ON A REVERSE ISOPERIMETRIC INEQUALITY FOR RELATIVE OUTER PARALLEL BODIES

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ABSTRACT. We show a reverse isoperimetric inequality within the class of relative outer parallel bodies, with respect to a general convex body E, along with its equality condition. Based on the convexity of the sequence of quermassintegrals of Minkowski sums we also prove further inequalities.

1. INTRODUCTION

Let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbb{R}^n$, i.e., the set of all non-empty compact convex subsets of $\mathbb{R}^n$. For two convex bodies $K, E \in \mathcal{K}^n$ and a non-negative real number λ, the volume $\mathrm{vol}(\cdot)$ (i.e., the Lebesgue measure) of the Minkowski sum $K + \lambda\, E$ is expressed as a polynomial of degree at most n in λ, and it is written as

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i. \tag{1.1}$$

This expression is called *Minkowski-Steiner formula* or *relative Steiner formula* of K. The coefficients $\mathrm{W}_i(K; E)$ are the *relative quermassintegrals* of K, and they are a special case of the more general defined *mixed volumes* for which we refer to [11, Chapter 6] and [16, Section 5.1]. In particular, we have $\mathrm{W}_0(K; E) = \mathrm{vol}(K)$, $\mathrm{W}_n(K; E) = \mathrm{vol}(E)$, $\mathrm{W}_i(\lambda_1 K; \lambda_2 E) = \lambda_1^{n-i}\lambda_2^i \mathrm{W}_i(K; E)$ for $\lambda_1, \lambda_2 \geq 0$ and, if E has dimension $\dim(E) = n$, $\mathrm{W}_i(K; E) = 0$ if and only if $\dim(K) \leq n - i - 1$. Moreover, $n\mathrm{W}_1(K; E)$ is the (relative) Minkowski content of K, which will be denoted by $\mathrm{S}(K; E)$. Quermassintegrals admit also a Steiner formula, namely

$$\mathrm{W}_k(K + \lambda E; E) = \sum_{i=0}^{n-k} \binom{n-k}{i} \mathrm{W}_{k+i}(K; E)\lambda^i, \tag{1.2}$$

for any $0 \leq k \leq n$.

A finite sequence of real numbers $(a_0, \ldots, a_m)$ is called concave (see e.g. [16, Section 7.4]) if

$$a_{i-1} - 2a_i + a_{i+1} \leq 0 \text{ for } i = 1, \ldots, m - 1, \tag{1.3}$$

or equivalently, if

$$a_0 - a_1 \leq a_1 - a_2 \leq \cdots \leq a_{m-1} - a_m.$$

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Moreover, a sequence $(a_0,\dots,a_m)$ is said to be convex if it satisfies the reversed inequality to (1.3), i.e., $a_{i-1}-2a_i+a_{i+1}\geq 0$, for $1\leq i\leq m-1$.

As a consequence of this, a concave sequence satisfies the following inequality [16, (7.59)]:

$$(k-j)a_i+(i-k)a_j+(j-i)a_k\leq 0 \tag{1.4}$$

for any $0\leq i<j<k\leq m$. In the latter, there is equality if and only if

$$a_{r-1}-2a_r+a_{r+1}=0 \text{ for } r=i+1,\dots,k-1.$$

When the sequence $(a_0,\dots,a_m)$ contains only positive numbers, it is called log-concave if the logarithm of the sequence, namely $(\log a_0,\dots,\log a_m)$ is concave, which is equivalent to the following inequalities [16, Section 7.4]:

$$a_i^2\geq a_{i-1}a_{i+1} \text{ for } i=1,\dots,m-1. \tag{1.5}$$

For a convex body $K\in\mathcal{K}^n$ the fundamental inequalities

$$\mathrm{W}_i(K;E)^2\geq \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E),$$

which are consequences of the more general Aleksandrov-Fenchel inequalities for mixed volumes (see e.g. [16, Section 7.3]) do yield that the quermassintegrals of K (relative to E) $(\mathrm{W}_0(K;E),\dots,\mathrm{W}_n(K;E))$ constitute a log-concave sequence, i.e., they satisfy (1.5). Furthermore, when dealing with n-dimensional convex bodies $K,E\in\mathcal{K}^n$ with a common projection onto a hyperplane, then the sequence of relative quermassintegrals $(\mathrm{W}_0(K;E),\dots,\mathrm{W}_n(K;E))$ is also concave and thus it satisfies (1.4) (see [16, Theorem 7.7.1 and (7.190)]).

Here we show that, moreover, the reverse form of (1.4) holds when assuming that E is a *summand* of K:

Theorem 1.1. *Let $M,E\in\mathcal{K}^n$ with $K=M+E$ and $\dim(E)=n$. Then, for every $0\leq i<j<k\leq n$,*

$$(k-j)\mathrm{W}_i(K;E)+(i-k)\mathrm{W}_j(K;E)+(j-i)\mathrm{W}_k(K;E)\geq 0. \tag{1.6}$$

Equality holds if and only if $\dim(M)\leq 1$.

By taking $i=0$, $j=1$ and $k=n$ in Theorem 1.1 we get the following result:

Corollary 1.1. *Let $M,E\in\mathcal{K}^n$ with $K=M+E$ and $\dim(E)=n$. Then*

$$\mathrm{vol}(K)\geq\frac{S(K;E)}{n-1}-\frac{\mathrm{vol}(E)}{n-1}.$$

Equality holds if and only if $\dim(M)\leq 1$.

The latter inequality can be regarded as a reverse form of the well-known (relative) *isoperimetric inequality* for n-dimensional convex bodies $K,E\in\mathcal{K}^n$ (also referred to in the literature as the *Minkowski first inequality*, see e.g. [16, Theorem 7.2.1]):

$$S(K;E)^n\geq n^n\mathrm{vol}(K)^{n-1}\mathrm{vol}(E),$$

and equality holds if and only if $K=rE$ (up to translations) for some $r>0$. Another result in this regard is the celebrated reverse isoperimetric inequality, due to Ball [1] (with equality conditions later supplied by Barthe [3]), in the classical setting (see also [2] and the references therein).

The case $E=\big(1/\lambda\big)B_n$ of the Euclidean ball of radius $1/\lambda$ (for $\lambda>0$) in both Theorem 1.1 and Corollary 1.1 has been recently obtained in [8]. There, using

Kubota's formula [16, (5.72)] and induction on the dimension, the authors derive the above relations for the classical quermassintegrals of K, i.e., when the relative body E is the Euclidean unit ball B_n. Here we show, by using the Steiner formula (1.2), that these relations can be extended to the setting of the so-called *Minkowski relative geometry*, this is, when the functionals of convex bodies are evaluated in relation to an arbitrary convex body E rather than B_n. This will be shown in Section 2 of the present paper. Next, in Section 3 we prove further inequalities involving quermassintegrals of Minkowski sums of convex bodies.

2. Relations for the quermassintegrals of relative outer parallel bodies

We start this section by showing that the sequence $(\mathrm{W}_0(K;E),\dots,\mathrm{W}_n(K;E))$ of (relative) quermassintegrals is convex when E is a summand of K. This will be proven by using the Steiner formula (1.2) jointly with the following property of the binomial coefficients of the numbers $n,k\in\mathbb{N}\cup\{0\}$:

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}, \tag{2.1}$$

where $\binom{n}{k}:=0$ whenever $k>n$.

Lemma 2.1. *Let $M,E\in\mathcal{K}^n$ with $K=M+E$ and $\dim(E)=n$. Then the sequence $(\mathrm{W}_0(K;E),\dots,\mathrm{W}_n(K;E))$ is convex, i.e., for every $1\leq i\leq n-1$,*

$$\mathrm{W}_{i-1}(K;E)-2\mathrm{W}_i(K;E)+\mathrm{W}_{i+1}(K;E)\geq 0. \tag{2.2}$$

Equality holds if and only if $\dim(M)\leq 1$.

Proof. By the Steiner formula for the quermassintegrals (1.2) we get

$$\begin{aligned}
&\mathrm{W}_{i-1}(K;E)-2\mathrm{W}_i(K;E)+\mathrm{W}_{i+1}(K;E)\\
&=\sum_{j=0}^{n-i+1}\binom{n-i+1}{j}\mathrm{W}_{i-1+j}(M;E)-2\sum_{j=0}^{n-i}\binom{n-i}{j}\mathrm{W}_{i+j}(M;E)\\
&+\sum_{j=0}^{n-i-1}\binom{n-i-1}{j}\mathrm{W}_{i+1+j}(M;E)=\mathrm{W}_{i-1}(M;E)+\mathrm{W}_i(M;E)(n-i-1)\\
&+\sum_{j=1}^{n-i-1}\binom{n-i-1}{j}\mathrm{W}_{i+j}(M;E)\left(\binom{n-i+1}{j+1}-2\binom{n-i}{j}+\binom{n-i-1}{j-1}\right),
\end{aligned}$$

and thus, from (2.1), we have that

$$\begin{aligned}
\mathrm{W}_{i-1}(K;E)-2\mathrm{W}_i(K;E)+\mathrm{W}_{i+1}(K;E)&=\mathrm{W}_{i-1}(M;E)+\mathrm{W}_i(M;E)(n-i-1)\\
&+\sum_{j=1}^{n-i-2}\binom{n-i-1}{j}\binom{n-i-1}{j+1}\mathrm{W}_{i+j}(M;E)\geq 0.
\end{aligned}$$

Moreover, from the latter identity we may assert that $\mathrm{W}_{i-1}(K;E)-2\mathrm{W}_i(K;E)+\mathrm{W}_{i+1}(K;E)=0$ if and only if $\mathrm{W}_{n-2}(M;E)=0$. This completes the proof. □

At this point we would like to compare this result with Bonnesen's inequality in the plane, which establishes that

$$\mathrm{W}_0(K;E)-2\mathrm{W}_1(K;E)\mathrm{r}(K;E)+\mathrm{W}_2(K;E)\mathrm{r}(K;E)^2\leq 0,$$

with equality if and only if $K = L + \mathrm{r}(K;E)E$ for $\dim L \leq 1$. Here $\mathrm{r}(K;E)$ is the *relative inradius* of K with respect to E, which is defined by

$$\mathrm{r}(K;E) = \sup\{r \geq 0 : \exists\, x \in \mathbb{R}^n \text{ with } x + r\,E \subset K\}.$$

Bonnesen proved this result for $E = B_2$ [6], being the proof of the general case due to Blaschke [4, Pages 33-36]. This inequality sharpens (in the plane) the Aleksandrov-Fenchel and isoperimetric inequalities, and there is no known generalization of it to higher dimension. Some Bonnesen style inequalities in arbitrary dimension n can be found in e.g. [5, 7, 9, 10, 14, 15] (see also [16, Notes for Section 7.2] and the references therein).

Remark 2.1. *We notice that if $K, E \in \mathcal{K}^2$ are such that $K = M + \mathrm{r}(K;E)E$, which implies that $\dim(M) \leq 1$, then Lemma 2.1 for $n = 2$ (and such a pair of convex bodies) is just a simple consequence of Bonnesen's inequality.*

Now we derive a more general result for three non-necessarily consecutive (relative) quermassintegrals. For the sake of clarity, we notice that along its proof we are following the steps of [16, Pages 399, 400] (cf. (1.4)) (see also [8]). From now on, and unless we say the opposite, $\mathrm{W}_i := \mathrm{W}_i(K;E)$.

Proof of Theorem 1.1. Let $0 \leq i < j < k \leq n$ be fixed. From Lemma 2.1 it follows that

$$\mathrm{W}_i - \mathrm{W}_{i+1} \geq \mathrm{W}_{j-1} - \mathrm{W}_j \geq \mathrm{W}_{k-1} - \mathrm{W}_k \geq 0. \tag{2.3}$$

We notice that the latter inequality follows from the fact that $K = M + E$ and the monotonicity and translation invariance of mixed volumes.

Thus,

$$\begin{aligned} \big(\mathrm{W}_j - \mathrm{W}_{j+1}\big) + \cdots + \big(\mathrm{W}_{k-1} - \mathrm{W}_k\big) &\leq (k-j)\big(\mathrm{W}_{j-1} - \mathrm{W}_j\big)\\ \big(\mathrm{W}_i - \mathrm{W}_{i+1}\big) + \cdots + \big(\mathrm{W}_{j-1} - \mathrm{W}_j\big) &\geq (j-i)\big(\mathrm{W}_{j-1} - \mathrm{W}_j\big) \end{aligned}$$

and hence,

$$0 \leq \frac{\mathrm{W}_j - \mathrm{W}_k}{k-j} \leq \mathrm{W}_{j-1} - \mathrm{W}_j \leq \frac{\mathrm{W}_i - \mathrm{W}_j}{j-i},$$

which yields (1.6). Moreover, (1.6) holds with equality if and only if the same holds for (2.2). Thus, the result follows from the equality case of Lemma 2.1. □

The following classical result provides us with a sufficient condition for a sequence of positive real numbers to be the (relative) quermassintegrals of certain convex bodies $K, E \in \mathcal{K}^n$:

Proposition 2.1 ([17])**.** *If the sequence of positive real numbers $(a_0, \ldots, a_n)$ is log-concave, then there exist simplices $K, E \in \mathcal{K}^n$ such that $\mathrm{W}_i(K;E) = a_i$, $0 \leq i \leq n$.*

A refined version of this result can be found in [13]. The most general problem of characterizing whether a finite sequence of non-negative numbers are the mixed volumes of $m \in \mathbb{N}$ convex bodies remains open, except the 2-dimensional case for $m = 3$, which was solved by Heine in [12].

Using Proposition 2.1 we may assure on one hand that there exist convex bodies in $\mathbb{R}^n$ for which (1.6) does not hold, and that there exist convex bodies, which are not Minkowski sums one of each other, satisfying (1.6), on the other hand. We collect the statement here, for the sake of completeness:

Proposition 2.2.

i) *There are convex bodies for which* (1.6) *does not hold.*
ii) *There exist pairs of convex bodies* $K, E \in \mathcal{K}^n$ *for which* (1.6) *does hold and such that* $K \neq M + E$ *for all* $M \in \mathcal{K}^n$.

Proof. The idea for both assertions is to use Proposition 2.1. To prove (i) it is enough to find a log-concave sequence of positive real numbers which is not convex. We consider $(1, 2, 1)$, whose log-concavity (cf. (1.5)) is clearly fulfilled and thus, it ensures the existence of planar simplices $K, E \in \mathcal{K}^2$ with $\mathrm{W}_0(K; E) = \mathrm{W}_2(K; E) = 1$ and $\mathrm{W}_1(K; E) = 2$, and for which (1.6) does not hold.

To prove (ii), we argue in the same way with the sequence $(3/4, 1/2, 1/4)$, which is both log-concave and convex. Since this sequence satisfies (2.2) with equality, and $\mathrm{vol}(K) = 3/4 \neq 1/4 = \mathrm{vol}(E)$, the simplex E cannot be a summand of the simplex K. □

Taking Theorem 1.1 into account, together with the log-concavity of the quermassintegrals of any convex body, it is natural to ask whether the log-concavity together with the convexity (cf. Theorem 1.1) of the sequence of quermassintegrals of a Minkowski sum could provide us with better inequalities or, eventually, equalities in some known inequalities. Unfortunately, so far we have no concluding answers to this issue.

3. Further inequalities

In the following we will write, for $K, E \in \mathcal{K}^n$,

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E) z^i$$

to denote the (relative) Steiner polynomial of K, regarded as a formal polynomial in a complex variable $z \in \mathbb{C}$. Notice that, for $z \geq 0$, $f_{K;E}(z)$ yields the volume of $K + zE$ (cf. (1.1)). Moreover, we consider the derivatives of Steiner polynomials in the variable z, namely

$$f_{K;E}^{(j)}(z) = n(n-1)\cdots(n-j+1) \sum_{k=0}^{n-j} \binom{n-j}{k} \mathrm{W}_{j+k}(K; E) z^k,$$

for $0 \leq j \leq n$. Of particular interest for our purposes will be the quantities $f_i^{(j)}$ given by

$$f_i^{(j)} = \sum_{k=0}^{n-j} \binom{n-j}{k} \mathrm{W}_{i+k}(K; E)(-1)^k \tag{3.1}$$

for any $0 \leq i \leq j \leq n$ (and fixed $K, E \in \mathcal{K}^n$), which can be regarded as formal extensions (up to the constant $n(n-1)\cdots(n-j+1)$) of the value at $z = -1$ of the j-th derivative of the Steiner polynomial, since

$$f_j^{(j)} = n(n-1)\cdots(n-j+1) f_{K;E}^{(j)}(-1).$$

Now we prove some generalizations of the inequalities we have dealt with in the previous section. To avoid straightforward (but a bit lengthy) computations throughout (the proof of) Proposition 3.1, first we collect here some direct relations for combinatorial numbers.

Lemma 3.1. *Let $N, I, m \in \mathbb{N}\cup\{0\}$. If $I \geq m$ then*

$$\binom{N}{I}-\binom{N-1}{I-1}\binom{m}{1}+\cdots+(-1)^{m-1}\binom{N-m+1}{I-m+1}\binom{m}{m-1}+(-1)^m\binom{N-m}{I-m}\geq 0.$$

Moreover, if $I < m$ then

$$\binom{N}{I}-\binom{N-1}{I-1}\binom{m}{1}+\cdots+(-1)^I\binom{N-I}{0}\binom{m}{I}\geq 0.$$

Proof. Assume first that $I \geq m$. Then, by using recursively (2.1), we have

$$\begin{aligned}&\binom{N}{I}-\binom{N-1}{I-1}\binom{m}{1}+\cdots+(-1)^{m-1}\binom{N-m+1}{I-m+1}\binom{m}{m-1}+(-1)^m\binom{N-m}{I-m}\\&=\binom{N-1}{I}-\binom{N-2}{I-1}\binom{m-1}{1}+\cdots+(-1)^{m-2}\binom{N-m+1}{I-m+2}\binom{m-1}{m-2}\\&+(-1)^{m-1}\binom{N-m}{I-m+1}=\cdots=\binom{N-m+1}{I}-\binom{N-m}{I-1}=\binom{N-m}{I}\geq 0.\end{aligned}$$

Now, if $I < m$, and again using (2.1) recursively we get that

$$\begin{aligned}&\binom{N}{I}-\binom{N-1}{I-1}\binom{m}{1}+\cdots+(-1)^I\binom{N-I}{0}\binom{m}{I}=\binom{N-1}{I}\\&-\binom{N-2}{I-1}\binom{m-1}{1}+\cdots+(-1)^{I-1}\binom{N-I+1}{1}\binom{m-1}{I-1}\\&+(-1)^I\binom{N-I-1}{0}\binom{m-1}{I}=\cdots=\binom{N-m+I}{I}-\binom{N-m+I-1}{I-1}\binom{I}{1}\\&+\cdots+(-1)^{I-1}\binom{N-m+1}{1}\binom{I}{I-1}+(-1)^I\binom{N-m}{0},\end{aligned}$$

which, by the previous case, is non-negative. This finishes the proof. □

We note that, in terms of the quantities $f_i^{(j)}$ (see (3.1)), Lemma 2.1 yields that

$$f_i^{(n-2)}\geq 0$$

for every $0 \leq i \leq n-2$. Here we extend these inequalities to any value of j, with $0 \leq i \leq j \leq n$.

Proposition 3.1. *Let $M, E \in \mathcal{K}^n$ with $K = M + E$ and $\dim(E) = n$. Then, for every $0 \leq i \leq j \leq n$,*

$$f_i^{(j)}\geq 0.$$

Equality holds if and only if $\dim(M) \leq n-j-1$.

Proof. From the Steiner formula for quermassintegrals (1.2), and writing $m = n-j$, we have

$$\begin{aligned}
f_i^{(j)} &= \sum_{k=0}^{m} \binom{m}{k} \left(\sum_{r=0}^{n-i-k} \binom{n-i-k}{r} \mathrm{W}_{i+k+r}(M;E) \right) (-1)^k \\
&= \sum_{r=0}^{n-i} \binom{n-i}{r} \mathrm{W}_{i+r}(M;E) - m \sum_{r=0}^{n-i-1} \binom{n-i-1}{r} \mathrm{W}_{i+1+r}(M;E) + \dots \\
&+ (-1)^m \sum_{r=0}^{n-i-m} \binom{n-i-m}{r} \mathrm{W}_{i+m+r}(M;E) \\
&= \sum_{r=0}^{n-i-m} \mathrm{W}_{i+m+r}(M;E) \left(\binom{n-i}{r+m}\binom{m}{0} - \binom{n-i-1}{r+m-1}\binom{m}{1} + \dots \right. \\
&\left. +(-1)^{m-1} \binom{n-i-m+1}{r+1}\binom{m}{m-1} + (-1)^m \binom{n-i-m}{r}\binom{m}{m} \right) \\
&+ \mathrm{W}_{i+m-1}(M;E) \left(\binom{n-i}{m-1}\binom{m}{0} - \binom{n-i-1}{m-2}\binom{m}{1} + \dots \right. \\
&\left. +(-1)^{m-2} \binom{n-i-m+2}{1}\binom{m}{m-2} + (-1)^{m-1} \binom{n-i-m+1}{0}\binom{m}{m-1} \right) \\
&+ \dots + \mathrm{W}_{i+1}(M;E) \left(\binom{n-i}{1}\binom{m}{0} - \binom{n-i-1}{0}\binom{m}{1} \right) + \mathrm{W}_i(M;E).
\end{aligned}$$

From Lemma 3.1 we obtain that all the summands in the above expression are non-negative. Moreover, since the sole non-zero coefficients involved are those with indexes between i and j, we have that $f_i^{(j)} = 0$ if and only if $\mathrm{W}_j(M;E) = 0$. This completes the proof. □

Lemma 3.2. *Let $M, E \in \mathcal{K}^n$ with $K = M + E$ and $\dim(E) = n$. Then, for every $0 \le i < j \le n$,*

$$f_i^{(j)} - f_{i+1}^{(j)} = f_i^{(j-1)} \ge 0.$$

Proof. By using (2.1) we get that

$$\begin{aligned}
f_i^{(j)} - f_{i+1}^{(j)} &= \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k \mathrm{W}_{i+k} - \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k \mathrm{W}_{i+1+k} \\
&= \mathrm{W}_i + \sum_{l=1}^{n-j} (-1)^l \mathrm{W}_{i+l} \left[\binom{n-j}{l} + \binom{n-j}{l-1} \right] + (-1)^{n-j+1} \mathrm{W}_{i+1+n-j} \\
&= \mathrm{W}_i + \sum_{l=1}^{n-j} (-1)^l \mathrm{W}_{i+l} \binom{n-j+1}{l} + (-1)^{n-j+1} \mathrm{W}_{i+1+n-j} = f_i^{(j-1)},
\end{aligned}$$

which, jointly with Proposition 3.1, finishes the proof. □

We conclude the paper by showing an extension of Theorem 1.1. Indeed, the latter is derived by setting $l = n$ in the following result.

Theorem 3.1. *Let $M, E \in \mathcal{K}^n$ with $K = M + E$ and $\dim(E) = n$. Then, for every $0 \le i < j < k \le l \le n$,*

$$(k-j) f_i^{(l)} + (i-k) f_j^{(l)} + (j-i) f_k^{(l)} \ge 0. \tag{3.2}$$

Equality holds if and only if $\dim(M) \leq n - l + 1$.

Proof. The proof is completely analogous to the proof of Theorem 1.1, interchanging every involved r-th quermassintegrals by $f_r^{(l)}$, and using Lemma 3.2 instead of Lemma 2.1. Finally, (3.2) holds with equality if and only if the same holds for

$$f_{j-1}^{(l)} - f_j^{(l)} = f_{j-1}^{(l-1)} \geq f_j^{(l-1)} = f_j^{(l)} - f_{j+1}^{(l)},$$

which is equivalent to $f_{j-1}^{(l-2)} \geq 0$. Then the result follows from the equality case of Proposition 3.1. □

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