ON GRÜNBAUM TYPE INEQUALITIES

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Abstract. Given a compact set $K \subset \mathbb{R}^n$ of positive volume, and fixing a hyperplane H passing through its centroid, we find a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, depending on the concavity nature of the function that gives the volumes of cross-sections (parallel to H) of K, where K^- denotes the intersection of K with a halfspace bounded by H. When K is convex, this inequality recovers a classical result by Grünbaum. To this respect, we also show that the log-concave case is the limit concavity assumption for such a generalization of Grünbaum's inequality.

1. Introduction

Let $K \subset \mathbb{R}^n$ be a compact set with positive volume $\mathrm{vol}(K)$ (i.e., with positive n-dimensional Lebesgue measure). The centroid of K is the affine-covariant point

$$\mathrm{g}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x \,\mathrm{d}x.$$

According to a classical result by Grünbaum [8], if K is convex with centroid at the origin, then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n, \tag{1.1}$$

where $K^- = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 0\}$ and $K^+ = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \geq 0\}$ represent the parts of K which are split by the hyperplane $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$, for any given $u \in \mathbb{S}^{n-1}$. Equality holds, for a fixed $u \in \mathbb{S}^{n-1}$, if and only if K is a cone in the direction u, i.e., the convex hull of $\{x\} \cup (K \cap (y + H))$, for some $x, y \in \mathbb{R}^n$.

Grünbaum's result was extended to the case of sections [5, 16] and projections [19] of compact convex sets, and generalized to the analytic setting of *log-concave* functions [15] (see also [2, Lemma 2.2.6]) and p-concave functions [16], for $p > 0$. Other Grünbaum type inequalities involving volumes

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of sections of compact convex sets through their centroid, later generalized to the case of classical and dual *quermassintegrals* in [18], can be found in [4, 13].

The underlying key fact in the original proof of (1.1) (see [8]) is the following classical result (see e.g. [2, Section 1.2.1] and also [14, Theorem 12.2.1]):

Theorem A (Brunn's concavity principle)**.** *Let $K \subset \mathbb{R}^n$ be a non-empty compact and convex set and let H be a hyperplane. Then, the function $f : H^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$ is $(1/(n-1))$-concave.*

In other words, for any given hyperplane H, the cross-sections volume function f to the power $1/(n-1)$ is concave on its support, which is equivalent (due to the convexity of K) to the well-known *Brunn-Minkowski inequality* (see (2.1)). Although this property cannot be in general enhanced, one can easily find compact convex sets for which f satisfies a stronger concavity, for a suitable hyperplane H; similarly, the Brunn-Minkowski inequality can be refined when dealing with restricted families of sets (see e.g. [10, 11] and the references therein). Thus, on the one hand, it is natural to wonder about a possible enhanced version of Grünbaum's inequality (1.1) for the family of those compact convex sets K such that (there exists a hyperplane H for which) f is p-concave, with $1/(n-1) < p$. On the other hand, one could expect to extend this inequality to compact sets K, not necessarily convex, for which f is p-concave (for some hyperplane H), with $p < 1/(n-1)$.

Observing that the equality case in Grünbaum's inequality (1.1) is characterized by cones, that is, those sets for which f is $(1/(n-1))$-affine (i.e., such that $f^{1/(n-1)}$ is an affine function), the following sets of revolution, associated to p-affine functions, arise as natural candidates to be the extremal sets, in some sense, of these inequalities.

Definition 1.1. *Let $p \in \mathbb{R}$ and let $c, \gamma, \delta > 0$ be fixed. Then*

i) *if $p \neq 0$, let $g_p : I \longrightarrow \mathbb{R}_{\geq 0}$ be the non-negative function given by $g_p(t) = c(t+\gamma)^{1/p}$, where $I = [-\gamma, \delta]$ if $p > 0$ and $I = (-\gamma, \delta]$ if $p < 0$;*

ii) *if $p = 0$, let $g_0 : (-\infty, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ be the non-negative function defined by $g_0(t) = ce^{\gamma t}$.*

Let $u \in \mathbb{S}^{n-1}$ be fixed. By C_p we denote the set of revolution whose section by the hyperplane $\{x \in \mathbb{R}^n : \langle x, u\rangle = t\}$ is an $(n-1)$-dimensional ball of radius $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ with axis parallel to u (see Figure 1). (We warn the reader that, in the following, we will use the word "radius" to refer to such a generating function $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ of the set C_p, for short.)

In other words, one may speculate whether, among all compact sets K with centroid at the origin such that f is p-concave (for some hyperplane H), C_p gives the infimum for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$. We note that, in

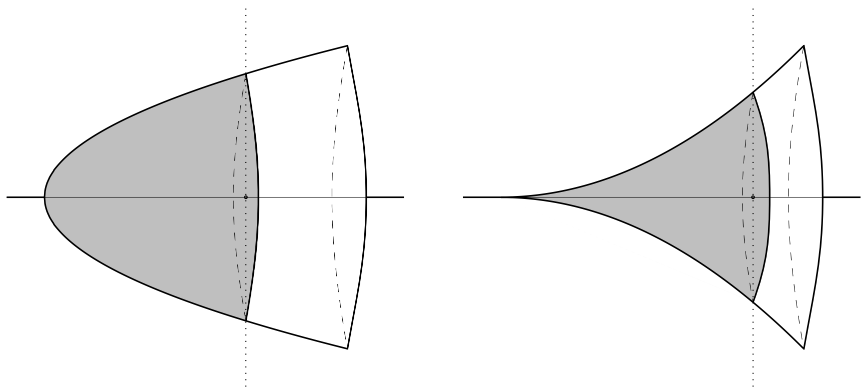

FIGURE 1. Sets C_p in $\mathbb{R}^3$, with centroid at the origin, and C_p^- (coloured), for $p = 1$ (left) and $p = 1/4$ (right).

this way, we would have a general family of inequalities depending on a real parameter p (with extremal sets varying continuously on it), and having Grünbaum's inequality (1.1) as the particular case $p = 1/(n-1)$.

Here we study the above-mentioned problem and show that it has a positive answer in the full range of $p \in [0,\infty]$ (in the following, $\sigma_{H^\perp}$ denotes the *Schwarz symmetrization* with respect to $H^\perp$; see Section 2 for the precise definition):

Theorem 1.1. *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane such that the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$ is p-concave, for some $p \in [0,\infty)$. If $p > 0$ then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{1.2}$$

with equality if and only if $\sigma_{H^\perp}(K) = C_p$. If $p = 0$ then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq e^{-1}. \tag{1.3}$$

The inequality is sharp; that is, the quotient $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ *comes arbitrarily close to e^{-1}.*

We point out that Theorem 1.1 can be obtained from recent involved results in the functional setting (more precisely, the case $p > 0$ is derived from [16, Theorem 1] whereas the case $p = 0$ follows from [15, Theorem in p. 746] -see also [2, Lemma 2.2.6]). Our goal here is to provide with a simpler geometric proof, inspired by the role of Brunn's concavity principle and comparing with the sets C_p, in the spirit of Grünbaum's approach in [8]. In this paper we also consider the range of $p \in [-\infty, 0)$ and we prove that $[0,\infty]$ is the largest set (where the parameter p lies) in which C_p provides us with the infimum value for such a Grünbaum type inequality.

The paper is organized as follows: in Section 2 we recall some preliminaries and we establish an auxiliary result that will be needed later on, whereas the proofs of our main results will be established in Section 3.

2. Background material and auxiliary results

We work in the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the standard inner product $\langle\cdot,\cdot\rangle$, and we write e_i to represent the i-th canonical unit vector. We denote by B_n the n-dimensional Euclidean (closed) unit ball and by $\mathbb{S}^{n-1}$ its boundary. Given a unit direction $u\in\mathbb{S}^{n-1}$, an orthonormal basis of $\mathbb{R}^n$ $(u_1,u_2,\dots,u_n)$ with $u_1=u$, and a vector $x\in\mathbb{R}^n$, we write $[x]_1$ for the first coordinate of x with respect to this basis. For any hyperplane $H=\{x\in\mathbb{R}^n:\langle x,u\rangle=c\}$, $c\in\mathbb{R}$, we represent by $H^-=\{x\in\mathbb{R}^n:\langle x,u\rangle\leq c\}$ and $H^+=\{x\in\mathbb{R}^n:\langle x,u\rangle\geq c\}$ the corresponding halfspaces bounded by H.

The Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathrm{G}(n,k)$, and for $H\in\mathrm{G}(n,k)$, the orthogonal projection of a subset $M\subset\mathbb{R}^n$ onto H is denoted by $M|H$, whereas the orthogonal complement of H is represented by $H^\perp$. The k-dimensional Lebesgue measure of M, provided that M is measurable, is denoted by $\mathrm{vol}_k(M)$ and we will omit the index k when it is equal to the dimension n of the ambient space. When integrating $\mathrm{d}x$ stands for $\mathrm{dvol}(x)$, and we write $\kappa_n=\mathrm{vol}(B_n)$.

Relating the volume of the Minkowski addition of two sets in terms of their volumes, one is led to the famous Brunn-Minkowski inequality (for extensive survey articles on this and related inequalities we refer the reader to [1, 7]; for a general reference on Brunn-Minkowski theory, we also refer to the updated monograph [17]). One form of it asserts that if K and L are non-empty compact convex subsets of $\mathbb{R}^n$, and $\lambda\in(0,1)$, then

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}\geq(1-\lambda)\mathrm{vol}(K)^{1/n}+\lambda\mathrm{vol}(L)^{1/n}, \tag{2.1}$$

with equality, if $\mathrm{vol}(K)\mathrm{vol}(L)>0$, if and only if K and L are homothetic. Here $+$ is used for the Minkowski sum, i.e., $A+B=\{a+b:\,a\in A,\,b\in B\}$ for any non-empty sets $A,B\subset\mathbb{R}^n$.

In other words, due to the convexity of K and L, the above result states that the function $\lambda\mapsto\mathrm{vol}\big((1-\lambda)K+\lambda L\big)$, $\lambda\in[0,1]$, is $(1/n)$-concave. We recall that a function $f:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq0}$ is p-concave, for $p\in\mathbb{R}\cup\{\pm\infty\}$, if

$$f\big((1-\lambda)x+\lambda y\big)\geq M_p\big(f(x),f(y),\lambda\big)$$

for all $x,y\in\mathbb{R}^n$ such that $f(x)f(y)>0$ and any $\lambda\in(0,1)$. Here M_p denotes the *p-mean* of two non-negative numbers a,b:

$$M_p(a,b,\lambda)=\begin{cases}\big((1-\lambda)a^p+\lambda b^p\big)^{1/p}, & \text{if } p\neq0,\pm\infty,\\ a^{1-\lambda}b^{\lambda} & \text{if } p=0,\\ \max\{a,b\} & \text{if } p=\infty,\\ \min\{a,b\} & \text{if } p=-\infty.\end{cases}$$

Note that if $p > 0$, then f is p-concave if and only if f^p is concave on its support $\{x \in \mathbb{R}^n : f(x) > 0\}$ and thus, in particular, 1-concave is just concave (on its support) in the usual sense. A 0-concave function is usually called log-concave whereas a $(-\infty)$-concave function is referred to as *quasi-concave*. Moreover, Jensen's inequality for means (see e.g. [9, Section 2.9] and [3, Theorem 1 p. 203]) implies that if $-\infty \leq p < q \leq \infty$ then

$$M_p(a, b, \lambda) \leq M_q(a, b, \lambda),$$

with equality for $ab > 0$ and $\lambda \in (0, 1)$ if and only if $a = b$; thus, a q-concave function is also p-concave, whenever $q > p$.

Another important technique in the original proof of (1.1) is the Schwarz symmetrization (see [12, Chapter IV], [6, Page 62]) of a compact set K, which is defined as follows: given a hyperplane $H \in \mathrm{G}(n, n-1)$, for any $x \in K|H^\perp$ let $B_{n-1}(x, r_x) \subset x+H$ be the $(n-1)$-dimensional Euclidean ball with center x and radius r_x such that $\mathrm{vol}_{n-1}\big(B_{n-1}(x, r_x)\big) = \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$; then $\sigma_{H^\perp}(K) = \bigcup_{x \in K|H^\perp} B_{n-1}(x, r_x)$ is the Schwarz symmetral of K with respect to $H^\perp$.

The aim of this paper is to provide with both a refinement and an extension of Grünbaum's inequality (1.1) in terms of the concavity nature of the cross-sections volume function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ defined by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$, for a given hyperplane $H \in \mathrm{G}(n, n-1)$, when dealing with compact sets $K \subset \mathbb{R}^n$. Although in general, when K is convex, f is $(1/(n-1))$-concave (see Theorem A), it is easy to find other examples of concavity. Indeed, given $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\} \in \mathrm{G}(n, n-1)$, $u \in \mathbb{S}^{n-1}$, and $H' \in \mathrm{G}(n, n-2)$ with $H' \subset H$, let $K_0 \subset H$ and $K_1 \subset u + H$ be compact sets with $K_0|H' = K_1|H'$. Then the Brunn-Minkowski inequality (2.1) admits the enhanced version $\mathrm{vol}_{n-1}\big((1-\lambda)K_0 + \lambda K_1\big) \geq (1-\lambda)\mathrm{vol}_{n-1}(K_0) + \lambda \mathrm{vol}_{n-1}(K_1)$ (see e.g. [10, Theorem A] and the references therein). This implies that, defining K as the convex hull of $K_0 \cup K_1$, the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is concave.

Remark 2.1. *The concavity nature of the cross-sections volume function f depends on the choice of the hyperplane H. Indeed, given $H_1 = \{x \in \mathbb{R}^3 : \langle x, \mathrm{e}_1\rangle = 0\}$ and $H_2 = \{x \in \mathbb{R}^3 : \langle x, \mathrm{e}_2\rangle = 0\}$, and considering the set*

$$C_1 = \big\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \in [0, 1],\ x_2^2 + x_3^2 \leq r(x_1)^2\big\}$$

of radius $r(t) = t^{1/2}$ (cf. Definition 1.1), we have $\mathrm{vol}_2\big(C_1 \cap (t\mathrm{e}_1 + H_1)\big) = \kappa_2 t$ and

$$\begin{aligned}\mathrm{vol}_2\big(C_1 \cap (t\mathrm{e}_2 + H_2)\big) &= \mathrm{vol}_2\big(\big\{x \in \mathbb{R}^3 : x_1 \in [t^2, 1],\ x_3^2 \leq r(x_1)^2 - t^2\big\}\big)\\ &= \frac{4}{3}(1 - t^2)^{3/2},\end{aligned}$$

for any $t \in [0,1]$. *Therefore, the function* $f_1 : H_1^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ *defined by* $\mathrm{vol}_2\big(C_1 \cap (x+H_1)\big)$ *is* 1*-concave whereas the function* $f_2 : H_2^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ *given by* $\mathrm{vol}_2\big(C_1 \cap (x+H_2)\big)$ *is not* 1*-concave.*

For the sake of simplicity, in the following we consider $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$, for a given direction $u \in \mathbb{S}^{n-1}$ that we extend to an orthonormal basis $(u_1, u_2, \dots, u_n)$ of $\mathbb{R}^n$, with $u_1 = u$. Moreover, given a compact set $K \subset \mathbb{R}^n$ with non-empty interior, we denote by $K(t) = K \cap (tu + H)$ for any $t \in \mathbb{R}$. We notice that, if $K|H^{\perp} \subset [au, bu]$, Fubini's theorem implies (provided that $a \leq 0$) that

$$\mathrm{vol}(K) = \int_a^b f(t)\,\mathrm{d}t \quad \text{and} \quad \mathrm{vol}(K^-) = \int_a^0 f(t)\,\mathrm{d}t, \tag{2.2}$$

where, as usual, we are identifying the linear subspace spanned by u with $\mathbb{R}$. Since the set $\{t \in \mathbb{R} : f(t) > 0\}$ is convex whenever f is quasi-concave, from now on we will assume, without loss of generality, that $f(t) > 0$ for all $t \in [a,b]$. Furthermore, by Fubini's theorem, we get

$$[\mathrm{g}(K)]_1 = \frac{1}{\mathrm{vol}(K)} \int_a^b t f(t)\,\mathrm{d}t \tag{2.3}$$

and thus, in particular, $a < [\mathrm{g}(K)]_1 < b$ (cf. (2.2)).

As mentioned in the introduction, the sets C_p associated to (cross-sections volume) functions that are p-affine (see Definition 1.1) seem to be possible extremal sets of such expected inequalities. So, we start by computing the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$ for the sets C_p.

Lemma 2.1. *Let* $p \in (-\infty,-1) \cup [0,\infty)$ *and let* $H \in \mathrm{G}(n, n-1)$ *be a hyperplane with unit normal vector* $u \in \mathbb{S}^{n-1}$. *Let* g_p *and* C_p*, with axis parallel to* u*, be as in Definition 1.1, for any fixed* $c, \gamma, \delta > 0$. *If* C_p *has centroid at the origin then*

$$\frac{\mathrm{vol}\big(C_p^-\big)}{\mathrm{vol}(C_p)} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{2.4}$$

where, if $p = 0$*, the above identity must be understood as*

$$\frac{\mathrm{vol}\big(C_0^-\big)}{\mathrm{vol}(C_0)} = \lim_{p\to 0^+} \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} = e^{-1}. \tag{2.5}$$

Proof. First we assume that $p \neq 0$ and show (2.4). On the one hand, by Fubini's theorem, we get

$$\mathrm{vol}(C_p) = \int_{-\gamma}^{\delta} g_p(t)\,\mathrm{d}t = \frac{c\,p(\delta+\gamma)^{(p+1)/p}}{p+1}.$$

On the other hand, from (2.3), we have

$$\big[\mathrm{g}(C_p)\big]_1 = \frac{1}{\mathrm{vol}(C_p)}\int_{-\gamma}^{\delta} t g_p(t)\,\mathrm{d}t = \frac{p+1}{p(\delta+\gamma)^{(p+1)/p}}\int_0^{\delta+\gamma}(s-\gamma)s^{1/p}\,\mathrm{d}s$$
$$= \frac{(p+1)(\delta+\gamma)}{2p+1} - \gamma.$$

Therefore, from the hypothesis $\mathrm{g}(C_p) = 0$, we obtain that $\gamma/(\delta+\gamma) = (p+1)/(2p+1)$, and hence

$$\frac{\mathrm{vol}\big(C_p^-\big)}{\mathrm{vol}(C_p)} = \frac{1}{\mathrm{vol}(C_p)}\int_{-\gamma}^{0} g_p(t)\,\mathrm{d}t = \left(\frac{\gamma}{\delta+\gamma}\right)^{(p+1)/p} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p},$$

as desired.

Now we assume that $p = 0$ and show (2.5). Again, by Fubini's theorem and (2.3), respectively, we get

$$\mathrm{vol}(C_0) = \int_{-\infty}^{\delta} g_0(t)\,\mathrm{d}t = \frac{ce^{\gamma\delta}}{\gamma}$$

and

$$\big[\mathrm{g}(C_0)\big]_1 = \frac{1}{\mathrm{vol}(C_0)}\int_{-\infty}^{\delta} t g_0(t)\,\mathrm{d}t = \delta - \frac{1}{\gamma}.$$

In particular, $\mathrm{g}(C_0) = 0$ implies that $\delta = 1/\gamma$, and hence

$$\frac{\mathrm{vol}\big(C_0^-\big)}{\mathrm{vol}(C_0)} = \frac{1}{\mathrm{vol}(C_0)}\int_{-\infty}^{0} g_0(t)\,\mathrm{d}t = e^{-1}.$$

This concludes the proof. □

Although the value $\big((p+1)/(2p+1)\big)^{(p+1)/p}$ obtained in (2.4) is also defined for any $p \in (-1/2, 0)$, the corresponding sets C_p present remarkable differences with those of the range $p \geq 0$, as we will see next. So, we will study this case separately.

To this aim, let $p \in (-1/2, 0)$ and let $\varepsilon > 0$ be fixed. Let $C_{p,\varepsilon}$ be the set of revolution, with axis parallel to u, of radius $(g_{p,\varepsilon}(t)/\kappa_{n-1})^{1/(n-1)}$ associated to the p-affine function $g_{p,\varepsilon} : [-\gamma+\varepsilon, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ given by $g_{p,\varepsilon}(t) = c(t+\gamma)^{1/p}$, for some $c, \gamma, \delta > 0$ (for our purpose we may assume that $\gamma > \varepsilon$).

On the one hand, by Fubini's theorem, we get

$$\mathrm{vol}(C_{p,\varepsilon}) = \int_{-\gamma+\varepsilon}^{\delta} g_p(t)\,\mathrm{d}t = \frac{c\,p\left((\delta+\gamma)^{(p+1)/p} - \varepsilon^{(p+1)/p}\right)}{p+1}.$$

Then we notice first that $\mathrm{vol}(C_{p,\varepsilon}) \to \infty$ as $\varepsilon \to 0^+$. On the other hand, from (2.3), we have

$$\begin{aligned}\big[\mathrm{g}(C_{p,\varepsilon})\big]_1 &= \frac{1}{\mathrm{vol}(C_{p,\varepsilon})}\int_{-\gamma+\varepsilon}^{\delta} t g_p(t)\,\mathrm{d}t\\ &= \frac{p+1}{p\left((\delta+\gamma)^{(p+1)/p}-\varepsilon^{(p+1)/p}\right)}\int_{\varepsilon}^{\delta+\gamma}(s-\gamma)s^{1/p}\,\mathrm{d}s\\ &= \frac{(p+1)\alpha(\varepsilon)}{2p+1}-\gamma,\end{aligned}$$

where

$$\alpha(\varepsilon) = \frac{(\delta+\gamma)^{(2p+1)/p}-\varepsilon^{(2p+1)/p}}{(\delta+\gamma)^{(p+1)/p}-\varepsilon^{(p+1)/p}}.$$

We note that $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, and moreover that $\alpha(\varepsilon) > 0$ because of the direct relation $-\gamma+\varepsilon \leq \big[\mathrm{g}(C_{p,\varepsilon})\big]_1$ jointly with $(p+1)/(2p+1) > 0$. Hence, we get

$$\begin{aligned}\frac{\mathrm{vol}\big(C_{p,\varepsilon}\cap\big(\mathrm{g}(C_{p,\varepsilon})+H\big)^+\big)}{\mathrm{vol}(C_{p,\varepsilon})} &= \frac{1}{\mathrm{vol}(C_{p,\varepsilon})}\int_{-\gamma+(p+1)\alpha(\varepsilon)/(2p+1)}^{\delta} g_p(t)\,\mathrm{d}t\\ &= \frac{(\delta+\gamma)^{(p+1)/p}-\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}\alpha(\varepsilon)^{(p+1)/p}}{(\delta+\gamma)^{(p+1)/p}-\varepsilon^{(p+1)/p}}.\end{aligned}$$

Therefore, although $\lim_{\varepsilon\to 0^+}\mathrm{vol}(C_{p,\varepsilon}) = \infty$, we have

$$\lim_{\varepsilon\to 0^+}\frac{\mathrm{vol}\big(C_{p,\varepsilon}\cap\big(\mathrm{g}(C_{p,\varepsilon})+H\big)^+\big)}{\mathrm{vol}(C_{p,\varepsilon})} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}. \tag{2.6}$$

Thus, the value $\big((p+1)/(2p+1)\big)^{(p+1)/p}$ is asymptotically attained by the sets $C_{p,\varepsilon}$. The main difference with the case $p \geq 0$ is that it is now reached by their parts given by the positive halfspace (with respect to the normal direction $u \in \mathbb{S}^{n-1}$) bounded by the hyperplane through their centroid.

3. Grünbaum type inequalities

Grünbaum's inequality (1.1) can also be expressed by saying that if K is a compact convex set, of positive volume, with centroid at the origin, then

$$\min\left\{\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)},\frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}\right\} \geq \left(\frac{n}{n+1}\right)^n.$$

We start this section by showing that, if the cross-sections volume function f associated to a compact set K is increasing in the direction of the normal vector of H, then the above minimum is attained at $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, independently of the concavity nature of f.

Proposition 3.1. *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let $H \in \mathrm{G}(n, n-1)$ be a hyperplane, with unit normal vector $u \in \mathbb{S}^{n-1}$, such that the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\bigl(K \cap (x+H)\bigr)$ is quasi-concave with $f(bu) = \max_{x\in H^\perp} f(x)$, where $[au, bu] = K|H^\perp$. Then*

$$\frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)} \geq \frac{1}{2}.$$

Proof. Let $g : [-\gamma, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ be the constant function given by $g(t) = f(0)$, where

$$\gamma = \frac{1}{f(0)} \int_a^0 f(t)\,\mathrm{d}t \quad \text{and} \quad \delta = \frac{1}{f(0)} \int_0^b f(t)\,\mathrm{d}t. \tag{3.1}$$

Since f is quasi-concave with $f(b) = \max_{t\in\mathbb{R}} f(t)$, f is increasing on $[a, b]$ and thus (from (3.1)) we have $a \leq -\gamma < 0 < b \leq \delta$. Hence, since $\mathrm{g}(K) = 0$ (and using (2.3)), from (3.1) we get

$$\begin{aligned}
f(0)\,\frac{\gamma^2 - \delta^2}{2} &= -\int_{-\gamma}^{\delta} t g(t)\,\mathrm{d}t = \int_a^b t f(t)\,\mathrm{d}t - \int_{-\gamma}^{\delta} t g(t)\,\mathrm{d}t \\
&= \int_a^{-\gamma} (t+\gamma) f(t)\,\mathrm{d}t + \int_{-\gamma}^{0} (t+\gamma)\bigl(f(t) - g(t)\bigr)\,\mathrm{d}t \\
&\quad + \int_0^b (t-b)\bigl(f(t) - g(t)\bigr)\,\mathrm{d}t + \int_b^{\delta} (t-b)\bigl(-g(t)\bigr)\,\mathrm{d}t \leq 0,
\end{aligned}$$

which yields $\gamma \leq \delta$, or equivalently $\mathrm{vol}(K^-) \leq \mathrm{vol}(K^+)$. This concludes the proof. □

We are now ready to prove our main theorem.

Proof of Theorem 1.1. First we assume that $p > 0$ and show (1.2). We assert that there exists a (p-affine) function $g_p : [-\gamma, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ given by $g_p(t) = c(t + \gamma)^{1/p}$, for some $\gamma, \delta, c > 0$, such that $g_p(0) = f(0)$,

$$\int_{-\gamma}^0 g_p(t)\,\mathrm{d}t = \int_a^0 f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^{\delta} g_p(t)\,\mathrm{d}t = \int_0^b f(t)\,\mathrm{d}t. \tag{3.2}$$

Indeed, taking

$$\gamma = \frac{p+1}{p f(0)} \int_a^0 f(t)\,\mathrm{d}t, \quad c = \frac{f(0)}{\gamma^{1/p}} \quad \text{and} \quad \delta = \left(\frac{p+1}{pc} \int_a^b f(t)\,\mathrm{d}t\right)^{p/(p+1)} - \gamma,$$

elementary computations show (3.2). We also note that, since

$$\gamma = \left(\frac{p+1}{pc} \int_a^0 f(t)\,\mathrm{d}t\right)^{p/(p+1)},$$

we actually have $\delta > 0$.

In other words, for the set of revolution C_p of radius $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$, we have $C_p(0) = \sigma_{H^\perp}\big(K(0)\big)$,

$$\mathrm{vol}\big(C_p^-\big) = \mathrm{vol}(K^-) \quad \text{and} \quad \mathrm{vol}\big(C_p^+\big) = \mathrm{vol}(K^+). \tag{3.3}$$

And thus, in particular, $\mathrm{vol}(C_p) = \mathrm{vol}(K)$.

From the concavity of f^p, together with the relations $g_p(0) = f(0)$ and (3.2), we get on the one hand that $-\gamma \le a < 0 < \delta \le b$. On the other hand, defining the functions $\bar{f}, \bar{g}_p : [-\gamma, b] \longrightarrow \mathbb{R}_{\ge 0}$ given by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in [a, b], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{g}_p(t) = \begin{cases} g_p(t) & \text{if } t \in [-\gamma, \delta], \\ 0 & \text{otherwise,} \end{cases}$$

we may conclude that there exists $x_0 \in [a, 0)$ such that $\bar{f}(t) \ge \bar{g}_p(t)$ for all $t \in [x_0, 0] \cup [\delta, b]$ and $\bar{f}(t) \le \bar{g}_p(t)$ otherwise (see Figure 2). Hence, since $\mathrm{g}(K) = 0$ (and using (2.3)), from (3.2) we have

$$\begin{aligned} -\int_{-\gamma}^{\delta} t g_p(t)\,\mathrm{d}t &= \int_a^b t f(t)\,\mathrm{d}t - \int_{-\gamma}^{\delta} t g_p(t)\,\mathrm{d}t = \int_{-\gamma}^{b} t\big(\bar{f}(t) - \bar{g}_p(t)\big)\,\mathrm{d}t \\ &= \int_{-\gamma}^{0} t\big(\bar{f}(t) - \bar{g}_p(t)\big)\,\mathrm{d}t + \int_0^b t\big(\bar{f}(t) - \bar{g}_p(t)\big)\,\mathrm{d}t \\ &= \int_{-\gamma}^{0} (t - x_0)\big(\bar{f}(t) - \bar{g}_p(t)\big)\,\mathrm{d}t + \int_0^b (t - \delta)\big(\bar{f}(t) - \bar{g}_p(t)\big)\,\mathrm{d}t \ge 0, \end{aligned}$$

with equality if and only if $f = g_p$. Thus, we have $[\mathrm{g}(C_p)]_1 \le 0$, and equality holds if and only if $f = g_p$. Then, from (3.3) and Lemma 2.1,

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}\big(C_p^-\big)}{\mathrm{vol}(C_p)} \ge \frac{\mathrm{vol}\big(C_p \cap \big(\mathrm{g}(C_p) + H\big)^-\big)}{\mathrm{vol}(C_p)} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p},$$

with equality if and only if $f = g_p$, that is, if and only if $\sigma_{H^\perp}(K) = C_p$.

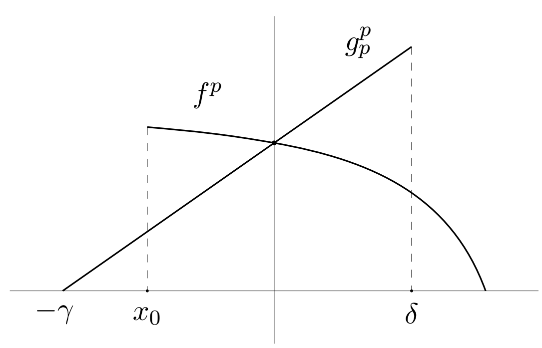

FIGURE 2. Relative position of the functions f^p and g_p^p.

Now we assume that $p = 0$ and show (1.3). We assert that there exists an exponential function $g_0 : (-\infty, \delta] \longrightarrow \mathbb{R}_{\ge 0}$ given by $g_0(t) = ce^{\gamma t}$, for some

$\gamma, \delta, c > 0$, such that $g_0(0) = f(0)$,

$$\int_{-\infty}^{0} g_0(t)\,\mathrm{d}t = \int_{a}^{0} f(t)\,\mathrm{d}t \quad \text{and} \quad \int_{0}^{\delta} g_0(t)\,\mathrm{d}t = \int_{0}^{b} f(t)\,\mathrm{d}t. \tag{3.4}$$

Straightforward computations show that the above relations are equivalent to take

$$c = f(0), \quad \gamma = f(0)\left(\int_{a}^{0} f(t)\,\mathrm{d}t\right)^{-1} \quad \text{and} \quad \delta = \frac{1}{\gamma}\log\left(\frac{\gamma}{f(0)}\int_{a}^{b} f(t)\,\mathrm{d}t\right);$$

note that, indeed, $\delta > 0$.

Again, the set of revolution C_0 of radius $(g_0(t)/\kappa_{n-1})^{1/(n-1)}$ satisfies that $C_0(0) = \sigma_{H^\perp}\big(K(0)\big)$,

$$\mathrm{vol}\big(C_0^-\big) = \mathrm{vol}(K^-) \quad \text{and} \quad \mathrm{vol}\big(C_0^+\big) = \mathrm{vol}(K^+), \tag{3.5}$$

and thus, in particular, $\mathrm{vol}(C_0) = \mathrm{vol}(K)$.

Now the concavity of $\log f$, jointly with the relations $g_0(0) = f(0)$ and (3.4), implies that $(g_0(t) \geq f(t)$ for all $t \in [0, \delta]$ and so) $\delta \leq b$. Moreover, for the functions $\bar{f}, \bar{g}_0 : (-\infty, b] \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in [a, b], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{g}_0(t) = \begin{cases} g_0(t) & \text{if } t \in (-\infty, \delta], \\ 0 & \text{otherwise,} \end{cases}$$

we conclude that there exists $x_0 \in [a, 0)$ such that $\bar{f}(t) \geq \bar{g}_0(t)$ for all $t \in [x_0, 0] \cup [\delta, b]$ and $\bar{f}(t) \leq \bar{g}_0(t)$ otherwise (cf. Figure 2). Arguing as in the case $p > 0$, using (3.4) and $\mathrm{g}(K) = 0$, we have that $[\mathrm{g}(C_0)]_1 \leq 0$. Then, from (3.5) and Lemma 2.1,

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}\big(C_0^-\big)}{\mathrm{vol}(C_0)} \geq \frac{\mathrm{vol}\Big(C_0 \cap \big(\mathrm{g}(C_0) + H\big)^-\Big)}{\mathrm{vol}(C_0)} = e^{-1}.$$

Finally we notice that if we consider an unbounded set L with centroid at the origin and such that $\sigma_{H^\perp}(L) = C_0$, for a given $g_0 : (-\infty, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ of the form $g_0(t) = ce^{\gamma t}$, with $\gamma, \delta, c > 0$, then $\mathrm{vol}(L^-)/\mathrm{vol}(L) = e^{-1}$ (cf. (2.5)). Hence, considering $K_a = L \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \geq a\}$, $a < \delta$, we have $[\mathrm{g}(K_a)]_1 \to 0$ and $\mathrm{vol}\big(K_a^-\big)/\mathrm{vol}(K_a) \to e^{-1}$, as $a \to -\infty$. This proves the final statement of the theorem. □

Note that the "limit case" $p = \infty$ in Theorem 1.1 is also trivially fulfilled. Indeed, if f is ∞-concave then f is constant on $[a, b]$ and thus $0 = [\mathrm{g}(K)]_1 = b + a$ (see (2.3)), which yields that $a = -b$ and hence

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{1}{2} = \lim_{p\to\infty}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

Remark 3.1. *Grünbaum's inequality* (1.1)*, jointly with its equality case, is collected in the case* $p = 1/(n-1)$ *of Theorem 1.1. Indeed, on the one hand, Theorem A implies that the cross-sections volume function* f *is* $(1/(n-1))$*-concave, and thus* (1.2) *yields* (1.1)*. On the other hand, regarding the equality case of* (1.1)*, we note that the fact that* f *is* $(1/(n-1))$*-affine,* *combined with the convexity of K jointly with the equality case of the Brunn-Minkowski inequality* (2.1)*, implies that K must be a cone in the direction of the normal vector of H.*

Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin, such that its cross-sections volume function f is p-concave, with respect to a given hyperplane H. Moreover, if $p \in (-\infty, -1) \cup (-1/2, \infty)$, we write for short

$$\alpha_p := \left(\frac{p+1}{2p+1}\right)^{(p+1)/p},$$

where, if $p = 0$, α_0 is the value that is obtained "by continuity", that is,

$$\alpha_0 = \lim_{p \to 0} \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} = e^{-1}.$$

In Theorem 1.1 we have shown that, whenever $p \geq 0$, α_p is a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, as a consequence of the fact that $[\mathrm{g}(C_p)]_1 \leq 0$ for the (suitable) set C_p such that $\mathrm{vol}\big(C_p^-\big) = \mathrm{vol}(K^-)$ and $\mathrm{vol}\big(C_p^+\big) = \mathrm{vol}(K^+)$. Next we point out that, in fact, these two conditions are equivalent.

Corollary 3.1. *Let $p \in (-\infty, -1) \cup [0, \infty)$ and let $H \in \mathrm{G}(n, n-1)$ be a hyperplane with unit normal vector $u \in \mathbb{S}^{n-1}$. Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. If C_p, given as in Definition 1.1, with axis parallel to u, is such that*

$$\mathrm{vol}\big(C_p^-\big) = \mathrm{vol}(K^-) \quad \text{and} \quad \mathrm{vol}\big(C_p^+\big) = \mathrm{vol}(K^+),$$

then the following assertions are equivalent:

(a) $\mathrm{vol}(K^-)/\mathrm{vol}(K) \geq \alpha_p$*;*

(b) $[\mathrm{g}(C_p)]_1 \leq 0$.

Proof. From Lemma 2.1, we have

$$\frac{\mathrm{vol}\big(C_p \cap \big(\mathrm{g}(C_p) + H\big)^-\big)}{\mathrm{vol}(C_p)} = \alpha_p.$$

Moreover, by hypothesis, we get

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}\big(C_p \cap H^-\big)}{\mathrm{vol}(C_p)}.$$

Therefore, the result now follows from the fact that, for any $x, y \in \mathbb{R}^n$ such that $\{x, y\}|H^\perp \subset C_p|H^\perp$, $\mathrm{vol}\big(C_p \cap (x+H)^-\big) \leq \mathrm{vol}\big(C_p \cap (y+H)^-\big)$ if and only if $[x]_1 \leq [y]_1$. □

Next we show that Theorem 1.1 cannot be extended to the range of $p \in (-\infty, -1)$. In fact, we prove a more general result:

Proposition 3.2. *Let $p \in (-\infty,-1)$. There exists no positive constant β_p such that*

$$\min\left\{\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}, \frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}\right\} \geq \beta_p$$

for all compact sets $K \subset \mathbb{R}^n$ with non-empty interior and with centroid at the origin, for which there exists $H \in \mathrm{G}(n, n-1)$ such that $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$, $x \in H^\perp$, is p-concave.

Proof. By Lemma 2.1, for any $q \in (-\infty,-1)$ we have $\alpha_q = \mathrm{vol}\big(C_q^-\big)/\mathrm{vol}(C_q)$, provided that C_q has centroid at the origin. Since $\alpha_q \to 1$ as $q \to -1^-$, we obtain

$$\lim_{q\to -1^-} \min\left\{\frac{\mathrm{vol}\big(C_q^-\big)}{\mathrm{vol}(C_q)}, \frac{\mathrm{vol}\big(C_q^+\big)}{\mathrm{vol}(C_q)}\right\} = \lim_{q\to -1^-}(1-\alpha_q) = 0.$$

The proof is now concluded from the fact that any q-concave function is also p-concave, whenever $q > p$. □

We conclude the paper by showing that the statement of Theorem 1.1 cannot be extended to the range of $p \in (-1/2, 0)$ either. To this aim, note that if $p < q$ are parameters for which β_p and β_q are such sharp lower bounds for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ (i.e., in the cases in which f is respectively p-concave and q-concave) then $\beta_p \leq \beta_q$, because every q-concave function is also p-concave. We notice however that, if $p \in (-1/2, 0)$, the value obtained by C_p is not α_p but $1-\alpha_p$ (cf. (2.6)), and then $1-\alpha_p \geq 1-\alpha_0 > 1/2$ for any $p \in (-1/2, 0)$ whereas $\alpha_p \leq 1/2$ for all $p \geq 0$.

Therefore, this fact (jointly with the case in which $p \in (-\infty,-1)$, collected in Proposition 3.2) gives that $[0,\infty]$ is the largest subset of the real line (with respect to set inclusion) for which C_p provides us with the infimum value for the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$, among all compact sets with (centroid at the origin and) p-concave cross-sections volume function. However, since α_p is increasing in the parameter p on $(-1/2,\infty)$, and $\alpha_p \to 0$ as $p \to (-1/2)^+$, it is still possible to expect α_p to be a lower bound for $\min\big\{\mathrm{vol}(K^-)/\mathrm{vol}(K), \mathrm{vol}(K^+)/\mathrm{vol}(K)\big\}$. Unfortunately, we do not know so far whether this issue has a positive answer or not.

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