REFINEMENTS OF THE BRUNN-MINKOWSKI INEQUALITY

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Abstract. Brunn-Minkowski theorem says that $\mathrm{vol}\bigl((1-\lambda)K+\lambda L\bigr)^{1/n}$, for K, L convex bodies, is a concave function in λ, and assuming a common hyperplane projection of K and L, it was proved that the volume itself is concave. In this paper we study refinements of Brunn-Minkowski inequality, in the sense of 'enhancing' the exponent, either when a common projection onto an $(n-k)$-plane is assumed or for particular families of sets. In the first case, we show that the expected result of concavity for the k-th root of the volume is not true, although other Brunn-Minkowski type inequalities can be obtained under the $(n-k)$-projection hypothesis. In the second case, we show that for p-tangential bodies, the exponent in Brunn-Minkowski inequality can be replaced by $1/p$.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$, and we denote by e_i the i-th canonical unit vector. The subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior is denoted by $\mathcal{K}^n_0$, and we write B_n for the n-dimensional Euclidean unit ball. The n-dimensional volume of a set $M \subsetneq \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (or $\mathrm{vol}_n(M)$ if the distinction of the dimension is useful) and, in particular, we write $\kappa_n = \mathrm{vol}(B_n)$, which takes the value

$$\kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}, \tag{1.1}$$

where Γ denotes the gamma function. With $\mathrm{int}\, M$ and $\mathrm{lin}\, M$ we represent its interior and linear hull, respectively. Finally, the set of all k-dimensional (linear) planes of $\mathbb{R}^n$ is denoted by $\mathcal{L}^n_k$, and for $H \in \mathcal{L}^n_k$, $K \in \mathcal{K}^n$, the orthogonal projection of K onto H is denoted by $K|H$ and with $H^\perp \in \mathcal{L}^n_{n-k}$ we represent the orthogonal complement of H.

Relating the volume with the Minkowski (vectorial) addition of convex bodies, one is led to the famous Brunn-Minkowski inequality. One form of

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it states that if $K, L \in \mathcal{K}^n$ and $0 \leq \lambda \leq 1$, then

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}, \tag{1.2}$$

i.e., the n-th root of the volume is a concave function. Equality for some $\lambda \in (0,1)$ holds if and only if K and L either lie in parallel hyperplanes or are homothetic.

Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond: for instance, its equivalent analytic version (Prékopa-Leindler inequality, see e.g. [8, Theorem 8.14]) and the fact that the convexity/compactness assumption can be 'weakened' to consider just Lebesgue measurable sets (see [12]), have allowed it to move in much wider fields. It implies very important inequalities as the isoperimetric and Urysohn inequalities (see e.g. [19, p. 318]) or even the Aleksandrov-Fenchel inequality (see e.g. [19, s. 6.3]), and it has been the starting point for new developments like the so called L_p-Brunn-Minkowski theory (see e.g. [13, 14]), a Brunn-Minkowski result for integer lattices (see [6]), or a reverse Brunn-Minkowski inequality (see e.g. [15]), among many others. It would not be possible to collect here all references regarding versions, applications and/or generalizations on Brunn-Minkowski inequality. So, for extensive and beautiful surveys on them we refer to [2, 5].

In [3, s. 50], linear refinements of the Brunn-Minkowski inequality are obtained for convex bodies having a common/equal volume hyperplane projection (see also [16] for compact sets and more recently [7, ss. 1.2.4]).

Theorem A ([3, 7]). *Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $K|H = L|H$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}(K) + \lambda \mathrm{vol}(L). \tag{1.3}$$

This is, the volume itself is a concave function.

Theorem B ([3, 7, 16]). *Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(L|H)$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}(K) + \lambda \mathrm{vol}(L).$$

The volume of the sum of two convex bodies $K, \lambda E \in \mathcal{K}^n$, $\lambda \geq 0$, has also a precise expression as a polynomial, namely

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^i, \tag{1.4}$$

known as the (relative) *Steiner formula* of K. The coefficients $\mathrm{W}_i(K;E)$ are the *relative quermassintegrals* of K, and they are a special case of the more general defined *mixed volumes* for which we refer to [19, s. 5.1]. In particular, we have $\mathrm{W}_0(K;E) = \mathrm{vol}(K)$, $\mathrm{W}_n(K;E) = \mathrm{vol}(E)$ and $\mathrm{W}_i(K;E) = \mathrm{W}_{n-i}(E;K)$. If $E = B_n$, (1.4) becomes the classical Steiner formula [21], and $\mathrm{W}_i(K;B_n)$, for short denoted by $\mathrm{W}_i(K)$, is the classical i-th quermass-integral of K. If the dimension $\dim K=2$, then $\mathrm{vol}(K)=\mathrm{A}(K)$ is the usual *area* of K and $2\mathrm{W}_1(K)=\mathrm{p}(K)$ is its *perimeter*.

Brunn-Minkowski inequality has a more general version for quermassintegrals: if $K,L,E\in\mathcal{K}^n$ and $0\leq\lambda\leq 1$, then, for all $i=0,\dots,n-2$,

$$\mathrm{W}_i\big((1-\lambda)K+\lambda L;E\big)^{1/(n-i)}\geq(1-\lambda)\mathrm{W}_i(K;E)^{1/(n-i)}+\lambda\mathrm{W}_i(L;E)^{1/(n-i)}, \tag{1.5}$$

whereas $\mathrm{W}_{n-1}\big((1-\lambda)K+\lambda L;E\big)=(1-\lambda)\mathrm{W}_{n-1}(K;E)+\lambda\mathrm{W}_{n-1}(L;E)$; in fact, there exist the most general version of Brunn-Minkowski inequality for mixed volumes (see [19, Theorem 6.4.3]).

Regarding Theorem A, Schneider proved in a very elegant way that even the most general Brunn-Minkowski inequality for mixed volumes (and thus, in particular, the Brunn-Minkowski inequality for quermassintegrals (1.5)) admits an improved version of this type, unifying different results in the literature about this topic ([18], see also [19, s. 6.7]): if $K,L\in\mathcal{K}^n$ are convex bodies such that there exists a hyperplane $H\in\mathcal{L}^n_{n-1}$ with $K|H=L|H$, then any mixed volume (quermassintegrals, volume) itself of the convex combination $(1-\lambda)K+\lambda L$ is a concave function in $\lambda\in[0,1]$.

At this point it is a natural question whether an analogous result to Theorem A, but with the suitable exponent, can be obtained if a *common projection onto an $(n-k)$-dimensional plane* is assumed. Thus, the following property would be a natural expected solution:

(1.6) *Let $k\in\{1,\dots,n\}$ and let $K,L\in\mathcal{K}^n$ be convex bodies such that there exists $H\in\mathcal{L}^n_{n-k}$ with $K|H=L|H$. Then for all $\lambda\in[0,1]$ it holds*

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/k}\geq(1-\lambda)\mathrm{vol}(K)^{1/k}+\lambda\mathrm{vol}(L)^{1/k}.$$

In this paper we show that this statement is not true. More precisely, we prove the following theorem:

Theorem 1.1. *For every $n\geq 3$, there exist convex bodies $K,L\in\mathcal{K}^n$, with a common $(n-2)$-dimensional projection $K|H=L|H$, $H\in\mathcal{L}^n_{n-2}$, such that, for all $\lambda\in(0,1)$,*

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/2}<(1-\lambda)\mathrm{vol}(K)^{1/2}+\lambda\mathrm{vol}(L)^{1/2}. \tag{1.7}$$

Therefore, either additional assumptions should be imposed in order to get (1.6) or, under that precise hypothesis, a different inequality can be obtained. In this sense, we get the following results.

Proposition 1.1. *Let $k\in\{1,\dots,n\}$ and let $K,L\in\mathcal{K}^n$ be convex bodies such that there exists $H\in\mathcal{L}^n_{n-k}$ with $K|H=L|H$. Then, for all $\lambda\in[0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)\geq(1-\lambda)^k\mathrm{vol}(K)+\lambda^k\mathrm{vol}(L). \tag{1.8}$$

As in the case of Theorems A and B, the same inequality (1.8) can be obtained when a condition on the volume of the projection is assumed.

Proposition 1.2. *Let $k \in \{1,\dots,n\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}^n_{n-k}$ with $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)^k \mathrm{vol}(K) + \lambda^k \mathrm{vol}(L).$$

An analogous result to the above one can be obtained if we replace the projection condition by a suitable section hypothesis (see Proposition 3.1).

We observe that the above relation (1.8) has inequality (1.3) as a particular case; however, Brunn-Minkowski inequality cannot be obtained from it (see Remark 3.1). Next theorem provides an extension of both inequalities (1.8) and (1.2) (see Remark 3.2). In order to state the result, we need the following additional notation, which will be used throughout all the paper: given $K \in \mathcal{K}^n$ and $H \in \mathcal{L}^n_{n-k}$, we will write, for any $u \in K|H$,

$$K(u) = \big\{x \in \mathbb{R}^k : (x,u) \in K\big\}. \tag{1.9}$$

Theorem 1.2. *Let $k \in \{1,\dots,n\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}^n_{n-k}$ with $K|H = L|H = U$. Then, for all $\lambda \in [0,1]$,*

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/k} \geq (1-\lambda) &\int_U \left(\frac{\mathrm{vol}_k(K(u))}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k} \mathrm{d}u \\ &+ \lambda \int_U \left(\frac{\mathrm{vol}_k(L(u))}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k} \mathrm{d}u.\end{aligned} \tag{1.10}$$

In Section 2 we give the counterexample to statement (1.6) by showing Theorem 1.1, whereas Section 3 is devoted to prove Propositions 1.1, 1.2 and Theorem 1.2. There, we also show some related results for (relative) quermassintegrals (Propositions 2.1 and 3.2).

Next, we wonder whether refinements of Brunn-Minkowski inequality of type (1.6) can be obtained for particular families of sets or under additional assumptions. In Section 4 we deal with this question, and show, among others, that it has a positive answer for the family of the so called p-tangential bodies (see Section 4 for the definition). In this case, also a refinement of the more general Brunn-Minkowski inequality for quermassintegrals (1.5) can be achieved.

Theorem 1.3. *Let K be a p-tangential body of $E \in \mathcal{K}^n_0$, $1 \leq p \leq n-1$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda E\big)^{1/p} \geq (1-\lambda)\mathrm{vol}(K)^{1/p} + \lambda \mathrm{vol}(E)^{1/p},$$

and equality holds if and only if $K = E$. Moreover,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda E; K\big)^{1/(p-i)} \geq (1-\lambda)\mathrm{W}_i(K;K)^{1/(p-i)} + \lambda \mathrm{W}_i(E;K)^{1/(p-i)},$$

$i = 0,\dots,p-1$, and equality holds for some fixed i, if and only if K is also an i-tangential body of E.

2. The counterexample

We start by showing a preliminary result which will be needed in the proof of Theorem 1.1.

Lemma 2.1. *The sequence* $\left(\kappa_n\kappa_{n-2}/\kappa_{n-1}^2\right)_{n\geq 2}$ *is strictly increasing and* $\lim_{n\to\infty}\kappa_n\kappa_{n-2}/\kappa_{n-1}^2 = 1$.

Proof. On the one hand, we consider the real functions $f_i:(0,\infty)\longrightarrow\mathbb{R}$, $i=1,2$, given by $f_1(x)=(x-1/2)\log x$ and $f_2(x)=\theta/(12x)$ for (any) fixed $0<\theta<1$ which will be suitably chosen later on. From the concavity of their first derivatives we get

$$2f_i'\left(x+\frac{1}{2}\right)-f_i'(x)-f_i'(x+1)=2\left[f_i'\left(x+\frac{1}{2}\right)-\frac{f_i'(x)+f_i'(x+1)}{2}\right]>0,$$

and hence, the real functions $h_i:(0,\infty)\longrightarrow\mathbb{R}$, $i=1,2$, given by

$$h_i(x)=2f_i\left(x+\frac{1}{2}\right)-f_i(x)-f_i(x+1)$$

are strictly increasing. Therefore, $e^{h_1(x)+h_2(x)}$ is also strictly increasing.

On the other hand, Stirling's formula for the gamma function $\Gamma(x)$ (see e.g. [1, p.24]) allows us to write

$$\frac{\Gamma\left(x+\frac{1}{2}\right)^2}{\Gamma(x)\Gamma(x+1)}=e^{h_1(x)+h_2(x)}$$

for a suitable $\theta\in(0,1)$ (see [1, (3.9)]). Thus, all together, and using (1.1), we can conclude that

$$\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2}=\frac{\Gamma\left(\frac{n-1}{2}+1\right)^2}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n-2}{2}+1\right)}=e^{h_1\left(\frac{n}{2}\right)+h_2\left(\frac{n}{2}\right)}$$

is strictly increasing in n. The last assertion comes from the fact that $\lim_{n\to\infty}(\kappa_{n-k}/\kappa_n)/(\kappa_{n-1}/\kappa_n)^k=1$ for all $k\geq 0$ (see [10, Lemma 3.1]); in particular, for $k=2$ we get the required result. □

In order to prove Theorem 1.1, we explicitly construct the convex bodies providing a counterexample for statement (1.6).

Proof of Theorem 1.1. Let $g(\lambda)=\mathrm{vol}\big((1-\lambda)K+\lambda L\big)$ and $f(\lambda)=g(\lambda)^{1/2}$. On the one hand, we observe that the reverse inequality to (1.7), namely,

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/2}\geq(1-\lambda)\mathrm{vol}(K)^{1/2}+\lambda\mathrm{vol}(L)^{1/2}$$

for convex bodies $K,L\in\mathcal{K}^n$ having a common $(n-2)$-dimensional projection, $K|H=L|H$, holds if and only if $f(\lambda)$ is a concave function on $[0,1]$. Indeed, since $K|H=L|H$, then $(1-\lambda_1)K|H+\lambda_1L|H=(1-\lambda_2)K|H+\lambda_2L|H$ for any $\lambda_1,\lambda_2\in[0,1]$, and thus the above inequality can be applied to the convex bodies $(1-\lambda_1)K+\lambda_1L,(1-\lambda_2)K+\lambda_2L$ in order to get the inequality $f\big((1-t)\lambda_1+t\lambda_2\big)\geq(1-t)f(\lambda_1)+tf(\lambda_2)$. Conversely, if f is a concave function on $[0,1]$ then we have, in particular, that $f(t)\geq(1-t)f(0)+tf(1)$, which gives the required inequality for the volume.

On the other hand, $f(\lambda)$ is concave if and only if

$$f''(\lambda)=\frac{1}{2}g(\lambda)^{-3/2}\left[g(\lambda)g''(\lambda)-\frac{1}{2}g'(\lambda)^2\right]\leq 0,$$

i.e., if and only if $F(\lambda)=g(\lambda)g''(\lambda)-(1/2)g'(\lambda)^2\leq 0$. Therefore, if we find two convex bodies $K,L\in\mathcal{K}^n$, having a common $(n-2)$-dimensional projection, and verifying that $F(\lambda)>0$ for all $\lambda\in[0,1]$, then inequality (1.7) will hold for all $\lambda\in(0,1)$.

Let $L=B_n$ and $K=M+B_n$, with $M\in\mathcal{K}_0^2$ lying in a 2-dimensional linear plane. On the one hand, it is clear that if $H=(\operatorname{lin}M)^{\perp}\in\mathcal{L}_{n-2}^n$ is the orthogonal complement of $\operatorname{lin}M$, then $K|H=B_n|H$.

On the other hand, Steiner formula (1.4) allows us to write

$$\begin{aligned}g(\lambda)&=\operatorname{vol}\bigl((1-\lambda)(M+B_n)+\lambda B_n\bigr)=\operatorname{vol}\bigl((1-\lambda)M+B_n\bigr)\\&=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(M)(1-\lambda)^{n-i},\end{aligned}$$

and since $\dim M=2$, the quermassintegrals $\mathrm{W}_i(M)$ take the values

$$\begin{aligned}&\mathrm{W}_i(M)=0,\quad i=0,\dots,n-3,\\&\mathrm{W}_{n-2}(M)=\frac{2\kappa_{n-2}}{n(n-1)}\mathrm{A}(M),\qquad \mathrm{W}_{n-1}(M)=\frac{\kappa_{n-1}}{2n}\mathrm{p}(M)\end{aligned}$$

(see e.g. [17, Property 3.1]), and therefore,

$$\begin{aligned}g(\lambda)&=\frac{n(n-1)}{2}\mathrm{W}_{n-2}(M)(1-\lambda)^2+n\mathrm{W}_{n-1}(M)(1-\lambda)+\kappa_n\\&=\kappa_{n-2}\mathrm{A}(M)(1-\lambda)^2+\frac{\kappa_{n-1}}{2}\mathrm{p}(M)(1-\lambda)+\kappa_n.\end{aligned}$$

Thus, it is an easy computation to check that

$$F(\lambda)=g(\lambda)g''(\lambda)-\frac{g'(\lambda)^2}{2}=\frac{1}{8}\Bigl[16\kappa_n\kappa_{n-2}\mathrm{A}(M)-\kappa_{n-1}^2\mathrm{p}(M)^2\Bigr],$$

and does not depend on λ. So, $F(\lambda)>0$ if and only if there exists a planar convex body M verifying that

$$\mathrm{p}(M)^2<16\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2}\mathrm{A}(M)\tag{2.1}$$

for all $n\geq 3$. We observe that $\kappa_n\kappa_{n-2}/\kappa_{n-1}^2$ is strictly increasing for $n\geq 2$ (see Lemma 2.1), and hence, since $n\geq 3$, we have

$$16\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2}>16\frac{\kappa_2\kappa_0}{\kappa_1^2}=4\pi=\frac{\mathrm{p}(B_2)^2}{\mathrm{A}(B_2)}.$$

Thus, the planar unit ball B_2 satisfies (2.1) for any value of the dimension. It finishes the proof. In fact, many planar convex bodies verify (2.1). □

An analogous argument also shows that the corresponding expected refinement for the Brunn-Minkowski inequality for quermassintegrals (1.5) (when $E = B_n$) is not possible:

Proposition 2.1. *Let $i \in \mathbb{N}$ be fixed. Then there exists $n_0 \geq i+3$ such that, for all $n \geq n_0$, there are convex bodies $K, L \in \mathcal{K}^n$, with a common $(n-2)$-dimensional projection, verifying that for all $\lambda \in (0,1)$,*

$$\mathrm{W}_i\big((1-\lambda)K + \lambda L\big)^{1/2} < (1-\lambda)\mathrm{W}_i(K)^{1/2} + \lambda \mathrm{W}_i(L)^{1/2}. \tag{2.2}$$

Proof. Let $g(\lambda) = \mathrm{W}_i\big((1-\lambda)K + \lambda L\big)$ and $f(\lambda) = g(\lambda)^{1/2}$. Arguing in the same way as in the proof of Theorem 1.1, we conclude that if we find two convex bodies $K, L \in \mathcal{K}^n$, having a common $(n-2)$-dimensional projection, n large enough, and verifying that $F(\lambda) = g(\lambda)g''(\lambda) - (1/2)g'(\lambda)^2 > 0$ for all $\lambda \in [0,1]$, then inequality (2.2) will hold for all $\lambda \in (0,1)$.

Again, let $L = B_n$ and $K = M + B_n$ with $\dim M = 2$, for which the projection condition is fulfilled. Similar computations as before show that $F(\lambda) > 0$ if and only if there exists a planar convex body M verifying that

$$\mathrm{p}(M)^2 < 16 \frac{n(n-i-1)}{(n-1)(n-i)} \frac{\kappa_n \kappa_{n-2}}{\kappa_{n-1}^2} \mathrm{A}(M). \tag{2.3}$$

It is easy to check that the function $n(n-i-1)/\big((n-1)(n-i)\big)$ is strictly increasing in n if $n \geq (i+1)/2$ (in particular, for $n \geq i+3$) for fixed i, and has limit 1 when n goes to infinity. Since $\kappa_n \kappa_{n-2}/\kappa_{n-1}^2$ is also increasing in the dimension and tends to 1 when $n \to \infty$ (see Lemma 2.1), the product of both functions is increasing and we get

$$\lim_{n \to \infty} 16 \frac{n(n-i-1)}{(n-1)(n-i)} \frac{\kappa_n \kappa_{n-2}}{\kappa_{n-1}^2} = 16 > 4\pi = \frac{\mathrm{p}(B_2)^2}{\mathrm{A}(B_2)}.$$

Thus, if $n_0 \in \mathbb{N}$ is the first value of the dimension such that the planar unit ball B_2 satisfies (2.3) (the above condition for the limit ensures that n_0 always exists), then the monotonicity shows that for all $n \geq n_0$, inequality (2.2) holds for $K = B_2 + B_n$ and $L = B_n$. □

We observe, for instance, that in the case $i = 1$, the value of the dimension from which inequality (2.2) holds is $n_0 = 5$.

In [7, Corollary 1.2.1] a similar result to Theorem B was proved but involving sections instead projections: if

$$\max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (H+x)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(L \cap (H+x)\big), \tag{2.4}$$

for $K, L \in \mathcal{K}^n$ and some hyperplane $H \in \mathcal{L}^n_{n-1}$, then

$$\mathrm{vol}\left(\frac{1}{2}(K+L)\right) \geq \frac{1}{2}\mathrm{vol}(K) + \frac{1}{2}\mathrm{vol}(L). \tag{2.5}$$

The same construction can be made in order to show that an analogous result for $(n-k)$-dimensional sections will be not true: indeed, since the convex bodies $K = B_2 + B_n$ and $L = B_n$ are symmetric with respect to the origin, for any $(n-k)$-plane H, the section $K\cap(H+x)$, $x\in H^\perp$, with maximum $(n-k)$-dimensional volume is the one through the origin, i.e., $K\cap H$, which coincides with the projection $K|H$ (and analogously for $L=B_n$). Therefore, choosing H as in the proof of Theorem 1.1, condition (2.4) is fulfilled, but we get that $\mathrm{vol}\bigl((1/2)(K+L)\bigr)^{1/2}<(1/2)\mathrm{vol}(K)^{1/2}+(1/2)\mathrm{vol}(L)^{1/2}$.

3. Refinements of Brunn-Minkowski inequality involving projections

Next we deal with Propositions 1.1 and 1.2. We point out that the proofs of these results follow the idea of the proofs of Theorems A and B in [7].

Proof of Proposition 1.1. Without loss of generality we may assume that H is the $(n-k)$-plane $H=\bigl\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_1=\cdots=x_k=0\bigr\}$, and for the sake of brevity we write, on the one hand,

$$U=K|H=L|H \quad\text{and}\quad M_\lambda=(1-\lambda)K+\lambda L.$$

Thus, $M_\lambda|H=(1-\lambda)(K|H)+\lambda(L|H)=U$, for all $\lambda\in[0,1]$. On the other hand, it is clear that for all $u\in U$ and any $x\in K(u), y\in L(u)$ (cf. (1.9)), it holds

$$\bigl((1-\lambda)x+\lambda y,u\bigr)=(1-\lambda)(x,u)+\lambda(y,u)\in M_\lambda,$$

and therefore, $(1-\lambda)K(u)+\lambda L(u)\subset M_\lambda(u)$. Thus, using Fubini's theorem and Brunn-Minkowski inequality (1.2), we get

$$\begin{aligned}
\mathrm{vol}\bigl((1-\lambda)K+\lambda L\bigr)&=\mathrm{vol}(M_\lambda)=\int_U \mathrm{vol}_k\bigl(M_\lambda(u)\bigr)\,\mathrm{d}u\\
&\geq\int_U \mathrm{vol}_k\bigl((1-\lambda)K(u)+\lambda L(u)\bigr)\,\mathrm{d}u\\
&\geq\int_U \Bigl((1-\lambda)\mathrm{vol}_k\bigl(K(u)\bigr)^{1/k}+\lambda\mathrm{vol}_k\bigl(L(u)\bigr)^{1/k}\Bigr)^k\mathrm{d}u\\
&\geq\int_U \Bigl((1-\lambda)^k\mathrm{vol}_k\bigl(K(u)\bigr)+\lambda^k\mathrm{vol}_k\bigl(L(u)\bigr)\Bigr)\,\mathrm{d}u\\
&=(1-\lambda)^k\mathrm{vol}(K)+\lambda^k\mathrm{vol}(L).
\end{aligned}$$

□

As in the case of Theorem B, the identity assumption on projections can be weakened to an equality between suitable $(n-k)$-dimensional volumes.

Proof of Proposition 1.2. Applying Schwarz symmetrization to the convex bodies K, L and $(1-\lambda)K+\lambda L$, with respect to the $(n-k)$-plane H, yields new convex bodies $K'=\sigma_H K$, $L'=\sigma_H L$ and $\sigma_H\bigl((1-\lambda)K+\lambda L\bigr)$ verifying

$$(1-\lambda)K'+\lambda L'\subset\sigma_H\bigl((1-\lambda)K+\lambda L\bigr)$$

(see [11, ch. IV]), and since Schwarz symmetrization preserves the volume, it suffices to prove that

$$\mathrm{vol}\bigl((1-\lambda)K'+\lambda L'\bigr)\geq(1-\lambda)^k\mathrm{vol}(K')+\lambda^k\mathrm{vol}(L').$$

Next, we notice that $K|H = K' \cap H$ and, moreover,

$$\mathrm{vol}_{n-k}\big(K' \cap H\big) = \max_{t\in H^\perp} \mathrm{vol}_{n-k}\big(K' \cap (t+H)\big),$$

and analogously for the convex body L. Then, applying again Schwarz symmetrization to the sets K', L', but now with respect to $H^\perp$, we get new convex bodies $\sigma_{H^\perp}K', \sigma_{H^\perp}L'$ verifying that

$$\big(\sigma_{H^\perp}K'\big)\,|H = \left(\frac{\mathrm{vol}_{n-k}(K|H)}{\kappa_{n-k}}\right)^{1/(n-k)} B_{n-k},$$
$$\big(\sigma_{H^\perp}L'\big)\,|H = \left(\frac{\mathrm{vol}_{n-k}(L|H)}{\kappa_{n-k}}\right)^{1/(n-k)} B_{n-k},$$

and since $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$, we obtain that

$$\big(\sigma_{H^\perp}K'\big)\,|H = \big(\sigma_{H^\perp}L'\big)\,|H.$$

Thus, we can apply Proposition 1.1 to the convex bodies $\sigma_{H^\perp}K', \sigma_{H^\perp}L'$ which, together with the facts that the volume is preserved and the inclusion $(1-\lambda)\sigma_{H^\perp}K' + \lambda\,\sigma_{H^\perp}L' \subset \sigma_{H^\perp}\big((1-\lambda)K' + \lambda L'\big)$ holds, yields

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K' + \lambda L'\big) &= \mathrm{vol}\Big(\sigma_{H^\perp}\big((1-\lambda)K' + \lambda L'\big)\Big)\\ &\geq \mathrm{vol}\Big((1-\lambda)\sigma_{H^\perp}K' + \lambda\,\sigma_{H^\perp}L'\Big)\\ &\geq (1-\lambda)^k\mathrm{vol}\big(\sigma_{H^\perp}K'\big) + \lambda^k\mathrm{vol}\big(\sigma_{H^\perp}L'\big)\\ &= (1-\lambda)^k\mathrm{vol}(K') + \lambda^k\mathrm{vol}(L'). \qquad \square\end{aligned}$$

Remark 3.1. *We observe that Brunn-Minkowski inequality* (1.2) *implies that* $\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)^n\mathrm{vol}(K) + \lambda^n\mathrm{vol}(L)$. *Therefore inequality* (1.8) *generalizes the above one for* $k = n$ *and* (1.3) *when* $k = 1$.

Inequality (1.8) can be also obtained if we replace the projection volume property $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$ by a section volume condition:

Proposition 3.1. *Let* $k \in \{1, \dots, n\}$ *and let* $K, L \in \mathcal{K}^n$ *be convex bodies such that there exists* $H \in \mathcal{L}^n_{n-k}$ *with*

$$\max_{x\in H^\perp} \mathrm{vol}_{n-k}\big(K \cap (x+H)\big) = \max_{x\in H^\perp} \mathrm{vol}_{n-k}\big(L \cap (x+H)\big).$$

Then, for all $\lambda \in [0,1]$,

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)^k\mathrm{vol}(K) + \lambda^k\mathrm{vol}(L).$$

This result generalizes [7, Corollary 1.2.1] to all $k \in \{1, \dots, n\}$. The proof is a direct consequence of Proposition 1.2 and the following lemma.

Lemma 3.1. *Let* $k \in \{1, \dots, n\}$ *and let* $H \in \mathcal{L}^n_{n-k}$. *The following statements are equivalent:*

i) *If* $K, L \in \mathcal{K}^n$ *satisfy that* $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$, *then inequality* (1.8) *holds for all* $\lambda \in [0,1]$.

ii) *If* $K,L\in\mathcal{K}^n$ *satisfy that*

$$\max_{x\in H^\perp}\mathrm{vol}_{n-k}\big(K\cap(x+H)\big)=\max_{x\in H^\perp}\mathrm{vol}_{n-k}\big(L\cap(x+H)\big),$$

then inequality (1.8) *holds for all* $\lambda\in[0,1]$.

Proof. First, we suppose i) and assume that

$$\max_{x\in H^\perp}\mathrm{vol}_{n-k}\big(K\cap(x+H)\big)=\max_{x\in H^\perp}\mathrm{vol}_{n-k}\big(L\cap(x+H)\big)=\nu.$$

Then the orthogonal projections onto H of the Schwarz symmetrals of K and L with respect to $H^\perp$, namely, $\sigma_{H^\perp}K$, $\sigma_{H^\perp}L$, are equal; more precisely, $\big(\sigma_{H^\perp}K\big)|H=(\nu/\kappa_{n-k})^{1/(n-k)}B_{n-k}=\big(\sigma_{H^\perp}L\big)|H$. Thus i), together with known properties of the Schwarz symmetrization (see [11, ch. IV]), yields to

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K+\lambda L\big)&=\mathrm{vol}\Big(\sigma_{H^\perp}\big((1-\lambda)K+\lambda L\big)\Big)\\&\geq\mathrm{vol}\big((1-\lambda)\sigma_{H^\perp}K+\lambda\,\sigma_{H^\perp}L\big)\\&\geq(1-\lambda)^k\mathrm{vol}\big(\sigma_{H^\perp}K\big)+\lambda^k\mathrm{vol}\big(\sigma_{H^\perp}L\big)\\&=(1-\lambda)^k\mathrm{vol}(K)+\lambda^k\mathrm{vol}(L).\end{aligned}$$

Conversely, we suppose ii) and assume that $\mathrm{vol}_{n-k}(K|H)=\mathrm{vol}_{n-k}(L|H)$. Then the Schwarz symmetrals $\sigma_H K$ and $\sigma_H L$ verify that

$$\begin{aligned}\max_{x\in H^\perp}\mathrm{vol}_{n-k}\big((\sigma_H K)\cap(x+H)\big)&=\mathrm{vol}_{n-k}\big((\sigma_H K)\cap H\big)=\mathrm{vol}_{n-k}(K|H)\\&=\mathrm{vol}_{n-k}(L|H)=\max_{x\in H^\perp}\mathrm{vol}_{n-k}\big((\sigma_H L)\cap(x+H)\big),\end{aligned}$$

and therefore, ii), together with known properties of the Schwarz symmetrization, yields to

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K+\lambda L\big)&=\mathrm{vol}\Big(\sigma_H\big((1-\lambda)K+\lambda L\big)\Big)\\&\geq\mathrm{vol}\big((1-\lambda)\sigma_H K+\lambda\,\sigma_H L\big)\\&\geq(1-\lambda)^k\mathrm{vol}(\sigma_H K)+\lambda^k\mathrm{vol}(\sigma_H L)\\&=(1-\lambda)^k\mathrm{vol}(K)+\lambda^k\mathrm{vol}(L).\end{aligned}$$ □

Next we prove Theorem 1.2, which generalizes both (1.2) and (1.3).

Proof of Theorem 1.2. Arguing in the same way as in the proof of Proposition 1.1, we get $(1-\lambda)K(u)+\lambda L(u)\subset M_\lambda(u)$ which, together with Brunn-Minkowski inequality (1.2), yields

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/k}&=\mathrm{vol}(M_\lambda)^{1/k}=\left(\int_U\mathrm{vol}_k\big(M_\lambda(u)\big)\,\mathrm{d}u\right)^{1/k}\\&\geq\left(\int_U\mathrm{vol}_k\big((1-\lambda)K(u)+\lambda L(u)\big)\,\mathrm{d}u\right)^{1/k}\\&\geq\left(\int_U\Big[(1-\lambda)\mathrm{vol}_k\big(K(u)\big)^{1/k}+\lambda\mathrm{vol}_k\big(L(u)\big)^{1/k}\Big]^k\,\mathrm{d}u\right)^{1/k}.\end{aligned}$$

Then, applying Hölder's inequality (see e.g. [8, Corollary 1.5]) to the functions $(1-\lambda)\mathrm{vol}_k\bigl(K(u)\bigr)^{1/k}+\lambda\mathrm{vol}_k\bigl(L(u)\bigr)^{1/k}$ and 1, we finally get

$$\begin{aligned}
&\mathrm{vol}\bigl((1-\lambda)K+\lambda L\bigr)^{1/k}\\
&\quad\geq\left(\int_U\left[(1-\lambda)\mathrm{vol}_k\bigl(K(u)\bigr)^{1/k}+\lambda\mathrm{vol}_k\bigl(L(u)\bigr)^{1/k}\right]^k\mathrm{d}u\right)^{1/k}\\
&\quad\geq\frac{1}{\mathrm{vol}_{n-k}(U)^{1-1/k}}\int_U\Bigl((1-\lambda)\mathrm{vol}_k\bigl(K(u)\bigr)^{1/k}+\lambda\mathrm{vol}_k\bigl(L(u)\bigr)^{1/k}\Bigr)\,\mathrm{d}u\\
&\quad=(1-\lambda)\int_U\left(\frac{\mathrm{vol}_k(K(u))}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k}\mathrm{d}u+\lambda\int_U\left(\frac{\mathrm{vol}_k(L(u))}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k}\mathrm{d}u.\square
\end{aligned}$$

Remark 3.2. *Theorem 1.2 generalizes both, Brunn-Minkowski inequality* (1.2) *and Theorem A. Indeed, if $k=1$ then* (1.10) *becomes inequality* (1.3)*; for $k=n$, then $U=\{0\}$ and hence,* $\mathrm{vol}_0(U)=1$ *and the integrals in* (1.10) *are just the volumes of K and L, respectively. Therefore,* (1.10) *gives* (1.2).

We conclude this section by showing that, for a particular relative quermassintegral, the expected refinement can be obtained. In order to prove it, we shortly need some notation on mixed volumes, i.e., the coefficients of the most general (n-variables) polynomial which is obtained when the volume of a linear combination $\lambda_1K_1+\cdots+\lambda_mK_m$, $K_i\in\mathcal{K}^n$ and $\lambda_i\geq 0$, $i=1,\dots,m$, is computed. As usual, $\mathrm{V}(K_1,\dots,K_n)$ denotes the mixed volume of the convex bodies $K_1,\dots,K_n\in\mathcal{K}^n$, and for the sake of brevity we will use the abbreviation $\bigl(K_1[r_1],\dots,K_m[r_m]\bigr)\equiv\bigl(K_1,\overset{(r_1)}{\dots},K_1,\dots,K_m,\overset{(r_m)}{\dots},K_m\bigr)$. In particular, it holds $\mathrm{W}_i(K_1;K_2)=\mathrm{V}\bigl(K_1[n-i],K_2[i]\bigr)$. For a deep study of mixed volumes we refer to [19, s. 5.1].

Proposition 3.2. *Let $k\in\{1,\dots,n-1\}$ and let $K,L\in\mathcal{K}^n$ be convex bodies such that there exists $H\in\mathcal{L}^n_{n-k}$ with $K|H=L|H$. Then, for any convex body $E_k\subsetneq H^{\perp}$ and all $\lambda\in[0,1]$,*

(3.1)
$$\mathrm{W}_{k-1}\bigl((1-\lambda)K+\lambda L;E_k\bigr)^{1/k}\geq(1-\lambda)\mathrm{W}_{k-1}(K;E_k)^{1/k}+\lambda\mathrm{W}_{k-1}(L;E_k)^{1/k}.$$

Proof. We observe that (3.1) holds for convex bodies K,L having a common $(n-k)$-projection if and only if $f(\lambda)=\mathrm{W}_{k-1}\bigl((1-\lambda)K+\lambda L;E_k\bigr)^{1/k}$ is a concave function on $[0,1]$ (see the proof of Theorem 1.1). So we have to see that $f''(\lambda)\leq 0$, and following the argument of the proof of [19, Theorem 6.4.3], it suffices to show this for $\lambda=0$. It can be checked that $f''(0)\leq 0$ if and only if

$$\begin{aligned}
(n-k)\mathrm{W}_{k-1}(K;E_k)\Bigl[&\mathrm{W}_{k-1}(K;E_k)-2\mathrm{V}\bigl(K[n-k],L,E_k[k-1]\bigr)\\
&+\mathrm{V}\bigl(K[n-k-1],L[2],E_k[k-1]\bigr)\Bigr]\\
-\left(1-\frac{1}{k}\right)(n-k+1)&\Bigl[\mathrm{V}\bigl(K[n-k],L,E_k[k-1]\bigr)-\mathrm{W}_{k-1}(K;E_k)\Bigr]^2\leq 0.
\end{aligned}$$

The second summand is clearly negative, and hence, we have to study the sign of the first one.

On the one hand, denoting for short $\mathcal{C} = \bigl(K[n-k-1], E_k[k-1]\bigr)$, it is well-known that, in particular,

$$\frac{\mathrm{V}(K[2],\mathcal{C})}{\mathrm{V}(K,M,\mathcal{C})^2} - \frac{2\mathrm{V}(K,L,\mathcal{C})}{\mathrm{V}(K,M,\mathcal{C})\mathrm{V}(L,M,\mathcal{C})} + \frac{\mathrm{V}(L[2],\mathcal{C})}{\mathrm{V}(L,M,\mathcal{C})^2} \leq 0 \tag{3.2}$$

for any convex body $M \in \mathcal{K}^n$ (see [19, Theorem 6.4.2]). On the other hand, denoting by $\vartheta^{(j)}$ the mixed volume computed in a j-dimensional affine subspace, and since $E_k \subsetneq H^{\perp}$, it holds (see [19, (5.3.23)])

$$\mathrm{W}_k(K;E_k) = \frac{1}{\binom{n}{k}}\mathrm{vol}_k(E_k)\mathrm{vol}_{n-k}(K|H) \qquad \text{and}$$

$$\begin{aligned}\mathrm{V}\bigl(K[n-k-1],L,E_k[k]\bigr) &= \mathrm{vol}_k(E_k)\mathrm{V}\left(K[n-k-1],L,\frac{E_k}{\mathrm{vol}_k(E_k)^{1/k}}[k]\right)\\ &= \frac{\mathrm{vol}_k(E_k)}{\binom{n}{k}}\vartheta^{(n-k)}\bigl(K|H[n-k-1],L|H\bigr),\end{aligned}$$

and using the projection assumption $K|H = L|H$, we get

$$\mathrm{V}\bigl(K[n-k-1],L,E_k[k]\bigr) = \frac{\mathrm{vol}_k(E_k)}{\binom{n}{k}}\mathrm{vol}_{n-k}(K|H) = \mathrm{W}_k(K;E_k),$$

i.e., $\mathrm{V}(L,E_k,\mathcal{C}) = \mathrm{V}(K,E_k,\mathcal{C})$. Then, (3.2) for $M = E_k$ yields

$$\mathrm{W}_{k-1}(K;E_k) - 2\mathrm{V}\bigl(K[n-k],L,E_k[k-1]\bigr) + \mathrm{V}\bigl(K[n-k-1],L[2],E_k[k-1]\bigr) \leq 0,$$

which shows that $f''(0) \leq 0$, as required. □

4. Brunn-Minkowski inequality for particular families of convex bodies

A convex body $K \in \mathcal{K}^n$ containing the convex body $E \in \mathcal{K}^n$ is called a *p-tangential body* of E, $p \in \{0,\dots,n-1\}$, if each support plane of K not supporting E contains only $(p-1)$-singular points of K [19, p. 76]. Here a boundary point x of K is said to be an r-singular point of K if the dimension of the normal cone in x is at least $n-r$. For further characterizations and properties of p-tangential bodies we refer to [19, s. 2.2].

So a 0-tangential body of E is just the body E itself and each p-tangential body of E is also a q-tangential body for $p < q \leq n-1$.

A 1-tangential body is usually called *cap-body*, which can be seen as the convex hull of E and countably many points such that the line segment joining any pair of those points intersects E.

The following theorem provides a characterization of n-dimensional p-tangential bodies in terms of the relative quermassintegrals.

Theorem 4.1 (Favard [4], [19, p. 367])**.** *Let $K,E \in \mathcal{K}^n_0$, $E \subset K$, and let $p \in \{0,\dots,n-1\}$. Then $\mathrm{W}_0(K;E) = \mathrm{W}_1(K;E) = \cdots = \mathrm{W}_{n-p}(K;E)$ if and only if K is a p-tangential body of E.*

In this section we improve Brunn-Minkowski inequality for the family of p-tangential bodies, i.e., we show Theorem 1.3, which is a direct consequence of the following slightly more general result.

Theorem 4.2. *Let $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_0$ and $s \in \{1, \dots, n\}$ be such that* $\mathrm{W}_s(K;E) = \mathrm{W}_{s+1}(K;E) = \cdots = \mathrm{W}_n(K;E)$. *Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda E\big)^{1/s} \geq (1-\lambda)\mathrm{vol}(K)^{1/s} + \lambda \mathrm{vol}(E)^{1/s} \tag{4.1}$$

and equality holds if and only if $K = E$. Moreover,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda E; E\big)^{1/(s-i)} \geq (1-\lambda)\mathrm{W}_i(K;E)^{1/(s-i)} + \lambda \mathrm{W}_i(E;E)^{1/(s-i)},$$

$i = 0, \dots, s-1$, and equality holds for some fixed i, if and only if K, E satisfy $\mathrm{W}_i(K;E) = \cdots = \mathrm{W}_n(K;E)$.

Indeed, if K is a p-tangential body of $E \in \mathcal{K}^n_0$, $1 \leq p \leq n-1$, then Favard's Theorem 4.1 ensures that

$$\mathrm{W}_0(K;E) = \mathrm{W}_1(K;E) = \cdots = \mathrm{W}_{n-p}(K;E) \neq 0,$$

and since $\mathrm{W}_j(K;E) = \mathrm{W}_{n-j}(E;K)$, Theorem 4.2 immediately implies Theorem 1.3.

Remark 4.1. *The condition* $\mathrm{int}\, E \neq \emptyset$ *cannot be removed, since it is needed that* $\mathrm{vol}(E) \neq 0$. *Indeed, taking $K = [0, \mathrm{e}_1] + [0, \mathrm{e}_2] + [0, 2\mathrm{e}_3] \in \mathcal{K}^3$ and $E = [0, \mathrm{e}_3]$, then* $\mathrm{W}_2(K;E) = \mathrm{W}_3(K;E) = 0$. *However, for every $\lambda \in (0,1)$, it holds*

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/2} &= (1-\lambda)(2-\lambda)^{1/2} \\ &< (1-\lambda)\sqrt{2} = (1-\lambda)\mathrm{vol}(K)^{1/2} + \lambda \mathrm{vol}(E)^{1/2}.\end{aligned}$$

Proof of Theorem 4.2. We will show the inequality (4.1) for the volume. The relations for the quermassintegrals, as well the corresponding equality cases, can be obtained analogously.

Using the well-known Aleksandrov-Fenchel inequality for quermassintegrals (see e.g. [19, s. 6.3]), namely,

$$\mathrm{W}_i(K;E)^2 \geq \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E), \quad 1 \leq i \leq n-1, \tag{4.2}$$

and since $\mathrm{W}_s(K;E) \neq 0$, we easily get that

$$\mathrm{W}_0(K;E) \leq \cdots \leq \mathrm{W}_{s-1}(K;E) \leq \mathrm{W}_s(K;E) = \cdots = \mathrm{W}_n(K;E). \tag{4.3}$$

Now we consider the polynomial function

$$f(\lambda) = \sum_{i=0}^{s} \binom{s}{i} \mathrm{W}_i(K;E)(1-\lambda)^{s-i}\lambda^i,$$

for $\lambda\in[0,1]$. On the one hand, we can write

$$\begin{aligned}
f(\lambda) &= \bigl[(1-\lambda)+\lambda\bigr]^{n-s}f(\lambda)\\
&= \left(\sum_{j=0}^{n-s}\binom{n-s}{j}(1-\lambda)^{n-s-j}\lambda^j\right)\left(\sum_{i=0}^{s}\binom{s}{i}\mathrm{W}_i(K;E)(1-\lambda)^{s-i}\lambda^i\right)\\
&= \sum_{k=0}^{n}\left(\sum_{i+j=k}\mathrm{W}_i(K;E)\binom{s}{i}\binom{n-s}{j}\right)(1-\lambda)^{n-k}\lambda^k,
\end{aligned}$$

and using (4.3) we get that

$$\begin{aligned}
\sum_{i+j=k}\mathrm{W}_i(K;E)\binom{s}{i}\binom{n-s}{j} &\leq \mathrm{W}_k(K;E)\sum_{i+j=k}\binom{s}{i}\binom{n-s}{j}\\
&= \mathrm{W}_k(K;E)\binom{n}{k}.
\end{aligned}$$

Therefore,

(4.4)

$$f(\lambda)^{1/s}\leq\left(\sum_{k=0}^{n}\binom{n}{k}\mathrm{W}_k(K;E)(1-\lambda)^{n-k}\lambda^k\right)^{1/s}=\mathrm{vol}\bigl((1-\lambda)K+\lambda E\bigr)^{1/s}.$$

On the other hand, since the coefficients of the polynomial $f(\lambda)$, namely, $\mathrm{W}_i(K;E)$ for $i=0,\dots,s$, are non-negative real numbers satisfying the Aleksandrov-Fenchel inequalities (4.2), a result of Shephard (see [20], [9, Lemma 2.1]) ensures that $\mathrm{W}_i(K;E)=\mathrm{W}_i^{(s)}(K_s;E_s)$ are the relative quermassintegrals in $\mathbb{R}^s$ of two convex bodies $K_s,E_s\in\mathcal{K}^s$, $i=0,\dots,s$. Then, using (4.4) and Brunn-Minkowski inequality (1.2) in $\mathbb{R}^s$, we conclude that

$$\begin{aligned}
\mathrm{vol}\bigl((1-\lambda)K+\lambda E\bigr)^{1/s} &\geq f(\lambda)^{1/s}=\mathrm{vol}_s\bigl((1-\lambda)K_s+\lambda E_s\bigr)^{1/s}\\
&\geq(1-\lambda)\mathrm{vol}_s(K_s)^{1/s}+\lambda\mathrm{vol}_s(E_s)^{1/s}\\
&=(1-\lambda)\mathrm{vol}(K)^{1/s}+\lambda\mathrm{vol}(E)^{1/s},
\end{aligned}$$

since $\mathrm{vol}_s(K_s)=\mathrm{W}_0^{(s)}(K_s;E_s)=\mathrm{W}_0(K;E)$ and $\mathrm{vol}_s(E_s)=\mathrm{W}_s^{(s)}(K_s;E_s)=\mathrm{W}_s(K;E)=\mathrm{W}_n(K;E)$.

Clearly, if $K=E$ then equality holds in (4.1). Conversely, if we have equality in (4.1), then equality holds in (4.3) for all quermassintegrals, i.e., $\mathrm{W}_0(K;E)=\cdots=\mathrm{W}_n(K;E)$. It implies that $K=E$. □

In order to conclude the paper, we make an observation regarding another family of convex bodies for which a refinement of Brunn-Minkowski inequality can be obtained, namely, $\mathcal{V}=\bigl\{K\in\mathcal{K}^n:\mathrm{vol}(K)=v\bigr\}$, for a fixed positive real number $v\in\mathbb{R}_{>0}$: if $K,L\in\mathcal{V}$, then the multiplicative version of Brunn-Minkowski inequality (see e.g. [8, Theorem 8.15]) leads to

$$\mathrm{vol}\bigl((1-\lambda)K+\lambda L\bigr)\geq\mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}=v=(1-\lambda)\mathrm{vol}(K)+\lambda\mathrm{vol}(L).$$

Thus, the following corollary has been proved.

Corollary 4.1. *Let* $K, L \in \mathcal{K}^n$ *with* $\mathrm{vol}(K) = \mathrm{vol}(L)$. *Then,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}(K) + \lambda\mathrm{vol}(L).$$

The above result can be also obtained as a consequence (for $k = 0$) of a more general refinement of Brunn-Minkowski inequality of type (1.6) for quermassintegrals.

Proposition 4.1. *Let* $k \in \{0, \dots, n-2\}$ *and let* $K, L \in \mathcal{K}^n$ *be convex bodies such that for all* $H \in \mathcal{L}^n_{n-k}$, $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$. *Then,*

$$\mathrm{W}_k\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{W}_k(K) + \lambda\mathrm{W}_k(L). \tag{4.5}$$

Proof. Kubota's integral recursion formula (see e.g. [19, p. 295, (5.3.27)] states, in particular, that, for any convex body $K \in \mathcal{K}^n$,

$$\mathrm{W}_k(K) = \frac{\kappa_n}{\kappa_{n-k}} \int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}(K|H)\,\mathrm{d}\mu(H),$$

where μ is the (rotationally invariant) Haar measure on the set $\mathcal{L}^n_{n-k}$ such that $\mu(\mathcal{L}^n_{n-k}) = 1$. Thus, since $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$, we immediately get that $\mathrm{W}_k(K) = \mathrm{W}_k(L)$, and moreover, using Brunn-Minkowski inequality in $\mathbb{R}^k$ we can conclude that

$$\begin{aligned}
&\int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}\Big(\big((1-\lambda)K + \lambda L\big)\big|H\Big)\,\mathrm{d}\mu(H) \\
&\quad = \int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}\big((1-\lambda)K|H + \lambda L|H\big)\,\mathrm{d}\mu(H) \\
&\quad \geq \int_{\mathcal{L}^n_{n-k}} \Big[(1-\lambda)\mathrm{vol}_{n-k}(K|H)^{1/(n-k)} + \lambda\mathrm{vol}_{n-k}(L|H)^{1/(n-k)}\Big]^{n-k}\mathrm{d}\mu(H) \\
&\quad = \int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}(K|H)\,\mathrm{d}\mu(H).
\end{aligned}$$

Therefore,

$$\mathrm{W}_k\big((1-\lambda)K + \lambda L\big) \geq \mathrm{W}_k(K) = (1-\lambda)\mathrm{W}_k(K) + \lambda\mathrm{W}_k(L). \qquad \square$$

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