Standing Wave Patterns

In practical applications it is very convenient to plot the magnitude of phasor voltage and phasor current along the transmission line. These are the standing wave patterns:

$$\text{Loss-less line}\begin{cases} |V(d)| = |V^+| \cdot |(1+\Gamma(d))| \\ |I(d)| = \left|\dfrac{V^+}{Z_0}\right| \cdot |(1-\Gamma(d))| \end{cases}$$

$$\text{Lossy line}\begin{cases} |V(d)| = |V^+ e^{\alpha d}| \cdot |(1+\Gamma(d))| \\ |I(d)| = \left|\dfrac{V^+ e^{\alpha d}}{Z_0}\right| \cdot |(1-\Gamma(d))| \end{cases}$$

The standing wave patterns provide the top envelopes that bound the time-oscillations of voltage and current along the line. In other words, the standing wave patterns provide the maximum values that voltage and current can ever establish at each location of the transmission line for given load and generator.

The standing wave pattern gives a clear representation of wave interference in a transmission line. The patterns present a succession of maxima and minima which repeat in space with a period of length $\lambda/2$, due to constructive or destructive interference between forward and reflected waves. The patterns for a loss-less line are periodic in space, repeating exactly with a $\lambda/2$ period.

Again, note that although we talk about maxima and minima of the standing wave pattern we are always examining a maximum of voltage or current that can be achieved at a transmission line location during any period of oscillation.

We limit now our discussion to the loss-less transmission line case where the generalized reflection coefficient varies as

$$\Gamma(\mathrm{d}) = \Gamma_R \exp(-j2\beta\,\mathrm{d}) = |\Gamma_R| \exp(j\phi) \exp(-j2\beta\,\mathrm{d})$$

Note that the magnitude of an exponential with imaginary argument is always unity, so

$$|\exp(j\phi)\exp(-j2\beta\,\mathrm{d})| = 1$$

and that in a loss-less line it is always true that, at any line location,

$$|\Gamma(\mathrm{d})| = |\Gamma_R|$$

When $\mathbf{d}$ increases, moving from load to generator, the generalized reflection coefficient on the complex plane moves clockwise on a circle with radius $|\Gamma_R|$ and is identified by the angle $\phi - 2\beta\,\mathbf{d}$.

The voltage standing wave pattern has a maximum at locations where the generalized reflection coefficient is real and positive

$$\Gamma(\mathrm{d}) = |\Gamma_R|$$

$$\exp(j\phi)\exp(-j2\beta\,\mathrm{d}) = 1 \qquad \Rightarrow \qquad |\phi - 2\beta\,\mathrm{d}| = 2n\pi$$

At these locations we have

$$|1+\Gamma(\mathrm{d})| = 1 + |\Gamma_R|$$

$$\Rightarrow V_{\max} = |V(\mathrm{d}_{\max})| = |V^+|\cdot(1+|\Gamma_R|)$$

The phase angle $\phi - 2\beta\,\mathrm{d}$ changes by an amount 2π, when moving from one maximum to the next. This corresponds to a distance between successive maxima of $\lambda/2$.

The voltage standing wave pattern has a minimum at locations where the generalized reflection coefficient is real and negative

$$\Gamma(\mathrm{d}) = -\left|\Gamma_R\right|$$

$$\exp\left(j\phi\right)\exp\left(-j2\beta\,\mathrm{d}\right) = -1 \qquad \Rightarrow \qquad \left|\phi - 2\beta\,\mathrm{d}\right| = \left(2n+1\right)\pi$$

At these locations we have

$$\left|1+\Gamma(\mathrm{d})\right| = 1 - \left|\Gamma_R\right|$$

$$\Rightarrow V_{\min} = \left|V(\mathrm{d}_{\min})\right| = \left|V^+\right| \cdot \left(1 - \left|\Gamma_R\right|\right)$$

Also when moving from one minimum to the next, the phase angle $\phi - 2\beta\,\mathrm{d}$ changes by an amount 2π. This again corresponds to a distance between successive minima of $\lambda/2$.

The voltage standing wave pattern provides immediate information on the transmission line circuit

- If the load is matched to the transmission line ($Z_R = Z_0$) the voltage standing wave pattern is flat, with value $|\,V^+\,|$.
- If the load is real and $Z_R > Z_0$, the voltage standing wave pattern starts with a maximum at the load.
- If the load is real and $Z_R < Z_0$, the voltage standing wave pattern starts with a minimum at the load.
- If the load is complex and $\mathrm{Im}(Z_R) > 0$ (inductive reactance), the voltage standing wave pattern initially increases when moving from load to generator and reaches a maximum first.
- If the load is complex and $\mathrm{Im}(Z_R) < 0$ (capacitive reactance), the voltage standing wave pattern initially decreases when moving from load to generator and reaches a minimum first.

Since in all possible cases

$$\left|\Gamma(\mathrm{d})\right| \leq 1$$

the voltage standing wave pattern

$$\left|V(\mathrm{d})\right| = \left|V^{+}\right| \cdot \left|\left(1+\Gamma(\mathrm{d})\right)\right|$$

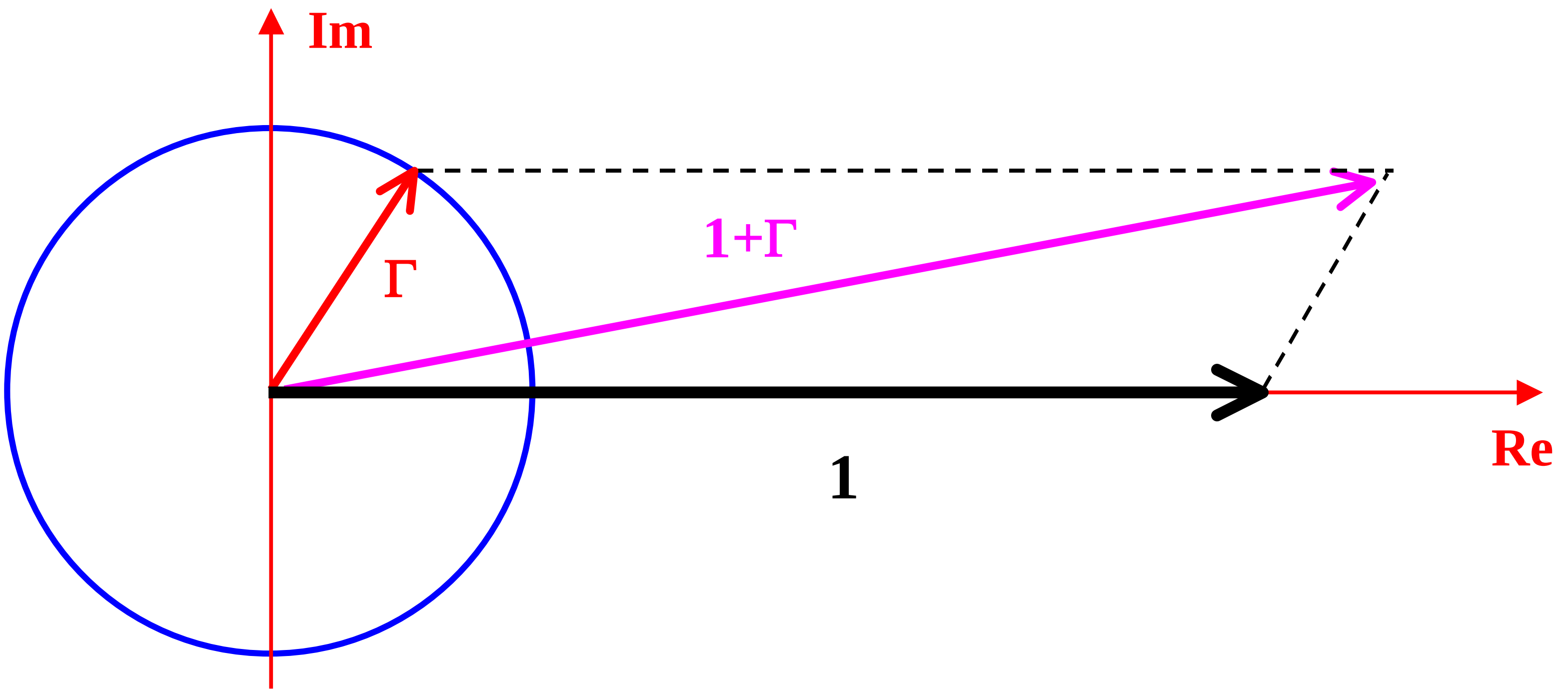
cannot exceed the value $2\,|V^{+}|$ in a loss-less transmission line. If the load is a short circuit, an open circuit, or a pure reactance, there is total reflection with

$$\left|\Gamma(\mathrm{d})\right| = 1$$

since the load cannot consume any power. The voltage standing wave pattern in these cases is characterized by

$$V_{\max} = 2\left|V^{+}\right| \qquad \text{and} \qquad V_{\min} = 0 \,.$$

The quantity $1 + \Gamma(d)$ is in general a complex number, that can be constructed as a vector on the complex plane. The number 1 is represented as $1 + j0$ on the complex plane, and it is just a vector with coordinates $(1,0)$ positioned on the Real axis. The reflection coefficient $\Gamma(d)$ is a complex number such that $|\Gamma(d)| \leq 1$.

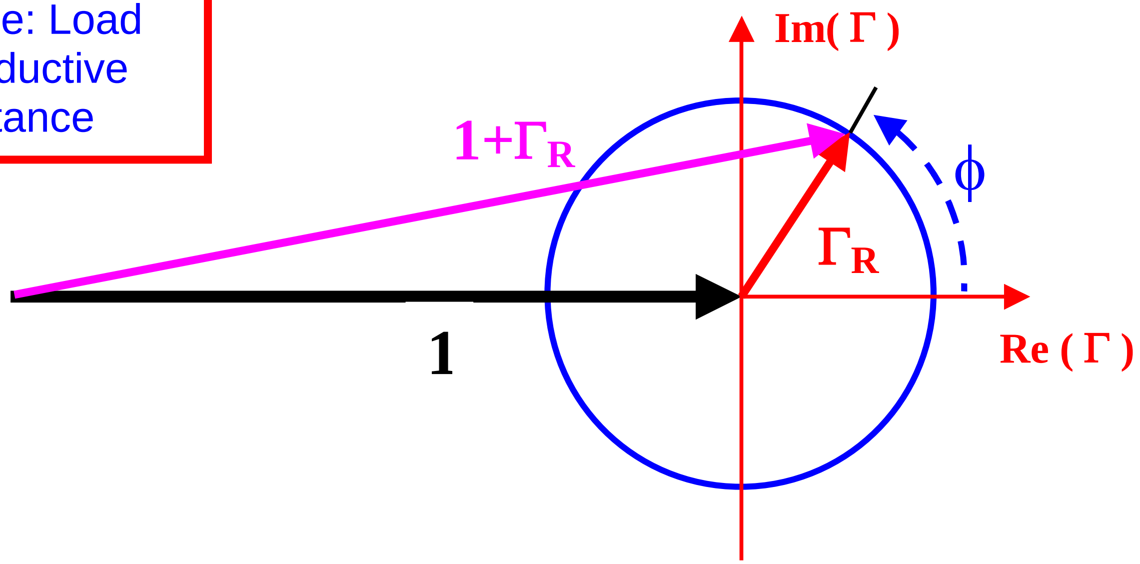


We can use a geometric construction to visualize the behavior of the voltage standing wave pattern

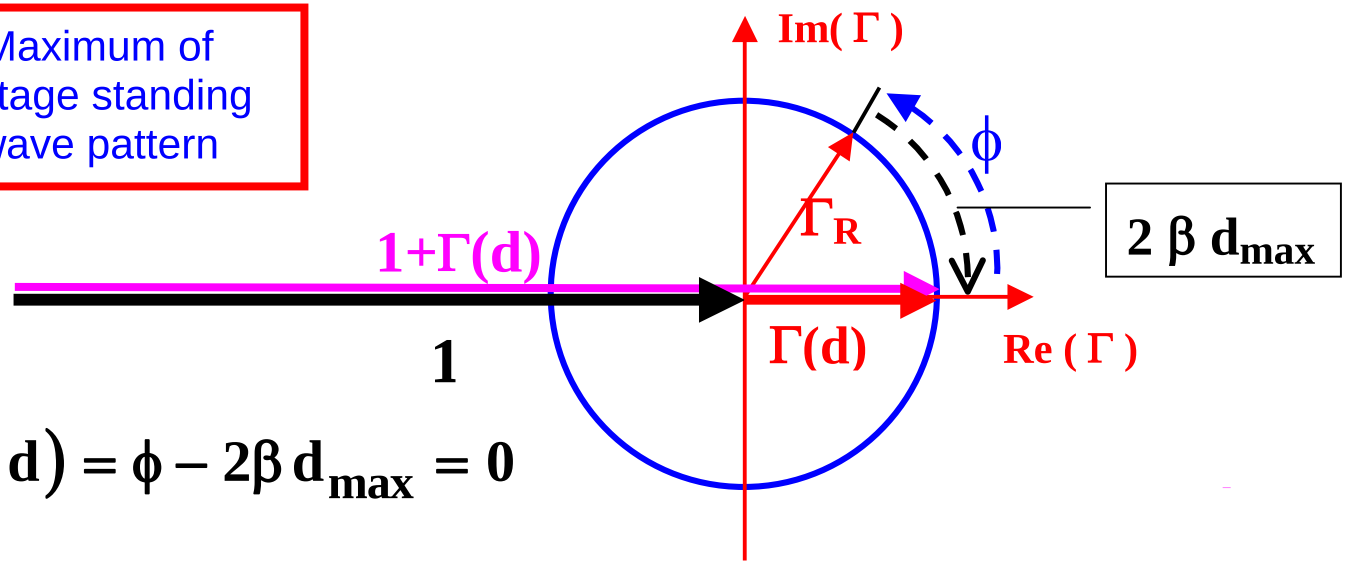
$$|V(\mathrm{d})| = |V^{+}| \cdot |(1+\Gamma(\mathrm{d}))|$$

simply by looking at a vector plot of $|(1+\Gamma(d))|$. $|V+|$ is just a scaling factor, fixed by the generator. For convenience, we place the reference of the complex plane representing the reflection coefficient in correspondence of the tip of the vector (1, 0).

Example: Load with inductive reactance

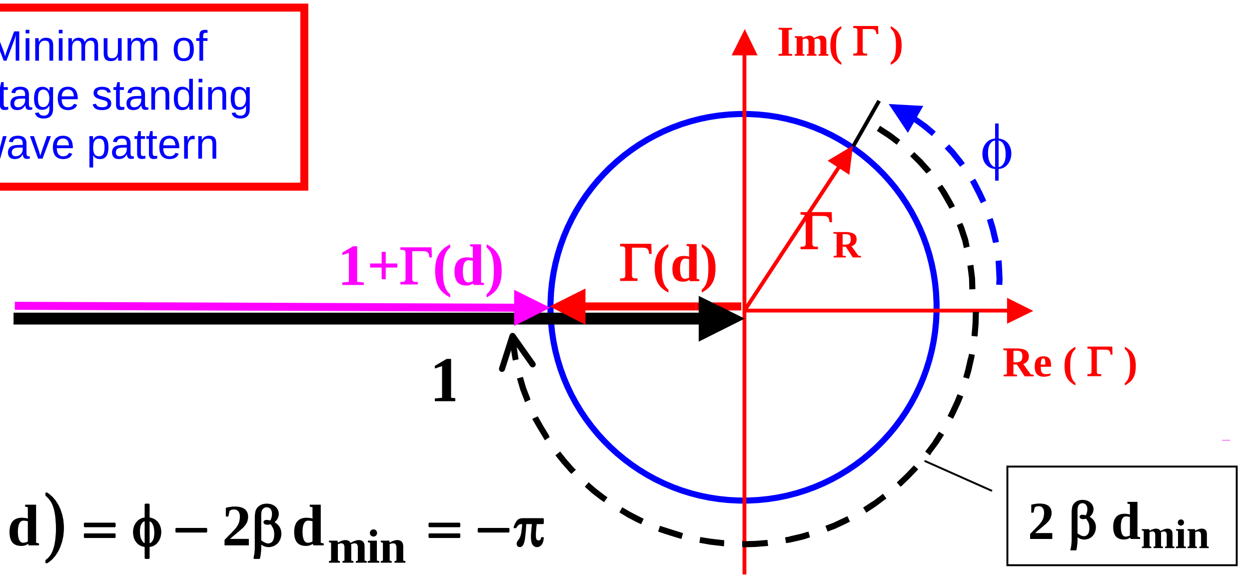


Maximum of
voltage standing
wave pattern



$$\angle\Gamma\left(\mathbf{d}\right)=\phi-2\beta\,\mathbf{d}_{\mathbf{max}}=0$$

Minimum of
voltage standing
wave pattern



$$\angle\Gamma\left(\mathbf{d}\right)=\phi-2\beta\,\mathbf{d}_{\mathbf{min}}=-\pi$$

The voltage standing wave ratio (VSWR) is an indicator of load matching which is widely used in engineering applications

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}$$

When the load is perfectly matched to the transmission line

$$\Gamma_R = 0 \qquad \Rightarrow \qquad VSWR = 1$$

When the load is a short circuit, an open circuit or a pure reactance

$$|\Gamma_R| = 1 \qquad \Rightarrow \qquad VSWR \to \infty$$

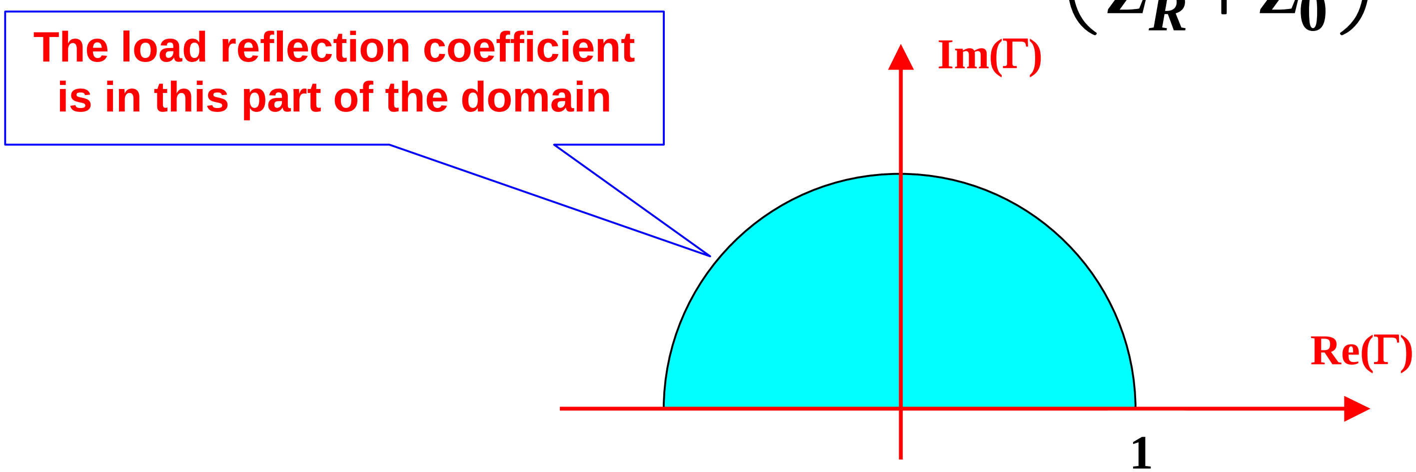
We have the following useful relation

$$|\Gamma_R| = \frac{VSWR - 1}{VSWR + 1}$$

Maxima and minima of the voltage standing wave pattern.

❑ Load with inductive reactance

$$\mathrm{Im}\left(Z_R\right) > 0 \qquad \Rightarrow \qquad \mathrm{Im}\left(\Gamma_R\right) = \mathrm{Im}\left(\frac{Z_R - Z_0}{Z_R + Z_0}\right) > 0$$

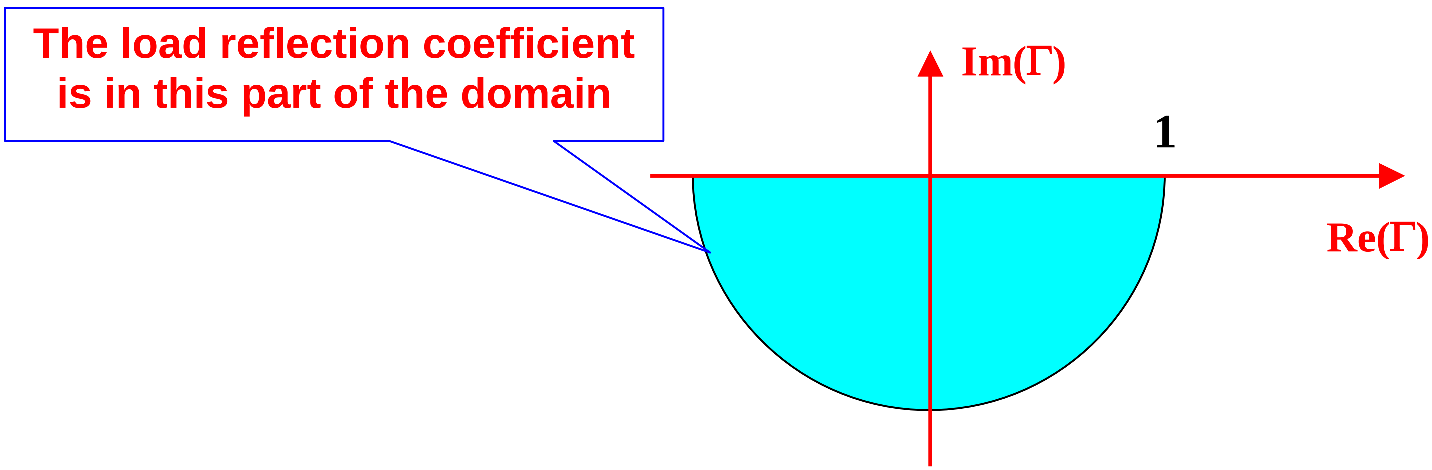


The first maximum of the voltage standing wave pattern is closest to the load, at location

$$\angle\Gamma\left(\mathrm{d}\right) = \phi - 2\beta\,\mathrm{d}_{\max} = 0 \qquad \Rightarrow \qquad \mathrm{d}_{\max} = \frac{\phi}{4\pi}\lambda$$

Load with capacitive reactance

$$\mathrm{Im}\left(Z_R\right)<0 \qquad \Rightarrow \qquad \mathrm{Im}\left(\Gamma_R\right)=\mathrm{Im}\left(\frac{Z_R-Z_0}{Z_R+Z_0}\right)<0$$



The first minimum of the voltage standing wave pattern is closest to the load, at location

$$\angle\Gamma(\mathrm{d})=\left|\phi-2\beta\,\mathrm{d}_{\min}\right|=\pi \qquad \Rightarrow \qquad \mathrm{d}_{\min}=\frac{\left|\pi-\phi\right|}{4\pi}\lambda$$

A measurement of the voltage standing wave pattern provides the locations of the first voltage maximum and of the first voltage minimum with respect to the load.

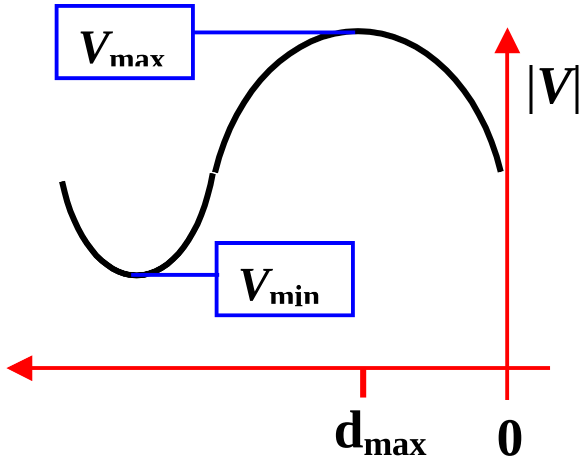
The ratio of the voltage magnitude at these points gives directly the voltage standing wave ratio (VSWR).

This information is sufficient to determine the load impedance Z_R , if the characteristic impedance of the transmission line Z_0 is known.

- **STEP 1: The VSWR provides the magnitude of the load reflection coefficient**

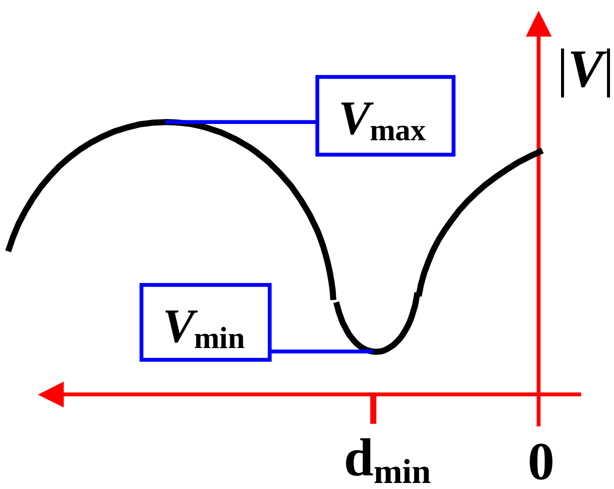
$$\left|\Gamma_R\right| = \frac{VSWR - 1}{VSWR + 1}$$

- **STEP 2: The distance from the load of the first maximum or minimum gives the phase ϕ of the load reflection coefficient.**



For an inductive reactance, a voltage maximum is closest to the load and

$$\phi = 2\beta\, d_{max} = \frac{4\pi}{\lambda} d_{max}$$



For a capacitive reactance, a voltage minimum is closest to the load and

$$\phi = -\pi + 2\beta\, d_{min} = -\pi + \frac{4\pi}{\lambda} d_{min}$$

- **STEP 3: The load impedance is obtained by inverting the expression for the reflection coefficient**

$$\Gamma_R = \left|\Gamma_R\right| \exp\left(j\phi\right) = \frac{Z_R - Z_0}{Z_R + Z_0}$$

$$\Rightarrow \quad Z_R = Z_0 \frac{1 + \left|\Gamma_R\right| \exp\left(j\phi\right)}{1 - \left|\Gamma_R\right| \exp\left(j\phi\right)}$$