ON PROPERTIES FOR m-POLYNOMIALS OF UNIT p-BALLS

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Abstract. In this paper we study properties of 'weighted' Steiner polynomials associated to the unit p-balls. We show that the corresponding functional can be bounded just by the last but one relative quermassintegral. Then we give a general asymptotic relation between the roots of Steiner polynomials and the above mentioned polynomials. These properties will be obtained as consequences of more general results for the so called **m**-polynomials.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in $\mathbb{R}^n$. The volume of a set $M \subsetneq \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (or $\mathrm{vol}_n(M)$ if the distinction of the dimension is needed). We write B_n^p to represent the unit p-ball associated to the p-norm $|\cdot|_p$, $1 \leq p \leq \infty$, and by $\kappa_n^p = \mathrm{vol}(B_n^p)$, which takes the value

$$\kappa_n^p = \frac{\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^n}{\Gamma\left(\frac{n}{p}+1\right)}, \tag{1.1}$$

where Γ denotes the gamma function (see e.g. [9, p. 11]). In the particular case $p = 2$, we write for short B_n to denote the n-dimensional unit ball and $\kappa_n = \mathrm{vol}(B_n)$. Finally, with $\mathrm{lin}\, M$ we represent the linear hull of $M \subsetneq \mathbb{R}^n$.

For convex bodies $K, E \in \mathcal{K}^n$ and a non-negative real number λ, the well-known *Steiner formula* states that the volume of the Minkowski sum $K + \lambda E$ can be expressed as a polynomial of degree (at most) n in the parameter λ,

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i; \tag{1.2}$$

here the coefficients $\mathrm{W}_i(K;E)$ are called the *relative quermassintegrals* of K with respect to E, and they are a special case of the more general defined mixed volumes (see e.g. [10, s. 5.1] or [2, s. 6.2]). In particular, it holds $\mathrm{W}_0(K;E) = \mathrm{vol}(K)$ and $\mathrm{W}_n(K;E) = \mathrm{vol}(E)$. If we have to distinguish the dimension in which the quermassintegrals are computed, we will write $\mathrm{W}_i^{(k)}$ to denote the i-th quermassintegral in $\mathbb{R}^k$.

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In 1973 [13] Wills introduced and studied the related functional

$$\sum_{i=0}^{n}\binom{n}{i}\frac{\mathrm{W}_i(K;B_n)}{\kappa_i}, \tag{1.3}$$

which has many interesting applications, e.g., in Discrete Geometry or for Gaussian random processes [11]. Many other nice properties of this functional, as well as relations with other measures, have been studied in the last years, see e.g. [3, 4, 8, 13, 14, 15]. In particular, in [3] Hadwiger showed, among others, the following integral representation:

$$\sum_{i=0}^{n}\binom{n}{i}\frac{\mathrm{W}_i(K;B_n)}{\kappa_i}=\int_{\mathbb{R}^n}e^{-\pi d(x,K)^2}\mathrm{d}x, \tag{1.4}$$

where $d(x,K)$ denotes the Euclidean distance between $x\in\mathbb{R}^n$ and K. Recently, generalizations of the previous identity (1.4) have been studied when the 'distance'

$$d_E(x,K)=\inf\{r\geq 0: x\in K+rE\}$$

between $x\in\mathbb{R}^n$ and K, relative to a convex body E, is considered. Then it can be proved, assuming (without loss of generality) that the origin 0 is a relative interior point of E, that (see [6] and [7])

$$\int_{K+\operatorname{lin}E}e^{-\pi d_E(x,K)^2}\mathrm{d}x=\sum_{i=0}^{n}\binom{n}{i}\frac{\mathrm{W}_i(K;E)}{\kappa_i}, \tag{1.5}$$

which is called the *relative Wills functional*. Now we consider the corresponding polynomials coming from (1.2) and (1.5) regarded as formal polynomials in a complex variable $z\in\mathbb{C}$, which we call the *relative Steiner and Wills polynomials*, respectively, and denote by

$$f_{K;E}(z)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)z^i,\qquad f_{K;E}^{g}(z)=\sum_{i=0}^{n}\binom{n}{i}\frac{\mathrm{W}_i(K;E)}{\kappa_i}z^i.$$

Notice that the (relative) Wills polynomial can be seen as a Steiner polynomial with certain 'weights'. This leads to consider the following definition: given a sequence $\mathbf{m}=(m_i)_{i\in\mathbb{N}}$ of positive real numbers, for each $n\in\mathbb{N}$ and any pair $K,E\in\mathcal{K}^n$ with dimension $\dim(K+E)=n$, let

$$f_{K;E}^{\mathbf{m}}(z)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\,m_i z^i,$$

which we call the $\mathbf{m}$-*polynomial* of K and E. If the weights m_i are the moments of some measure μ on the non-negative real line $\mathbb{R}_{\geq 0}$, namely, if

$$m_i=m_i(\mu)=\int_0^\infty t^i\,\mathrm{d}\mu(t),\quad i=0,\dots,n,$$

then it can be shown that the functional $\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\,m_i(\mu)$ corresponding to the polynomial $f_{K;E}^{\mu}(z)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\,m_i(\mu)z^i$ has also an integral expression of the form

$$\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\,m_i(\mu)=\int_{K+\operatorname{lin}E}G\bigl(d_E(x,K)\bigr)\,\mathrm{d}x,$$

where $G(t)=\mu\bigl([t,\infty)\bigr)$, $t\in\mathbb{R}_{\geq 0}$ (see [6] and [7]).

In [6] we have investigated the structure of the roots of the family of **m**-polynomials of convex bodies when **m** is associated to a given measure μ on the non-negative real line $\mathbb{R}_{\geq 0}$.

A particulary interesting case of **m**-polynomial associated to a measure on $\mathbb{R}_{\geq 0}$ is the following one. Let $G_p(t) = e^{-C_p t^p}$ be the function associated to the measure

$$\mu_p(A) = \int_A p\, C_p\, e^{-C_p t^p} t^{p-1}\, \mathrm{d}t$$

on the non-negative real line $\mathbb{R}_{\geq 0}$, where $C_p = \bigl(2\Gamma(1/p+1)\bigr)^p$. Then it can be checked that the moments $m_i(\mu_p) = 1/\kappa_i^p$, $i = 0, \dots, n$ (see Lemma 2.3). Therefore, the **m**-polynomials associated to the measure μ_p, which we call μ_p-polynomials, are given by

$$f_{K;E}^{\mu_p}(z) = \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_i(K;E)}{\kappa_i^p} z^i, \quad K, E \in \mathcal{K}^n.$$

Here we are mainly interested in studying several properties of the μ_p-polynomial $f_{K;B_n^p}^{\mu_p}(z)$, $K \in \mathcal{K}^n$. First we show that the corresponding functional in K obtained when $z = 1$ can be bounded just by the last but one relative quermassintegral.

Theorem 1.1. *For any convex body $K \in \mathcal{K}^n$ and all $1 \leq p \leq \infty$, it holds*

$$\sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_i(K;B_n^p)}{\kappa_i^p} \leq e^{n \mathrm{W}_{n-1}(K;B_n^p)/\kappa_{n-1}^p}.$$

This property will be obtained as a consequence of a more general inequality for **m**-polynomials (see Proposition 3.1). These results will be proved in Section 3. Finally, in Section 4, we give a general asymptotic relation involving the roots of Steiner polynomials and **m**-polynomials, and then we particularize it to provide the connection between the roots of the Steiner polynomial and the μ_p-polynomials $f_{K;B_n^p}^{\mu_p}(z)$.

Theorem 1.2. *For $s \in \mathbb{N}$ fixed, let $K \in \mathcal{K}^s$ and let $\nu_1, \dots, \nu_s$ be the roots of $f_{K;B_s^p}^{\mu_p}(z)$, $1 \leq p \leq \infty$. Embedding $K \subsetneq \mathbb{R}^n$, $n \geq s$, let $\gamma_{1,n}, \dots, \gamma_{s,n}$ be the non-zero roots of $f_{K;B_n^p}(z)$. Then, reordering if necessary, it holds*

$$\lim_{n\to\infty} \frac{\kappa_n^p}{\kappa_{n-1}^p} \gamma_{i,n} = \nu_i, \quad i = 1, \dots, s.$$

2. Some preliminary results

In this section we collect several results which will be needed in the proofs of the main theorems. The proof of the first lemma includes the construction of an special family of gauge bodies which will be used in the following.

Lemma 2.1. *Let $[a,b]$ be a closed interval in $\mathbb{R}$ containing the origin 0 and let $r : [a,b] \longrightarrow [0,\infty)$ be a continuous concave (and not identically zero) function. Then there exists a sequence of convex bodies $\{E_n\}_{n\in\mathbb{N}}$ with $\dim E_n = n$, such that*

$$\frac{\mathrm{vol}_{n-k}(E_{n-k})}{\mathrm{vol}_n(E_n)} \frac{\mathrm{vol}_n(E_n)^k}{\mathrm{vol}_{n-1}(E_{n-1})^k} = \frac{\left(\int_a^b r(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1} \int_a^b r(t)^{n-k+i}\mathrm{d}t}, \quad 2 \leq k \leq n.$$

Proof. We consider the family of convex bodies inductively defined by

$$E_0 = \{0\}, \quad E_1 = [a,b], \quad E_n = \bigcup_{t\in[a,b]} \Big(r(t)E_{n-1} \times \{t\}\Big). \tag{2.1}$$

From the concavity and the continuity of $r(t)$, it is easy to see that E_n is, in fact, a convex body in $\mathbb{R}^n$, and since $r(t)$ is not identically zero, $\dim E_n = n$. Moreover, we have that, for all $0 \le k \le n$,

$$\begin{aligned}\mathrm{vol}_n(E_n) &= \int_a^b \mathrm{vol}_{n-1}\big(r(t)E_{n-1}\big)\mathrm{d}t = \mathrm{vol}_{n-1}(E_{n-1}) \int_a^b r(t)^{n-1}\mathrm{d}t \\ &= \cdots = \mathrm{vol}_{n-k}(E_{n-k}) \prod_{i=0}^{k-1} \int_a^b r(t)^{n-k+i}\mathrm{d}t\end{aligned}$$

which gives the required identity. □

If for some fixed $s \in \mathbb{N}$, the limits

$$\lim_{n\to\infty} \frac{\left(\int_a^b r(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1}\int_a^b r(t)^{n-k+i}\mathrm{d}t}$$

exist and are positive, $k = 2, \dots, s$, then we define (see Lemma 2.1)

$$\lambda_k = \lim_{n\to\infty} \frac{\left(\int_a^b r(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1}\int_a^b r(t)^{n-k+i}\mathrm{d}t} = \lim_{n\to\infty} \frac{\mathrm{vol}_{n-k}(E_{n-k})\,\mathrm{vol}_n(E_n)^k}{\mathrm{vol}_n(E_n)\,\mathrm{vol}_{n-1}(E_{n-1})^k} > 0, \tag{2.2}$$

$k = 2, \dots, s$, and $\lambda_0 = \lambda_1 = 1$.

For $1 \le p < \infty$, we consider the function

$$r_p : [-1,1] \longrightarrow [0,\infty) \quad \text{given by} \quad r_p(t) = \big(1 - |t|^p\big)^{1/p}. \tag{2.3}$$

We observe that the family of unit p-balls

$$B_n^p = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \le 1 \right\},$$

can be derived from (2.1) using the function r_p. Next lemma shows that r_p satisfies the limit condition defining λ_k (cf. (2.2)).

Lemma 2.2. *For all $k \ge 1$,*

$$\lim_{n\to\infty} \frac{\left(\int_{-1}^1 r_p(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1}\int_{-1}^1 r_p(t)^{n-k+i}\mathrm{d}t} = 1.$$

Proof. First we observe that, for any $i \ge 0$,

$$\begin{aligned}\int_{-1}^1 r_p(t)^i\,\mathrm{d}t &= 2\int_0^1 (1-t^p)^{i/p}\,\mathrm{d}t = \frac{4}{p}\int_0^{\pi/2} (\cos s)^{(2i/p)+1}(\sin s)^{(2/p)-1}\,\mathrm{d}s \\ &= \frac{2}{p}\,\mathrm{B}\left(\frac{i}{p}+1, \frac{1}{p}\right) = \frac{2i}{p(i+1)} \frac{\Gamma(\frac{i}{p})\Gamma(\frac{1}{p})}{\Gamma(\frac{i+1}{p})},\end{aligned}$$

where B denotes the beta function (see e.g. [12, p. 215]). Then, it is an easy computation to check that, for all $k\geq 1$,

$$\frac{\left(\int_{-1}^{1} r_p(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1}\int_{-1}^{1} r_p(t)^{n-k+i}\mathrm{d}t}=\frac{\left(\frac{n-1}{n}\right)^k\left(\Gamma\left(\frac{n-1}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)^k}{\left(\frac{n-k}{n}\right)\left(\Gamma\left(\frac{n-k}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)},$$

and since $\lim_{n\to\infty}\bigl((n-1)/n\bigr)^k/\bigl((n-k)/n\bigr)=1$, it suffices to prove that

$$\lim_{n\to\infty}\frac{\left(\Gamma\left(\frac{n-1}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)^k}{\Gamma\left(\frac{n-k}{p}\right)/\Gamma\left(\frac{n}{p}\right)}=1. \tag{2.4}$$

Stirling's formula for the gamma function (see e.g. [1, p. 24]) yields the asymptotic formula

$$\lim_{n\to\infty}\frac{\Gamma(x_n)}{\sqrt{2\pi}\,\left(\frac{x_n}{e}\right)^{x_n}\,\frac{1}{\sqrt{x_n}}}=1,$$

when the sequence $(x_n)_{n\in\mathbb{N}}\to\infty$ if n goes to ∞. Therefore we get

$$\begin{aligned}\lim_{n\to\infty}\frac{\left(\Gamma\left(\frac{n-1}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)^k}{\Gamma\left(\frac{n-k}{p}\right)/\Gamma\left(\frac{n}{p}\right)}&=\lim_{n\to\infty}\left(\frac{\left(\frac{n-1}{e}\right)^{(n-1)/p}\frac{1}{\sqrt{n-1}}}{\left(\frac{n}{e}\right)^{\frac{n}{p}}\frac{1}{\sqrt{n}}}\right)^k\frac{\left(\frac{n}{e}\right)^{n/p}\frac{1}{\sqrt{n}}}{\left(\frac{n-k}{e}\right)^{\frac{n-k}{p}}\frac{1}{\sqrt{n-k}}}\\&=\lim_{n\to\infty}\frac{(n-1)^{(n-1)k/p}\,n^{n/p}}{(n-k)^{(n-k)/p}\,n^{nk/p}}=1.\end{aligned}$$ □

Next result shows that the (inverse of the) volumes of unit p-balls can be obtained as the moments of a certain measure.

Lemma 2.3. *Let μ_p be the measure on the non-negative real line $\mathbb{R}_{\geq 0}$ associated to the function $G_p(t)=e^{-C_pt^p}$, $t\geq 0$, with $C_p=\bigl(2\Gamma(1/p+1)\bigr)^p$, $1\leq p<\infty$. Then the moments $m_i(\mu_p)=1/\kappa_i^p$, $i\geq 0$.*

Proof. It is just an easy computation to check that (see (1.1))

$$\begin{aligned}m_i(\mu_p)&=p\,C_p\int_0^\infty t^i\,e^{-C_pt^p}t^{p-1}\,\mathrm{d}t=\frac{1}{C_p^{i/p}}\int_0^\infty s^{i/p}e^{-s}\,\mathrm{d}s\\&=\frac{\Gamma\left(\frac{i}{p}+1\right)}{\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^i}=\frac{1}{\kappa_i^p}.\end{aligned}$$ □

Remark 2.1. If $p=\infty$, the corresponding measure μ_∞ is the discrete one given by $\mu_\infty\bigl(\{1/2\}\bigr)=1$, $\mu_\infty\bigl(\mathbb{R}_{\geq 0}\setminus\{1/2\}\bigr)=0$, for which $m_i(\mu_\infty)=1/\kappa_i^\infty$.

3. On inequalities for m-polynomials

The well-known inequalities

$$\mathrm{W}_i(K;E)^2\geq \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E),\quad 1\leq i\leq n-1, \tag{3.1}$$

particular cases of the Aleksandrov-Fenchel inequality (see e.g. [10, s. 6.3]), will be the main ingredient for the proof of the following result. It generalizes the inequality obtained in [8] for the Wills functional (1.3), namely, that $\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;B_n)/\kappa_i\leq e^{n\mathrm{W}_{n-1}(K;B_n)/\kappa_{n-1}}$. The proof follows the idea of the one in [8].

Proposition 3.1. *Let* $\mathbf{m} = (m_i)_{i\in\mathbb{N}}$ *be a sequence of positive real numbers such that* $\big((n+1)m_n^2/(nm_{n-1}m_{n+1})\big)_{n\in\mathbb{N}}$ *is a decreasing sequence and with*

$$\lambda = \lim_{n\to\infty}\frac{(n+1)}{n}\frac{m_n^2}{m_{n-1}m_{n+1}} > 0.$$

Then, denoting by $C(\lambda) = (1/\lambda)^{n(n-1)/2}$ *if* $0<\lambda<1$*, and* $C(\lambda)=1$ *otherwise, it holds*

$$f_{K;E}^{\mathbf{m}}(1) \le m_n \mathrm{vol}(E)\, C(\lambda)\, e^{n\, m_{n-1}\mathrm{W}_{n-1}(K;E)/(m_n \mathrm{vol}(E))}. \tag{3.2}$$

Proof. For the sake of brevity we will write $\widetilde{\mathrm{W}}_r = \binom{n}{n-r}\mathrm{W}_{n-r}(K;E)m_{n-r}$. Then, by the Aleksandrov-Fenchel inequalities (3.1) we get

$$\widetilde{\mathrm{W}}_r^2 \ge \frac{r+1}{r}\frac{(n-r+1)m_{n-r}^2}{(n-r)m_{n-r-1}m_{n-r+1}}\widetilde{\mathrm{W}}_{r-1}\widetilde{\mathrm{W}}_{r+1},$$

and the monotonicity hypothesis yields $\widetilde{\mathrm{W}}_r^2 \ge \big((r+1)/r\big)\lambda\, \widetilde{\mathrm{W}}_{r-1}\,\widetilde{\mathrm{W}}_{r+1}$. Thus

$$\frac{\widetilde{\mathrm{W}}_r}{\widetilde{\mathrm{W}}_{r+1}} \ge \frac{r+1}{r}\,\lambda\,\frac{\widetilde{\mathrm{W}}_{r-1}}{\widetilde{\mathrm{W}}_r} \ge \frac{r+1}{r-1}\,\lambda^2\,\frac{\widetilde{\mathrm{W}}_{r-2}}{\widetilde{\mathrm{W}}_{r-1}} \ge \cdots \ge \lambda^r\,(r+1)\frac{\widetilde{\mathrm{W}}_0}{\widetilde{\mathrm{W}}_1},$$

and consequently

$$\widetilde{\mathrm{W}}_r \le \widetilde{\mathrm{W}}_0\,\frac{1}{\lambda^{r(r-1)/2}}\,\frac{1}{r!}\left(\frac{\widetilde{\mathrm{W}}_1}{\widetilde{\mathrm{W}}_0}\right)^r \le \widetilde{\mathrm{W}}_0\,C(\lambda)\,\frac{1}{r!}\left(\frac{\widetilde{\mathrm{W}}_1}{\widetilde{\mathrm{W}}_0}\right)^r.$$

Therefore, summing in r, for $r=0,\dots,n$, we obtain

$$f_{K;E}^{\mathbf{m}}(1) \le \widetilde{\mathrm{W}}_0\,C(\lambda)\,e^{\widetilde{\mathrm{W}}_1/\widetilde{\mathrm{W}}_0}.$$

□

Remark 3.1. The sequence $\mathbf{m}=(1)_{n\in\mathbb{N}}$ trivially verifies the conditions of Proposition 3.1 and hence, Steiner polynomials satisfy a (3.2)-type inequality, namely,

$$f_{K;E}(1) \le \mathrm{vol}(E)\,e^{n\mathrm{W}_{n-1}(K;E)/\mathrm{vol}(E)}.$$

Next we show that μ_p-polynomials also verify a (3.2)-type inequality, i.e., we prove Theorem 1.1.

Proof of Theorem 1.1. Since $f_{K;B_n^p}^{\mu_p}(z) = \sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;B_n^p)/\kappa_i^p z^i$, we have to check that the conditions of Proposition 3.1 are satisfied for the sequence $(1/\kappa_n^p)_{n\in\mathbb{N}}$, $1\le p\le\infty$.

First we notice that for $p=\infty$ we get

$$\frac{n+1}{n}\frac{\kappa_{n-1}^\infty\kappa_{n+1}^\infty}{(\kappa_n^\infty)^2} = \frac{n+1}{n}\frac{2^{n-1}2^{n+1}}{(2^n)^2} = \frac{n+1}{n},$$

which is clearly a decreasing sequence and $\lim_{n\to\infty}(n+1)/n = 1$. So, we assume $1\le p<\infty$. On the one hand, it is easy to check that (cf. (1.1))

$$\frac{n+1}{n}\frac{\kappa_{n-1}^p\kappa_{n+1}^p}{(\kappa_n^p)^2} = \frac{n}{n-1}\frac{\Gamma\left(\frac{n}{p}\right)^2}{\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n+1}{p}\right)},$$

and using (2.4) for $k=2$ we get that it converges to 1 when n goes to ∞. Therefore $\lambda=1$ and so $C(\lambda)=1$.

So, it remains to be studied the monotonicity of the above sequence, which, for convenience, can be also rewritten as

$$\frac{n+1}{n}\frac{\kappa_{n-1}^p\kappa_{n+1}^p}{(\kappa_n^p)^2}=\frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{n}{p}+1-\frac{1}{p}\right)\Gamma\left(\frac{n}{p}+\frac{1}{p}\right)}. \tag{3.3}$$

In order to do it, we consider the real functions $f_i:(0,\infty)\longrightarrow\mathbb{R}$, $i=1,2$, given by $f_1(x)=(x-1/2)\log x$ and $f_2(x)=\theta/(12x)$ for a fixed $0<\theta<1$ which will be suitably chosen later on. The concavity of their first derivatives, f_i', $i=1,2$, together with the Mean-Value Theorem, allows to deduce that, in both cases $1\leq p\leq 2$ and $p\geq 2$, it holds

$$f_i'(x)+f_i'(x+1)-f_i'\left(x+\frac{1}{p}\right)-f_i'\left(x+1-\frac{1}{p}\right)<0.$$

Hence, the real functions $h_i:(0,\infty)\longrightarrow\mathbb{R}$, $i=1,2$, given by

$$h_i(x)=f_i(x)+f_i(x+1)-f_i\left(x+\frac{1}{p}\right)-f_i\left(x+1-\frac{1}{p}\right),$$

are strictly decreasing, which implies that $e^{h_1(x)+h_2(x)}$ is so. Now, Stirling's formula for the gamma function $\Gamma(x)$ (see e.g. [1, p. 24]) allows to write

$$\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1-\frac{1}{p})\Gamma(x+\frac{1}{p})}=e^{h_1(x)+h_2(x)}$$

for a suitable $\theta\in(0,1)$ (see [1, (3.9)]). Thus, all together, we can conclude that the sequence in (3.3) is strictly decreasing in n.

Therefore, all conditions in Proposition 3.1 are satisfied, and thus, inequality (3.2) for $E=B_n^p$ and $\mathbf{m}=(1/\kappa_n^p)_{n\in\mathbb{N}}$ shows that

$$f_{K;B_n^p}^{\mu_p}(1)\leq e^{n\mathrm{W}_{n-1}(K;B_n^p)/\kappa_{n-1}^p},$$

as desired. □

4. (Asymptotically) relating the roots of Steiner and m-polynomials

In this section we state and prove one of the main theorems in the paper. From it, a consequence for particular **m**-polynomials involving the unit p-balls will be obtained.

Theorem 4.1. *Let $s\in\mathbb{N}$ and $r:[a,b]\longrightarrow[0,\infty)$ be a continuous concave (non zero) function, $0\in[a,b]$, such that λ_k exists, $0\leq k\leq s$ (cf. (2.2)). Let $K\in\mathcal{K}^s$ and $\mathbf{m}=\big(\lambda_{s-i}/\mathrm{vol}_i(E_i)\big)_{i\in\mathbb{N}}$, with $m_i=0$ for $i>s$, and E_j defined by (2.1). Embedding $K\subsetneq\mathbb{R}^n$, $n>s$, let $\gamma_{1,n},\dots,\gamma_{s,n}$ be the non-zero roots of $f_{K;E_n}(z)$ and let $\nu_1,\dots,\nu_s$ be the roots of the **m**-polynomial $f_{K;E_s}^{\mathbf{m}}(z)$. Then, reordering if necessary, it holds*

$$\lim_{n\to\infty}\frac{\mathrm{vol}_n(E_n)}{\mathrm{vol}_{n-1}(E_{n-1})}\gamma_{i,n}=\nu_i,\quad i=1,\dots,s.$$

Proof. For $t\in\mathbb{R}$, let $H(t)=\big\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_n=t\big\}$. We may assume without loss of generality that $K\subsetneq H(0)$. Then,

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E_n)\lambda^i &= \mathrm{vol}(K+\lambda E_n)\\ &=\int_{\lambda a}^{\lambda b}\mathrm{vol}_{n-1}\big((K+\lambda E_n)\cap H(t)\big)\mathrm{d}t\\ &=\int_{\lambda a}^{\lambda b}\mathrm{vol}_{n-1}\Big(K+\lambda\, r\big(\tfrac{t}{\lambda}\big)\,E_{n-1}\Big)\mathrm{d}t\\ &=\sum_{i=0}^{n-1}\binom{n-1}{i}\mathrm{W}_i^{(n-1)}(K;E_{n-1})\int_{\lambda a}^{\lambda b}\lambda^i\, r\big(\tfrac{t}{\lambda}\big)^i\,\mathrm{d}t\\ &=\sum_{i=0}^{n-1}\binom{n-1}{i}\mathrm{W}_i^{(n-1)}(K;E_{n-1})\frac{1}{\mathrm{vol}_i(E_i)}\int_{\lambda a}^{\lambda b}\mathrm{vol}_i\Big(\lambda\, r\big(\tfrac{t}{\lambda}\big)E_i\Big)\mathrm{d}t\\ &=\sum_{i=0}^{n-1}\binom{n-1}{i}\mathrm{W}_i^{(n-1)}(K;E_{n-1})\frac{\mathrm{vol}_{i+1}(E_{i+1})}{\mathrm{vol}_i(E_i)}\,\lambda^{i+1},\end{aligned}$$

and identifying coefficients of both polynomials, we get that

$$\mathrm{W}_i(K;E_n)=\frac{i\,\mathrm{vol}_i(E_i)}{n\,\mathrm{vol}_{i-1}(E_{i-1})}\,\mathrm{W}_{i-1}^{(n-1)}(K;E_{n-1}).$$

Thus, using the above relation recursively, we obtain

$$\mathrm{W}_{n-s+j}(K;E_n)=\frac{\mathrm{vol}_{n-s+j}(E_{n-s+j})}{\binom{n}{n-s+j}}\,\frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)},\quad j=0,\dots,s,$$

and, since $\dim K=s$, the (relative) Steiner polynomial takes the form

$$\begin{aligned}f_{K;E_n}(z)&=z^{n-s}\sum_{j=0}^{s}\binom{n}{n-s+j}\mathrm{W}_{n-s+j}(K;E_n)z^j\\ &=z^{n-s}\sum_{j=0}^{s}\mathrm{vol}_{n-s+j}(E_{n-s+j})\frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)}z^j.\end{aligned}$$

The rest of the argument is similar to the one in [5, Theorem 1.3]; we include it here for completeness.

Then, for all $i=1,\dots,s$, $\gamma_{i,n}$ is a (non-zero) root of $f_{K;E_n}(z)$ if and only if the complex number $\tilde{\gamma}_{i,n}=\big(\mathrm{vol}_n(E_n)/\mathrm{vol}_{n-1}(E_{n-1})\big)\gamma_{i,n}$ satisfies

$$\sum_{j=0}^{s}\frac{\mathrm{vol}_{n-s+j}(E_{n-s+j})}{\mathrm{vol}_n(E_n)}\left(\frac{\mathrm{vol}_{n-1}(E_{n-1})}{\mathrm{vol}_n(E_n)}\right)^j\frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)}\,\tilde{\gamma}_{i,n}^j=0,$$

or equivalently, dividing by $\big(\mathrm{vol}_{n-1}(E_{n-1})/\mathrm{vol}_n(E_n)\big)^s$, if and only if $\tilde{\gamma}_{i,n}$ is a root of the polynomial

$$\sum_{j=0}^{s}\frac{\mathrm{vol}_{n-(s-j)}\big(E_{n-(s-j)}\big)\mathrm{vol}_n(E_n)^{s-j}}{\mathrm{vol}_n(E_n)\mathrm{vol}_{n-1}(E_{n-1})^{s-j}}\frac{\binom{s}{j}\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)}z^j$$
$$= z^s+\frac{s\,\mathrm{W}_{s-1}^{(s)}(K;E_s)}{\mathrm{vol}_{s-1}(E_{s-1})}z^{s-1}+\sum_{j=0}^{s-2}\beta_{s-j,n}\frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)}z^j$$

where, for the sake of brevity we are writing, for each $k=2,\dots,s$,

$$\beta_{k,n}=\frac{\mathrm{vol}_{n-k}\big(E_{n-k}\big)\mathrm{vol}_n(E_n)^k}{\mathrm{vol}_n(E_n)\mathrm{vol}_{n-1}(E_{n-1})^k}.$$

By assumption (cf. (2.2)), $\lim_{n\to\infty}\beta_{k,n}=\lambda_k$, $k=2,\dots,s$, which shows that the pointwise limit

$$\lim_{n\to\infty}\left(z^s+\frac{s\,\mathrm{W}_{s-1}^{(s)}(K;E_s)}{\mathrm{vol}_{s-1}(E_{s-1})}z^{s-1}+\sum_{j=0}^{s-2}\beta_{s-j,n}\frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)}z^j\right)=f_{K;E_s}^{\mathbf{m}}(z).$$

This, together with the fact that the roots of a polynomial are continuous functions of the coefficients, concludes the proof. □

As a direct consequence of Theorem 4.1 for unit p-balls we obtain Theorem 1.2. In a sense, this result is saying that for high dimension n, the (n-dimensional) Steiner polynomial $f_{K;B_n^p}(z)=\sum_{i=n-s}^{n}\binom{n}{i}\mathrm{W}_i(K;B_n^p)z^i$ of a convex body K with fixed dimension $\dim K=s$ 'behaves as' its μ_p-polynomial $f_{K;B_s^p}^{\mu_p}(z)=\sum_{i=0}^{s}\binom{s}{i}\mathrm{W}_i^{(s)}(K;B_s^p)/\kappa_i^p z^i$.

Proof of Theorem 1.2. For $1\le p<\infty$, let r_p be the function given by (2.3), which yields the unit p-balls via (2.1), i.e., $E_i=B_i^p$.

Then, Lemma 2.2 ensures that $\lambda_k=1$ for all $k\ge 1$ (cf. (2.2)), and thus we can apply Theorem 4.1 to get the required result. Notice that now, $m_i=1/\kappa_i^p$, $i=1,\dots,s$.

Finally, we deal with $p=\infty$. In that case B_n^∞ is the n-dimensional regular cube with edge-length 2, and hence

$$\frac{\kappa_n^\infty}{\kappa_{n-1}^\infty}\gamma_{i,n}=\frac{2^n}{2^{n-1}}\gamma_{i,n}=2\gamma_{i,n}.$$

Now we observe that, since $\dim K=s$, identifying K with its canonical embedding in $\mathrm{e}_{s+1}^\perp\subsetneq\mathbb{R}^{s+1}$ (e_i denotes the i-th canonical unit vector), then

$$\sum_{i=1}^{s+1}\binom{s+1}{i}\mathrm{W}_i^{(s+1)}(K;B_{s+1}^\infty)\lambda^i=\mathrm{vol}_{s+1}(K+\lambda B_{s+1}^\infty)=2\lambda\mathrm{vol}_s(K+\lambda B_s^\infty)$$
$$=2\lambda\sum_{i=0}^{s}\binom{s}{i}\mathrm{W}_i^{(s)}(K;B_s^\infty)\lambda^i,$$

and identifying coefficients of both polynomials we get

$$\binom{s+1}{i}\mathrm{W}_i^{(s+1)}(K;B_{s+1}^\infty)=2\binom{s}{i-1}\mathrm{W}_{i-1}^{(s)}(K;B_s^\infty),\quad i=1,\dots,s+1.$$

Iterating this embedding-process till $K \subsetneq \mathbb{R}^n$, we finally get the identities

$$\binom{n}{i}\mathrm{W}_i(K;B_n^\infty) = 2^{n-s}\binom{s}{i-(n-s)}\mathrm{W}^{(s)}_{i-(n-s)}(K;B_s^\infty), \quad i=n-s,\dots,n,$$

and hence,

$$\begin{aligned}
f_{K;B_n^\infty}(z) &= \sum_{i=n-s}^{n}\binom{n}{i}\mathrm{W}_i(K;B_n^\infty)z^i = z^{n-s}\sum_{i=n-s}^{n}\binom{n}{i}\mathrm{W}_i(K;B_n^\infty)z^{i-n+s} \\
&= z^{n-s}\sum_{i=n-s}^{n}2^{n-s}\binom{s}{i-(n-s)}\mathrm{W}^{(s)}_{i-(n-s)}(K;B_s^\infty)z^{i-n+s} \\
&= (2z)^{n-s}\sum_{j=0}^{s}\binom{s}{j}\mathrm{W}^{(s)}_j(K;B_s^\infty)z^j \\
&= (2z)^{n-s}\sum_{j=0}^{s}\binom{s}{j}\frac{\mathrm{W}^{(s)}_j(K;B_s^\infty)}{2^j}(2z)^j \\
&= (2z)^{n-s}\sum_{j=0}^{s}\binom{s}{j}\frac{\mathrm{W}^{(s)}_j(K;B_s^\infty)}{\kappa_j^\infty}(2z)^j = (2z)^{n-s}f^{\mu_\infty}_{K;B_s^\infty}(2z).
\end{aligned}$$

Therefore, $2\gamma_{i,n} = \nu_i$. It concludes the proof. □

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