ON DISCRETE BRUNN-MINKOWSKI AND ISOPERIMETRIC TYPE INEQUALITIES

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ABSTRACT. We show that the lattice point enumerator $\mathrm{G}_n(\cdot)$ satisfies

$$\mathrm{G}_n\big(tK + sL + (-1, \lceil t+s \rceil)^n\big)^{1/n} \geq t\mathrm{G}_n(K)^{1/n} + s\mathrm{G}_n(L)^{1/n}$$

for any $K, L \subset \mathbb{R}^n$ bounded sets with integer points and all $t, s \geq 0$.

We also prove that a certain family of compact sets, extending that of cubes $[-m, m]^n$, with $m \in \mathbb{N}$, minimizes the functional $\mathrm{G}_n(K + t[-1,1]^n)$, for any $t \geq 0$, among those bounded sets $K \subset \mathbb{R}^n$ with given positive lattice point enumerator.

Finally, we show that these new discrete inequalities imply the corresponding classical Brunn-Minkowski and isoperimetric inequalities for non-empty compact sets.

1. Introduction

The classical *Brunn-Minkowski inequality* for non-empty compact sets $K, L \subset \mathbb{R}^n$ asserts that if $t, s \geq 0$ then

$$\mathrm{vol}(tK + sL)^{1/n} \geq t\mathrm{vol}(K)^{1/n} + s\mathrm{vol}(L)^{1/n}. \tag{1.1}$$

Here $\mathrm{vol}(\cdot)$ denotes the n-dimensional Lebesgue measure and $+$ is used for the Minkowski sum, i.e., $A + B = \{a + b : a \in A, b \in B\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$. Moreover, rA stands for the set $\{ra : a \in A\}$ for any $r \geq 0$ and we write $x + A$ for $\{x\} + A$, where $x \in \mathbb{R}^n$.

The Brunn-Minkowski inequality has become not only a cornerstone of the Brunn-Minkowski theory (for which we refer the reader to the updated monograph [22]) but also a powerful tool in other related fields of mathematics. Moreover, it quickly yields other well-known inequalities, such as *Urysohn's inequality* (see e.g. [22, page 382]), it has inspired new engaging related results, such as a reverse Brunn-Minkowski inequality (see [16]), and it has been the starting point for new extensions and generalizations (see e.g. [22, Chapter 9]). For extensive survey articles on this and other related inequalities we refer the reader to [2, 5].

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In [6] Gardner and Gronchi obtained a powerful discrete analogue of the following form of the Brunn-Minkowski inequality, in the setting of $\mathbb{Z}^n$ with the cardinality $|\cdot|$: $\mathrm{vol}(K+L) \geq \mathrm{vol}(B_K + B_L)$, where B_K and B_L denote centered Euclidean balls of the same volume as K and L, respectively. More precisely, they proved that if A, B are finite subsets of the integer lattice $\mathbb{Z}^n$, with dimension $\dim B = n$, then

$$|A+B| \geq \left|D^B_{|A|} + D^B_{|B|}\right|. \tag{1.2}$$

Here $D^B_{|A|}, D^B_{|B|}$ are *B-initial segments*: for $m \in \mathbb{N}$, D^B_m is the set of the first m points of $\mathbb{Z}^n_+$ in the so-called "B-order", which is a particular order defined on $\mathbb{Z}^n_+$ depending only on the cardinality of B. For both a proper definition and a deep study of it we refer the reader to [6]. As consequences of (1.2), they also get two additional engaging discrete Brunn-Minkowski type inequalities (improving previous results obtained by Ruzsa in [20, 21]):

$$|A+B|^{1/n} \geq |A|^{1/n} + \frac{1}{(n!)^{1/n}}\big(|B|-n\big)^{1/n}$$

and, if $|B| \leq |A|$, then

$$|A+B| \geq |A| + (n-1)|B| + \big(|A|-n\big)^{(n-1)/n}\big(|B|-n\big)^{1/n} - \frac{n(n-1)}{2}.$$

More recently, different discrete analogues of the Brunn-Minkowski inequality have been obtained, including the case of its classical form (cf. (1.1)) for the cardinality [7, 11, 13], functional extensions of it [8, 12, 13, 14] and versions for the *lattice point enumerator* $\mathrm{G}_n(\cdot)$ [8, 13], which is given by $\mathrm{G}_n(M) = |M \cap \mathbb{Z}^n|$. In this respect, [13, Theorem 2.1] reads as follows:

Theorem A. *Let $\lambda \in (0,1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Then*

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}.$$

Equality is attained if $K = [0,a]^n$ and $L = [0,b]^n$ are cubes with a, b, $(1-\lambda)a + \lambda b \in \mathbb{Z}$.

Here we show that it is possible to extend the previous inequality to the case of arbitrary $t, s \geq 0$ (cf. (1.1)), that is, not necessarily such that $t+s=1$. More precisely, in Section 2 we show the following result (here $\lceil x \rceil$ represents the ceiling function of x, namely, the least integer greater than or equal to x):

Theorem 1.1. *Let $t, s \geq 0$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets such that $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$. Then*

$$\mathrm{G}_n\big(tK + sL + (-1, \lceil t+s \rceil)^n\big)^{1/n} \geq t\mathrm{G}_n(K)^{1/n} + s\mathrm{G}_n(L)^{1/n}. \tag{1.3}$$

Equality is attained, when $t+s \in \mathbb{Z}$, if $K = [0,a]^n$ and $L = [0,b]^n$ are cubes with $a, b, ta+sb \in \mathbb{Z}$.

Probably the most outstanding and striking conclusion from the Brunn-Minkowski inequality (1.1) is the following long-standing result: the classical *isoperimetric inequality*. Its form for convex bodies in $\mathbb{R}^n$, i.e., for non-empty compact convex subsets of the n-dimensional Euclidean space, states that the volume $\mathrm{vol}(\cdot)$ and surface area $\mathrm{S}(\cdot)$ (Minkowski content) of any n-dimensional convex body K satisfy

$$\left(\frac{\mathrm{S}(K)}{\mathrm{S}(B_n)}\right)^n \geq \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_n)}\right)^{n-1}, \tag{1.4}$$

where B_n denotes the Euclidean (closed) unit ball. In other words, Euclidean balls minimize the surface area among those convex bodies with prescribed positive volume.

There exist various facets of the isoperimetric inequality (see e.g. [22, Section 7.2] and the references therein), having different ramifications into other settings such as its versions in the spherical and hyperbolic spaces (see e.g. [3]). The isoperimetric inequality has been the starting point for new engaging related results, such as a reverse isoperimetric inequality (see [1]), and it has led to various remarkable consequences not only in geometry but also in analysis (see e.g. [4]). For an extensive survey article on this inequality we refer the reader to [17].

The isoperimetric inequality (1.4) admits the following "*neighbourhood* form" (see e.g. [15, Proposition 14.2.1]): for any n-dimensional convex body $K \subset \mathbb{R}^n$, and all $t \geq 0$, we have

$$\mathrm{vol}(K + tB_n) \geq \mathrm{vol}(rB_n + tB_n), \tag{1.5}$$

where rB_n, $r > 0$, is a ball of the same volume as K. In fact, by subtracting $\mathrm{vol}(K) = \mathrm{vol}(rB_n)$, dividing both sides of (1.5) by t, and taking limits as $t \to 0^+$, one immediately gets (1.4) from (1.5).

The neighbourhood $K + tB_n$, $t \geq 0$, of the n-dimensional convex body K coincides with the set of all points of $\mathbb{R}^n$ having (Euclidean) distance from K at most t. Exchanging the role of the unit ball B_n in (1.5) by another (n-dimensional) convex body $E \subset \mathbb{R}^n$, i.e., changing the involved "distance", one is naturally led to the fact

$$\mathrm{vol}(K + tE) \geq \mathrm{vol}(rE + tE) \tag{1.6}$$

for all $t \geq 0$, where again $r > 0$ is such that rE has the same volume as K. Thus, the advantage of using the volume of a neighbourhood of K, instead of its surface area, is that it can be extended to other spaces in which the latter notion makes no sense; it just suffices to consider a metric and a measure on the given space.

A relevant example of a space in which an isoperimetric inequality in this form holds is the n-dimensional discrete cube $\{0, 1\}^n$ (see e.g. [15, Section 14.2]). Similar inequalities also hold in other discrete metric spaces, in the settings of combinatorics and graph theory (for which we refer the reader to [10]). Recently, in [18], a discrete isoperimetric inequality has been derived for the integer lattice $\mathbb{Z}^n$ endowed with the L_∞ norm and the cardinality measure $|\cdot|$ (see also [9] for a related result in the case of the L_1 norm, where the author uses a method based on the solvability of a certain finite difference equations problem). To this aim, a suitable extension of *lattice cubes* (i.e., the intersection of cubes $[a,b]^n$ with $\mathbb{Z}^n$) is considered: we will call these sets *extended lattice cubes*, which will be denoted by $\mathcal{I}_r$ (see Definition 2.2), for any $r \in \mathbb{N}$. In fact, when $r = m^n$ for some $m \in \mathbb{N}$, $\mathcal{I}_r$ turns out to be a *lattice cube*. Thus, the authors show that such sets $\mathcal{I}_r$ minimize the cardinality of the suitable neighbourhood among all non-empty sets of fixed cardinality r. More precisely, [18, Theorem 1], combined with [18, Lemma 1], leads to the following discrete analogue of (1.6):

Theorem B ([18])**.** *Let $A \subset \mathbb{Z}^n$ be a non-empty finite set and let $r \in \mathbb{N}$ be such that $|\mathcal{I}_r| = |A|$. Then*

$$\left|A + \left((m[-1,1]^n) \cap \mathbb{Z}^n\right)\right| \geq \left|\mathcal{I}_r + \left((m[-1,1]^n) \cap \mathbb{Z}^n\right)\right| \tag{1.7}$$

for all $m \in \mathbb{N}$.

Indeed, the authors prove the above theorem for $m = 1$; a brief additional argument allows to state it for all $m \in \mathbb{N}$ (see Section 2 for the proper explanation).

Here, we are interested in studying an analogue of the above discrete isoperimetric inequality in the setting of arbitrary non-empty bounded sets in $\mathbb{R}^n$ endowed with (the L_∞ norm and) the lattice point enumerator $\mathrm{G}_n(\cdot)$. In this way, one may consider neighbourhoods of a given set at any distance $t \geq 0$, not necessarily integer (cf. (1.7)). In Section 2 we show that such extremal sets will be the *extended cubes* $\mathcal{C}_r$ (see Definition 2.2), which satisfy $\mathcal{C}_r \cap \mathbb{Z}^n = \mathcal{I}_r$ (and thus $\mathrm{G}_n(\mathcal{C}_r) = |\mathcal{I}_r| = r$) and are furthermore characterized as the largest sets for which $\mathcal{C}_r + (-1,1)^n = \mathcal{I}_r + (-1,1)^n$, for any $r \in \mathbb{N}$:

Theorem 1.2. *Let $K \subset \mathbb{R}^n$ be a bounded set with $\mathrm{G}_n(K) > 0$ and let $r \in \mathbb{N}$ be such that $\mathrm{G}_n(\mathcal{C}_r) = \mathrm{G}_n(K)$. Then*

$$\mathrm{G}_n\left(K + t[-1,1]^n\right) \geq \mathrm{G}_n\left(\mathcal{C}_r + t[-1,1]^n\right) \tag{1.8}$$

for all $t \geq 0$.

Finally, in Section 3 we show that both the classical Brunn-Minkowski inequality (1.1) and the isoperimetric inequality (1.6), in the setting of non-empty compact sets, can be derived as a consequence of these new discrete inequalities for the lattice point enumerator $\mathrm{G}_n(\cdot)$:

Theorem 1.3. *The discrete Brunn-Minkowski inequality* (1.3) *implies the classical Brunn-Minkowski inequality* (1.1) *for non-empty compact sets.*

Theorem 1.4. *The discrete isoperimetric inequality* (1.8) *implies the classical isoperimetric inequality* (1.6)*, with $E = [-1,1]^n$, for non-empty compact sets.*

2. Brunn-Minkowski and isoperimetric inequalities for $\mathrm{G}_n(\cdot)$

2.1. **A Brunn-Minkowski type inequality for the lattice point enumerator.** This subsection is mainly devoted to Theorem 1.1, shown by induction on the dimension. We start by proving the following result, which will be used to obtain the one-dimensional case of Theorem 1.1. Although it turns out to be a particular case of the corresponding analogue of [13, Lemma 2.1] (by suitably replacing the parameters $(1-\lambda)$ and λ by t and s, respectively), we include its proof here to make the paper more self-contained.

Lemma 2.1. *Let $t,s\geq 0$ and let $K,L,M\subset\mathbb{R}$ be non-empty sets such that $tK+sL\subset M$. If $M=\bigcup_{i=1}^{r}[a_i,b_i]$, with $a_i,b_i\in\mathbb{Z}$ for all $i=1,\dots,r$, is a finite union of pairwise disjoint compact intervals with integer extremes then*

$$\mathrm{G}_1(M)+r(t+s-1)\geq t\mathrm{G}_1(K)+s\mathrm{G}_1(L).$$

From now on, by $\lfloor x\rfloor$ we will denote the floor function of the real number x, i.e., the greatest integer less than or equal to x.

Proof. We prove the result by induction on r. For the case $r=1$, i.e., when $M=[a_1,b_1]$ is a (non-empty) compact interval (with $a_1,b_1\in\mathbb{Z}$), we have on the one hand that $\mathrm{G}_1(M)=b_1-a_1+1$. Moreover, denoting by $a=\inf K$, $b=\sup K$, $c=\inf L$ and $d=\sup L$, we clearly get $\mathrm{G}_1(K)\leq \mathrm{G}_1([a,b])=\lfloor b\rfloor-\lceil a\rceil+1$ and $\mathrm{G}_1(L)\leq\mathrm{G}_1([c,d])=\lfloor d\rfloor-\lceil c\rceil+1$. On the other hand, the inclusion $tK+sL\subset M$ implies that $b_1\geq t\lfloor b\rfloor+s\lfloor d\rfloor$ and $a_1\leq t\lceil a\rceil+s\lceil c\rceil$, and thus $b_1-a_1\geq t\left(\lfloor b\rfloor-\lceil a\rceil\right)+s(\lfloor d\rfloor-\lceil c\rceil)$. Altogether, we get $\mathrm{G}_1(M)-1\geq t(\mathrm{G}_1(K)-1)+s(\mathrm{G}_1(L)-1)$, showing the case $r=1$.

So, we suppose that the inequality is true for $r\geq 1$ and assume that $M=\cup_{i=1}^{r+1}[a_i,b_i]$, where $b_i<a_{i+1}$ for all $1\leq i\leq r$.

Denoting by $M_1=[a_1,b_1]$ and $M_2=\cup_{i=2}^{r+1}[a_i,b_i]$, we may assume, without loss of generality, that $M_1\cap(tK+sL)\neq\emptyset$. Hence, we may define $m=\sup\bigl(M_1\cap(tK+sL)\bigr)$ and then, since K and L are bounded (because $tK+sL\subset M$), there exist $k\in\operatorname{cl}K$ and $l\in\operatorname{cl}L$ such that $t\,k+s\,l=m$. Thus, considering the sets $K_1=\{x\in K: x\leq k\}$, $K_2=K\setminus K_1$, $L_1=\{x\in L: x\leq l\}$ and $L_2=L\setminus L_1$, we have that $tK_1+sL_1\subset M_1$ and $tK_2+sL_2\subset M_2$. Therefore, applying the induction hypothesis (and taking into account that M_1 are M_2 are disjoint), we get

$$\begin{aligned}\mathrm{G}_1(M)+(r+1)(t+s-1)&=\mathrm{G}_1(M_1)+(t+s-1)+\mathrm{G}_1(M_2)+r(t+s-1)\\&\geq t\mathrm{G}_1(K_1)+s\mathrm{G}_1(L_1)+t\mathrm{G}_1(K_2)+s\mathrm{G}_1(L_2)\\&=t\mathrm{G}_1(K)+s\mathrm{G}_1(L),\end{aligned}$$

as desired. □

The following result yields the case $n=1$ of Theorem 1.1 and, as previously announced, it will be used to derive (1.3).

Lemma 2.2. *Let $t,s \geq 0$ and let $K,L \subset \mathbb{R}$ be non-empty bounded sets. Then*

$$\mathrm{G}_1\Big(tK+sL+\big(-1,\lceil t+s\rceil\big)\Big) \geq t\mathrm{G}_1(K)+s\mathrm{G}_1(L). \tag{2.1}$$

Equality is attained, when $t+s\in\mathbb{Z}$, if $K=[0,a]$ and $L=[0,b]$ are intervals with $a,b,ta+sb\in\mathbb{Z}$.

Proof. Let $M=\bigcup_{x\in tK+sL}\big[\lfloor x\rfloor,\lceil x\rceil\big]$. Clearly, since K and L are bounded, M is a finite union of compact disjoint intervals, say $M=\bigcup_{i=1}^r[a_i,b_i]$ for some $a_i,b_i\in\mathbb{Z}$, $i=1,\dots,r$.

For $I=\big\{1\leq i<r : a_{i+1}-b_i\leq\lceil t+s-1\rceil\big\}$, let $M'=M\cup\big(\bigcup_{i\in I}[b_i,a_{i+1}]\big)$ and let $M''=M'+\big[0,\lceil t+s-1\rceil\big]=M+\big[0,\lceil t+s-1\rceil\big]$. From Lemma 2.1 we obtain

$$\begin{aligned}
\mathrm{G}_1(M'') &= \mathrm{G}_1(M')+\big(r-|I|\big)\big(\lceil t+s-1\rceil\big)\\
&\geq \mathrm{G}_1(M')+\big(r-|I|\big)(t+s-1)\\
&\geq t\mathrm{G}_1(K)+s\mathrm{G}_1(L).
\end{aligned}$$

This yields (2.1) since $M\cap\mathbb{Z}=\big(tK+sL+(-1,1)\big)\cap\mathbb{Z}$ and

$$M''\cap\mathbb{Z}=\Big(tK+sL+\big(-1,\lceil t+s-1\rceil+1\big)\Big)\cap\mathbb{Z}.$$

Finally, in order to show that equality may be attained (for some $t,s\geq 0$), it is enough to consider $a,b,t,s>0$ such that $a,b,t+s,ta+sb\in\mathbb{Z}$, and take $K=[0,a]$ and $L=[0,b]$, for which we have

$$tK+sL+\big(-1,\lceil t+s\rceil\big)=(-1,ta+sb+t+s),$$

and thus

$$\mathrm{G}_1\Big(tK+sL+\big(-1,\lceil t+s\rceil\big)\Big)=t(a+1)+s(b+1)=t\mathrm{G}_1(K)+s\mathrm{G}_1(L). \qquad \square$$

Before proving (the general case of) Theorem 1.1, we need to state an auxiliary result and additional notation. First we collect the following lemma, which can be regarded as a discrete counterpart of the well-known *Cavalieri Principle* for the lattice point enumerator (see [13] and the references therein).

Lemma 2.3. [13, Corollary 2.1] *Let $\Omega\subset\mathbb{R}^n$ be a bounded set, let $f:\mathbb{R}^n\longrightarrow[0,\infty)$ and set $f(\Omega\cap\mathbb{Z}^n)\subset\{k_0,k_1,\dots,k_r\}$ with $0=k_0<k_1<\dots<k_r$. Then*

$$\sum_{x\in\Omega\cap\mathbb{Z}^n}f(x)=\sum_{i=1}^r(k_i-k_{i-1})\mathrm{G}_n\big(\{x\in\Omega : f(x)\geq k_i\}\big).$$

Now, we denote by e_i the i-th canonical unit vector and we set (x,y) for the open segment with endpoints $x,y\in\mathbb{R}^n$. Moreover, given a non-empty bounded set $M\subset\mathbb{R}^n$ and $\tau\in\mathbb{R}$, we denote by $M(\tau)$ the hyperplane section of M at height τ (in the direction of e_n), i.e.,

$$M(\tau)=\big\{x\in\mathbb{R}^{n-1} : (x,\tau)\in M\big\}.$$

Next we show the general case of Theorem 1.1. The proof follows the ideas of [11, Theorem 2.1] and [13, Theorem 2.1]; we include it here for the sake of completeness.

Proof of Theorem 1.1. We will show (1.3) by (finite) induction on the dimension n. The case $n = 1$ is collected in (2.1). So, we will suppose that the inequality is true for $n-1$.

We first observe that for all $x, y \in \mathbb{R}$,

$$(tK + sL)\,(tx + sy) \supset tK(x) + sL(y).$$

Then, for any $x, y \in \mathbb{R}$ such that $K(x), L(y) \neq \emptyset$, applying the induction hypothesis (i.e., (1.3) in $\mathbb{R}^{n-1}$) and setting $C_{n-1} = \sum_{i=1}^{n-1}\bigl(-e_i, \lceil t+s\rceil e_i\bigr)$, we get that

$$\begin{aligned} &\mathrm{G}_{n-1}\bigl((tK + sL + C_{n-1})(tx + sy)\bigr) \\ &\qquad \geq \mathrm{G}_{n-1}\Bigl(tK(x) + sL(y) + \bigl(-1, \lceil t+s\rceil\bigr)^{n-1}\Bigr) \\ &\qquad \geq \Bigl(t\mathrm{G}_{n-1}\bigl(K(x)\bigr)^{1/(n-1)} + s\mathrm{G}_{n-1}\bigl(L(y)\bigr)^{1/(n-1)}\Bigr)^{n-1}. \end{aligned} \tag{2.2}$$

For the sake of brevity we denote by

$$\begin{aligned} N &= tK + sL + C_{n-1}, & M &= tK + sL + \bigl(-1, \lceil t+s\rceil\bigr)^n, \\ a &= \max_{x\in\mathbb{Z}} \mathrm{G}_{n-1}\bigl(K(x)\bigr), & b &= \max_{x\in\mathbb{Z}} \mathrm{G}_{n-1}\bigl(L(x)\bigr). \end{aligned}$$

Since $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$, we have that $a > 0$ and $b > 0$. Without loss of generality, we may assume that t, s are not both identically zero, and then we define

$$c = \Bigl(ta^{1/(n-1)} + sb^{1/(n-1)}\Bigr)^{n-1} \quad \text{and} \quad \theta = \frac{s\, b^{1/(n-1)}}{c^{1/(n-1)}}.$$

Finally, for $S = K, L, N$ or M, we denote by $f_S : \mathbb{R} \longrightarrow [0, \infty)$ the functions given by

$$\begin{aligned} f_K(x) &= \frac{\mathrm{G}_{n-1}\bigl(K(x)\bigr)}{a}, & f_L(x) &= \frac{\mathrm{G}_{n-1}\bigl(L(x)\bigr)}{b}, \\ f_N(x) &= \frac{\mathrm{G}_{n-1}\bigl(N(x)\bigr)}{c}, & f_M(x) &= \frac{\mathrm{G}_{n-1}\bigl(M(x)\bigr)}{c}. \end{aligned}$$

Using (2.2) we get

$$\begin{aligned} \mathrm{G}_{n-1}\bigl(N(tx + sy)\bigr) &\geq \Bigl(t\mathrm{G}_{n-1}\bigl(K(x)\bigr)^{1/(n-1)} + s\mathrm{G}_{n-1}\bigl(L(y)\bigr)^{1/(n-1)}\Bigr)^{n-1} \\ &= c\left(t\frac{a^{1/(n-1)} f_K(x)^{1/(n-1)}}{c^{1/(n-1)}} + s\frac{b^{1/(n-1)} f_L(y)^{1/(n-1)}}{c^{1/(n-1)}}\right)^{n-1} \\ &= c\Bigl((1-\theta) f_K(x)^{1/(n-1)} + \theta f_L(y)^{1/(n-1)}\Bigr)^{n-1} \\ &\geq c \min\bigl\{f_K(x), f_L(y)\bigr\}. \end{aligned}$$

Thus, we have obtained the functional inequality

$$f_N(tx+sy)\geq \min\bigl\{f_K(x),f_L(y)\bigr\}. \tag{2.3}$$

Now we observe, on the one hand, that the superlevel sets

$$\bigl\{x\in\mathbb{R}: f_K(x)\geq\tau\bigr\},\qquad \bigl\{x\in\mathbb{R}: f_L(x)\geq\tau\bigr\}$$

are non-empty for all $\tau\in[0,1]$. On the other hand, (2.3) implies that

$$\bigl\{x\in\mathbb{R}: f_N(x)\geq\tau\bigr\}\supset t\bigl\{x\in\mathbb{R}: f_K(x)\geq\tau\bigr\}+s\bigl\{x\in\mathbb{R}: f_L(x)\geq\tau\bigr\},$$

and thus, applying Lemma 2.2, we obtain

$$\begin{aligned}&\mathrm{G}_1\Bigl(\bigl\{x\in\mathbb{R}: f_N(x)\geq\tau\bigr\}+\bigl(-1,\lceil t+s\rceil\bigr)\Bigr)\\ &\quad\geq t\mathrm{G}_1\Bigl(\bigl\{x\in\mathbb{R}: f_K(x)\geq\tau\bigr\}\Bigr)+s\mathrm{G}_1\Bigl(\bigl\{x\in\mathbb{R}: f_L(x)\geq\tau\bigr\}\Bigr)\end{aligned} \tag{2.4}$$

for all $\tau\in[0,1]$. Now, since $N\subset M$, then $f_N(x)\leq f_M(x)$ for every $x\in\mathbb{R}$ and so

$$\bigl\{x\in\mathbb{R}: f_N(x)\geq\tau\bigr\}\subset\bigl\{x\in\mathbb{R}: f_M(x)\geq\tau\bigr\}.$$

Moreover, from

$$C_{n-1}+\bigl(-e_n,\lceil t+s\rceil e_n\bigr)=\bigl(-1,\lceil t+s\rceil\bigr)^n,$$

we have $N+\bigl(-e_n,\lceil t+s\rceil e_n\bigr)=M$ and hence

$$\bigl\{x\in\mathbb{R}: f_M(x)\geq\tau\bigr\}\supset\bigl\{x\in\mathbb{R}: f_N(x)\geq\tau\bigr\}+\bigl(-1,\lceil t+s\rceil\bigr). \tag{2.5}$$

Finally, set $\{k_0,k_1,\dots,k_r\}\supset f_K(\mathbb{Z})\cup f_L(\mathbb{Z})\cup f_N(\mathbb{Z})\cup f_M(\mathbb{Z})$, with $0=k_0<k_1<\dots<k_r$ where, for some $s\in\{1,\dots,r\}$,

$$k_s=\max_{m\in\mathbb{Z}}f_K(m)=\max_{m\in\mathbb{Z}}f_L(m)=1.$$

Then, using (2.4) and (2.5), together with Lemma 2.3 we obtain

$$
\begin{aligned}
\mathrm{G}_n(M) &= \sum_{m\in\mathbb{Z}} \mathrm{G}_{n-1}\bigl(M(m)\bigr) = \sum_{m\in\mathbb{Z}} c f_M(m)\\
&= c\sum_{i=1}^{r}(k_i-k_{i-1})\,\mathrm{G}_1\Bigl(\bigl\{x\in\mathbb{R} : f_M(x)\geq k_i\bigr\}\Bigr)\\
&\geq c\sum_{i=1}^{r}(k_i-k_{i-1})\,\mathrm{G}_1\Bigl(\bigl\{x\in\mathbb{R} : f_N(x)\geq k_i\bigr\}+\bigl(-1,\lceil t+s\rceil\bigr)\Bigr)\\
&\geq c\sum_{i=1}^{s}(k_i-k_{i-1})\Bigl[t\mathrm{G}_1\Bigl(\bigl\{x\in\mathbb{R} : f_K(x)\geq t\bigr\}\Bigr)\\
&\qquad\qquad + s\mathrm{G}_1\Bigl(\bigl\{x\in\mathbb{R} : f_L(x)\geq t\bigr\}\Bigr)\Bigr]\\
&= c\left(t\sum_{m\in\mathbb{Z}} f_K(m) + s\sum_{m\in\mathbb{Z}} f_L(m)\right) = c\left(t\frac{\mathrm{G}_n(K)}{a}+s\frac{\mathrm{G}_n(L)}{b}\right)\\
&\geq \left[\Bigl(ta^{1/(n-1)}\Bigr)^{1-1/n}\left(\frac{t}{a}\mathrm{G}_n(K)\right)^{1/n} + \Bigl(sb^{1/(n-1)}\Bigr)^{1-1/n}\left(\frac{s}{b}\mathrm{G}_n(L)\right)^{1/n}\right]^n\\
&= \Bigl(t\mathrm{G}_n(K)^{1/n}+s\mathrm{G}_n(L)^{1/n}\Bigr)^n,
\end{aligned}
$$

where the last inequality follows from Hölder's inequality.

In order to prove that equality may be attained, we consider $a,b,t,s>0$ such that $a,b,t+s,ta+sb\in\mathbb{Z}$ and take $K=[0,a]^n$ and $L=[0,b]^n$, for which we have

$$tK+sL+\bigl(-1,\lceil t+s\rceil\bigr)^n=\bigl(-1,ta+sb+\lceil t+s\rceil\bigr)^n.$$

Therefore

$$
\begin{aligned}
\mathrm{G}_n\Bigl(tK+sL+\bigl(-1,\lceil t+s\rceil\bigr)^n\Bigr)^{1/n} &= t(a+1)+s(b+1)\\
&= t\mathrm{G}_n(K)^{1/n}+s\mathrm{G}_n(L)^{1/n}. \qquad \square
\end{aligned}
$$

Theorem 1.1 allows us to obtain a property for the lattice point enumerator that resembles the homogeneity of the volume:

Corollary 2.1. *Let $t\geq 0$ and let $K\subset\mathbb{R}^n$ be a non-empty bounded set. Then*

$$\mathrm{G}_n\Bigl(tK+\bigl(-1,\lceil t\rceil\bigr)^n\Bigr)\geq t^n\mathrm{G}_n(K).$$

Proof. We may assume, without loss of generality, that $\mathrm{G}_n(K)>0$. Then the result follows from Theorem 1.1 for $s=0$ and $L=\{0\}$. □

2.2. **An isoperimetric type inequality for the lattice point enumerator.** Let K be a non-empty bounded set with

$$\mathrm{G}_n(K)=\mathrm{G}_n\bigl(r[-1,1]^n\bigr)=(2r+1)^n$$

for some $r \in \mathbb{N}$. Then, from Theorem 1.1 for $t = 1$ and $s \in \mathbb{N}$,

$$\begin{aligned}\mathrm{G}_n\Big(K + s[-1,1]^n + \big(-1,\lceil 1+s\rceil\big)^n\Big) &\geq \Big(\mathrm{G}_n(K)^{1/n} + s\mathrm{G}_n\big([-1,1]^n\big)^{1/n}\Big)^n \\ &= (2r+3s+1)^n \\ &= \mathrm{G}_n\Big(r[-1,1]^n + s[-1,1]^n + \big(-1,\lceil 1+s\rceil\big)^n\Big),\end{aligned}$$

which gives a particular discrete analogue of (1.6). Here we will show how such a type of inequality can be extended to the case of any bounded set K, i.e., with an arbitrary amount of integer points, and any $s \geq 0$; in other words, we will prove Theorem 1.2.

To this aim, first we need to give some definitions and point out some important facts. Given a vector $u = (u_1 \dots, u_n) \in \mathbb{Z}^n$ and fixing $i_u \in \{1, \dots, n\}$, we will write

$$u' = (u_1 \dots, u_{i_u-1}, u_{i_u+1}, \dots, u_n) \in \mathbb{Z}^{n-1}.$$

With this notation, in [18] the following well-order $\prec$ on $\mathbb{Z}^n$ is defined:

Definition 2.1. *If $n = 1$ we define the order $\prec$ given by*

$$0 \prec 1 \prec -1 \prec 2 \prec -2 \prec \dots \prec m \prec -m \prec \dots$$

For $n \geq 2$ we set, for $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$,

$$m_w = \max_{\prec}\{w_i : i = 1, \dots, n\} \quad \text{and} \quad i_w = \min\big\{i : w_i = m_w\big\},$$

and we define $\prec$ recursively as follows: for any $u, v \in \mathbb{Z}^n$ with $u \neq v$,

i) *if $m_u \prec m_v$ then $u \prec v$;*
ii) *if $m_u = m_v$ then $u \prec v$ if either $i_v < i_u$ or ($i_v = i_u$ and) $u' \prec v'$.*

Moreover, we write $u \preceq v$ if either $u \prec v$ or $u = v$.

This order will allow us to define the extended lattice cube $\mathcal{I}_r$ of r points as the initial segment in $\mathbb{Z}^n$ with respect to $\prec$. To define the sets $\mathcal{C}_r$, which will be referred to as *extended cubes*, first we need the following definition, which can be seen as a particular case of the family of *weakly unconditional sets*, first introduced in [19] (we refer the reader to this work for further properties and relations of them with certain Brunn-Minkowski type inequalities): for any non-empty finite set $A \subset \mathbb{R}^n$, we write

$$\mathcal{C}_A = \big\{(\lambda_1 x_1, \dots, \lambda_n x_n) \in \mathbb{R}^n : (x_1, \dots, x_n) \in A, \lambda_i \in [0,1] \text{ for } i = 1, \dots, n\big\}$$

(see Figure 1).

Definition 2.2. *Let $r \in \mathbb{N}$. By $\mathcal{I}_r$ we denote the initial segment in $(\mathbb{Z}^n, \prec)$ of length r, i.e., the set of the first r points with respect to the order $\prec$ on $\mathbb{Z}^n$ (see Figure 2, left). Moreover, by $\mathcal{C}_r$ we denote the set given by $\mathcal{C}_r := \mathcal{C}_{\mathcal{I}_r}$ (see Figure 2, right).*

We note that if $r = m^n$ for some $m \in \mathbb{N}$ then $\mathcal{I}_r$ is indeed a lattice cube. More precisely, $\mathcal{I}_r = \{-m/2+1, -m/2+2, \dots, m/2-1, m/2\}^n$ if m is even and $\mathcal{I}_r = \big\{-(m-1)/2, -(m-1)/2+1, \dots, (m-1)/2, (m-1)/2\big\}^n$ if m

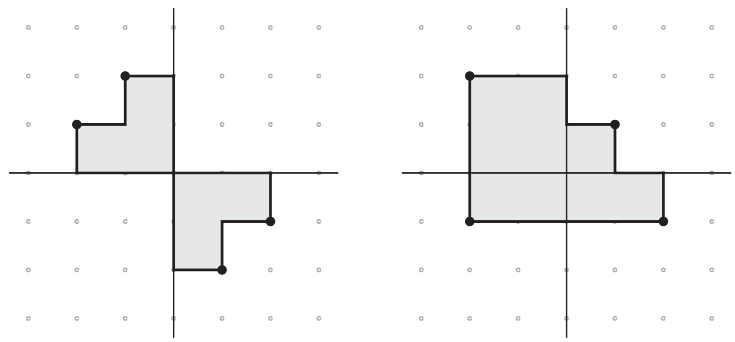

FIGURE 1. Sets $\mathcal{C}_A \subset \mathbb{R}^2$ for different finite sets $A \subset \mathbb{Z}^2$.

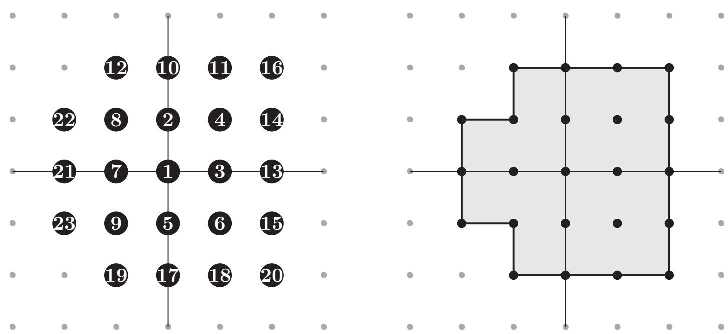

FIGURE 2. The extended lattice cube $\mathcal{I}_{23}$ in $\mathbb{Z}^2$ (left) and the corresponding extended cube $\mathcal{C}_{23}$ in $\mathbb{R}^2$ (right).

is odd (cf. Figure 2, left). This further implies that $\mathcal{C}_r$ is a cube whenever $r = m^n$ for some $m \in \mathbb{N}$.

In [18, Theorem 1]) it was proven that Theorem B holds for $m = 1$. Next we show that [18, Theorem 1]), in combination with [18, Lemma 1] (which ensures that $\mathcal{I}_r + \{-1, 0, 1\}^n$ is also an extended lattice cube, for any $r \in \mathbb{N}$), indeed implies Theorem B for all $m \in \mathbb{N}$. We include the proof here to make the paper more self-contained:

Proof of Theorem B. First, we write

$$r_m = \Big|A + \big((m[-1,1]^n) \cap \mathbb{Z}^n\big)\Big| \quad \text{and} \quad s_m = \Big|\mathcal{I}_r + \big((m[-1,1]^n) \cap \mathbb{Z}^n\big)\Big|$$

for any $m \in \mathbb{N}$, where $r \in \mathbb{N}$ is such that $|\mathcal{I}_r| = |A|$. The case $m = 1$ of (1.7) is collected in [18, Theorem 1]). So, let $m \geq 2$ and assume that Theorem B holds for $m - 1$. On the one hand, from [18, Theorem 1] applied to the set $A + \Big(\big((m-1)[-1,1]^n\big) \cap \mathbb{Z}^n\Big)$, we have

$$r_m = \Big|A + \Big(\big((m-1)[-1,1]^n\big) \cap \mathbb{Z}^n\Big) + \{-1,0,1\}^n\Big| \geq \big|\mathcal{I}_{r_{m-1}} + \{-1,0,1\}^n\big|. \tag{2.6}$$

On the other hand, from [18, Lemma 1] we immediately get

$$\mathcal{I}_r + \Big(\big((m-1)[-1,1]^n\big) \cap \mathbb{Z}^n\Big) = \mathcal{I}_{s_{m-1}}. \tag{2.7}$$

Hence, from the induction hypothesis we obtain $r_{m-1} \geq s_{m-1}$ and then $\mathcal{I}_{r_{m-1}} \supset \mathcal{I}_{s_{m-1}}$. Therefore, this inclusion jointly with (2.7) and (2.6) implies that $r_m \geq s_m$, as desired. □

Now we are in a position to prove Theorem 1.2. To this aim, we will relate the extended cubes $\mathcal{C}_r$ to the lattice ones $\mathcal{I}_r$.

Proof of Theorem 1.2. First we claim that

$$\mathcal{C}_r + (-1,1)^n = \mathcal{I}_r + (-1,1)^n \tag{2.8}$$

for all $r \in \mathbb{N}$. To show it, let $r \in \mathbb{N}$ and let $z = (z_1, \ldots, z_n) \in \mathcal{C}_r \cap \mathbb{Z}^n$. Then, there exists $x = (x_1, \ldots, x_n) \in \mathcal{I}_r$ such that $z_i = \lambda_i x_i$ for some $\lambda_i \in [0,1]$, $i = 1, \ldots, n$, which implies that $z_i \preceq x_i$ for all $i = 1, \ldots, n$ and thus $z \preceq x$. Hence, $z \in \mathcal{I}_r$ and the relation $\mathcal{C}_r \cap \mathbb{Z}^n \subset \mathcal{I}_r$ infers. Since the reverse inclusion trivially follows from the definition of $\mathcal{C}_r$, we have that $\mathcal{C}_r \cap \mathbb{Z}^n = \mathcal{I}_r$ for all $r \in \mathbb{N}$. This further implies that $\mathcal{C}_{\{x\}} + (-1,1)^n \subset \mathcal{I}_r + (-1,1)^n$ for all $x \in \mathcal{I}_r$ and thus, since $\mathcal{C}_r = \bigcup_{x \in \mathcal{I}_r} \mathcal{C}_{\{x\}}$, we obtain (2.8).

Now, let $\lambda = t - \lfloor t \rfloor \in [0,1)$ be the decimal part of $t \geq 0$. From (2.8) we have $\mathcal{C}_r + [-\lambda, \lambda]^n \subset \mathcal{I}_r + (-1,1)^n$ and then, by adding the cube $\lfloor t \rfloor [-1,1]^n$, we get $\mathcal{C}_r + t[-1,1]^n \subset \mathcal{I}_r + \big(\lfloor t \rfloor + 1\big)(-1,1)^n$. Hence, since $\mathcal{I}_r \subset \mathbb{Z}^n$, we have

$$\mathrm{G}_n\big(\mathcal{C}_r + t[-1,1]^n\big) \leq \left|\mathcal{I}_r + \Big(\big(\lfloor t \rfloor [-1,1]^n\big) \cap \mathbb{Z}^n\Big)\right|.$$

Thus, from Theorem B applied to the set $K \cap \mathbb{Z}^n$, we get

$$\begin{aligned} \mathrm{G}_n\big(K + t[-1,1]^n\big) &\geq \mathrm{G}_n\big((K \cap \mathbb{Z}^n) + t[-1,1]^n\big) \\ &= \left|(K \cap \mathbb{Z}^n) + \Big(\big(\lfloor t \rfloor [-1,1]^n\big) \cap \mathbb{Z}^n\Big)\right| \\ &\geq \left|\mathcal{I}_r + \Big(\big(\lfloor t \rfloor [-1,1]^n\big) \cap \mathbb{Z}^n\Big)\right| \geq \mathrm{G}_n\big(\mathcal{C}_r + t[-1,1]^n\big), \end{aligned}$$

as desired. □

Remark 2.1. *From the proof of the previous result we note that the role of the extended cubes $\mathcal{C}_r$ could be played by other sets L_r, with $\mathrm{G}_n(L_r) = r$, such that $L_r + (-1,1)^n \subset \mathcal{I}_r + (-1,1)^n$. However, $\mathcal{C}_r$ are the largest sets (with respect to set inclusion) satisfying this property (cf.* (2.8)*). Indeed, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x + (-1,1)^n \subset \mathcal{I}_r + (-1,1)^n$, it is enough to consider the point $y = (y_1, \ldots, y_n) \in \big(x + (-1,1)^n\big) \cap \mathbb{Z}^n \subset \big(\mathcal{I}_r + (-1,1)^n\big) \cap \mathbb{Z}^n = \mathcal{I}_r$ given by*

$$y_i = \begin{cases} \lceil x_i \rceil & \text{if } x_i > 0, \\ 0 & \text{if } x_i = 0, \\ \lfloor x_i \rfloor & \text{otherwise,} \end{cases}$$

which yields $x \in \mathcal{C}_{\{y\}} \subset \mathcal{C}_r$.

Remark 2.2. *Theorems 1.1 and 1.2 can be extended to the setting of an arbitrary n-dimensional lattice* $\Lambda \subset \mathbb{R}^n$*. Indeed, if* $\mathcal{B} = \{v_1 \dots, v_n\}$ *is a basis of* Λ*, we may consider* $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *the linear (bijective) map given by* $\varphi(x) = \sum_{i=1}^n x_i v_i$ *for any* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$*. Then, denoting by* $\mathrm{G}_\Lambda(M) = |M \cap \Lambda|$*, Theorem 1.1 implies that*

$$\mathrm{G}_\Lambda\Big(tK + sL + \varphi\big((-1, \lceil t+s \rceil)^n\big)\Big)^{1/n} \geq t\mathrm{G}_\Lambda(K)^{1/n} + s\mathrm{G}_\Lambda(L)^{1/n}$$

for any non-empty bounded sets $K, L \subset \mathbb{R}^n$ *with* $\mathrm{G}_\Lambda(K)\mathrm{G}_\Lambda(L) > 0$ *and all* $t, s \geq 0$*. Analogously, Theorem 1.2 yields*

$$\mathrm{G}_\Lambda\Big(K + t\varphi\big([-1,1]^n\big)\Big) \geq \mathrm{G}_\Lambda\Big(\varphi(\mathcal{C}_r) + t\varphi\big([-1,1]^n\big)\Big)$$

for any bounded set $K \subset \mathbb{R}^n$ *with* $\mathrm{G}_\Lambda(K) > 0$ *and all* $t \geq 0$*, where* $r \in \mathbb{N}$ *is such that* $\mathrm{G}_\Lambda\big(\varphi(\mathcal{C}_r)\big) = \mathrm{G}_\Lambda(K)$*.*

3. From the discrete versions to the continuous ones

We first fix some additional notation that will be used throughout the rest of the paper. For each $m \in \mathbb{N}$, we denote by $\mathrm{G}_{m,n}(\cdot)$ the lattice point enumerator with respect to the lattice $2^{-m}\mathbb{Z}^n$, that is,

$$\mathrm{G}_{m,n}(L) = \big|L \cap (2^{-m}\mathbb{Z}^n)\big| = \big|(2^m L) \cap \mathbb{Z}^n\big| = \mathrm{G}_n\big(2^m L\big),$$

for any $L \subset \mathbb{R}^n$. Moreover, for each $m \in \mathbb{N}$, we write $\mathrm{R}^m = [0, 2^{-m})^n \subset \mathbb{R}^n$ and $\mathrm{O}^m = \mathrm{R}^m - \mathrm{R}^m = (-2^{-m}, 2^{-m})^n$. Finally, for any compact set $M \subset \mathbb{R}^n$ and each $m \in \mathbb{N}$ we denote by

$$M_m = \big\{z \in 2^{-m}\mathbb{Z}^n : (z + \mathrm{R}^m) \cap M \neq \emptyset\big\},$$

for which we clearly have

$$M \subset M_m + \mathrm{R}^m \subset M + \mathrm{O}^m. \tag{3.1}$$

Next we will prove that (the neighbourhood form of) the isoperimetric inequality (1.6) for compact sets, when $E = [-1,1]^n$, can be obtained as a consequence of the discrete inequality (1.8). Although the underlying idea is nothing but successively shrinking the lattice and then approximating the volume by means of the lattice point enumerator, we will include here all the details of the proof for the sake of completeness.

To this aim we show the following auxiliary result. Before, we make a final observation: in the rest of this section, we will take limits provided that they exist; in fact, their existence can be checked using standard arguments in every case and we omit them.

Lemma 3.1. *Let* $K \subset \mathbb{R}^n$ *be a non-empty compact set. If* $\{p_m\}_{m\in\mathbb{N}} \subset \mathbb{N} \cup \{0\}$ *is a sequence satisfying* $(2p_m + 1)^n \leq |K_m| < (2p_m + 3)^n$ *then*

$$\lim_{m\to\infty} \frac{p_m}{2^m} = \frac{\mathrm{vol}(K)^{1/n}}{2}. \tag{3.2}$$

Proof. First we show that $\lim_{m\to\infty} 2^{-mn}|K_m| = \mathrm{vol}(K)$. Using (3.1) we have

$$\mathrm{vol}(K) \le \mathrm{vol}(K_m + \mathrm{R}^m) \le \mathrm{vol}(K + \mathrm{O}^m).$$

This, together with the identity $\mathrm{vol}(K_m + \mathrm{R}^m) = 2^{-mn}|K_m|$ and the fact that $\{K + \mathrm{O}^m\}_{m\in\mathbb{N}}$ is a decreasing sequence with

$$\bigcap_{m=1}^{\infty}(K + \mathrm{O}^m) = K,$$

shows that

$$\mathrm{vol}(K) \le \lim_{m\to\infty}\frac{|K_m|}{2^{mn}} \le \lim_{m\to\infty}\mathrm{vol}(K+\mathrm{O}^m) = \mathrm{vol}\left(\bigcap_{m=1}^{\infty}(K+\mathrm{O}^m)\right) = \mathrm{vol}(K).$$

Furthermore, from

$$\lim_{m\to\infty}\frac{(2p_m+1)^n}{2^{mn}} \le \lim_{m\to\infty}\frac{|K_m|}{2^{mn}} = \mathrm{vol}(K),$$

we infer the existence of a constant $c > 0$ such that $p_m < 2p_m + 1 < 2^m c$ for all $m \in \mathbb{N}$. Thus, applying that $(x+2)^n - x^n \le 3^n x^{n-1}$ for any $x \ge 1$, we have

$$\begin{aligned}0 \le |K_m| - (2p_m+1)^n &< (2p_m+3)^n - (2p_m+1)^n \\ &\le 3^n(2p_m+1)^{n-1} \le 3^n(2^{m+1}c+1)^{n-1},\end{aligned}$$

and since we may assume, without loss of generality, that $c \ge 1/4$, then

$$\begin{aligned}0 \le |K_m| - (2p_m+1)^n < 3^n(2^{m+1}c+1)^{n-1} &\le 3^n(2^{m+1}c+2^{m+1}c)^{n-1} \\ &= 2^{mn-m+2n-2}3^n c^{n-1}.\end{aligned}$$

Hence,

$$0 \le \lim_{m\to\infty}\frac{|K_m| - (2p_m+1)^n}{2^{mn}} \le \lim_{m\to\infty} 2^{-m+2n-2}3^n c^{n-1} = 0.$$

Finally, we have

$$\begin{aligned}\frac{\mathrm{vol}(K)^{1/n}}{2} &= \frac{1}{2}\left(\lim_{m\to\infty}\frac{|K_m|}{2^{mn}}\right)^{1/n} \\ &= \frac{1}{2}\left(\lim_{m\to\infty}\frac{|K_m| - (2p_m+1)^n}{2^{mn}} + \lim_{m\to\infty}\frac{(2p_m+1)^n}{2^{mn}}\right)^{1/n} \\ &= \frac{1}{2}\lim_{m\to\infty}\frac{2p_m+1}{2^m} = \lim_{m\to\infty}\frac{p_m}{2^m},\end{aligned}$$

which shows (3.2). This concludes the proof. □

We observe that, considering the partition

$$\Big\{\big[(2k+1)^n, (2k+3)^n\big) \cap \mathbb{N}\Big\}_{k\in\mathbb{N}\cup\{0\}}$$

of $\mathbb{N}$, then the relation $(2p_m+1)^n \le |K_m| < (2p_m+3)^n$ given in Lemma 3.1 uniquely determines such a sequence $\{p_m\}_{m\in\mathbb{N}} \subset \mathbb{N}\cup\{0\}$.

We conclude this section by proving Theorem 1.4.

Proof of Theorem 1.4. Let $\{p_m\}_{m\in\mathbb{N}} \subset \mathbb{N}\cup\{0\}$ be a sequence satisfying the conditions of Lemma 3.1 and, for the sake of brevity, we write $r_m = (2p_m+1)^n$. Since

$$\begin{aligned}\mathrm{G}_n\bigl(2^m K_m + 2^m \mathrm{R}^m\bigr) &= \mathrm{G}_{m,n}(K_m + \mathrm{R}^m) = |K_m| \\ &\geq (2p_m+1)^n = \bigl|\mathcal{I}_{r_m}\bigr| = \mathrm{G}_n\bigl(\mathcal{C}_{r_m}\bigr),\end{aligned}$$

applying (1.8) we get

$$\begin{aligned}\mathrm{G}_{m,n}\bigl(K_m + \mathrm{R}^m + t[-1,1]^n\bigr) &= \mathrm{G}_n\bigl(2^m K_m + 2^m \mathrm{R}^m + 2^m t[-1,1]^n\bigr) \\ &\geq \mathrm{G}_n\bigl(\mathcal{C}_{r_m} + 2^m t[-1,1]^n\bigr) \\ &= \mathrm{G}_{m,n}\bigl(2^{-m}\mathcal{C}_{r_m} + t[-1,1]^n\bigr)\end{aligned}$$

for all $m \in \mathbb{N}$. Therefore

(3.3)
$$\lim_{m\to\infty} \frac{\mathrm{G}_{m,n}\bigl(K_m + \mathrm{R}^m + t[-1,1]^n\bigr)}{2^{mn}} \geq \lim_{m\to\infty} \frac{\mathrm{G}_{m,n}\bigl(2^{-m}\mathcal{C}_{r_m} + t[-1,1]^n\bigr)}{2^{mn}}.$$

Applying again (3.1) to the set $M = K_m + \mathrm{R}^m + t[-1,1]^n$, we get

$$\begin{aligned}K_m + \mathrm{R}^m + t[-1,1]^n &\subset \bigl(K_m + \mathrm{R}^m + t[-1,1]^n\bigr)_m + \mathrm{R}^m \\ &\subset K_m + \mathrm{R}^m + t[-1,1]^n + \mathrm{O}^m \subset K + t[-1,1]^n + 2\mathrm{O}^m\end{aligned}$$

and then

$$\begin{aligned}\frac{\mathrm{G}_{m,n}\bigl(K_m + \mathrm{R}^m + t[-1,1]^n\bigr)}{2^{mn}} &\leq \frac{\Bigl|\bigl(K_m + \mathrm{R}^m + t[-1,1]^n\bigr)_m\Bigr|}{2^{mn}} \\ &= \mathrm{vol}\Bigl(\bigl(K_m + \mathrm{R}^m + t[-1,1]^n\bigr)_m + \mathrm{R}^m\Bigr) \\ &\leq \mathrm{vol}\bigl(K + t[-1,1]^n + 2\mathrm{O}^m\bigr).\end{aligned}$$

Since $\bigl\{K + t[-1,1]^n + 2\mathrm{O}^m\bigr\}_{m\in\mathbb{N}}$ is a decreasing sequence with

$$\bigcap_{m\in\mathbb{N}} \bigl(K + t[-1,1]^n + 2\mathrm{O}^m\bigr) = K + t[-1,1]^n,$$

we have

$$\lim_{m\to\infty} \mathrm{vol}\bigl(K + t[-1,1]^n + 2\mathrm{O}^m\bigr) = \mathrm{vol}\bigl(K + t[-1,1]^n\bigr).$$

Therefore

$$\lim_{m\to\infty} \frac{\mathrm{G}_{m,n}\bigl(K_m + \mathrm{R}^m + t[-1,1]^n\bigr)}{2^{mn}} \leq \mathrm{vol}\bigl(K + t[-1,1]^n\bigr). \tag{3.4}$$

Finally, we note that

$$\mathrm{G}_{m,n}\bigl(2^{-m}\mathcal{C}_{r_m} + t[-1,1]^n\bigr) = \bigl(2(p_m + t_m) + 1\bigr)^n,$$

where $t_m := \lfloor 2^m t \rfloor$ for all $m \in \mathbb{N}$ (which clearly satisfies that $t_m/2^m \to t$ as $m \to \infty$). Thus, writing $r = \mathrm{vol}(K)^{1/n}/2$ and applying Lemma 3.1, we get

$$\begin{aligned}\lim_{m\to\infty} \frac{\mathrm{G}_{m,n}\bigl(2^{-m}\mathcal{C}_{r_m} + t[-1,1]^n\bigr)}{2^{mn}} &= \lim_{m\to\infty}\left(\frac{2(p_m+t_m)+1}{2^m}\right)^n \\ &= \bigl(2(r+t)\bigr)^n = \mathrm{vol}\bigl(r[-1,1]^n + t[-1,1]^n\bigr).\end{aligned}$$

This, together with (3.3) and (3.4), shows (1.6), as desired. □

To conclude the paper, we point out that in [13, Corollary 2.3] it was shown that the discrete inequality (1.3) with $t = (1-\lambda)$ and $s = \lambda$ (for $\lambda \in (0,1)$) implies the corresponding form of the classical Brunn-Minkowski inequality (i.e., for such values of t and s). Since the latter is equivalent, by homogeneity, to (1.1) for arbitrary $t, s \geq 0$, Theorem 1.3 then infers. Furthermore, a direct proof of this result can be given in a way similar to what is performed within the proof of Theorem 1.4, by directly approximating the volume by means of the lattice point enumerator, as the lattice shrinks.

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