p-DIFFERENCE: A COUNTERPART OF MINKOWSKI DIFFERENCE IN THE FRAMEWORK OF THE FIREY-BRUNN-MINKOWSKI THEORY

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Abstract. As a substraction counterpart of the well-known p-sum of convex bodies, we introduce the notion of p-difference. We prove several properties of the p-difference, introducing also the notion of p-(inner) parallel bodies. We prove an analog of the concavity of the family of classical parallel bodies for the p-parallel ones, as well as the continuity of this new family, in its definition parameter. Further results on inner parallel bodies are extended to p-inner ones; for instance, we show that tangential bodies are characterized as the only convex bodies such that their p-inner parallel bodies are homothetic copies of them.

1. Preliminaries

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., non-empty compact convex sets in the Euclidean space $\mathbb{R}^n$, endowed with the standard scalar product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $|\cdot|$, and let $\mathcal{K}^n_0$ be the subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin. Let B_n be the n-dimensional unit ball and $\{e_1,\dots,e_n\}$ the canonical basis in $\mathbb{R}^n$.

We will denote by $h(K,u)=\max\bigl\{\langle x,u\rangle : x\in K\bigr\}$ the support function of $K\in\mathcal{K}^n$ in the direction u of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. For a set $M\subseteq\mathbb{R}^n$, let $\operatorname{bd} M$, $\operatorname{cl} M$, $\operatorname{int} M$ and $\operatorname{relint} M$ denote its boundary, closure, interior and relative interior. The convex hull and the positive hull of M are represented by $\operatorname{conv} M$ and $\operatorname{pos} M$. The dimension of M, i.e., the dimension of its affine hull, is denoted by $\dim M$. If M is measurable, we write $\operatorname{vol}(M)$ to denote its volume, that is, its n-dimensional Lebesgue measure. Finally, for $u,v\in\mathbb{R}^n$, the notation $[u,v]$ stands for the convex hull of $\{u,v\}$, i.e., the line segment with end points u,v.

The vectorial or Minkowski addition of non-empty sets in $\mathbb{R}^n$ is defined as

$$A+B=\{a+b : a\in A, b\in B\}.$$

We refer the reader to the books [5, 11] for a detailed study of the same.

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Minkowski difference (though it was not introduced by Minkowski) can be regarded as the substraction counterpart of the Minkowski sum: for two sets $A, B \subseteq \mathbb{R}^n$, the Minkowski difference of A and B is defined by

$$A \sim B = \{x \in \mathbb{R}^n : B + x \subseteq A\},$$

this is, $A \sim B$ is the largest set such that $(A \sim B) + B \subseteq A$ (if $B \neq \emptyset$). If $B = \emptyset$, then we set $A \sim B = \mathbb{R}^n$.

In 1962 Firey introduced the following generalization of the classical Minkowski addition (see [4]). For $p \geq 1$ and $K, E \in \mathcal{K}_0^n$ the p-sum (or L_p sum) of K and E is the convex body $K +_p E \in \mathcal{K}_0^n$ whose support function is given by

$$h(K +_p E, u) = \Big(h(K,u)^p + h(E,u)^p\Big)^{1/p}, \tag{1.1}$$

for all u in $\mathbb{S}^{n-1}$. When $p = 1$, the latter defines the usual Minkowski sum whereas for $p = \infty$, $h(K +_\infty E, u) = \max\big\{h(K,u), h(E,u)\big\}$, i.e.,

$$K +_\infty E = \mathrm{conv}(K \cup E) \tag{1.2}$$

(see Figure 1). We notice also that when combining the p-sum with the scalar multiplication $\mu K = \{\mu x : x \in K\}$, the following fact holds:

$$\mu K +_p \mu E = \mu(K +_p E), \tag{1.3}$$

for all $K, E \in \mathcal{K}_0^n$, $\mu > 0$ and $p \geq 1$.

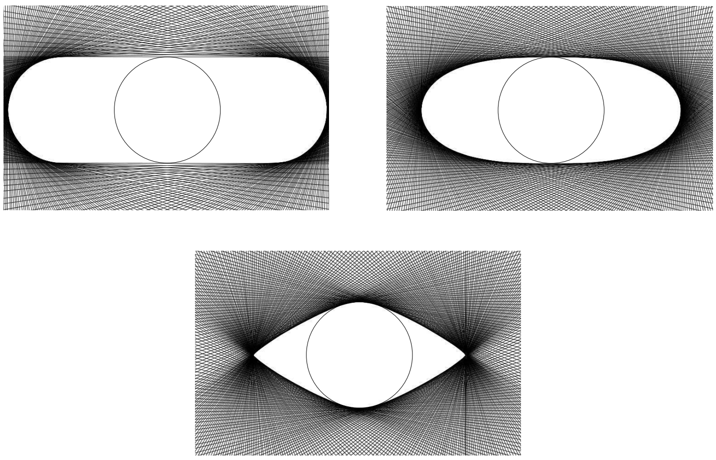

FIGURE 1. $[-2e_1, 2e_1] +_p B_2$, where $p = 1$, 1.5, 10.

Moreover, in [4, Theorem 1] it is shown that

$$K +_q E \subseteq K +_p E \text{ for all } 1 \leq p \leq q. \tag{1.4}$$

We would like to observe a straightforward *geometrically important* difference between the Minkowski sum and the p-sum: the loss of the translation invariance.

As we shall see later, Minkowski difference gives rise to the notion of inner parallel bodies, a notion which has many applications in the geometry of convex bodies: according to [11, Section 7.5] "some of the deeper investigations of inequalities for mixed volumes make essential use of the method of inner parallel bodies"; we refer the reader to [11, Note 2 for Section 7.5] for further applications of inner parallel bodies.

The works [8, 9] of Lutwak, where a systematic study of means of convex bodies is taken up, constitute the outstanding rising of the nowadays known as Brunn-Minkowski-Firey theory. In the last years many important developments of this theory have come out; for further details, as well as detailed bibliography on the topic we refer the reader to [11, Chapter 9] and the references therein.

Our aim in this work is to introduce a new operation, the *p-difference* of convex bodies, which emulates the Minkowski difference in the framework of the L_p-Brunn-Minkowski theory: for $K, E \in \mathcal{K}_0^n$, $E \subseteq K$, we will define the p-difference of K and E as the largest convex body $K \sim_p E \in \mathcal{K}_0^n$ such that $(K \sim_p E) +_p E \subseteq K$. Then, in Section 2 we will show that the following equality of sets holds, providing two different expressions for the p-difference of K and E:

$$K \sim_p E = \bigcup_{M \in \mathcal{F}^p_{K;E}} M = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p} \Big\}$$

where $\mathcal{F}^p_{K;E} = \{ M \in \mathcal{K}_0^n : \ M +_p E \subseteq K \}$.

Next we prove suitable extensions of properties of the Minkowski substraction for the p-difference. That is the content of Section 2.

Later, in Section 3 we discuss the necessity of introducing a subfamily of $\mathcal{K}_0^n$ where to work with the p-difference. We prove that the natural notion of kernel (see Section 2 for details) of a convex body in the context of the p-difference satisfies appropriate "p-versions" of the classical properties. More precisely we prove that the *p-kernel* of K with respect to E may be larger than the classical one, but remains always lower dimensional, i.e., contained in a hyperplane. Finally, in the last section we introduce the notion of p-inner parallel body, showing, among others, a concavity property of the family of p-parallel bodies and the characterization of sets for which their p-inner parallel bodies are homothetic copies of them.

2. p-DIFFERENCE OF CONVEX BODIES. DEFINITION AND FIRST PROPERTIES

There are several equivalent definitions of Minkowski difference, all of which turn out to be equivalent (see [11, p. 146]). Next we provide the two of them, which, in our investigation, settle down the basis for the definition of p-difference of convex bodies.

On the one hand, as mentioned already in the introduction, the Minkowski difference of two sets $A, B \in \mathbb{R}^n$ can be defined by

$$A \sim B = \{x \in \mathbb{R}^n : B + x \subseteq A\}. \tag{2.1}$$

On the other hand, if $\psi : \mathbb{S}^{n-1} \to [0, \infty)$ is a non-negative continuous function, the *Wulff-shape* or *Aleksandrov body* of ψ is the set

$$\mathrm{WS}(\psi) = \bigcap_{u \in \mathbb{S}^{n-1}} \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \le \psi(u)\bigr\}.$$

For further details about Wulff-shapes we refer the reader to [11, Section 7.5] and the references therein.

Remark 2.1. *It is easy to see (*[11, Section 7.5]*) that for any such* ψ, $\mathrm{WS}(\psi)$ *is a convex body containing the origin, and moreover,*

$$h\bigl(\mathrm{WS}(\psi), \cdot\bigr) \le \psi(\cdot).$$

Thus, for convex bodies $K, E \in \mathcal{K}^n$, the Minkowski difference can be defined as the Wulff-shape of the function $\psi(u) = h(K, u) - h(E, u)$:

$$K \sim E = \bigcap_{u \in \mathbb{S}^{n-1}} \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \le h(K, u) - h(E, u)\bigr\}. \tag{2.2}$$

Unlike what happens with the Minkowski sum and the p-sum, the support function of the Minkowski difference of K, E cannot be given, in general, by an easy combination of the support functions of K and E. Nevertheless, Remark 2.1 provides a bound for the support function of the Minkowski difference, namely,

$$h(K \sim E, u) \le h(K, u) - h(E, u) \quad \text{ for all } u \in \mathbb{S}^{n-1}.$$

We would like to point out that, in general, there is no equality in the above inequality relating support functions (cf. Lemma 2.1 ii)).

Next lemma collects some useful rules which relate Minkowski addition and substraction.

Lemma 2.1 ([11, p. 147])**.** *Let* $A, B, C \subseteq \mathbb{R}^n$. *Then*

i) $(A + B) \sim B \supseteq A$. *If* $A, B \in \mathcal{K}^n$, *then there is equality.*
ii) $(A \sim B) + B \subseteq A$, *if* $B \neq \emptyset$. *If* $A, B \in \mathcal{K}^n$, *equality holds if and only if* B *is a summand of* A, *i.e., if there exists* $D \in \mathcal{K}^n$ *with* $A = D + B$.
iii) $(A \sim B) + C \subseteq (A + C) \sim B$.
iv) $(A \sim B) \sim C = A \sim (B + C)$.
v) $A + B \subseteq C$ *if and only if* $A \subseteq C \sim B$.

We have seen two equivalent geometric constructions giving rise to the Minkowski difference, namely, equations (2.1) and (2.2). The first definition makes use of the natural connection of the substraction with the Minkowski sum: $K \sim E$ *should be maximal among all convex bodies which (Minkowski) added to* E *keep the result within* K. The second definition takes advantage of the connection of the Minkowski sum with the support function via a Wulff-shape, since differences of support functions need not be support functions.

These two constructions settle down the basis for describing the p-difference of convex bodies. As mentioned in the introduction we define this operation in the following way.

Definition 2.1. *Let $K, E \in \mathcal{K}_0^n$, $E \subseteq K$, and let $p \geq 1$. The* p-difference *of K and E is the largest convex body $K \sim_p E \in \mathcal{K}_0^n$ such that*

$$(K \sim_p E) +_p E \subseteq K. \tag{2.3}$$

On the one hand, it is clear from the above definition that

$$K \sim_p E = \bigcup_{M \in \mathcal{F}^p_{K;E}} M, \tag{2.4}$$

where

$$\mathcal{F}^p_{K;E} = \Big\{ M \in \mathcal{K}_0^n : \; M +_p E \subseteq K \Big\},$$

because $\bigcup_{M \in \mathcal{F}^p_{K;E}} M$ is a convex body. Indeed, if $K_1, K_2 \in \mathcal{F}^p_{K;E}$ then also $\operatorname{conv}(K_1 \cup K_2) \in \mathcal{F}^p_{K;E}$, i.e., the above union is a convex set. Now given a sequence of points $(x_n)_n \subset \bigcup_{M \in \mathcal{F}^p_{K;E}} M$ with $\lim_{n\to\infty} x_n = x$, there exists a sequence $(M_n)_n \subseteq \mathcal{F}^p_{K;E}$ with $x_n \in M_n$. By Blaschke Selection Theorem (see e.g. [11, Theorem 1.8.6]) we can choose $(M_n)_n$ to be convergent to a convex body $\overline{M}$, and it is clear that $\overline{M} \in \mathcal{F}^p_{K;E}$. Therefore, $x \in \overline{M} \subseteq \bigcup_{M \in \mathcal{F}^p_{K;E}} M$.

Taking (1.2) into account, it is easy to check that $K \sim_\infty E = K$, and for $p = 1$ we obviously obtain the classical Minkowski difference of K and E.

On the other hand, and looking back to (2.2), one expects that such a kind of expression also works for the p-difference, in order to use the powerful Wulff-shape structure and its connection with the support function. The following theorem shows that this is the case. First we will assume that $1 \leq p < \infty$. The case $p = \infty$ will be treated later.

Theorem 2.1. *Let $1 \leq p < \infty$ and let $K, E \in \mathcal{K}_0^n$ with $E \subseteq K$. Then,*

$$K \sim_p E = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p} \Big\}. \tag{2.5}$$

Proof. We show (2.5) using the already known expression (2.4) for $K \sim_p E$. Let

$$L = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p} \Big\}.$$

Remark 2.1 ensures that $h(L,u) \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p}$ and so we have $h(L,u)^p + h(E,u)^p \leq h(K,u)^p$ for all $u \in \mathbb{S}^{n-1}$. It yields $L \subseteq K \sim_p E$.

Conversely, if $x \in K \sim_p E$, then there exists $M \in \mathcal{F}^p_{K;E}$ such that $x \in M$, and from the definition of $\mathcal{F}^p_{K;E}$, we obtain that

$$h(M,u)^p + h(E,u)^p \leq h(K,u)^p \quad \text{for all } \; u \in \mathbb{S}^{n-1}.$$

It implies that $\langle x,u\rangle \leq \big(h(K,u)^p - h(E,u)^p\big)^{1/p}$ for all $u\in\mathbb{S}^{n-1}$, i.e., $x\in L$, which shows the reverse inclusion and concludes the proof. □

We observe that by Remark 2.1, $K\sim_p E$ is a convex body whose support function satisfies

$$h(K\sim_p E,u) \leq \big(h(K,u)^p - h(E,u)^p\big)^{1/p}. \tag{2.6}$$

For $p=\infty$, the right-hand side in the defining inequality in (2.5) shall be seen as the limit when p goes to infinity. Then, the case $p=\infty$ is not achieved in the above result as the following example shows.

Example 2.1. *Denoting by $C_n=[-1,1]^n$ the n-dimensional cube of edge-length 2 centered at the origin, we have*

$$\bigcap_{u\in\mathbb{S}^{n-1}} \left\{x\in\mathbb{R}^n : \langle x,u\rangle \leq \lim_{p\to\infty}\big(h(C_n,u)^p - h(B_n,u)^p\big)^{1/p}\right\} = \{0\},$$

whereas $C_n\sim_\infty B_n = C_n$ using (2.4)*.*

The problem relies on the fact that $h(K,u)=h(E,u)$ for some $u\in\mathbb{S}^{n-1}$ provokes a *devastating geometrical* effect on the intersection expression in (2.5), whereas it is almost unseen by the union used in (2.4). Indeed, if $h(K,u)=h(E,u)$ holds for some $u\in\mathbb{S}^{n-1}$ then

$$\lim_{p\to\infty}\big(h(K,u)^p - h(E,u)^p\big)^{1/p} = 0.$$

However, if $\operatorname{bd}K\cap\operatorname{bd}E=\emptyset$, as $E\subseteq\operatorname{int}K$, we have

$$\lim_{p\to\infty}\big(h(K,u)^p - h(E,u)^p\big)^{1/p} = h(K,u)$$

obtaining that

$$\bigcap_{u\in\mathbb{S}^{n-1}} \left\{x\in\mathbb{R}^n : \langle x,u\rangle \leq \lim_{p\to\infty}\big(h(K,u)^p - h(E,u)^p\big)^{1/p}\right\} = K.$$

Remark 2.2. *From now on, we set*

$$\big(h(K,u)^p - h(E,u)^p\big)^{1/p} = h(K,u) \quad \text{for } p=\infty,$$

which is the limit when $p\to\infty$ except if $h(K,u)=h(E,u)\neq 0$. With this convention, Theorem 2.1 remains true for $p=\infty$ too.

Next we state the p-analogue of Lemma 2.1, whose proof is a direct application of (2.5) and (1.1). Item i) of Lemma 2.1 will be included in Proposition 2.1.

Lemma 2.2. *Let $K,E,M\in\mathcal{K}^n_0$ and $p\geq 1$. Then, assuming the suitable inclusions among the sets,*

i) *$(K\sim_p E)+_p E\subseteq K$. Equality holds if and only if E is a p-summand of K, i.e., if there exists $L\in\mathcal{K}^n_0$ with $K=L+_p E$.*
ii) *$(K\sim_p E)+_p M\subseteq (K+_p M)\sim_p E$.*
iii) *$(K\sim_p E)\sim_p M = K\sim_p (E+_p M)$.*

iv) $K +_p E \subseteq M$ *if and only if* $K \subseteq M \sim_p E$.

Further properties of the p-difference are collected in the following.

Proposition 2.1. *Let* $K, E \in \mathcal{K}_0^n$. *Then, for* $1 \le p < \infty$,

$$(K +_p E) \sim_p E = K. \tag{2.7}$$

For all $p \ge 1$ *and* $0 \le \varepsilon \le 1$,

$$K \sim_p \varepsilon K = (1 - \varepsilon^p)^{1/p} K. \tag{2.8}$$

If $E \subseteq K$ *then, for all* $p \ge 1$ *and* $\lambda > 0$,

$$\lambda(K \sim_p E) = (\lambda K) \sim_p (\lambda E). \tag{2.9}$$

Proof. Since $h\big((K +_p E) \sim_p E, u\big)^p \le h(K, u)^p$ for all $u \in \mathbb{S}^{n-1}$ (cf. (2.6)), we obtain that $(K +_p E) \sim_p E \subseteq K$. Now, Lemma 2.2 iv) for $M = K +_p E$ yields $K \subseteq (K +_p E) \sim_p E$, and thus, $K = (K +_p E) \sim_p E$.

In order to prove (2.8), we first notice that the definition of p-sum (1.1) implies that $(1 - \varepsilon^p)^{1/p} K +_p \varepsilon K = K$. Then, by (2.7) we get the result.

Now we show (2.9). Taking support functions and using (2.6), it is immediate to see that $\big(\lambda(K \sim_p E)\big) +_p \lambda E \subseteq \lambda K$, which yields the inclusion

$$\lambda\big(K \sim_p E\big) \subseteq (\lambda K) \sim_p \big(\lambda E\big).$$

Then, applying this relation to λK, λE and $1/\lambda$, we finally get

$$K \sim_p E \subseteq \frac{1}{\lambda}\Big[(\lambda K) \sim_p \big(\lambda E\big)\Big] \subseteq K \sim_p E. \qquad \square$$

The following lemma is an easy consequence of (1.4) and (2.4).

Lemma 2.3. *Let* $K, E \in \mathcal{K}_0^n$, $E \subseteq K$, *and let* $1 \le p \le q \le \infty$. *Then*

$$K \sim_p E \subseteq K \sim_q E. \tag{2.10}$$

Remark 2.3. *We observe that the inclusion* (2.10) *may be strict, as relation* (2.8) *shows, because the map* $t \mapsto (1-\varepsilon^t)^{1/t}$, $0 \le \varepsilon \le 1$, *is strictly increasing.*

Finally we deal with the continuity of this new operation in $\mathcal{K}_0^n$. It is known (see [11, Remark 3.1.12]) that Minkowski substraction is not continuous with respect to the Hausdorff metric δ_H (see [11, Section 1.8] for the definition). Next we prove that the same holds for the p-difference of convex bodies, for any $1 < p < \infty$. For $p = \infty$, the continuity holds trivially.

Proposition 2.2. *Let* $1 < p < \infty$. *The* p*-difference is not continuous with respect to the Hausdorff metric.*

Proof. We consider the convex bodies

$$K = \operatorname{conv}\Big(B_2 \cup \big\{(2,1)^\intercal, (2,-1)^\intercal\big\}\Big),$$
$$K_i = \operatorname{conv}\Big(B_2 \cup \big\{(2,1)^\intercal, (2,-1+1/i)^\intercal\big\}\Big), \quad i \in \mathbb{N}.$$

Clearly, K_i converges to K with respect to the Hausdorff metric in $\mathcal{K}_0^2$. Indeed, it can be seen that $\delta_H(K_i, K) \le 1/i$.

On the one hand, we have (see (2.10)) $K \sim_p B_2 \supseteq K \sim B_2 = [0, e_1]$ for all $p > 1$. On the other hand, we claim that $K_i \sim_p B_2 = \{0\}$ for every $i \in \mathbb{N}$ and all $p > 1$, and hence we could conclude that $K_i \sim_p B_2$ does not converge to $K \sim_p B_2$, as required.

In order to prove the claim, let $i \in \mathbb{N}$ and suppose, by contradiction, that there exists $u = (a, b)^\intercal \in K_i \sim_p B_2$, $u \neq 0$, which yields $[0, u] +_p B_2 \subseteq K_i$. If $b \neq 0$ then

$$h\big([0, u] +_p B_2, \pm e_2\big) = \big(1 + |b|^p\big)^{1/p} > 1 = h(K_i, e_2),$$

where the sign for e_2 is chosen accordingly to the sign of b. Clearly it is not possible, and therefore $b = 0$, i.e., $u = ae_1$. Now, if $a < 0$ then

$$h\big([0, ae_1] +_p B_2, -e_1\big) = \big(1 + |a|^p\big)^{1/p} > 1 = h\big(K_i, -e_1\big),$$

again a contradiction. Hence, $a > 0$.

Let $u_i = (\cos\theta_i, \sin\theta_i)^\intercal \in \mathbb{S}^1$ be the unit normal vector to K_i at the "inclined bottom edge", i.e., the unique vector on $\mathbb{S}^1$ with coordinates $\cos\theta_i > 0$, $\sin\theta_i < 0$ (see Figure 2).

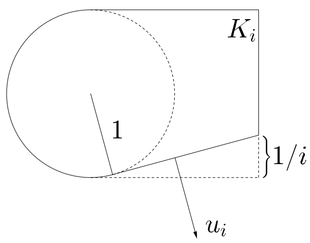

FIGURE 2. The p-difference is not continuous.

Then we have

$$h\big([0, ae_1] +_p B_2, u_i\big) = (1 + a^p \cos^p\theta_i)^{1/p} > 1 = h(B_2, u_i) = h(K_i, u_i),$$

which is impossible. Therefore, $K_i \sim_p B_2 = \{0\}$. □

We conclude the section with a brief observation on a Brunn-Minkowski type inequality. The famous Brunn-Minkowski inequality states that for $K, E \in \mathcal{K}^n$,

$$\mathrm{vol}(K + E)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(E)^{1/n}.$$

Using the monotonicity of the volume together with basic properties relating the Minkowski sum and difference (see Lemma 2.1, ii)) one obtains the following Brunn-Minkowski inequality for the Minkowski difference:

$$\mathrm{vol}(K \sim E)^{1/n} \leq \mathrm{vol}(K)^{1/n} - \mathrm{vol}(E)^{1/n}. \tag{2.11}$$

In the setting of the Brunn-Minkowski-Firey theory, the p-Brunn-Minkowski inequality, i.e., a Brunn-Minkowski type inequality for the p-sum, establishes that if $K, E \in \mathcal{K}^n_0$, $1 \leq p < \infty$, then

$$\mathrm{vol}(K +_p E)^{p/n} \geq \mathrm{vol}(K)^{p/n} + \mathrm{vol}(E)^{p/n} \tag{2.12}$$

(see e.g. [11, Theorem 9.1.3]). Taking into account (2.12), the inclusion (2.3) provides, in a straightforward manner, a Brunn-Minkowski type inequality for the p-difference of two convex bodies (cf. (2.11)):

Proposition 2.3. *Let $K, E \in \mathcal{K}_0^n$ with $E \subseteq K$ and let $1 \leq p < \infty$. Then*

$$\mathrm{vol}(K \sim_p E)^{p/n} \leq \mathrm{vol}(K)^{p/n} - \mathrm{vol}(E)^{p/n}.$$

Proof. Combining (2.3) with the monotonicity of the volume, we obtain that

$$\mathrm{vol}\big((K \sim_p E) +_p E\big)^{p/n} \leq \mathrm{vol}(K)^{p/n}.$$

Now (2.12) together with (1.3) yields

$$\mathrm{vol}\big((K \sim_p E) +_p E\big)^{p/n} \geq \mathrm{vol}(K \sim_p E)^{p/n} + \mathrm{vol}(E)^{p/n}.$$

Joining both inequalities we finally obtain that

$$\mathrm{vol}(K \sim_p E)^{p/n} \leq \mathrm{vol}(K)^{p/n} - \mathrm{vol}(E)^{p/n}. \qquad \square$$

The case $p = \infty$ leads to a trivial inequality.

3. p-inradius and p-kernel

From now on we shall always assume that $p \neq \infty$.

When dealing with the Minkowski difference, the notions of inradius and kernel play a prominent role (see e.g. [2, 6, 10] and the references therein). For two convex bodies $K, E \in \mathcal{K}^n$, the *relative inradius* $\mathrm{r}(K; E)$ of K with respect to E is defined by

$$\mathrm{r}(K; E) = \max\big\{r \geq 0 : x + rE \subseteq K \text{ for some } x \in \mathbb{R}^n\big\},$$

whereas $\ker(K; E) = K \sim \mathrm{r}(K; E)E$ is the set of relative incenters of K, usually called *kernel* of K with respect to E. The dimension of $\ker(K; E)$ is strictly less than n (see [3, p. 59]). Moreover, the (relative) *inradius at the origin* (cf. [1]) is defined as

$$\rho(K; E) = \max\{\rho \geq 0 : \rho E \subseteq K\}.$$

Regarding (any of) the definitions of p-difference, one would be tempted to introduce, for $K, E \in \mathcal{K}_0^n$, $E \subseteq K$, an analogue of the relative inradius, i.e., a *p-inradius of K relative to E* as

$$\max\big\{r \geq 0 : M +_p rE \subseteq K \text{ for some } M \in \mathcal{K}_0^n\big\}.$$

However, it is immediate to see that the above number, for $p > 1$, coincides with the (relative) inradius at the origin $\rho(K; E)$. Indeed, if there exists $M \in \mathcal{K}_0^n$ such that $M +_p \rho E \subseteq K$, then

$$\rho E = \{0\} +_p \rho E \subseteq M +_p \rho E \subseteq K,$$

as claimed.

We observe that since the "naturally defined" p-inradius does not depend on p, and since, in general, $\mathrm{r}(K; E) \neq \rho(K; E)$, in order to develop a structured and systematic study of the p-difference, also valid for $p = 1$, we have the *heuristic necessity* of introducing a subfamily of $\mathcal{K}^n_0$ where also the trivial cases are avoided.

Thus, for $E \in \mathcal{K}^n_0$, we define the subfamily, strongly depending on the *geometry* of the body $E \in \mathcal{K}^n_0$, given by

$$\begin{aligned}\mathcal{K}^n_{00}(E) &= \bigl\{K \in \mathcal{K}^n_0 : \mathrm{r}(K;E) = \rho(K;E)\bigr\}\\ &= \bigl\{K \in \mathcal{K}^n_0 : 0 \in \ker(K;E)\bigr\}.\end{aligned} \tag{3.1}$$

The last equality of sets follows easily: if $0 \in \ker(K;E)$ then $\mathrm{r}(K;E)E \subseteq K$, and thus $\mathrm{r}(K;E) \leq \rho(K;E)$, being the reverse inequality a direct consequence of the definition of inradius; conversely, if $\mathrm{r}(K;E) = \rho(K;E)$ then $\mathrm{r}(K;E)E \subseteq K$, which implies that $0 \in \ker(K;E)$.

For $K \in \mathcal{K}^n_{00}(E)$ we define the *p-kernel of K with respect to E* as

$$\ker_p(K;E) = K \sim_p \mathrm{r}(K;E)E.$$

Then, using (2.10) it follows that, for $1 \leq p \leq q < \infty$,

$$\ker_p(K;E) \subseteq \ker_q(K;E). \tag{3.2}$$

As in the case of the usual kernel, for which

$$\dim\bigl(\ker(K;E)\bigr) \leq n-1 \tag{3.3}$$

([3, p. 59])), the following result shows that for any value of $1 \leq p < \infty$, the p-kernel of $K \in \mathcal{K}^n_{00}(E)$ is always a degenerated convex body.

Proposition 3.1. *Let $K \in \mathcal{K}^n_{00}(E)$. Then, for any $1 \leq p < \infty$,*

$$\dim\bigl(\ker_p(K;E)\bigr) \leq n-1.$$

Proof. Without loss of generality, we may assume that $\mathrm{r}(K;E) = 1$. Then $U = \bigl\{u \in \mathbb{S}^{n-1} : h(K,u) = h(E,u)\bigr\} \neq \emptyset$. We observe that if we show that

$$\dim \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} \leq n-1, \tag{3.4}$$

then, using (2.5) we would get that

$$K \sim_p E \subseteq \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\},$$

which would finish the proof. Therefore, we have to prove (3.4).

Thus we assume, by contradiction, that

$$\dim \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} = n.$$

Let $v_1,\dots,v_n \in \mathbb{R}^n$ be n linearly independent vectors so that

$$A = \mathrm{pos}\{v_1,\dots,v_n\} \subseteq \mathrm{int} \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\},$$

and let $u_1,\dots,u_n \in \mathbb{S}^{n-1}$ be n unit vectors such that

$$\bigcap_{u\in\{u_1,\dots,u_n\}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} = \bigcap_{u\in\mathrm{pos}\{u_1,\dots,u_n\}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} = A.$$

Denoting by $\widetilde{U} = \mathrm{pos}\{u_1, \ldots, u_n\}$, we clearly have that $U \subseteq \mathrm{relint}\, \widetilde{U}$. Thus,

$$\varepsilon = \min\Big\{h(K,u) - h(E,u) : u \in \mathrm{cl}\big(\mathbb{S}^{n-1}\backslash\widetilde{U}\big)\Big\}$$

is a positive real number and hence,

$$\begin{aligned}
A \cap \varepsilon B_n &= \left(\bigcap_{u\in\widetilde{U}}\{x\in\mathbb{R}^n : \langle x,u\rangle \le 0\}\right) \cap \left(\bigcap_{u\in\mathbb{S}^{n-1}}\{x\in\mathbb{R}^n : \langle x,u\rangle \le \varepsilon\}\right)\\
&= \left(\bigcap_{u\in\widetilde{U}}\{x\in\mathbb{R}^n : \langle x,u\rangle \le 0\}\right) \cap \left(\bigcap_{u\in\mathbb{S}^{n-1}\backslash\widetilde{U}}\{x\in\mathbb{R}^n : \langle x,u\rangle \le \varepsilon\}\right)\\
&\subseteq \bigcap_{u\in\mathbb{S}^{n-1}}\Big\{x\in\mathbb{R}^n : \langle x,u\rangle \le h(K,u) - h(E,u)\Big\} = K \sim E.
\end{aligned}$$

This implies that $K \sim E$ has interior points (cf. (3.3)), a contradiction. It shows (3.4) and hence the proposition. □

But moreover, for a given $K \in \mathcal{K}^n_{00}(E)$, the dimension of the p-kernel depends on the parameter p. Before stating in a precise way this property, we need the following result, which allows to determine directly the inradius and the p-kernel in a special situation.

Lemma 3.1. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$. If $K = L +_p E$ with $L \in \mathcal{K}^n_0$, $\dim L < \dim(L+E)$, then $\mathrm{r}(K;E) = 1$ and $\ker_p(K;E) = L$.*

Proof. Since $E \subseteq L +_p E = K$, then $\mathrm{r}(K;E) \ge 1$. Moreover, by (1.4) we have that $L +_p E \subseteq L + E$, and since $\dim L < \dim(L+E)$, we get $1 \le \mathrm{r}(K;E) \le \mathrm{r}(L+E;E) = 1$, i.e., $\mathrm{r}(K;E) = 1$. Finally, by (2.7),

$$\ker_p(K;E) = K \sim_p \mathrm{r}(K;E)E = K \sim_p E = (L +_p E) \sim_p E = L. \qquad \square$$

Proposition 3.2. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ and $1 \le p \le q < \infty$. Then*

$$\dim\big(\ker_p(K;E)\big) \le \dim\big(\ker_q(K;E)\big).$$

The inequality may be strict.

Proof. The statement is an immediate consequence of (3.2). The following example shows that the inequality may be strict.

Let $1 \le p < q < \infty$ and we take the convex body $K = [-e_1, e_1] +_q B_n$. Then, Lemma 3.1 ensures that $\ker_q(K;B_n) = K \sim_q B_n = [-e_1, e_1]$, and we claim that $\ker_p(K;B_n) = K \sim_p B_n = \{0\}$, which would show the statement.

Since $K \sim_p B_n \subseteq K \sim_q B_n = [-e_1, e_1]$ (see Lemma 2.3), we suppose, by contradiction, that there exists $\lambda e_1 \in K \sim_p B_n$ with $0 < \lambda \le 1$. It implies that $[0, \lambda e_1] \subseteq K \sim_p B_n$, i.e., $[0, \lambda e_1] +_p B_n \subseteq K$, and then

$$h\big([0,\lambda e_1] +_p B_n, u\big)^p \le h(K,u)^p = h\big([-e_1,e_1] +_q B_n, u\big)^p$$

for all $u \in \mathbb{S}^{n-1}$. In particular, taking

$$u = \Big(\lambda^{p/(q-p)}, \big(1 - \lambda^{2p/(q-p)}\big)^{1/2}, 0, \ldots, 0\Big)^{\intercal} \in \mathbb{S}^{n-1},$$

the above inequality becomes $\lambda^{pq/(q-p)}+1\leq\left(\lambda^{pq/(q-p)}+1\right)^{p/q}$, which is a contradiction because $p<q$ and $\lambda>0$.

Due to the symmetry, the same argument shows that for all $-1\leq\lambda<0$, $\lambda e_1\notin K\sim_p B_n$. Therefore, $K\sim_p B_n=\{0\}$, as claimed. □

4. p-inner parallel bodies

For two convex bodies $K,E\in\mathcal{K}^n$, and a non-negative real number λ the *outer parallel body of K (relative to E) at distance λ* is the Minkowski sum $K+\lambda E$. For $-\mathrm{r}(K;E)\leq\lambda\leq 0$ the *inner parallel body of K (relative to E) at distance $|\lambda|$* is the Minkowski difference $K\sim|\lambda|\,E$. Inner parallel bodies and their properties have been studied in [2, 6, 10], among others.

Joining outer and inner parallel bodies, for a fixed $E\in\mathcal{K}^n$, the full system of relative parallel bodies of K is defined as

$$K_\lambda=\begin{cases} K\sim|\lambda|E & \text{if } -\mathrm{r}(K;E)\leq\lambda\leq 0,\\ K+\lambda E & \text{if } \qquad 0\leq\lambda<\infty.\end{cases}$$

Obviously $K_0=K$ and $K_{-\mathrm{r}(K;E)}=\ker(K;E)$. Moreover, for $K,L\in\mathcal{K}^n$ and arbitrary $\mu\geq-\mathrm{r}(K;E)$, $\sigma\geq-\mathrm{r}(L;E)$, the rule

$$K_\mu+L_\sigma\subseteq(K+L)_{\mu+\sigma} \tag{4.1}$$

is valid (see [11, (3.20)]). As a consequence, a very useful property of the full system of relative parallel bodies of a convex set is obtained: the full system $\mu\mapsto K_\mu$ is concave with respect to inclusion (see [11, Lemma 3.1.13]), i.e.,

$$(1-\lambda)K_\mu+\lambda K_\sigma\subseteq K_{(1-\lambda)\mu+\lambda\sigma}.$$

In this section we define a *full system of p-parallel bodies* of K for $1<p<\infty$ and prove, in the above spirit, several properties of such a system. Since we will work with convex bodies lying in $\mathcal{K}^n_{00}(E)$, the lower bound for the parameters will be always (minus) the classical relative inradius (cf. (3.1)).

Let $E\in\mathcal{K}^n_0$ and $K\in\mathcal{K}^n_{00}(E)$. We define the *full system of p-parallel bodies* of K relative to E, $1\leq p<\infty$, as follows.

Definition 4.1. *Let $K\in\mathcal{K}^n_{00}(E)$. Then, for any $1\leq p<\infty$,*

$$K^p_\lambda=\begin{cases} K\sim_p|\lambda|E & \textit{if } -\mathrm{r}(K;E)\leq\lambda\leq 0,\\ K+_p\lambda E & \textit{if } \qquad 0\leq\lambda<\infty.\end{cases}$$

We will refer to K^p_λ as the p-inner *(respectively,* p-outer*) parallel body of K at distance $|\lambda|$ relative to E.*

4.1. **On the continuity and the concavity of the family of p-parallel bodies.** Next we show that, similarly as in the case $p=1$, the full system $\mu\mapsto K^p_\mu$ is, say, $+_p$-concave, with respect to set inclusion. First we introduce some notation for the p-sum of two real numbers, which will play an important role in the following. Since negative real numbers are allowed, this definition extends (up to a constant) the classical p-mean of positive real numbers (see [7]).

Let $+_p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ denote the binary operation defined by

$$a +_p b = \begin{cases} \operatorname{sgn}_2(a,b)\left(|a|^p + |b|^p\right)^{1/p} & \text{if } ab \geq 0, \\ \operatorname{sgn}_2(a,b)\left(\max\{|a|,|b|\}^p - \min\{|a|,|b|\}^p\right)^{1/p} & \text{if } ab \leq 0, \end{cases} \tag{4.2}$$

being $\operatorname{sgn}_2 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ the function given by

$$\operatorname{sgn}_2(a,b) = \begin{cases} \operatorname{sgn}(a) = \operatorname{sgn}(b) & \text{if } ab > 0, \\ \operatorname{sgn}(a) & \text{if } ab \leq 0 \text{ and } |a| \geq |b|, \\ \operatorname{sgn}(b) & \text{if } ab \leq 0 \text{ and } |a| < |b|, \end{cases}$$

where, as usual, sgn denotes the sign function.

We notice that for $ab \geq 0$, this definition corresponds, up to maybe a signed constant, to the classical p-mean ([7, Chapter II]) and does not correspond to any of the more general ϕ-means considered in [7, Chapter III].

We point out two (easily proved) facts about this operation, which will be used throughout the rest of the section without further special mention:

i) $a +_p b = b +_p a$ for all $a, b \in \mathbb{R}$, i.e., $+_p$ is commutative.
ii) $(a +_p b) +_p c = a +_p (b +_p c) = (a +_p c) +_p b$ for all $a, b, c \in \mathbb{R}$, i.e., $+_p$ is associative.

In the setting of the Brunn-Minkowski-Firey theory, given a convex body $K \in \mathcal{K}_0^n$, a *p-scalar multiplication* is usually defined by

$$\lambda \cdot K = \lambda^{1/p} K \qquad \text{for } \lambda \geq 0$$

(see e.g. [11, p. 490]). We use the analogous notation to the above one in order to define, for $\lambda \geq 0$ and $a \in \mathbb{R}$, the numbers product

$$\lambda \cdot a = \lambda^{1/p} a.$$

With the above notation, the following result on the *p-sum* of (arbitrary) real numbers shows that its behavior fits in the context of convexity.

Lemma 4.1. *Let $a, b \in \mathbb{R}$, $a \leq b$ and $\lambda \in [0,1]$. Then, for all $p \geq 1$,*

$$(1-\lambda) \cdot a +_p \lambda \cdot b \in [a, b].$$

Proof. First, if $ab \geq 0$ then

$$\begin{aligned} (1-\lambda) \cdot a +_p \lambda \cdot b &= \left[(1-\lambda)^{1/p} a\right] +_p \left[\lambda^{1/p} b\right] \\ &= \operatorname{sgn}_2\left((1-\lambda)^{1/p} a, \lambda^{1/p} b\right)\left((1-\lambda)|a|^p + \lambda|b|^p\right)^{1/p} \\ &= \operatorname{sgn}(a)\left((1-\lambda)|a|^p + \lambda|b|^p\right)^{1/p}, \end{aligned}$$

and thus, in both cases $a \geq 0$ and $a \leq 0$, we get, from the above identity,

$$a \leq (1-\lambda) \cdot a +_p \lambda \cdot b \leq b,$$

as required.

So, we assume $ab \leq 0$, i.e., $a \leq 0 \leq b$, and we distinguish two cases.

- If $(1-\lambda)^{1/p}|a| \geq \lambda^{1/p}|b|$, then
$$\operatorname{sgn}_2\left((1-\lambda)^{1/p}a, \lambda^{1/p}b\right) = \operatorname{sgn}\left((1-\lambda)^{1/p}a\right) = -1,$$
and therefore (see (4.2)),
$$\begin{aligned}(1-\lambda)\cdot a +_p \lambda\cdot b &= \operatorname{sgn}_2\left((1-\lambda)^{1/p}a, \lambda^{1/p}b\right)\left((1-\lambda)|a|^p - \lambda|b|^p\right)^{1/p}\\ &= -\left((1-\lambda)|a|^p - \lambda|b|^p\right)^{1/p} \leq 0 \leq b\end{aligned}$$
and
$$(1-\lambda)\cdot a +_p \lambda\cdot b \geq -\left((1-\lambda)|a|^p\right)^{1/p} = -(1-\lambda)^{1/p}|a| \geq -|a| = a.$$
- If $(1-\lambda)^{1/p}|a| \leq \lambda^{1/p}|b|$, then
$$\operatorname{sgn}_2\left((1-\lambda)^{1/p}a, \lambda^{1/p}b\right) = \operatorname{sgn}\left(\lambda^{1/p}b\right) = 1,$$
which yields
$$(1-\lambda)\cdot a +_p \lambda\cdot b = \left(\lambda|b|^p - (1-\lambda)|a|^p\right)^{1/p} \leq \lambda^{1/p}|b| \leq |b| = b.$$
Obviously we also have $(1-\lambda)\cdot a +_p \lambda\cdot b \geq 0 \geq a$. □

The defined p-sum of real numbers (4.2) turns out to be the right operation in order to describe the behavior of the system of p-parallel bodies, as the following proposition shows. The proof is analogous to the proof of (4.1) (see [11, pp. 148–149]), just interchanging the Minkowski sum and difference of convex bodies by the p-sum and p-difference, and the usual sum of real numbers by the p-sum defined in (4.2).

Proposition 4.1. *For $E \in \mathcal{K}^n_0$, let $K, L \in \mathcal{K}^n_{00}(E)$, $-\mathrm{r}(K;E) \leq \mu < \infty$ and $-\mathrm{r}(L;E) \leq \sigma < \infty$. Then, for all $1 \leq p < \infty$, we have*
$$K^p_\mu +_p L^p_\sigma \subseteq (K +_p L)^p_{\mu +_p \sigma}. \tag{4.3}$$

As we already noticed when dealing with the p-difference, its combination with the p-sum is not necessarily commutative if the difference is taken first (cf. Lemma 2.2 and (2.7)). Next result shows how this fact is translated into the setting of p-parallel bodies.

Proposition 4.2. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$, and let $\lambda, \mu \geq 0$. The following relations hold for any $1 \leq p < \infty$:*
- i) $\left(K^p_\lambda\right)^p_\mu = K^p_{\lambda +_p \mu}$.
- ii) $\left(K^p_{-\lambda}\right)^p_\mu \subseteq K^p_{(-\lambda)+_p\mu}$ *if* $\lambda \leq \mathrm{r}(K;E)$.
- iii) $\left(K^p_{-\lambda}\right)^p_{-\mu} = K^p_{(-\lambda)+_p(-\mu)}$ *if* $\lambda^p + \mu^p \leq \mathrm{r}(K;E)^p$.
- iv) $\left(K^p_\lambda\right)^p_{-\mu} = K^p_{\lambda+_p(-\mu)}$ *if* $\mu \leq \mathrm{r}(K;E) +_p \lambda$.
- v) $\lambda K^p_\sigma = \left(\lambda K\right)^p_{\lambda\sigma}$ *for all* $-\mathrm{r}(K;E) \leq \sigma < \infty$.

Proof. For $p = 1$, i.e., for the usual relative parallel bodies, these relations can be found in [6].

Items i), ii) and iii) follow directly from the definition of p-sum, relation (4.3) with $L = \{0\}$ and Lemma 2.2 iii), respectively, taking into account that $\lambda E +_p \mu E = (\lambda +_p \mu)E$.

To prove iv) we notice first that if $\lambda \geq \mu$ then, by i),

$$K^p_{\lambda +_p (-\mu)} +_p \mu E = K^p_{[\lambda +_p (-\mu)] +_p \mu} = K^p_\lambda,$$

and using (2.7) we obtain $K^p_{\lambda +_p (-\mu)} = (K^p_\lambda)^p_{-\mu}$.

Now if $\lambda < \mu$, ii) yields

$$K^p_{\lambda +_p (-\mu)} +_p \mu E \subseteq K^p_{[\lambda +_p (-\mu)] +_p \mu} = K^p_\lambda,$$

and again from (2.7) we deduce that $K^p_{\lambda +_p (-\mu)} \subseteq (K^p_\lambda)^p_{-\mu}$. Moreover, using Lemma 2.2 ii) and (2.7), we obtain

$$\begin{aligned}(K^p_\lambda)^p_{-\mu} +_p |\lambda +_p (-\mu)|E &= (K^p_\lambda \sim_p \mu E) +_p |\lambda +_p (-\mu)|E \\ &\subseteq (K^p_\lambda +_p |\lambda +_p (-\mu)|E) \sim_p \mu E \\ &= K^p_{\lambda +_p |\lambda +_p (-\mu)|} \sim_p \mu E \\ &= K^p_\mu \sim_p \mu E = K,\end{aligned}$$

which shows the opposite inclusion $(K^p_\lambda)^p_{-\mu} \subseteq K^p_{\lambda +_p (-\mu)}$.

Finally, v) is straightforward from the definition of p-sum if $\sigma \geq 0$, and a direct consequence of (2.9) if $\sigma \leq 0$. □

From (4.3) and Proposition 4.2 v) we obtain the following result (cf. [11, Lemma 3.1.13]).

Theorem 4.1. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$. The full system of p-parallel sets of K relative to E, $1 \leq p < \infty$, is $+_p$-concave with respect to inclusion, i.e., for $\lambda \in [0, 1]$ and $\mu, \sigma \in \big[-\mathrm{r}(K; E), \infty\big)$,*

$$(1 - \lambda) \cdot K^p_\mu +_p \lambda \cdot K^p_\sigma \subseteq K^p_{(1-\lambda)\cdot\mu +_p \lambda\cdot\sigma}.$$

Proof. We notice that, by Lemma 4.1, $(1 - \lambda) \cdot \mu +_p \lambda \cdot \sigma \geq -\mathrm{r}(K; E)$. Then

$$\begin{aligned}(1 - \lambda) \cdot K^p_\mu +_p \lambda \cdot K^p_\sigma &= \big((1 - \lambda)^{1/p} K^p_\mu\big) +_p \big(\lambda^{1/p} K^p_\sigma\big) \\ &= \big[(1 - \lambda)^{1/p} K\big]^p_{(1-\lambda)^{1/p}\mu} +_p \big[\lambda^{1/p} K\big]^p_{\lambda^{1/p}\sigma} \\ &\subseteq K^p_{(1-\lambda)\cdot\mu +_p \lambda\cdot\sigma}.\end{aligned}$$

□

Next we show that the full system of p-parallel bodies is continuous in the parameter λ with respect to the Hausdorff metric (cf. Proposition 2.2).

Proposition 4.3. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ and let $1 \leq p < \infty$. Then the function $\Phi : \big[-\mathrm{r}(K; E), \infty\big) \longrightarrow \mathcal{K}^n_0$ given by $\Phi(\lambda) = K^p_\lambda$ is continuous with respect to the Hausdorff metric in $\mathcal{K}^n_0$.*

Proof. Let $\{\lambda_i\}_{i=1}^{\infty} \subseteq \big[-\mathrm{r}(K;E),\infty\big)$ be such that $\lim_{i\to\infty}\lambda_i=\lambda$. We have to prove that $\lim_{i\to\infty}\Phi(\lambda_i)=\Phi(\lambda)$. We notice first that

$$\Phi(\lambda_i)=K^p_{\lambda_i}=\mathrm{WS}(\varphi_{\lambda_i}),$$
$$\Phi(\lambda)=K^p_{\lambda}=\mathrm{WS}(\varphi_{\lambda}),$$

where $\varphi_\mu:\mathbb{S}^{n-1}\longrightarrow[0,\infty)$ is the (continuous) function given by

$$\varphi_\mu(u)=\Big(h(K,u)^p+\mathrm{sgn}(\mu)|\mu|^p h(E,u)^p\Big)^{1/p}.$$

From the continuity of the functions $\varphi_{\lambda_i},\varphi_\lambda$ and the compactness of $\mathbb{S}^{n-1}$ we deduce that there exist constants $M_{\lambda_i},M_\lambda\geq 0$, $i\in\mathbb{N}$, such that

$$\varphi_{\lambda_i}\big(\mathbb{S}^{n-1}\big)=[0,M_{\lambda_i}],\quad \varphi_{\lambda}\big(\mathbb{S}^{n-1}\big)=[0,M_{\lambda}].$$

Since $\lim_{i\to\infty}\lambda_i=\lambda$, the sequence $\{\lambda_i\}_{i=1}^{\infty}$ is bounded, and then there exists a constant $M>0$ such that $M\geq M_\lambda$ and $M\geq M_{\lambda_i}$, $i\in\mathbb{N}$.

If $\lambda\neq 0$, then $\mathrm{sgn}(\lambda_i)=\mathrm{sgn}(\lambda)$ for i large enough, whereas if $\lambda=0$, then $\varphi^p_{\lambda_i}-\varphi^p_0=\mathrm{sgn}(\lambda_i)|\lambda_i|^p h(E,\cdot)^p$. Therefore we have, in both cases, that

$$\big\|\varphi^p_{\lambda_i}-\varphi^p_\lambda\big\|_\infty=\Big\|\mathrm{sgn}(\lambda_i)\big(|\lambda_i|^p-|\lambda|^p\big)h(E,\cdot)^p\Big\|_\infty=\Big||\lambda_i|^p-|\lambda|^p\Big|\big\|h(E,\cdot)^p\big\|_\infty,$$

for i large enough, and thus, $\lim_{i\to\infty}\big\|\varphi^p_{\lambda_i}-\varphi^p_\lambda\big\|_\infty=0$.

Since the function $[0,M]\longrightarrow\mathbb{R}$ given by $t\mapsto t^{1/p}$, is uniformly continuous, then $\lim_{i\to\infty}\|\varphi_{\lambda_i}-\varphi_\lambda\|_\infty=0$. Now, [11, Lemma 7.5.2] for $\Omega=\mathbb{S}^{n-1}$ implies that $\lim_{i\to\infty}\mathrm{WS}(\varphi_{\lambda_i})=\mathrm{WS}(\varphi_\lambda)$, as desired. □

4.2. ***p*-inner parallel bodies for special families of sets.** As it occurs when dealing with the p- and the Minkowski sums, for which the first one happens to be more difficult to *visualize*, the p-difference is, in general, also more difficult to deal with than the Minkowski difference. However, there are particular cases in which the p-difference is easy to determine. In this subsection we deal with special families of convex bodies, for which p-parallel bodies can be explicitly determined.

Tangential bodies can be defined in several equivalent ways; here we will use the following one: a convex body $K\in\mathcal{K}^n$ containing a convex body $E\in\mathcal{K}^n$, is called a *tangential body* of E, if through each boundary point of K there exists a support hyperplane to K that also supports E. We notice that if K is a tangential body of E, then $\mathrm{r}(K;E)=1$. The n-dimensional cube is an example of this type of bodies for $E=B_n$. For an exhaustive study of the more general defined p-tangential bodies we refer to [11, Section 2.2 and p. 149].

The p-inner parallel bodies of a tangential body can be easily obtained (see Figure 3, cf. (2.8)).

Proposition 4.4. *Let $E\in\mathcal{K}^n_0$ and let $K\in\mathcal{K}^n_0$ be a tangential body of E. Then, for all $1\leq p<\infty$ and any $\lambda\in[0,1]$,*

$$K^p_{-\lambda}=(1-\lambda^p)^{1/p}K. \tag{4.4}$$

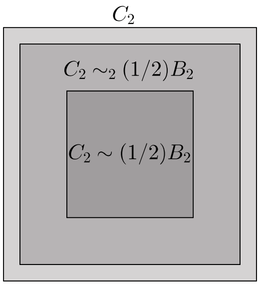

FIGURE 3. 1 and 2-difference of the square C_2 and the ball $(1/2)B_2$.

Proof. Let $\mathcal{U} \subseteq \mathbb{S}^{n-1}$ be the set of those outer normal vectors for which the support hyperplane to K also supports E. Clearly, $h(K,u) = h(E,u)$ for all $u \in \mathcal{U}$ and

$$K = \bigcap_{u \in \mathcal{U}} \big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K,u)\big\}.$$

Therefore we get, on the one hand,

$$K^p_{-\lambda} = K \sim_p \lambda E = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq \big(h(K,u)^p - \lambda^p h(E,u)^p\big)^{1/p}\Big\}$$
$$\subseteq \bigcap_{u \in \mathcal{U}} \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq (1-\lambda^p)^{1/p} h(K,u)\Big\} = (1-\lambda^p)^{1/p} K.$$

On the other hand, since $E \subseteq K$, then

$$h\big((1-\lambda^p)^{1/p}K, u\big)^p = (1-\lambda^p)h(K,u)^p \leq h(K,u)^p - \lambda^p h(E,u)^p$$

for all $u \in \mathbb{S}^{n-1}$, and hence

$$K^p_{-\lambda} = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq \big(h(K,u)^p - \lambda^p h(E,u)^p\big)^{1/p}\Big\}$$
$$\supseteq \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h\big((1-\lambda^p)^{1/p}K, u\big)\Big\} = (1-\lambda^p)^{1/p} K.$$

It concludes the proof. □

Moreover, tangential bodies can be characterized by (4.4), i.e., as the only convex bodies such that their p-inner parallel bodies are homothetic copies of them (see Figure 3). The case $p = 1$ was proved by Schneider, see [11, Lemma 3.1.14]. In order to prove it we need the following auxiliary result, which shows that p-inner parallel bodies are strongly related to the classical inner ones when dealing with dilations. For the sake of brevity we will assume that $\mathrm{r}(K;E) = 1$.

Proposition 4.5. *Let $K, E \in \mathcal{K}^n_0$ with $E \subseteq K$ and $\mathrm{r}(K;E) = 1$. Let $1 \leq p < \infty$ and $\lambda \in [0,1]$. If $K^p_{-\lambda} = \theta K$ for some $\theta \in [0,1]$, then $\theta = (1-\lambda^p)^{1/p}$ and $K_{-(1-\theta)} = \theta K$.*

Proof. First we prove that

$$\text{if } K^p_{-\lambda} = \theta K \text{ for } 0 \leq \lambda \leq 1, \text{ then } \theta = (1-\lambda^p)^{1/p}. \tag{4.5}$$

Indeed, since $(1-\lambda^p)^{1/p}K +_p \lambda E \subseteq (1-\lambda^p)^{1/p}K +_p \lambda K = K$, then we get $(1-\lambda^p)^{1/p}K \subseteq K \sim_p \lambda E = \theta K$, which yields $\theta \geq (1-\lambda^p)^{1/p}$. Moreover, since $\mathrm{r}(K;E) = 1$, there exists $u \in \mathbb{S}^{n-1}$ such that $h(K,u) = h(E,u) > 0$ (see [3, p. 59]). Therefore,

$$\begin{aligned}\theta h(K,u) = h(\theta K,u) = h(K \sim_p \lambda E, u) &\leq \bigl(h(K,u)^p - \lambda^p h(E,u)^p\bigr)^{1/p} \\ &= (1-\lambda^p)^{1/p} h(K,u),\end{aligned}$$

and since $h(K,u) > 0$, we get $\theta \leq (1-\lambda^p)^{1/p}$, which shows (4.5).

Now we prove the proposition.

First we observe that $\theta K + (1-\theta)E \subseteq \theta K + (1-\theta)K = K$, which yields

$$\theta K \subseteq K \sim (1-\theta)E = K_{-(1-\theta)},$$

and we assume, by contradiction, that there exists $x \in (K \sim (1-\theta)E)\backslash\theta K$. In particular, $x \notin \theta K = K^p_{-\lambda}$, and so (cf. (2.5)) there is $u_x \in \mathbb{S}^{n-1}$ such that

$$\langle x, u_x\rangle > \bigl(h(K,u_x)^p - \lambda^p h(E,u_x)^p\bigr)^{1/p}. \tag{4.6}$$

Moreover, since $x + (1-\theta)E \subseteq K$, taking support functions we get

$$\langle x, u_x\rangle + (1-\theta)h(E,u_x) \leq h(K,u_x), \tag{4.7}$$

and joining both inequalities (4.6) and (4.7), we obtain

$$\bigl(h(K,u_x)^p - \lambda^p h(E,u_x)^p\bigr)^{1/p} < h(K,u_x) - (1-\theta)h(E,u_x). \tag{4.8}$$

We notice that $h(K,u_x) > 0$ (cf. (4.6), since $h(K,u_x) \geq \langle x,u_x\rangle > 0$), and thus, writing $\alpha = h(E,u_x)/h(K,u_x) \in [0,1]$, inequality (4.8) becomes

$$(1-\lambda^p\alpha^p)^{1/p} < 1 - (1-\theta)\alpha. \tag{4.9}$$

In order to get the contradiction, let $f(\alpha) = (1-\lambda^p\alpha^p)^{1/p}$ defined on $[0,1]$. It can be checked that $f''(\alpha) \leq 0$, i.e., f is a concave function, with $f(0) = 1$ and $f(1) = (1-\lambda^p)^{1/p} = \theta$ (cf. (4.5)), which implies that $f(\alpha) \geq 1-(1-\theta)\alpha$ for all $\alpha \in [0,1]$. It contradicts (4.9), and shows the result. □

Remark 4.1. *Proposition 4.5 says that there is a bijection between p-inner parallel bodies and the inner parallel bodies of K, when they all are homothetic to K, given by*

$$K^p_{-\lambda} \longleftrightarrow K_{-1+(1-\lambda^p)^{1/p}}.$$

Theorem 4.2. *Let $K, E \in \mathcal{K}^n_0$, $\operatorname{int} E \neq \emptyset$, with $E \subseteq K$ and $\mathrm{r}(K;E) = 1$. Let $1 \leq p < \infty$ and $\lambda \in [0,1]$. Then K is a tangential body of E if and only if $K^p_{-\lambda}$ is homothetic to K.*

Proof. If K is a tangential body of E, then $K^p_{-\lambda} = (1-\lambda^p)^{1/p}K$ is homothetic to K (see Proposition 4.4). Conversely, if $K^p_{-\lambda} = \theta K$ for some $\theta \in [0,1]$, then by Proposition 4.5 we get $K_{-(1-\theta)} = \theta K$ with $\theta = (1-\lambda^p)^{1/p}$. Schneider's result [11, Lemma 3.1.14] shows that K is a tangential body of E. □

Other convex bodies for which their p-inner parallel bodies can be easily determined are those which are obtained as p-outer parallel bodies of a lower dimensional set.

Proposition 4.6. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ be given by $K = L +_p \mu E$, with $L \in \mathcal{K}^n_0$, $\dim L < \dim(L+E)$, and $\mu \geq 0$. Then*

$$K^p_\lambda = L +_p (\mu +_p \lambda)E,$$

for all $\lambda \in \big[-\mu, \infty\big)$.

Proof. For $\lambda \geq 0$ the result follows directly from the definitions of p-sums of convex bodies and numbers.

If $-\mu \leq \lambda \leq 0$ and since $\mathrm{r}(L;E) = 0$, we can use Proposition 4.2 iv) in order to obtain that

$$K^p_\lambda = K \sim_p |\lambda|E = (L +_p \mu E) \sim_p |\lambda|E = L +_p (\mu +_p \lambda)E. \qquad \square$$

We notice moreover that in this case $\ker_p(K;E) = L$ and $\mathrm{r}(K;E) = \mu$ (see Lemma 3.1). Besides, if we remove the assumption $\dim L < \dim(L+E)$, then the result also holds, but in the appropriate range of λ.

Proposition 4.6 indicates that for some convex bodies, the whole family of p-parallel bodies is made of only *p-outer parallel bodies*. We ask again whether it is possible to characterize the convex bodies satisfying such a property, i.e., whether a converse for the proposition is also true. This question will be answered in Theorem 4.3.

When $p = 1$, this fact was studied by Sangwine-Yager, who introduced, for a given convex body K and the particular case $E = B_n$, the set

$$S = \Big\{\tau \in \big[-\mathrm{r}(K;B_n), \infty\big) : K_\tau + (\lambda - \tau)B_n = K_\lambda, \text{ for all } \lambda \geq \tau\Big\},$$

proving in [10, Lemma 1.5] that it is always a left-hand closed interval, namely, that there exists $\sigma \in \big[-\mathrm{r}(K;B_n), 0\big]$ such that $S = [\sigma, \infty)$.

The natural extension of S to the case $p \geq 1$ is considered in the next. For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ and let $1 \leq p < \infty$. We define the set

$$S_p = \Big\{\tau \in \big[-\mathrm{r}(K;E), \infty\big) : K^p_\tau +_p \big(\lambda +_p (-\tau)\big)E = K^p_\lambda, \text{ for all } \lambda \geq \tau\Big\}.$$

Although S_p strongly depends on the convex bodies K, E, we shall write just S_p for short. We observe that when $E = B_n$ and $p = 1$ we recover S. We also notice that, in general, $S_p \neq S_q$ if $p \neq q$ and thus, in particular, $S_p \neq S$ for all $p > 1$. Indeed, for $K = [-e_1, e_1] +_q B_n$, Proposition 4.6 yields $S_q = [-1, \infty)$; however, for $p < q$, since $\ker_p(K;B_n) = \{0\}$ (cf. proof of Proposition 3.2), if $S_p = S_q$ it would be $K = \ker_p(K;B_n) +_p B_n = B_n$, which is not possible.

The following result shows, as it happened in Sangwine-Yager's result ([10, Lemma 1.5]), that S_p is also a left-hand closed interval for all $p\geq 1$.

Lemma 4.2. *For $E\in\mathcal{K}^n_0$, let $K\in\mathcal{K}^n_{00}(E)$ and let $1\leq p<\infty$. Then there exists $\sigma\in\big[-\mathrm{r}(K;E),0\big]$ such that $S_p=[\sigma,\infty)$.*

Proof. First, we prove that $[0,\infty)\subseteq S_p$. Clearly

$$K^p_0+_p\big(\lambda+_p(-0)\big)E=K+_p\lambda E=K^p_\lambda\quad\text{ for all }\lambda\geq 0,$$

and so $0\in S_p$. Now let $\tau>0$ and $\lambda\geq\tau$. Since $\lambda+_p(-\tau)=(\lambda^p-\tau^p)^{1/p}>0$, Proposition 4.2 i) ensures that

$$K^p_\tau+\big(\lambda+_p(-\tau)\big)E=K^p_{\tau+_p(\lambda+_p(-\tau))}=K^p_\lambda.$$

Hence, $\tau\in S_p$ for all $\tau\geq 0$ and thus, $[0,\infty)\subseteq S_p$, as required.

Next we see that if $\tau\in S_p$, $\tau<0$, then $[\tau,0)\subseteq S_p$. Indeed, for such a value $\tau\in S_p$, let $\mu\in(\tau,0)$ and $\lambda\geq\mu$. Clearly, both numbers

$$\mu+_p(-\tau)=\big(|-\tau|^p-|\mu|^p\big)^{1/p}>0\quad\text{ and}$$
$$\lambda+_p(-\mu)=\begin{cases}\big(\lambda^p+|-\mu|^p\big)^{1/p} & \geq 0\quad\text{ if }\lambda\geq 0,\\ \big(|-\mu|^p-|\lambda|^p\big)^{1/p} & \geq 0\quad\text{ if }\lambda<0,\end{cases}$$

and thus, since $\tau\in S_p$ and $\lambda\geq\mu>\tau$, we get, from Proposition 4.2 i) and the commutativity and associativity of the $+_p$ operation, that

$$\begin{aligned}K^p_\mu+_p\big(\lambda+_p(-\mu)\big)E&=\Big[K^p_\tau+_p\big(\mu+_p(-\tau)\big)E\Big]+_p\big(\lambda+_p(-\mu)\big)E\\&=K^p_\tau+_p\Big[\big(\mu+_p(-\tau)\big)+_p\big(\lambda+_p(-\mu)\big)\Big]E\\&=K^p_\tau+_p\big(\lambda+_p(-\tau)\big)E=K^p_\lambda\end{aligned}$$

for all $\lambda\geq\mu$. Hence, $\mu\in S_p$.

At this point, we have shown that if $\tau\in S_p$ then $[\tau,\infty)\subseteq S_p$. Finally, let $\sigma=\inf S_p$, which clearly satisfies $-\mathrm{r}(K;E)\leq\sigma\leq 0$. We have to prove that $\sigma\in S_p$. For $\lambda>\sigma$ let $\{\tau_i\}_{i=1}^\infty\subseteq S_p$ be a decreasing sequence with $\lim_{i\to\infty}\tau_i=\sigma$ and $\tau_1\leq\lambda$. Since $\tau_i\in S_p$ for all $i\in\mathbb{N}$ we have

$$K^p_{\tau_i}+_p\big(\lambda+_p(-\tau_i)\big)E=K^p_\lambda,$$

and taking limits when $i\to\infty$, the continuity of the full system of p-parallel bodies (Proposition 4.3) ensures that

$$K^p_\sigma+_p\big(\lambda+_p(-\sigma)\big)E=K^p_\lambda$$

for all $\lambda\geq\sigma$, i.e., $\sigma\in S_p$. It concludes the proof. □

Lemma 4.2 and Proposition 4.6 allow to determine the convex bodies for which S_p is maximal.

Theorem 4.3. *For $E\in\mathcal{K}^n_0$ let $K\in\mathcal{K}^n_{00}(E)$, and let $1\leq p<\infty$. Then $K=\ker_p(K;E)+_p\mathrm{r}(K;E)E$ if and only if $S_p=\big[-\mathrm{r}(K;E),\infty\big)$.*

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