A new derivation of two image interpolation formulas from sampling theorems[1]

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Abstract

In this paper we introduce two digital zoom methods based on sampling theory and we study their mathematical foundation. The first one ('sinc interpolation') is commonly used by the image processing community. The second one, as far as the authors know, is new. Both approaches to the final formulas are also new, being formal, but intuitive at the same time.

1 Introduction

Image zooming is a direct application of image interpolation procedures. In fact, a zoom can be easily seen as a homogeneous scaling of the image. Many image interpolation methods have been proposed in the literature [8]. On the other hand, those methods are only justified intuitively, and they do not have a mathematical foundation.

The main aim of this paper is to show how the construction of zooms of digital gray-level images (a natural application) can be seen as a consequence of the well-known digital and analog uniform sampling theorems in dimension two. These theorems are used widely in signal processing and in interpolation (for some applications of these theorems in dimension one we recommend to see [2], [3], [1]), and are the base of digital and analog signal processing.

In section 2 the basic background on digital and analog images is given, together with a formal definition of image interpolation. The next two sections (3 and 4) contain the main results of this work: in section 3, a zooming procedure (sinc interpolation) is derived from the digital uniform sampling theorem, obtaining some formal properties. In section 4, the classical analog sampling theorem of Shannon-Whittaker-Kotelnikov is used, in dimension two, to build a new image zooming procedure. The last section contains some examples of the algorithm performance, empirically comparing both approaches.

2 Preliminaries

2.1 Digital images

A digital image is an array of gray-level values. These values (sometimes called 'samples') are a discrete representation of a continuous function (an analog image), after being applied the process of sampling and quantization. In particular, it is usually assumed that there are only 256 gray levels, so that, given a real number $h \in [0, 1]$ the gray level assigned to h is the $k-$th gray level if $h \in \left[\frac{k}{256}, \frac{k+1}{256}\right)$ for $k = 0, 1, \ldots, 254$ and the 256-th gray level corresponds with

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$h \in \left[\frac{255}{256}, 1\right]$. Moreover, the gray level scale is such that 1 corresponds to the white color and 0 corresponds to the black one.

From a formal point of view, we can think of a digital image of size $N \times M$ as a matrix $I = (I(n,m))_{n=0,\dots,N-1}^{m=0,\dots,M-1}$ of real or complex numbers, being the unique digital images we can visualize those with all entries belonging to the real interval $[0,1]$. This model allow us to identify the set of digital images with the complex vector space

$$\begin{aligned} \ell^2(\mathbb{Z}_N \times \mathbb{Z}_M) &= \Big\{ I : \{0,\dots,N-1\} \times \{0,\dots,M-1\} \to \mathbb{C} : I \text{ is a map} \Big\} \\ &= \Big\{ I : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} : I \text{ is a map and } \forall (k,l) \in \mathbb{Z} \times \mathbb{Z}, \\ &\quad I(k+N,l) = I(k,l) = I(k,l+M) \Big\}, \end{aligned}$$

which is naturally equipped with the following scalar product:

$$\langle I, J \rangle = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} I(n,m)\,\overline{J(n,m)},$$

where the bar denotes complex conjugation. From now on, and for simplicity, we will only consider square images, so that $N = M$.

The standard basis of the space $\ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ is given by

$$\mathbf{T} = \{\mathbf{T}_{i,j}\}_{0 \le i,\, j < N},$$

where

$$\mathbf{T}_{i,j}(n,m) = 0 \text{ if } (n,m) \neq (i,j) \text{ and } \mathbf{T}_{i,j}(i,j) = 1,$$

so that any digital image, when viewed as a two-dimensional signal in the so called "time domain", is given by the expression

$$I = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\,\mathbf{T}_{i,j}.$$

Now, as it occurs for digital one-dimensional signals, there exists another special orthogonal basis which is naturally interpreted in terms of "frequencies". This basis is given by $\mathbf{F} = \{\mathbf{Exp}_{k,l}\}_{0 \le k,l < N}$, where

$$\mathbf{Exp}_{k,l}(n,m) = e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

The usual notation for a digital image I when viewed in the "frequency domain", is

$$I = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \widehat{I}(n,m)\,\mathbf{Exp}_{n,m}.$$

and the map $\mathcal{F} : \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ given by

$$\mathcal{F}\Big((I(n,m))_{0 \le n,m < N} \Big) = \Big(\widehat{I}(n,m) \Big)_{0 \le n,m < N}$$

is the so called discrete Fourier transform (DFT). This map is realized by the formula

$$\widehat{I}(k,l) = \frac{\langle I, \mathbf{Exp}_{k,l}\rangle}{\|\mathbf{Exp}_{k,l}\|^2} = \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} I(n,m)\, e^{\frac{-2\pi \mathbf{i}(kn+lm)}{N}}.$$

Moreover, the following inversion formula holds

$$I(k,l) = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

Now, using the periodicity of the elements of $\ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, one can also introduce negative frequencies and rewrite the inversion formula as follows

$$I(k,l) = \sum_{n=-N/2}^{N/2}\sum_{m=-N/2}^{N/2} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

Finally, we say that the image I is band-limited with band-size $M < N/2$, and we write this as $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$, if

$$I(k,l) = \sum_{n=-M}^{M}\sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

2.2 Analog images

Analog images are the elements of $L^2(\mathbb{R}^2)$. Moreover, we will say that $f \in L^2(\mathbb{R}^2)$ is band limited with band size $\leq M$ if

$$\forall |\xi|, |\nu| > M, \hat{f}(\xi,\nu) = 0,$$

where

$$\hat{f}(\xi,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-2\pi \mathbf{i}(x\xi + y\nu))dxdy$$

denotes the Fourier transform of f. Obviously, these images are precisely those satisfying the formula

$$f(x,y) = \int_{-M}^{M}\int_{-M}^{M} \widehat{f}(\xi,\nu)\exp(2\pi \mathbf{i}(x\xi + y\nu))d\xi d\nu$$

2.3 Image interpolation

Image interpolation [8] is the process of determining the unknown values of an image at positions lying between some known values, called samples. This task is often achieved by fitting a continuous function through the discrete input samples.

Interpolation methods are required in various tasks in image processing and computer vision such as image generation, compression, and zooming. In fact, the last one can be considered as a special case of interpolation, where the zoomed image results from interpolation at certain uniformly distributed samples which are taken to coincide with the original image. This will be our approach in this paper.

The most usual methods to obtain an analog image $f(x, y)$ by using interpolation are expressed as the convolution of the image samples $f_s(k, l)$ with a continuous 2D filter H_{2D}, which is called interpolation kernel:

$$f(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} u(k, l) H_{2D}(x - k, y - l)$$

Usually the interpolation kernel is selected to have the following properties [8]:

(a) Separability: $H_{2D}(x, y) = H_1(x)H_2(y)$.

(b) Symmetry for the separated kernels: $H_i(-x) = H_i(x)$, for $i \in \{1, 2\}$.

(c) Image invariance: $H_i(0) = 1$, and $H_i(x) = 0, \forall |x| = 1, 2, \dots$ and $i \in \{1, 2\}$.

(d) Partition of the unity condition:

$$\sum_{k=-\infty}^{\infty} H_i(d + k) = 1, \forall d : 0 \leq d < 1, \text{ and } i \in \{1, 2\}.$$

Conditions (a) and (b) are needed to avoid computational complexity. With property (c), we do not modify original image samples. Separated kernels that fulfill (c) are called interpolators, and those which does not verify that, are named approximators. The condition (d) implies that the brightness of the image is not altered when the image is interpolated, i.e. the energy (the standard ℓ^2 norm) of the image remains unchanged after the interpolation.

3 Digital sampling and zoom

Let $I(n, m)\, n, m \in \{0 \leq n, m \leq N-1\}$ be a digital image of size $N \times N$. It is quite natural to ask for a simple algorithm to zoom this image into another image J of size $dN \times dN$ for $d = 2, 3, \dots$ (we would say that J is a $(d \times 100)\%$ zoom of I). Clearly, there are several ways to zoom a digital image and all of them imply a certain amount of arbitrariness, since the original image I only gives information about the zoomed image at the points (kd, ld), $k, l \in \{0, \dots, N-1\}$, where the identities $J(kd, ld) = I(k, l)$ are assumed. Thus, we will have a $(d \times 100)\%$ zoom of I as soon as we define a process to reconstruct J at the other points of the array $\{0, \dots, dN-1\}^2$. More precisely, given $E \subseteq \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ and $F \subseteq \ell^2(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$ two spaces of digital images of size $N \times N$ and $dN \times dN$ respectively, we say that a map

$$Z : E \to F$$

defines a d-zoom process if for all $I \in E$ and all $(k, l) \in \{0, \dots, N-1\}^2$, we have that

$$Z(I)(kd, ld) = I(k, l).$$

All the standard algorithms used to zoom digital images lie into this notion. Of course, this definition is too general because it allows too many zoom processes. For example, defining

$Z(I)(t,s) = 1$ whenever $(t,s) \notin \{1, d, 2d, \dots d(N-1)\}^2$ would be considered "a very poor zoom" of I. Thus, it is usual to define zooms via some average process which takes into account the topological or morphological properties of the image I. The most popular methods of zooming are *nearest neighbor interpolation* and *pixel replication* (see [6] for details). These are fast methods and very easy to implement. On the other hand, they produce an undesirable checkerboard effect when applied to get a high factor of magnification. Because of this, several other zooming methods based on the use of bilinear interpolation or low degree spline interpolation have been proposed (see [6] for instance).

We recall now the digital uniform sampling theorem (see [7], [1] for the proof in dimensions two and one, respectively).

Theorem 1 (Digital uniform sampling theorem) *Let $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$ be a digital image of size $N \times N$ and limited band size M. Let d be a divisor of N and let us assume that $d(2M+1) \leq N$. Then I is completely determined by its samples $I(kd, ld)$, $0 \leq k, l \leq r := N/d-1$. In particular, the following synthesis formula holds*

$$I(n,m) = d^2 \sum_{i=0}^{r} \sum_{j=0}^{r} I(di, dj)\, \mathrm{sinc}_M(n-di)\, \mathrm{sinc}_M(m-dj) \tag{1}$$

with $n, m = 0, \dots, N-1$, and where $\mathrm{sinc}_M(n) = \frac{\sin(\pi(2M+1)n/N)}{N \sin(\pi n/N)}$ for $n \neq 0$ and $\mathrm{sinc}_M(0) = \frac{2M+1}{N}$.

Taking into account that the only imposed restriction by our definition of a d-zoom process Z is given by the knowledge of some sampling values of the zoomed image $J = Z(I)$, it follows that a natural question is to study wether Theorem 1 is applicable in order to recover all the entries of J from the known samples. More precisely, we would like to know if J can be chosen as a band-limited digital image, and for which band-size we can guarantee a unique $J = Z(I)$ verifying $J(kd, ld) = I(k, l)$. This is solved by the following result:

Theorem 2 *Let $M < \frac{N}{2}$, $d \in \{2, 3, \dots\}$ and let us assume that $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$. Then there exists a unique band-limited digital image $J \in \mathcal{B}_M(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$ satisfying $J(kd, ld) = I(k, l)$, $k, l \in \{0, \dots, N-1\}$. In particular, there exists a unique d-zoom process (which will be called a sampling d-zoom) $Z_S : \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N) \to \mathcal{B}_M(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$.*

Proof. Let $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$ be a band-limited digital image of size N and band-size $M < \frac{N}{2}$. We have

$$\begin{aligned} I(k,l) &= \sum_{n=-M}^{M} \sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}} \\ &= \sum_{n=-M}^{M} \sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}((kd)n+(ld)m)}{dN}} \\ &= Z_S(I)(kd, ld), \end{aligned}$$

where

$$Z_S(I)(t,s) := \sum_{n=-M}^{M} \sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(tn+sm)}{dN}}. \tag{2}$$

Obviously, Z_S is well defined. Moreover, we can use Theorem 1 to guarantee both the uniqueness of Z_S and the fact that

$$Z_S(I)(n,m) = d^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, \mathrm{sinc}_M(n-di)\, \mathrm{sinc}_M(m-dj), \tag{3}$$

where $n,m = 0,\ldots,dN-1$, since $d(2M+1) \leq dN$. □

For an arbitrary digital image $I \in \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ we will have $M = N/2$, so that $d(2M+1) > dN$ violates one of the assumptions of Theorem 1. This is the reason because we have restricted our attention to the space $\mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$ in Theorem 2. Now, in practice this is just a formalism because images from $\ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ are very well approximable by images from $\mathcal{B}_{[N/2]-1}(\mathbb{Z}_N \times \mathbb{Z}_N)$, $[N/2]$ being the integer part of $N/2$. Indeed, they are visually identical. Moreover, we can also use formula (2) instead of (3) to construct Z_S for arbitrary digital images I.

Remark 3 It is interesting to note that the map $Z_S : \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N) \to \mathcal{B}_M(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$ is an isometry. Thus, we have proved that there exists just one way to introduce a d-zoom between these spaces, and this zoom is also an isometry. On the other hand, if one wants to improve the quality of this zoom, it is natural to look for signals with a bigger band size than the original one. This may be difficult since it is not clear how to introduce the new frequencies from the information given by the old ones.

4 Analog sampling and zoom

The classical analog uniform sampling theorem, in dimension two, reads as follows (see [7], [9]):

Theorem 4 (2-Dimensional analog sampling theorem) *Let $f(x,y)$ be an analog image of finite band size $M < \infty$. Then*

$$f(x,y) = 4M^2 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f\Big(\frac{k}{2M}, \frac{l}{2M}\Big)\, \mathrm{sinc}(2Mx-k)\, \mathrm{sinc}(2My-l)$$

In practice, this theorem implies that, for $f(x,y)$ an analog image of finite band size $M < \infty$, the partial sums

$$P_N(x,y) = 4M^2 \sum_{k=-N}^{N} \sum_{l=-N}^{N} f\Big(\frac{k}{2M}, \frac{l}{2M}\Big)\, \mathrm{sinc}(2Mx-k)\, \mathrm{sinc}(2My-l) \tag{4}$$

are good approximations of $f(x,y)$ inside the square $[\frac{-N}{2M}, \frac{N}{2M}] \times [\frac{-N}{2M}, \frac{N}{2M}]$. In fact, these approximation should be visually good except near the border of the square, where some waves will distort the original image.

Now, the analog d-zoom of $f(x,y)$ is obviously given by the scaling $g(x,y) = f(\frac{x}{d}, \frac{y}{d})$, so that

$$\begin{aligned} g(x,y) &= \int_{-M}^{M}\int_{-M}^{M} \widehat{f}(\xi,\tau) e^{2\pi \mathbf{i}(\frac{x}{d}\xi + \frac{y}{d}\tau)} d\xi d\tau \\ &= d^2 \int_{-M/d}^{M/d}\int_{-M/d}^{M/d} \widehat{f}(d\cdot u, d\cdot v) e^{2\pi\mathbf{i}(xu+yv)}\, du dv. \end{aligned}$$

It follows that $g(x,y)$ is band limited with band size M/d, so that

$$g(x,y) = \frac{4M^2}{d^2} \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g\Big(\frac{kd}{2M}, \frac{ld}{2M}\Big)\operatorname{sinc}\Big(\frac{2M}{d}x - k\Big)\operatorname{sinc}\Big(\frac{2M}{d}y - l\Big) \tag{5}$$

and the partial sums

$$Q_N(x,y) = P_N\Big(\frac{x}{d}, \frac{y}{d}\Big) \tag{6}$$

are good approximations of $g(x,y)$ inside the square $[\frac{-Nd}{2M}, \frac{Nd}{2M}] \times [\frac{-Nd}{2M}, \frac{Nd}{2M}]$.

Let us now assume that $I \in \ell^2(\mathbb{Z}_{2N+1} \times \mathbb{Z}_{2N+1})$ is a digital image which has been constructed by sampling the analog image $f(x,y)$ of finite band size $M < \infty$ on the square $[\frac{-N}{2M}, \frac{N}{2M}] \times [\frac{-N}{2M}, \frac{N}{2M}]$ exactly at the Nyquist rate, so that

$$I(k,l) = f\Big(\frac{k-N}{2M}, \frac{l-N}{2M}\Big),\ k,l = 0,\ldots,2N. \tag{7}$$

If we denote by J a digital d-zoom of I we have that

$$J(kd, ld) = I(k,l) = g\Big(\frac{d(k-N)}{2M}, \frac{d(l-N)}{2M}\Big), \tag{8}$$

so that these samples are enough to recover $g(x,y)$ approximately inside the square $[\frac{-Nd}{2M}, \frac{Nd}{2M}] \times [\frac{-Nd}{2M}, \frac{Nd}{2M}]$. In particular, using 7 and 6, the formula

$$\begin{aligned} J(n,m) &= Q_N\Big(\frac{n-dN}{2M}, \frac{m-dN}{2M}\Big) \qquad (9) \\ &= \frac{4M^2}{d^2}\sum_{k=-N}^{N}\sum_{l=-N}^{N} I(k+N, l+N)\operatorname{sinc}\Big(\frac{n}{d} - N - k\Big)\operatorname{sinc}\Big(\frac{m}{d} - N - l\Big) \end{aligned}$$

defines a reasonable digital d-zoom of I.

The previous discussion is summarized on the following result:

Theorem 5 *Let $f(x,y)$ be an analog image of finite band size $M < \infty$ and let us set*

$$I(k,l) = f\Big(\frac{k-N}{2M}, \frac{l-N}{2M}\Big),\ k,l = 0,\ldots,2N.$$

Then,

$$J(n,m) = \frac{4M^2}{d^2} \sum_{k=-N}^{N} \sum_{l=-N}^{N} I(k+N, l+N)\, \mathrm{sinc}\Big(\frac{n}{d} - N - k\Big)\, \mathrm{sinc}\Big(\frac{m}{d} - N - l\Big)$$

defines a digital d-zoom of I.

Thus, it seems natural to normalize the zoomed image to another whose entries belong to the interval $[0,1]$. Surprisingly, this defines a digital zoom $Z_A(I)$ which is independent of the value of M and is given by the formula

$$Z_A(I) = \frac{J - \min(J)U}{\max(J)} = \frac{E - \min(E)U}{\max(E)}, \tag{10}$$

where E is defined by $E = \frac{1}{M^2}J$, and $U \in \ell^2(\mathbb{Z}_{d(2N+1)} \times \mathbb{Z}_{d(2N+1)})$ is given by $U(i,j) = 1$ for all i, j. It is important to note that E satisfies $E = E(I)$ (i.e., M has no role for the computation of the entries of E).

Particularly this is the digital zoom we wanted to introduce in this section.

Remark 6 It should be noticed that, in practice, the images are not finite band sized, in general, so that a high band size M is needed to get a good approximation of them, according our procedure. This fact implies that our assumption that we have a digital image which has been constructed by sampling an analog image at the Nyquist rate is not a reasonable one, since each pixel covers a square of size bigger than $1/2M$. Moreover, for analog finite band sized images, it is a main problem to know their exact band size M. These objections have motivated the normalized version of the zoom, previously introduced.

Remark 7 In general, the zoomed image defined by (10) is not a band limited digital signal. This could be used to improve the high frequency content of the sampling zoom given by (3). Moreover, the pictures that one visualizes when drawing the frequency content of the zoomed images are highly nonlinear and unpredictable. This should put some light on the difficulty of the problem of improving the sampling d-zoom mentioned at the very end of the section above.

5 A few examples

We have implemented in Matlab 7.0 the algorithm for sampling d-zoom of arbitrary images by using formula (3). The algorithm, for $I \in \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, takes a piece T of I and uses the sampling d-zoom Z_S to produce the image $J = Z_S(T)$. Moreover, we have implemented another algorithm which takes the entire image and, after zooming the whole picture with the d-zoom Z_S, extracts the desired fragment. Finally, we have also implemented the "analog" zoom Z_A given by formula (10). In this case we have tested too the zoom on the fragment of the image, and on the entire image.

We show the two algorithms working over a fragment of two well known test images in computer vision: "Lena" and "Living room" (size $= 512 \times 512$). We have compared our approach with the simpler one, pixel replication, using $d = 8$ for both cases, showing original images too.

(a) Original image with fragment marked

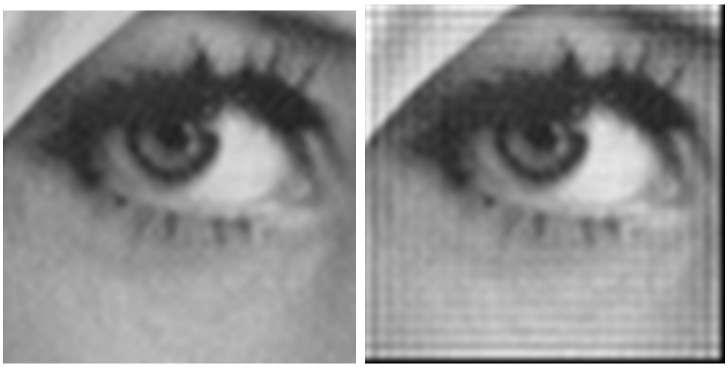
(b) Fragment zoomed with pixel replication

Figure 1: “Lena” image

(a) Using all the information from the original image

(b) Using only information from the fragment

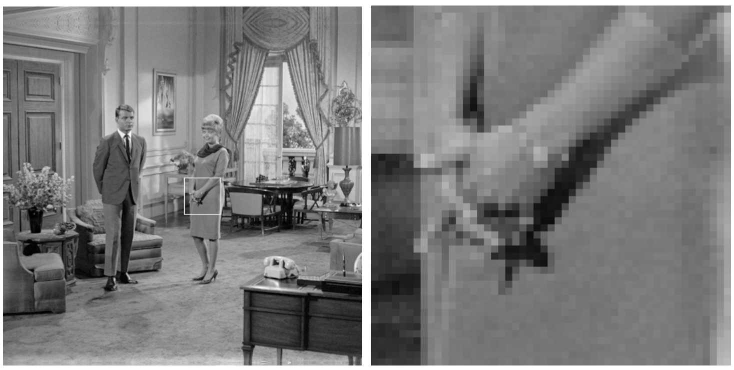
Figure 2: “Lena” image, fragment zoomed with Z_S for $d = 8$



(a) Using all the information from the original image

(b) Using only information from the fragment

Figure 3: “Lena” image, fragment zoomed with Z_A for $d = 8$



(a) Original image with fragment marked

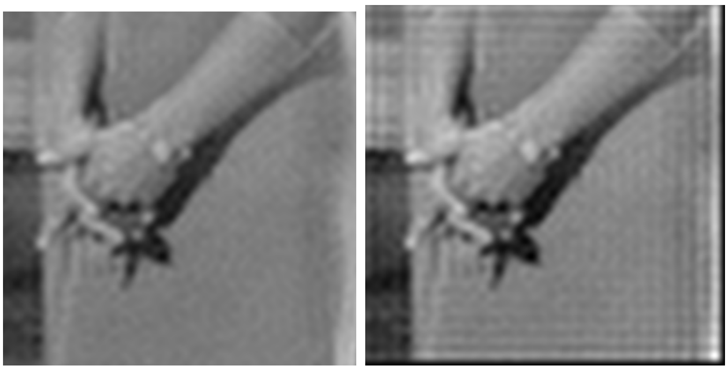
(b) Fragment zoomed with pixel replication

Figure 4: “Living room” image

(a) Using all the information from the original image

(b) Using only information from the fragment

Figure 5: “Living room” image, fragment zoomed with Z_S for $d = 8$



(a) Using all the information from the original image

(b) Using only information from the fragment

Figure 6: “Living room” image, fragment zoomed with Z_A for $d = 8$

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