}\Big)^n.$$

So, in order to reverse the above inequality we must add to $\mathrm{C}^n_{r_1}$, say, a suitable amount of points, such that the new set $\bar{\mathrm{C}}^n_{r_1}$ satisfies

$$\big|\bar{\mathrm{C}}^n_{r_1} + \mathrm{C}^n_{r_2}\big| \geq (r_1+r_2+2)^n. \tag{2.1}$$

In this spirit, in Section 4 we prove the following theorem.

Theorem 2.1. *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$\big|\bar{A} + B\big|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{2.2}$$

and equality holds when both A and B are lattice cubes.

Here $\bar{A}$ is an *extension* of A obtained by adding new integer points, by means of a recursive procedure, as follows. If $\Lambda \subset \mathbb{Z}^k$ (finite), $k \in \{1, \dots, n\}$, for each $m \in \mathbb{Z}^r$, $r \in \{1, \dots, k-1\}$, we write $\Lambda(m)$ to represent the section of Λ at m orthogonal to the coordinate plane $\mathbb{R}e_{k-r+1} + \cdots + \mathbb{R}e_k$, i.e.,

$$\Lambda(m) = \big\{p \in \mathbb{Z}^{k-r} : (p, m) \in \Lambda\big\}.$$

Next, for $r = 1$, let $m_0 \in \pi_k(\Lambda)$ be such that $\big|\Lambda(m_0)\big| = \max_m \big|\Lambda(m)\big|$. Certainly the integer m_0 providing the maximum section is not necessarily unique. In that case, one can choose arbitrarily any of the possibilities. In order to establish a criterion for the construction we set

$$m_0 = \max\Big\{m' \in \pi_k(\Lambda) : \big|\Lambda(m')\big| = \max_m \big|\Lambda(m)\big|\Big\}.$$

Finally, we define the function

$$\sigma_k : \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\} \longrightarrow \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\}$$

given by

$$\sigma_k(\Lambda) = \begin{cases} \Lambda \cup \{\max \Lambda + 1\} & \text{if } k = 1, \\ \Lambda \cup \Big(\Lambda(m_0) \times \big\{\max\{\pi_k(\Lambda)\} + 1\big\}\Big) & \text{if } k > 1; \end{cases}$$

i.e., σ_k acts on Λ just adding the maximum section $\Lambda(m_0)$ to the set in the position $\max\{\pi_k(\Lambda)\} + 1$. As before this choice is irrelevant, and the maximum section $\Lambda(m_0)$ can be placed at any $m \notin \pi_k(\Lambda)$.

We are now ready to recursively define $\bar{A}$ for our original set $A \subset \mathbb{Z}^n$. In a first step, we construct a new set A_1^+ by means of its sections: $A_1^+ = \sigma_n(A)$ (see Figure 1).

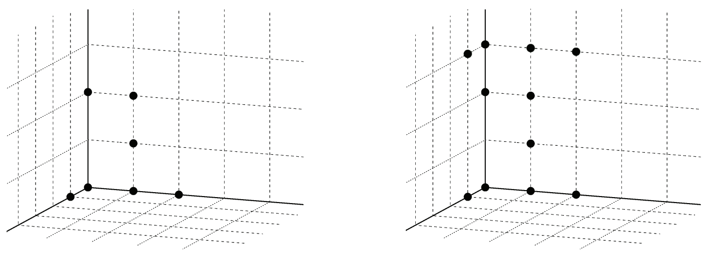

FIGURE 1. A discrete set A (left) and the set A_1^+ (right).

In the second one we take (see Figure 2)

$$A_2^+ = \bigcup_{m \in \pi_n(A_1^+)} \Big(\sigma_{n-1}\big(A_1^+(m)\big) \times \{m\}\Big).$$

In the k-th step, $k \geq 2$, we have

$$A_k^+ = \bigcup_{m \in \pi_{n-k+2,\dots,n}(A_{k-1}^+)} \Big(\sigma_{n-k+1}\big(A_{k-1}^+(m)\big) \times \{m\}\Big).$$

Then we define $\bar{A} = A_n^+$.

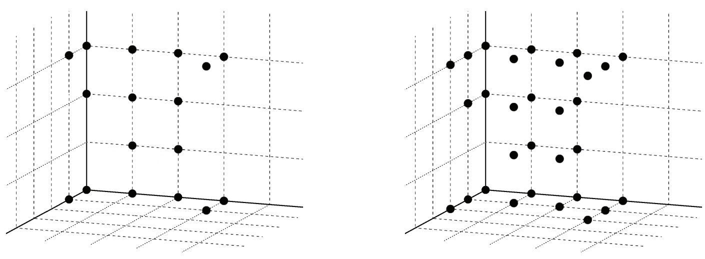

FIGURE 2. The sets A_2^+ (left) and $\bar{A} = A_3^+$ (right) for the discrete set A in Figure 1.

In the case of a lattice cube we have $\bar{\mathrm{C}}_{r_1}^n = \mathrm{C}_{r_1+1}^n$. Therefore $\bar{\mathrm{C}}_{r_1}^n + \mathrm{C}_{r_2}^n = \mathrm{C}_{r_1+r_2+1}^n$, and thus (2.1) holds with equality.

We note the recursive nature of the construction of $\bar{A}$, in which the action of adding the maximum section to the given set is repeatedly used onto every successive section of the original set A. Therefore, the following two properties are evident:

$$\text{i) } \pi_n\bigl(\bar{A}\bigr) = \pi_n(A_1^+) \qquad \text{and} \qquad \text{ii) } \bar{A}(m) = \overline{A_1^+(m)}. \tag{2.3}$$

In Section 5 we will show that the number of additional points in $|A|$ is somehow controlled. Moreover, upper and lower bounds for the ratio $\bigl|\bar{A}\bigr|/|A|$ and the difference $\bigl|\bar{A}\bigr| - |A|$ can be provided. In the first case only the dimension will play a role, whereas for the difference it will depend on the structure and the cardinality of A. We prove the following proposition:

Proposition 2.1. *Let $n \geq 1$ and let $A \subset \mathbb{Z}^n$ be finite and non-empty. Then*

$$1 \leq \frac{|\bar{A}|}{|A|} \leq 2^n \tag{2.4}$$

and

$$2^n - 1 \leq \bigl|\bar{A}\bigr| - |A| \leq \prod_{i=1}^{n}\Bigl(\bigl|\pi_i(A)\bigr| + 1\Bigr) - \prod_{i=1}^{n}\bigl|\pi_i(A)\bigr|. \tag{2.5}$$

In general these bounds cannot be improved.

Remark 2.1. The set $\bar{A}$ can be different (both its structure and cardinality) when either the role of the coordinate axes is interchanged in its construction, or if we use a different criterion for the choice of m_0, or even if we add as a "doubled" maximum section an arbitrary point set with the same cardinality. In any case, the number of additional points is controlled (see Proposition 2.1). Moreover, the above choices for the construction of $\bar{A}$ are not relevant for the proofs of the results. Thus, in order to bound from above $|A|^{1/n} + |B|^{1/n}$ in Theorem 2.1, one can choose in the definition of $\bar{A}$ the options (for m_0 and the axis order) making $\bigl|\bar{A} + B\bigr|$ minimum, which surely will depend on the original sets A and B.

We also note that although the cardinality of $\bar{A}$ is obviously enlarged, in many cases the difference between $\bigl|\bar{A} + B\bigr|$ and $|A+B|$ may be not too big; see Example 3.2, where indeed one has $\bigl|\bar{A} + B\bigr| = |A + B|$.

2.2. **Transforming one set by removing points.** Similarly, instead of adding points to the original (finite) set $A \subset \mathbb{Z}^n$, $A \neq \emptyset$, we may *reduce* it to define a new set $\mathrm{r}(A)$ in such a way that

$$\mathrm{r}\bigl(\bar{A}\bigr) = A. \tag{2.6}$$

To this aim, first we define the function

$$\delta_k : \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\} \longrightarrow \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\}$$

given by

$$\delta_k(\Lambda) = \begin{cases} \Lambda \setminus \{\max \Lambda\} & \text{if } k = 1, \\ \Lambda \setminus \Bigl(\Lambda(m_0) \times \bigl\{m_0\bigr\}\Bigr) & \text{if } k > 1; \end{cases}$$

i.e., δ_k acts on Λ just removing the maximum section $\Lambda(m_0)$ from the set. To complete the picture we set $\delta_k(\emptyset)=\emptyset$. In this way, δ_k is the left inverse function of σ_k.

Now, for $1\le k<n$, we write

$$A_k^- = \bigcup_{m\in\pi_{k+1,\dots,n}(A_{k-1}^-)} \Big(\delta_k\big(A_{k-1}^-(m)\big)\times\{m\}\Big),$$

with $A_0^-=A$ (see Figure 3). Then we define

$$\mathrm{r}(A)=\delta_n\big(A_{n-1}^-\big).$$

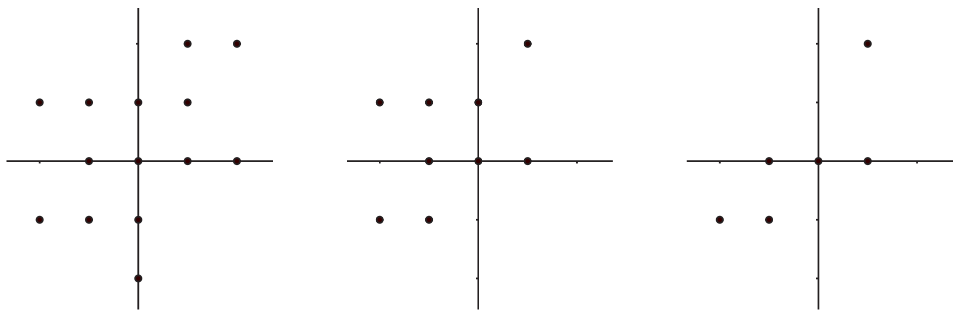

FIGURE 3. Transforming a discrete set A (left) into $\mathrm{r}(A)$ (right).

We note on the one hand that, from the definition of $\mathrm{r}(A)$, (2.6) holds because $\delta_k\big(\sigma_k(\Lambda)\big)=\Lambda$ for all $k=1,\dots,n$. On the other hand, since there are different ways to construct $\overline{\mathrm{r}(A)}$, (cf. Remark 2.1), it is possible to add every successive maximum section in such a way that

$$\overline{\mathrm{r}(A)}\subset A. \tag{2.7}$$

Remark 2.2. We observe that $\mathrm{r}(A)$ might be the empty set. Actually, $\mathrm{r}(A)\neq\emptyset$ necessarily implies that $|A|\ge 2^n$. Indeed, A_{n-1}^- must contain at least two points to assure that $\mathrm{r}(A)\neq\emptyset$; this yields that at least four points belong to A_{n-2}^- and, recursively, that $|A|\ge 2^n$.

Using this technique, in Section 4 we prove the following theorem.

Theorem 2.2. *Let $A,B\subset\mathbb{Z}^n$ be finite, $A,B\neq\emptyset$. Then*

$$|A+B|^{1/n}\ge\big|\mathrm{r}(A)\big|^{1/n}+|B|^{1/n}, \tag{2.8}$$

and equality holds when both A and B are lattice cubes.

In fact we prove that the discrete inequalities (2.2) and (2.8) are equivalent (see Proposition 4.1).

In [12], Matolcsi and Ruzsa consider the sum set $A+kB=A+B+\overset{(k)}{\dots}+B$, and provide a lower bound for its cardinality when $\dim B=n$ and $A\subset\operatorname{conv}B$. In [2], Böröczky, Santos and Serra characterize the sets A and B for which equality holds. As a direct consequence of Theorem 2.2 another bound for the cardinality $|A+kB|$ can be obtained, without additional conditions on the sets A and B:

Corollary 2.1. *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$|A+kB|^{1/n} \geq |A|^{1/n} + k\big|\mathrm{r}(B)\big|^{1/n}.$$

Equality holds when A and B are lattice cubes.

In Section 6 we also show that the classical Brunn-Minkowski inequality (1.1) for compact sets can be obtained as a consequence of the discrete version (2.8):

Theorem 2.3. *The discrete Brunn-Minkowski inequality* (2.8) *implies the classical Brunn-Minkowski inequality* (1.1).

We note that it is not possible to directly obtain any of the above discrete Brunn-Minkowski inequalities from the classical one (1.1) by using the method of replacing the points by suitable compact sets. As pointed out by Gardner and Gronchi in [6, pp. 3996–3997],

> *it is worth remarking that the obvious idea of replacing the points in the two finite sets by small congruent balls and applying the classical Brunn-Minkowski inequality to the resulting compact sets is doomed to failure. The fact that the sum of two congruent balls is a ball of twice the radius introduces an extra factor of 1/2 that renders the resulting bound weaker than even the trivial bound (11) below.*

We clarify that (11) in [6] coincides with (4.1) of the present paper.

3. On the different Brunn-Minkowski type inequalities

Before starting the proofs of our main theorems, we observe that inequalities (2.2) and (1.2) (or even (1.3)) are not comparable. For instance, if $A = B = \{0, \mathrm{e}_1, \mathrm{e}_2\}$, then $D^B_{|A|} = A$ and $D^B_{|B|} = B$, and obviously equality holds in (1.2), but we have a strict inequality in (2.2). Therefore (1.2) provides a stronger bound than (2.2). However, if $A = B = \mathrm{C}^2_2$ then $A+B = \mathrm{C}^2_4$ and, moreover, $D^B_{|A|} = D^B_{|B|}$ is the lattice simplex $\mathrm{conv}\{0, 7\mathrm{e}_1, \mathrm{e}_2\} \cap \mathbb{Z}^2$ (we refer the reader to [6, Section 5] for the construction, see Figure 4).

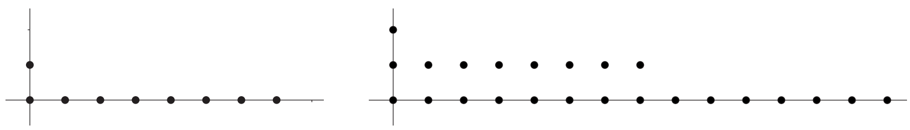

FIGURE 4. $D^B_{|A|}$ for $A = B = \mathrm{C}^2_2$ (left) and $D^B_{|A|} + D^B_{|B|}$ (right).

Hence

$$|A+B| = 25 > \left|D^B_{|A|} + D^B_{|B|}\right| = 24,$$

whereas we have equality in (2.2). In this case, the bound provided by (2.2) is stronger than (1.2) (or (1.3)).

It can also be easily seen, by just considering the corresponding extremal sets, that the bound of Matolcsi and Ruzsa in [12] for $|A+kB|$ and the one provided by Corollary 2.1 are not comparable.

As mentioned in the introduction, in most cases the classical Brunn-Minkowski inequality for the cardinality is not satisfied. Also lattice cubes or elongated simplices do not verify it. There are however particular sets or families of sets for which the inequality keeps its usual form, and so (2.2) would give a weaker bound. Nevertheless, as Example 3.2 will show, (2.2) may turn out to be a useful tool in order to prove the classical Brunn-Minkowski inequality for certain sets. This section is also devoted to studying the few examples we could find at this respect.

Example 3.1. For finite $A, B \subset \mathbb{Z}^n$, the relation

$$|A+B| \leq |A|\,|B|$$

trivially holds (see e.g. [17, Chapter 2]), and it is easy to check that equality holds if and only if any point of $A+B$ has a unique expression as a sum of a point of A and a point of B. Under this assumption, i.e., if $|A+B| = |A|\,|B|$, and furthermore, if $|A|, |B| \geq 2^n$ (they are large enough), then A, B satisfy a classical Brunn-Minkowski type inequality:

$$|A+B|^{1/n} = |A|^{1/n}|B|^{1/n} \geq \max\left\{2|A|^{1/n}, 2|B|^{1/n}\right\} \geq |A|^{1/n} + |B|^{1/n}.$$

Example 3.2. We recall that a compact convex set $K \subset \mathbb{R}^n$ is called *unconditional* if for any $(x_1, \dots, x_n) \in K$ then $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in K$ for all $\varepsilon_i \in \{-1, 1\}$, $i = 1, \dots, n$. We consider the following sets. Given an unconditional compact convex set $K \subset \mathbb{R}^n$, let $A = (K \setminus \bigcup_{i=1}^n H_i) \cap \mathbb{Z}^n$. Furthermore, let $B \subset \mathbb{Z}^n$ (finite) satisfy the following condition: if $(x_1, \dots, x_n) \in B$, there exist $\varepsilon_i \in \{-1, +1\}$, $i = 1, \dots, n$, such that $(x_1, \dots, x_{i-1}, x_i + \varepsilon_i, x_{i+1}, \dots, x_n) \in B$ for all $i = 1, \dots, n$ (see Figure 5).

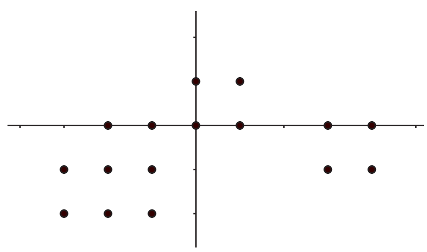

FIGURE 5. An example of a set B in the above construction.

As mentioned in Remark 2.1, there are different ways of constructing $\bar{A}$. In the case of $A = (K \setminus \bigcup_{i=1}^n H_i) \cap \mathbb{Z}^n$, we place the successive maximum sections on the coordinates hyperplanes. In this way we even have

$$\bar{A} + B = A + B.$$

Indeed, given $x = (x_1, \ldots, x_n) \in \bar{A} \setminus A$ and $b = (b_1, \ldots, b_n) \in B$ let $I \subset \{1, \ldots, n\}$ be such that $x_i = 0$ if $i \in I$ and $x_i \neq 0$ otherwise. On one hand, there exist $\varepsilon_i \in \{-1, 0, +1\}$, $i = 1, \ldots, n$, such that $(b_1+\varepsilon_1, \ldots, b_n+\varepsilon_n) \in B$, and so that $\varepsilon_i = 0$ if and only if $i \notin I$. On the other hand, and denoting by $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, we have that $x - \varepsilon \in A$ because K is unconditional. Then $x + b = (x - \varepsilon) + (b + \varepsilon) \in A + B$, which shows that, $\bar{A} + B \subset A + B$. The reverse inclusion in obvious. Therefore, although we are adding points in the construction of $\bar{A}$, *the cardinality of* $\bar{A} + B$ *does not increase* (with respect to that of $A + B$). Hence, Theorem 2.1 yields

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

We also note that for the constructed set A, the above inequality does not hold for arbitrary B.

4. Proofs of the discrete Brunn-Minkowski inequalities

We start this section by recalling the simple inequality

$$|A + B| \geq |A| + |B| - 1, \tag{4.1}$$

for finite subsets A, B in $\mathbb{Z}^n$ (see e.g. [17, Chapter 2]). Since the cardinality $|\cdot|$ is translation invariant, we can assume that both the maximum point of A and the minimum point of B in the lexicographical order are at the origin of coordinates. Then it is clear that $A + B \supset A \cup B$, and hence $|A + B| \geq |A \cup B| = |A| + |B| - 1$.

We observe that (4.1) provides, in particular, a 1-dimensional discrete Brunn-Minkowski inequality.

Before the proof of Theorem 2.1 we state two auxiliary results. The first one may be regarded as a discrete counterpart of the layer cake formula.

Lemma 4.1. *Let $\Omega \subset \mathbb{Z}$ be finite and let $f : \Omega \longrightarrow \mathbb{Z}_{\geq 0}$. Then*

$$\sum_{m \in \Omega} f(m) = \sum_{t=1}^{\max_\Omega f} \Big|\big\{m \in \Omega : f(m) \geq t\big\}\Big|.$$

Proof. Let $N = \max_{m \in \Omega} f(m)$, and we consider variables x_i, $i = 1, \ldots, N$. Then we have the relation

$$\sum_{m \in \Omega} \big(x_1 + x_2 + \cdots + x_{f(m)}\big) = \sum_{t=1}^{N} x_t \Big|\big\{m \in \Omega : f(m) \geq t\big\}\Big|,$$

because the variable x_t appears in the left-hand side expression if and only if $f(m) \geq t$. Then, setting $x_1 = \cdots = x_N = 1$, we get the result. □

Lemma 4.2. *Let $\Omega \subset \mathbb{Z}$ be finite and let $f : \Omega \longrightarrow \mathbb{Z}_{\geq 0}$. Then, for any $r, N \in \mathbb{Z}_{>0}$, we have*

$$r \sum_{t=1}^{N} \Big|\big\{m \in \Omega : f(m) \geq t\big\}\Big| = \sum_{t=\frac{1}{rN}, \frac{2}{rN}, \ldots, 1} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right|.$$

Proof. First we rewrite

$$\sum_{t=\frac{1}{rN},\frac{2}{rN},\dots,1}\left|\left\{m\in\Omega:\frac{f(m)}{N}\geq t\right\}\right| = \sum_{t=\frac{1}{rN},\dots,\frac{1}{N}}\left|\left\{m\in\Omega:\frac{f(m)}{N}\geq t\right\}\right| + \sum_{t=\frac{r+1}{rN},\dots,\frac{2}{N}}\left|\left\{m\in\Omega:\frac{f(m)}{N}\geq t\right\}\right| + \dots + \sum_{t=\frac{(N-1)r+1}{rN},\dots,1}\left|\left\{m\in\Omega:\frac{f(m)}{N}\geq t\right\}\right|.$$

We note that, for each of the above sums, i.e., for all $i=0,\dots,N-1$,

$$\begin{aligned}\sum_{t=\frac{ir+1}{rN},\dots,\frac{i+1}{N}}\left|\left\{m\in\Omega:\frac{f(m)}{N}\geq t\right\}\right| &= \sum_{t=\frac{ir+1}{r},\dots,i+1}\left|\left\{m\in\Omega:f(m)\geq t\right\}\right| \\ &= \sum_{t=i+\frac{1}{r},\dots,i+1}\left|\left\{m\in\Omega:f(m)\geq i+1\right\}\right| \\ &= r\left|\left\{m\in\Omega:f(m)\geq i+1\right\}\right|,\end{aligned}$$

and thus we can conclude that

$$\sum_{t=\frac{1}{rN},\frac{2}{rN},\dots,1}\left|\left\{m\in\Omega:\frac{f(m)}{N}\geq t\right\}\right| = r\sum_{i=0}^{N-1}\left|\left\{m\in\Omega:f(m)\geq i+1\right\}\right|.$$

This proves the result. □

Now we are in a position to prove our main result.

Proof of Theorem 2.1. We will show (2.2) by (finite) induction on the dimension n. The case $n=1$ is a direct consequence of (4.1):

$$\left|\bar{A}+B\right|\geq\left|\bar{A}\right|+|B|-1=|A|+|B|.$$

So, we will suppose that the inequality is true for $n-1$. We first observe that for all $m_1,m_2\in\mathbb{Z}$, it is clear that

$$\left(\bar{A}+B\right)(m_1+m_2)\supset\bar{A}(m_1)+B(m_2).$$

Then, taking $m_1\in\pi_n\bigl(\bar{A}\bigr)=\pi_n(A_1^+)$ (cf. (2.3) i)) and $m_2\in\pi_n(B)$, and applying induction hypothesis (i.e., (2.2) in $\mathbb{Z}^{n-1}$), we get (see also (2.3) ii))

$$\begin{aligned}\left|\left(\bar{A}+B\right)(m_1+m_2)\right| &\geq\left|\bar{A}(m_1)+B(m_2)\right|=\left|\overline{A_1^+(m_1)}+B(m_2)\right| \\ &\geq\left(\left|A_1^+(m_1)\right|^{1/(n-1)}+\left|B(m_2)\right|^{1/(n-1)}\right)^{n-1}.\end{aligned}\tag{4.2}$$

For the sake of brevity we denote by

$$c_A=\max_{m\in\mathbb{Z}}\left|A(m)\right|>0,\qquad c_B=\max_{m\in\mathbb{Z}}\left|B(m)\right|>0,$$

and let

$$c=\left(c_A^{1/(n-1)}+c_B^{1/(n-1)}\right)^{n-1} \quad \text{and} \quad \theta=\frac{c_B^{1/(n-1)}}{c_A^{1/(n-1)}+c_B^{1/(n-1)}}\in(0,1).$$

We observe that $c_{A_1^+}=c_A$. Furthermore, let $p,q\in\mathbb{N}$ satisfying

$$\frac{p}{q}\in\mathbb{Q} \quad \text{satisfy} \quad \frac{p}{q}\leq c. \tag{4.3}$$

Finally, for $M=A,A_1^+,B$ or $\bar{A}+B$, we denote by $f_M:\mathbb{Z}\longrightarrow\mathbb{Q}_{\geq 0}$ the functions given by

$$f_A(m)=\frac{\bigl|A(m)\bigr|}{c_A},\quad f_{A_1^+}(m)=\frac{\bigl|A_1^+(m)\bigr|}{c_A},\quad f_B(m)=\frac{\bigl|B(m)\bigr|}{c_B},\quad \text{and}$$
$$f_{\bar{A}+B}(m)=\frac{q}{p}\left|(\bar{A}+B)(m)\right|.$$

Using (4.2) we get

$$\begin{aligned}\left|\left(\bar{A}+B\right)(m_1+m_2)\right| &\geq \left(\bigl|A_1^+(m_1)\bigr|^{1/(n-1)}+\bigl|B(m_2)\bigr|^{1/(n-1)}\right)^{n-1}\\ &= c\left(\frac{c_A^{1/(n-1)}}{c^{1/(n-1)}}f_{A_1^+}(m_1)^{1/(n-1)}+\frac{c_B^{1/(n-1)}}{c^{1/(n-1)}}f_B(m_2)^{1/(n-1)}\right)^{n-1}\\ &= c\left((1-\theta)f_{A_1^+}(m_1)^{1/(n-1)}+\theta f_B(m_2)^{1/(n-1)}\right)^{n-1}\\ &\geq c\,\min\bigl\{f_{A_1^+}(m_1),f_B(m_2)\bigr\}\geq\frac{p}{q}\min\bigl\{f_{A_1^+}(m_1),f_B(m_2)\bigr\}.\end{aligned}$$

Thus, we have obtained the functional inequality

$$f_{\bar{A}+B}(m_1+m_2)\geq\min\bigl\{f_{A_1^+}(m_1),f_B(m_2)\bigr\}. \tag{4.4}$$

Now we observe, on one hand, that the super-level sets

$$\bigl\{m\in\mathbb{Z}:f_A(m)\geq t\bigr\},\quad \bigl\{m\in\mathbb{Z}:f_{A_1^+}(m)\geq t\bigr\},\quad \bigl\{m\in\mathbb{Z}:f_B(m)\geq t\bigr\}$$

are non-empty for all $t\in[0,1]$ and, moreover, the definition of A_1^+ yields

$$\left|\bigl\{m\in\mathbb{Z}:f_{A_1^+}(m)\geq t\bigr\}\right|=\left|\bigl\{m\in\mathbb{Z}:f_A(m)\geq t\bigr\}\right|+1.$$

On the other hand, (4.4) implies that

$$\bigl\{m\in\mathbb{Z}:f_{\bar{A}+B}(m)\geq t\bigr\}\supset\bigl\{m\in\mathbb{Z}:f_{A_1^+}(m)\geq t\bigr\}+\bigl\{m\in\mathbb{Z}:f_B(m)\geq t\bigr\},$$

and then, using (4.1) for $n=1$ and the above identity, we get

$$\begin{aligned}&\left|\bigl\{m\in\mathbb{Z}:f_{\bar{A}+B}(m)\geq t\bigr\}\right|\\ &\quad\geq\left|\bigl\{m\in\mathbb{Z}:f_{A_1^+}(m)\geq t\bigr\}\right|+\left|\bigl\{m\in\mathbb{Z}:f_B(m)\geq t\bigr\}\right|-1\\ &\quad=\left|\bigl\{m\in\mathbb{Z}:f_A(m)\geq t\bigr\}\right|+\left|\bigl\{m\in\mathbb{Z}:f_B(m)\geq t\bigr\}\right|.\end{aligned} \tag{4.5}$$

We also observe that the cardinality of $|\bar{A}+B|$ can be expressed as

$$|\bar{A}+B| = \sum_{m\in\mathbb{Z}} \left|(\bar{A}+B)(m)\right| = \sum_{m\in\mathbb{Z}} \frac{p}{q} f_{\bar{A}+B}(m),$$

where we write the sum over $\mathbb{Z}$ for the sake of brevity. Analogously,

$$|A| = \sum_{m\in\mathbb{Z}} c_A f_A(m) \quad \text{and} \quad |B| = \sum_{m\in\mathbb{Z}} c_B f_B(m).$$

Lemma 4.1 applied to the (integer) function $f(m) = p\, f_{\bar{A}+B}(m)$ leads to

$$|\bar{A}+B| = \frac{1}{q}\sum_{m\in\mathbb{Z}} p\, f_{\bar{A}+B}(m) = \frac{1}{q}\sum_{t=1}^{p\,\max_{\mathbb{Z}} f_{\bar{A}+B}} \left|\left\{m\in\mathbb{Z} : p\, f_{\bar{A}+B}(m)\geq t\right\}\right|,$$

and since $\max_{m\in\mathbb{Z}} f_{\bar{A}+B}(m)\geq 1$ by (4.4), we get

$$|\bar{A}+B| \geq \frac{1}{q}\sum_{t=1}^{p} \left|\left\{m\in\mathbb{Z} : p\, f_{\bar{A}+B}(m)\geq t\right\}\right|.$$

Let $c' = p\,c_A c_B$. Applying Lemma 4.2 to the above sum for $N = p$ and $r = c_A c_B$, and then using (4.5), we obtain

(4.6)

$$\begin{aligned} |\bar{A}+B| &\geq \frac{1}{q}\frac{1}{c_A c_B}\sum_{t=\frac{1}{c'},\frac{2}{c'},\dots,1} \left|\left\{m\in\mathbb{Z} : f_{\bar{A}+B}(m)\geq t\right\}\right| \\ &\geq \frac{1}{q}\frac{1}{c_A c_B}\sum_{t=\frac{1}{c'},\dots,1} \left[\left|\left\{m\in\mathbb{Z} : f_A(m)\geq t\right\}\right| + \left|\left\{m\in\mathbb{Z} : f_B(m)\geq t\right\}\right|\right]. \end{aligned}$$

Now, Lemma 4.2 for $N = c_A$, $r = p\,c_B$ and Lemma 4.1 yield

$$\begin{aligned} \sum_{t=\frac{1}{c'},\dots,1} \left|\left\{m\in\mathbb{Z} : f_A(m)\geq t\right\}\right| &= p\,c_B \sum_{t=1}^{c_A} \left|\left\{m\in\mathbb{Z} : c_A f_A(m)\geq t\right\}\right| \\ &= p\,c_B \sum_{m\in\mathbb{Z}} c_A f_A(m) = p\,c_B|A|, \end{aligned} \tag{4.7}$$

and analogously (now $N = c_B$ and $r = p\,c_A$ in Lemma 4.2),

$$\sum_{t=\frac{1}{c'},\dots,1} \left|\left\{m\in\mathbb{Z} : f_B(m)\geq t\right\}\right| = p\,c_A|B|. \tag{4.8}$$

Then, (4.6), (4.7) and (4.8) together, give

$$|\bar{A}+B| \geq \frac{1}{q}\frac{1}{c_A c_B}\Big(p\,c_B|A| + p\,c_A|B|\Big) = \frac{p}{q}\left(\frac{|A|}{c_A} + \frac{|B|}{c_B}\right). \tag{4.9}$$

Since (4.9) holds for any rational number $p/q\leq c$ (cf. (4.3)), by a limit procedure we also get inequality (4.9) for the real positive number c. And then, applying the (reverse) Hölder inequality (see e.g. [3, Theorem 1, p. 178]) with parameters $1/n$ and $-1/(n-1)$, we conclude that

$$\left|\bar{A}+B\right| \geq c\left(\frac{|A|}{c_A}+\frac{|B|}{c_B}\right) \geq \left(|A|^{1/n}+|B|^{1/n}\right)^n.$$

Finally we prove that the inequality is sharp. Indeed, let A, B be the lattice cubes $A=\mathrm{C}^n_{m_1}$ and $B=\mathrm{C}^n_{m_2}$. Then $\bar{A}=\mathrm{C}^n_{m_1+1}$, and hence $\bar{A}+B=\mathrm{C}^n_{m_1+m_2+1}$. Therefore,

$$\left|\bar{A}+B\right| = (m_1+m_2+2)^n = \left(|A|^{1/n}+|B|^{1/n}\right)^n. \qquad \square$$

The following corollary is a direct consequence of Theorem 2.1.

Corollary 4.1. *Let A, B be finite subsets of $\mathbb{Z}^n$, $A, B \neq \emptyset$. Then*

$$|A+B| \geq \left(|A|^{1/n}+|B|^{1/n}\right)^n - \left|\left(\bar{A}+B\right)\setminus(A+B)\right|.$$

We observe that our approach involves not only finite sets of $\mathbb{Z}^n$, but can be extended to general (finite) sets of $\mathbb{R}^n$ by suitably defining $\bar{A}$. Therefore, Theorem 2.1 can be stated for any finite (non-empty) set of $\mathbb{R}^n$.

We conclude this section by proving the second version of the discrete Brunn-Minkowski inequality.

Proof of Theorem 2.2. If $\mathrm{r}(A)=\emptyset$ then the inequality $|A+B|^{1/n} \geq |B|^{1/n}$ trivially holds. So we assume that $\mathrm{r}(A) \neq \emptyset$. In this case, Theorem 2.1 applied to the sets $\mathrm{r}(A)$ and B, together with (2.7) yields

$$|A+B|^{1/n} \geq \left|\overline{\mathrm{r}(A)}+B\right|^{1/n} \geq \left|\mathrm{r}(A)\right|^{1/n}+|B|^{1/n}.$$

The equality case is a consequence of the equality case in Theorem 2.1. $\square$

Moreover, it is easy to see that (2.2) and (2.8) are equivalent:

Proposition 4.1. *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then* (2.2) *and* (2.8) *are equivalent.*

Proof. In the proof of Theorem 2.2 we have already proved that (2.2) implies (2.8). In order to prove the converse we just have to note that the operator $\mathrm{r}(\cdot)$ has been defined in such a way that $\mathrm{r}\left(\bar{A}\right)=A$ for any $A \neq \emptyset$ (cf. (2.6)). Therefore, applying (2.8) to $\bar{A}$ and B we get

$$\left|\bar{A}+B\right|^{1/n} \geq \left|\mathrm{r}\left(\bar{A}\right)\right|^{1/n}+|B|^{1/n} = |A|^{1/n}+|B|^{1/n}. \qquad \square$$

5. Bounding the cardinality of the set $\bar{A}$

Let $A \subset \mathbb{Z}^n$ be finite and non-empty. In this section we will show that the number of additional points in $\bar{A}$ cannot be too large, and depends only on (the structure of) A and on the dimension.

If we intend to control the number of points that we add to A, first we have to determine how many new points we have in the first step A_1^+. Clearly,

$$\big|\pi_i(A_1^+)\big| = \begin{cases} \big|\pi_i(A)\big| & \text{for } i = 1, \dots, n-1, \\ \big|\pi_n(A)\big| + 1 & \text{for } i = n. \end{cases} \tag{5.1}$$

Proposition 5.1. *Let $n \geq 2$ and let $A \subset \mathbb{Z}^n$ be finite and non-empty. Then*

$$|A_1^+| - |A| \leq \prod_{i=1}^{n-1} \big|\pi_i(A)\big|. \tag{5.2}$$

Proof. Since $A \subset A_1^+$, then $|A_1^+| - |A| = |A_1^+ \backslash A| = \max_{m \in \mathbb{Z}} \big|A(m)\big|$, so it suffices to prove that for all $m \in \mathbb{Z}$,

$$\big|A(m)\big| \leq \prod_{i=1}^{n-1} \big|\pi_i(A)\big|, \tag{5.3}$$

which follows from the (discrete) Loomis-Whitney inequality: it can be seen by replacing each point in $A(m)$ by a small cube with edges parallel to the coordinate lines $\mathbb{R}e_i$, $i = 1, \dots, n$, that

$$\big|A(m)\big| \leq \prod_{i=1}^{n-1} \Big|\pi_i\big(A(m)\big)\Big|$$

(see e.g. [7, Section 5] and the references within). This shows (5.3). □

In order to establish the announced upper bounds for the cardinality of $\bar{A}$, we also need the following general identity for natural numbers. For the sake of brevity we set the meaningless products to be 1. We use this convention here and throughout the rest of the paper.

Lemma 5.1. *Let $a_1, \dots, a_n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n} (a_i + 1) \right) = \prod_{i=1}^{n} (a_i + 1) - \prod_{i=1}^{n} a_i.$$

Proof. A recursive procedure shows that

$$\begin{aligned}
&\prod_{i=1}^{n} a_i + \sum_{k=1}^{n} \left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n} (a_i + 1) \right) \\
&\quad = \prod_{i=1}^{n} a_i + \prod_{i=1}^{n-1} a_i + \sum_{k=2}^{n} \left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n} (a_i + 1) \right) \\
&\quad = \prod_{i=1}^{n-1} a_i (a_n + 1) + \prod_{i=1}^{n-2} a_i (a_n + 1) + \sum_{k=3}^{n} \left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n} (a_i + 1) \right) \\
&\quad = \prod_{i=1}^{n-2} a_i \prod_{i=n-1}^{n} (a_i + 1) + \sum_{k=3}^{n} \left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n} (a_i + 1) \right) = \cdots = \prod_{i=1}^{n} (a_i + 1).
\end{aligned}$$

□

We are now ready to provide the bounds for the cardinality of $\bar{A}$: we prove Proposition 2.1.

Proof of Proposition 2.1. First we prove (2.4). It is clear that $|A_1^+| \leq 2|A|$, and equality holds if and only if $\bigl|\pi_n(A)\bigr| = 1$. Moreover, $|A_{i+1}^+| \leq 2|A_i^+|$ for all $i = 1, \dots, n-1$, with equality if and only if $\bigl|\pi_{n-i}(A_i^+(m))\bigr| = 1$ for every $m \in \pi_{n-i+1,\dots,n}(A_i^+)$. Then we get $\bigl|\bar{A}\bigr|/|A| \leq 2^n$, and equality holds if and only if A is a singleton.

Finally we observe that the ratio $\bigl|\bar{A}\bigr|/|A|$ may be, nevertheless, arbitrarily small, as it is shown by considering $A = \mathrm{C}_r^n$, $r \in \mathbb{N}$. In this case,

$$\frac{\bigl|\bar{A}\bigr|}{|A|} = \frac{|\mathrm{C}_{r+1}^n|}{|\mathrm{C}_r|} = \left(1 + \frac{1}{r+1}\right)^n,$$

which tends to 1 when $r \to \infty$. This shows the lower bound in (2.4) as well as its tightness.

Next we prove (2.5). The lower bound is trivial, and equality holds if and only if A is a singleton. For the upper bound, if $n = 1$ then $\bigl|\bar{A}\bigr| = |A| + 1$, and (2.5) trivially holds. Therefore we assume that $n \geq 2$.

We observe that, in order to construct $\bar{A}$, we first add the new points corresponding to A_1^+, then the new points of $\sigma_{n-1}\bigl(A_1^+(m)\bigr)$ for each $m \in \pi_n(A_1^+)$, and so on. Therefore:

1st step: By (5.2) we add, at most, $\prod_{i=1}^{n-1}\bigl|\pi_i(A)\bigr|$ points.

2nd step: Using again (5.2), we can assure that we add $\bigl|\pi_n(A_1^+)\bigr| = \bigl|\pi_n(A)\bigr| + 1$ times (cf. (5.1)), at most, $\prod_{i=1}^{n-2}\bigl|\pi_i(A)\bigr|$ points.

kth step: In short, for $k = 1, \dots, n$ we are adding, at most,

$$\prod_{i=1}^{n-k}\bigl|\pi_i(A)\bigr| \prod_{i=n-k+2}^{n}\Bigl(\bigl|\pi_i(A)\bigr| + 1\Bigr) \tag{5.4}$$

new points.

Altogether, and using Lemma 5.1, we conclude that

$$\begin{aligned}\bigl|\bar{A}\bigr| - |A| &\leq \sum_{k=1}^{n}\left(\prod_{i=1}^{n-k}\bigl|\pi_i(A)\bigr| \prod_{i=n-k+2}^{n}\Bigl(\bigl|\pi_i(A)\bigr| + 1\Bigr)\right)\\ &= \prod_{i=1}^{n}\Bigl(\bigl|\pi_i(A)\bigr| + 1\Bigr) - \prod_{i=1}^{n}\bigl|\pi_i(A)\bigr|.\end{aligned}$$

In order to show that the upper bound in (2.5) may be attained it is enough to consider a lattice orthogonal box A (see Figure 6). □

6. From the discrete version to the continuous one

For each $k \in \mathbb{N}$, we consider the family of all (closed) cubes of edge-length 2^{-k}, with vertices in the lattice $2^{-k}\mathbb{Z}^n$. This family tessellates the whole space, i.e., covers $\mathbb{R}^n$ and its elements have disjoint interiors.

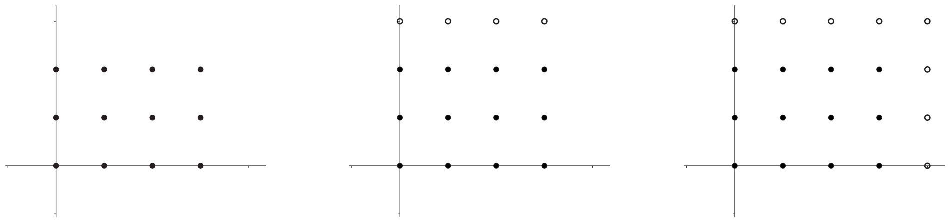

FIGURE 6. The upper bound in (2.5) is sharp: a lattice box A (left), A_1^+ (middle) and $\bar{A}$ (right).

Definition 6.1. *Let $K \subset \mathbb{R}^n$ be a compact set. The k-*discretization *of K, $k \in \mathbb{N}$, is defined as*

$$K_k = \left\{x \in 2^{-k}\mathbb{Z}^n : \left(x + \left[0, 2^{-k}\right]^n\right) \cap K \neq \emptyset\right\}.$$

Given a compact set $K \subset \mathbb{R}^n$, a standard straightforward computation shows that

$$K = \bigcap_{k=1}^{\infty} \left(K_k + \left[0, 2^{-k}\right]^n\right).$$

This, together with the fact that

$$\mathrm{vol}\left(\bigcap_{k=1}^{\infty} \left(K_k + \left[0, 2^{-k}\right]^n\right)\right) = \lim_{k\to\infty} \mathrm{vol}\left(K_k + \left[0, 2^{-k}\right]^n\right)$$

because $\left\{K_k + \left[0, 2^{-k}\right]^n\right\}_k$ is a decreasing sequence (see e.g. [4, Proposition 1.2.5 (b)]), allows to deduce the following result:

Lemma 6.1. *Let $K \subset \mathbb{R}^n$ be a non-empty compact set. Then*

$$\mathrm{vol}(K) = \lim_{k\to\infty} \frac{|K_k|}{2^{kn}}.$$

We conclude the paper by proving that the classical Brunn-Minkowski inequality for compact sets can be obtained as a consequence of Theorem 2.1.

Proof of Theorem 2.3. For each $k \in \mathbb{N}$, let K_k, L_k be the k-discretizations of K, L, respectively. Since K and L are compact, both K_k, L_k are finite sets and we can use (2.8) to deduce that, for any $k \in \mathbb{N}$, we have

$$|K_k + L_k|^{1/n} \geq \left|\mathrm{r}(K_k)\right|^{1/n} + |L_k|^{1/n}.$$

Therefore

$$\lim_{k\to\infty} \left(\frac{|K_k + L_k|}{2^{kn}}\right)^{1/n} \geq \lim_{k\to\infty} \left(\frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}}\right)^{1/n} + \lim_{k\to\infty} \left(\frac{|L_k|}{2^{kn}}\right)^{1/n}. \tag{6.1}$$

Now, for $k \in \mathbb{N}$, we define the set

$$F_k = \left(K_k + \left[0, 2^{-k}\right]^n\right) + \left(L_k + \left[0, 2^{-k}\right]^n\right).$$

It is clear that $F_1 \supset F_2 \supset \dots$ and, moreover,

$$K + L = \bigcap_{k=1}^{\infty} F_k.$$

Hence

$$\mathrm{vol}(K+L) = \mathrm{vol}\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mathrm{vol}(F_k)$$

(see e.g. [4, Proposition 1.2.5 (b)]) and then, from

$$F_k = K_k + L_k + \left[0, 2^{-k+1}\right]^n \supset K_k + L_k + \left[0, 2^{-k}\right]^n$$

we obtain

$$\mathrm{vol}(K+L)^{1/n} = \lim_{k\to\infty} \mathrm{vol}(F_k)^{1/n} \geq \lim_{k\to\infty} \left(\frac{|K_k + L_k|}{2^{kn}}\right)^{1/n}.$$

Now, using (6.1) and Lemma 6.1 we immediately get

$$\begin{aligned}\mathrm{vol}(K+L)^{1/n} &\geq \lim_{k\to\infty} \left(\frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}}\right)^{1/n} + \lim_{k\to\infty} \left(\frac{\left|L_k\right|}{2^{kn}}\right)^{1/n} \\ &= \lim_{k\to\infty} \left(\frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}}\right)^{1/n} + \mathrm{vol}(L)^{1/n}.\end{aligned}$$

Thus, in order to finish the proof, it suffices to show that

$$\lim_{k\to\infty} \frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}} = \mathrm{vol}(K). \tag{6.2}$$

For the sake of brevity we denote by $K_{k,i} = (K_k)^-_{i-1} \backslash (K_k)^-_i$, $i = 1, \dots, n$, i.e., the set of all points removed from K_k in the i-th step of the construction of $\mathrm{r}(K_k)$. Then it is clear that

$$\mathrm{vol}\left(K_{k,i} + \left[0, 2^{-k}\right]^n\right) = \mathrm{vol}\left(\pi_{(i)}(K_{k,i}) + \left[0, 2^{-k}\right]^n\right)$$

and hence

$$\begin{aligned}\frac{|K_{k,i}|}{2^{kn}} &= \mathrm{vol}\left(K_{k,i} + \left[0, 2^{-k}\right]^n\right) = \mathrm{vol}\left(\pi_{(i)}(K_{k,i}) + \left[0, 2^{-k}\right]^n\right) \\ &\leq \mathrm{vol}\left(\pi_{(i)}(K_k) + \left[0, 2^{-k}\right]^n\right) = \frac{\left|\pi_{(i)}(K_k)\right|}{2^{kn}}.\end{aligned}$$

Since K is compact, $\left(\pi_{(i)}(K)\right)_k = \pi_{(i)}(K_k)$, and then Lemma 6.1 yields

$$0 = \mathrm{vol}\left(\pi_{(i)}(K)\right) = \lim_{k\to\infty} \frac{\left|\left(\pi_{(i)}(K)\right)_k\right|}{2^{kn}} = \lim_{k\to\infty} \frac{\left|\pi_{(i)}(K_k)\right|}{2^{kn}} \geq \lim_{k\to\infty} \frac{|K_{k,i}|}{2^{kn}},$$

which implies that

$$\lim_{k\to\infty} \frac{|K_{k,i}|}{2^{kn}} = 0.$$

With Lemma 6.1 again, this shows that

$$\lim_{k\to\infty}\frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}}=\lim_{k\to\infty}\frac{|K_k|-\left|K_k\backslash\mathrm{r}(K_k)\right|}{2^{kn}}=\lim_{k\to\infty}\frac{|K_k|-\sum_{i=1}^n|K_{k,i}|}{2^{kn}}$$
$$=\lim_{k\to\infty}\frac{|K_k|}{2^{kn}}-\sum_{i=1}^n\lim_{k\to\infty}\frac{|K_{k,i}|}{2^{kn}}=\mathrm{vol}(K).$$

This proves (6.2) and concludes the proof. □

Acknowledgement. We thank the referees for many helpful suggestions and remarks. We also thank Prof. J. L. Torrea for bringing to our attention the question on how to get the classical Brunn-Minkowski inequality from the discrete one.

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Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100-Murcia, Spain
E-mail address: mhcifre@um.es
E-mail address: david.iglesias@um.es

Departamento de Matemáticas, Universidad de León, Campus de Vegazana, 24071-León, Spain
E-mail address: jyepn@unileon.es