New preservation properties for stochastic orderings and aging classes under the formation of order statistics and systems

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The distorted distributions were introduced in Yaari's dual theory of choice under risk (Econometrica 55 (1987):95–115).

The *distorted distribution* (DD) associated to a distribution function (DF) $F$ and to an increasing continuous distortion function $q : [0, 1] \rightarrow [0, 1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \quad (1.1)$$

If $q$ is strictly increasing, then $F$ and $F_q$ have the same support.

For the reliability functions (RF) $\overline{F} = 1 - F$, $\overline{F}_q = 1 - F_q$, we have

$$\overline{F}_q(t) = \overline{q}(\overline{F}(t)), \quad (1.2)$$

where $\overline{q}(u) = 1 - q(1 - u)$ is the dual distortion function.
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where \( \overline{q}(u) = 1 - q(1 - u) \) is the **dual distortion function**.
The generalized distorted distribution (GDD) associated to \( n \) DF \( F_1, \ldots, F_n \) and to an increasing continuous multivariate distortion function \( Q : [0,1]^n \rightarrow [0,1] \) such that \( Q(0,\ldots,0) = 0 \) and \( Q(1,\ldots,1) = 1 \), is

\[
F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \tag{1.3}
\]

If \( Q \) is strictly increasing and \( F_1, \ldots, F_n \) have the same support, then \( F_Q \) also has the same support.

For the RF we have

\[
\bar{F}_Q(t) = \overline{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)), \tag{1.4}
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Generalized Distorted Distributions (GDD)

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The PHR (Cox) model associated to a RF $\bar{F}$ is

$$\bar{F}_\alpha(t) = (\bar{F}(t))^{\alpha} = \bar{q}(\bar{F}(t))$$

for $\alpha > 0$. $\bar{F}_\alpha$ a DD with $\bar{q}(u) = u^\alpha$ and $q(u) = 1 - (1 - u)^\alpha$.

The hazard (failure) rate function is defined by $h(t) = f(t)/\bar{F}(t)$ where $f$ is the PDF.

Under the PHR model, $h_\alpha(t) = \alpha h(t)$.

The proportional reversed hazard rate (PRHR) model is

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Order statistics (OS)

- $X_1, \ldots, X_n$ IID~$F$ random variables.
- $X_1, \ldots, X_n$ exchangeable (EXC), i.e., for any permutation $\sigma$
  \[ (X_1, \ldots, X_n) =_{ST} (X_{\sigma(1)}, \ldots, X_{\sigma(n)}). \]
- $(X_1, \ldots, X_n)$ is an arbitrary random vector with
  \[ F(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) \]
  \[ \overline{F}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n). \]
- Let $X_{1:n}, \ldots, X_{n:n}$ be the associated OS.
- Let $F_{i:n}(t) = \Pr(X_{i:n} \leq t)$ be the DF.
- Let $\overline{F}_{i:n}(t) = \Pr(X_{i:n} > t)$ be the RF.
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$$ (X_1, \ldots, X_n) = ST (X_{\sigma(1)}, \ldots, X_{\sigma(n)}). $$

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Distorted Distribution Representation-IID case

In the IID case, we have

\[ F_{i:n}(t) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} F_{j:j}(t) = q_{i:n}(F(t)), \quad (1.5) \]

(see David and Nagaraja 2003, p. 46) where

\[ F_{j:j}(t) = \Pr(X_{j:j} \leq t) = \Pr(\max(X_1, \ldots, X_j) \leq t) = F^j(t) \]

and

\[ q_{i:n}(u) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} u^j \]

is a strictly increasing polynomial in \([0, 1]\).

Both \(F_{j:j}\) and \(F_{i:n}\) are DD from \(F\).
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Both \( F_{j:j} \) and \( F_{i:n} \) are DD from \( F \).
The upper OS $X_{j:j}$ (lifetime of the parallel system) satisfies the PRHR model with $\alpha = j$ since

$$F_{j:j}(t) = \Pr(X_{j:j} \leq t) = \Pr(\max(X_1, \ldots, X_j) \leq t) = (F(t))^j.$$ 

The lower OS $X_{1:j}$ (lifetime of the series system) satisfies the PHR model

$$\overline{F}_{1:j}(t) = \Pr(X_{1:j} \leq t) = \Pr(\min(X_1, \ldots, X_j) > t) = (\overline{F}(t))^j.$$ 

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In the EXC case the left hand side of (1.5) holds with

\[ F_{j:j}(t) = \Pr(\max(X_1, \ldots, X_j) \leq t) = F(t, \ldots, t, \infty, \ldots, \infty). \]

The copula representation for \( F \) is

\[ F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \]

where \( F_i(t) = \Pr(X_i \leq t) \) and \( C \) is the copula.

In the EXC case, \( F_1 = \cdots = F_n = F \) and

\[ F_{j:j}(t) = C(F(t), \ldots, F(t), 1, \ldots, 1) = q_{j:j}^{C}(F(t)) \]

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The copula representation for \( F \) is
\[ F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \quad (1.6) \]
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Both \( F_{j:j} \) and \( F_{i:n} \) are DD from \( F \).
In the general case

\[ F_{i:n}(t) = \Pr(X_{i:n} \leq t) = \Pr \left( \bigcup_{j=1}^{r} \{ X_{j}^{C} \leq t \} \right) \]

where \( X_{j}^{C} = \max_{k \in C_j} X_k \) and \( |C_j| = i, j = 1, \ldots, r, r = \binom{n}{i} \).

Then

\[ F_{i:n}(t) = \sum_{j=1}^{r} \Pr(X_{j}^{C} \leq t) - \sum_{j \neq k} \Pr(X_{j}^{C} \cup X_{k}^{C} \leq t) + \ldots \pm \Pr(X_{1}^{C} \cup \ldots \cup X_{r}^{C} \leq t) \]

By using the copula representation (1.6)

\[ F^{A}(t) = \Pr(X^{A} \leq t) = \Pr(\max_{j \in A} X_j \leq t) = C(F_1(x_1^{A}), \ldots, F_n(x_n^{A})) \]

where \( x_{i}^{A} = t \) if \( i \in A \) and \( x_{i}^{A} = \infty \) if \( i \notin A, A \subseteq \{1, \ldots, n\} \).
In the general case

\[ F_{i:n}(t) = \Pr(X_{i:n} \leq t) = \Pr \left( \bigcup_{j=1}^{r} \{ X^{C_j} \leq t \} \right) \]

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By using the copula representation (1.6)

\[ F^A(t) = \Pr(X^A \leq t) = \Pr(\max_{j \in A} X_j \leq t) = C(F_1(x^A_1), \ldots, F_n(x^A_n)), \]

where \( x^A_i = t \) if \( i \in A \) and \( x^A_i = \infty \) if \( i \notin A, A \subseteq \{1, \ldots, n\} \).
In the general case

\[ F_{i:n}(t) = \Pr(X_{i:n} \leq t) = \Pr \left( \bigcup_{j=1}^{r} \{X_C \leq t\} \right) \]

where \( X_C = \max_{k \in C_j} X_k \) and \( |C_j| = i, j = 1, \ldots, r, r = \binom{n}{i} \).

Then

\[
F_{i:n}(t) = \sum_{j=1}^{r} \Pr(X_C \leq t) - \sum_{j \neq k} \Pr(X_{C_j \cup C_k} \leq t) + \ldots \pm \Pr(X_{C_1 \cup \ldots \cup C_r} \leq t)
\]

By using the copula representation (1.6)

\[ F^A(t) = \Pr(X^A \leq t) = \Pr(\max_{j \in A} X_j \leq t) = C(F_1(x^A_1), \ldots, F_n(x^A_n)), \]

where \( x^A_i = t \) if \( i \in A \) and \( x^A_i = \infty \) if \( i \notin A \), \( A \subseteq \{1, \ldots, n\} \).
Therefore

\[ F^A(t) = Q_A^C(F_1(t), \ldots, F_n(t)) \]

for all \( A \subseteq \{1, \ldots, n\} \), where \( Q_A^C(u_1, \ldots, u_n) = C(u_1^A, \ldots, u_n^A) \)
and \( u_i^A = u_i \) if \( i \in A \) and \( u_i^A = 1 \) if \( i \notin A \).

So

\[
F_{i:n}(t) = \sum_{j=1}^{r} Q_{C_j}^C(F_1(t), \ldots, F_n(t)) - \sum_{j \neq k} Q_{C_j \cup C_k}^C(F_1(t), \ldots, F_n(t)) \\
+ \cdots \pm Q_{C_1 \cup \cdots \cup C_r}^C(F_1(t), \ldots, F_n(t)) \\
= Q_{i:n}^C(F_1(t), \ldots, F_n(t)).
\]

Both \( F^A \) and \( F_{i:n} \) are GDD from \( F_1, \ldots, F_n \).
Both are DD when \( F_1 = \cdots = F_n \) (ID).
Therefore

\[ F^A(t) = Q^C_A(F_1(t), \ldots, F_n(t)) \]

for all \( A \subseteq \{1, \ldots, n\} \), where \( Q^C_A(u_1, \ldots, u_n) = C(u^A_1, \ldots, u^A_n) \)
and \( u^A_i = u_i \) if \( i \in A \) and \( u^A_i = 1 \) if \( i \notin A \).

So

\[
F_{i:n}(t) = \sum_{j=1}^{r} \left( \sum_{Q^C_{C_j} \subseteq C_k} Q^C_{C_j} \right) (F_1(t), \ldots, F_n(t)) - \sum_{j \neq k} Q^C_{C_j \cup C_k} (F_1(t), \ldots, F_n(t)) \\
+ \cdots \pm Q^C_{C_1 \cup \ldots \cup C_r} (F_1(t), \ldots, F_n(t)) \\
= Q^C_{i:n} (F_1(t), \ldots, F_n(t)).
\]

Both \( F^A \) and \( F_{i:n} \) are GDD from \( F_1, \ldots, F_n \).

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for all \( A \subseteq \{1, \ldots, n\} \), where \( Q_C^A(u_1, \ldots, u_n) = C(u_1^A, \ldots, u_n^A) \)
and \( u_i^A = u_i \) if \( i \in A \) and \( u_i^A = 1 \) if \( i \notin A \).

So

\[
F_{i:n}(t) = \sum_{j=1}^{r} Q_C^{C_j}(F_1(t), \ldots, F_n(t)) - \sum_{j \neq k} Q_C^{C_j \cup C_k}(F_1(t), \ldots, F_n(t)) \\
+ \cdots \pm Q_C^{C_1 \cup \ldots \cup C_r}(F_1(t), \ldots, F_n(t)) \\
= Q_C^{C_i}(F_1(t), \ldots, F_n(t)).
\]

Both \( F^A \) and \( F_{i:n} \) are GDD from \( F_1, \ldots, F_n \).

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So

\[
F_{i:n}(t) = \sum_{j=1}^{r} Q_{C_j}^C(F_1(t), \ldots, F_n(t)) - \sum_{j \neq k} Q_{C_j \cup C_k}^C(F_1(t), \ldots, F_n(t)) \\
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\]

Both \( F^A \) and \( F_{i:n} \) are GDD from \( F_1, \ldots, F_n \).

Both are DD when \( F_1 = \cdots = F_n \) (ID).
An example-General case

Let us consider $X_{2:3}$, then $C_1 = \{1, 2\}$, $C_2 = \{1, 3\}$, $C_3 = \{2, 3\}$

$$F_{2:3}(t) = \Pr\left(\{X^{1,2} \leq t\} \cup \{X^{1,3} \leq t\} \cup \{X^{2,3} \leq t\}\right)$$

$$= \Pr\left(\{X^{1,2} \leq t\}\right) + \Pr\left(\{X^{1,3} \leq t\}\right) + \Pr\left(\{X^{2,3} \leq t\}\right)$$

$$- 2 \Pr\left(\{X^{1,2,3} \leq t\}\right)$$

$$= \mathbf{F}(t, t, \infty) + \mathbf{F}(t, \infty, t) + \mathbf{F}(\infty, t, t) - 2\mathbf{F}(t, t, t)$$

$$= C(F_1(t), F_2(t), 1) + C(F_1(t), 1, F_3(t)) + C(1, F_2(t), F_3(t))$$

$$- 2C(F_1(t), F_2(t), F_3(t)) = Q_{2:3}^C(F_1(t), F_2(t), F_3(t)),$$

where

$$Q_{2:3}^C(u_1, u_2, u_3) = C(u_1, u_2, 1) + C(u_1, 1, u_3) + C(1, u_2, u_3) - 2C(u_1, u_2, u_3).$$
An example—Particular cases

- In the EXC case, we get
  \[
  F_{2:3}(t) = C(F(t), F(t), 1) + C(F(t), 1, F(t)) + C(1, F(t), F(t)) \\
  - 2C(F(t), F(t), F(t)) \\
  = 3C(F(t), F(t), 1) - 2C(F(t), F(t), F(t)) = q_{2:3}^C(F(t)),
  \]
  where \( q_{2:3}^C(u) = 3C(u, u, 1) - 2C(u, u, u) \).

- In the IID case, for \( q_{2:3}(u) = 3u^2 - 2u^3 \), we have
  \[
  F_{2:3}(t) = F^2(t) - 3F^3(t) = q_{2:3}(F(t)).
  \]

- In the INID case, we get
  \[
  F_{2:3}(t) = F_1(t)F_2(t) + F_1(t)F_3(t) + F_2(t)F_3(t) - 2F_1(t)F_2(t)F_3(t) \\
  = Q_{2:3}(F_1(t), F_2(t), F_3(t)),
  \]
  where \( Q_{2:3}(u_1, u_2, u_3) = u_1u_2 + u_1u_3 + u_2u_3 - 2u_1u_2u_3 \).
In the EXC case, we get

\[ F_{2:3}(t) = C(F(t), F(t), 1) + C(F(t), 1, F(t)) + C(1, F(t), F(t)) - 2C(F(t), F(t), F(t)) \\
= 3C(F(t), F(t), 1) - 2C(F(t), F(t), F(t)) = q^C_{2:3}(F(t)), \]

where \( q^C_{2:3}(u) = 3C(u, u, 1) - 2C(u, u, u) \).

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= Q_{2:3}(F_1(t), F_2(t), F_3(t)), \]

where \( Q_{2:3}(u_1, u_2, u_3) = u_1u_2 + u_1u_3 + u_2u_3 - 2u_1u_2u_3. \)
In the EXC case, we get

\[ F_{2:3}(t) = C(F(t), F(t), 1) + C(F(t), 1, F(t)) + C(1, F(t), F(t)) - 2C(F(t), F(t), F(t)) = 3C(F(t), F(t), 1) - 2C(F(t), F(t), F(t)) = q_C^{2:3}(F(t)), \]

where \( q_C^{2:3}(u) = 3C(u, u, 1) - 2C(u, u, u). \)

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where \( Q_{2:3}(u_1, u_2, u_3) = u_1u_2 + u_1u_3 + u_2u_3 - 2u_1u_2u_3. \)
A coherent system is

\[ \phi = \phi(x_1, \ldots, x_n) : \{0, 1\}^n \rightarrow \{0, 1\}, \]

where \( x_i \in \{0, 1\} \) (it represents the state of the \( i \)th component) and where \( \phi \) (which represents the state of the system) is increasing in \( x_1, \ldots, x_n \) and strictly increasing in \( x_i \) for at least a point \( (x_1, \ldots, x_n) \), for all \( i = 1, \ldots, n \).

- If \( X_1, \ldots, X_n \) are the component lifetimes, then there exists \( \psi \) such that the system lifetime \( T = \psi(X_1, \ldots, X_n) \).
- \( X_{1:n}, \ldots, X_{n:n} \) are the lifetimes of \( k \)-out-of-\( n \) systems.
- \( X_{1:n} \) is the series system lifetime and \( X_{n:n} \) is the parallel system lifetime.
Coherent systems

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Coherent systems- IID and EXC case

- Samaniego (IEEE TR, 1985), IID case:

\[ \bar{F}_T(t) = \sum_{i=1}^{n} p_i \bar{F}_{i:n}(t), \]  \hspace{1cm} (1.7)

where \( p_i = \Pr(T = X_{i:n}). \)

- \( p = (p_1, \ldots, p_n) \) is the signature of the system.
- IID case: \( p_i \) only depends on \( \phi \)

\[ p_i = \frac{\left| \{ \sigma : \phi(x_1, \ldots, x_n) = x_{i:n}, \text{ when } x_{\sigma(1)} < \ldots < x_{\sigma(n)} \} \right|}{n!} \]  \hspace{1cm} (1.8)

- Navarro, Samaniego, Balakrishnan and Bhathacharya (NRL, 2008), (1.7) holds for EXC r.v. when \( p \) is given by (1.8).
- In both cases \( \bar{F}_T \) is a DD from \( \bar{F}. \)
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Generalized mixture representations

- Navarro, Ruiz and Sandoval (CSTM, 2007), EXC case:

\[ \bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t). \quad (1.9) \]

- \( a = (a_1, \ldots, a_n) \) is the minimal signature of \( T \).
- \( a_i \) only depends on \( \phi \) but can be negative and so (1.9) is called a generalized mixture.

- In the IID case:

\[ \bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_i(t) = \bar{q}_\phi(\bar{F}(t)), \quad (1.10) \]

\[ \bar{q}_\phi(x) = \sum_{i=1}^{n} a_i x^i \] is the domination (reliability) polynomial.
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\bar{q}_\phi(x) = \sum_{i=1}^{n} a_i x^i \text{ is the domination (reliability) polynomial.}
\]
A path set of $T$ is a set $P \subseteq \{1, \ldots, n\}$ such that if all the components in $P$ work, then the system works.

A minimal path set of $T$ is a path set which does not contains other path sets.

If $P_1, \ldots, P_r$ are the minimal path sets of $T$, then $T = \max_{j=1,\ldots,r} X_{P_j}$, where $X_P = \min_{i \in P} X_i$ and

$$
\bar{F}_T(t) = \Pr \left( \max_{j=1,\ldots,r} X_{P_j} > t \right) = \Pr \left( \bigcup_{j=1}^r \{X_{P_j} > t\} \right) = \sum_{i=1}^r \bar{F}_{P_i}(t) - \sum_{i \neq j} \bar{F}_{P_i \cup P_j}(t) + \cdots \pm \bar{F}_{P_1 \cup \cdots \cup P_r}(t)
$$

where $\bar{F}_P(t) = \Pr(X_P > t)$. 
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Jorge Navarro, E-mail: jorgenav@um.es
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$$

where $\overline{F}_P(t) = \Pr(X_P > t)$. 
Coherent systems-General case

- The copula representation for the RF of \((X_1, \ldots, X_n)\) is
  \[ F(x_1, \ldots, x_n) = K(F_1(x_1), \ldots, F(x_n)), \]
  where \(F_i(t) = \Pr(X_i > t)\) and \(K\) is the survival copula.

- Then
  \[ F_P(t) = Q_{P,K}(F_1(t), \ldots, F_n(t)), \]
  where \(Q_{P,K}(u_1, \ldots, u_n) = K(u_1^P, \ldots, u_n^P)\) and \(u_i^P = u_i\) for \(i \in P\) and \(u_i^P = 1\) for \(i \notin P\).

- Therefore, from the minimal path set repres., we get
  \[ F_T(t) = Q_{\phi,K}(F_1(t), \ldots, F_n(t)). \]

- In the ID case
  \[ F_T(t) = q_{\phi,K}(F(t)). \quad (1.11) \]
Coherent systems-General case

The copula representation for the RF of \((X_1, \ldots, X_n)\) is

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\bar{F}(x_1, \ldots, x_n) = K(\bar{F}_1(x_1), \ldots, \bar{F}(x_n)),
\]

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\[
\bar{F}_P(t) = Q_{P,K}(\bar{F}_1(t), \ldots, \bar{F}_n(t)),
\]

where \(Q_{P,K}(u_1, \ldots, u_n) = K(u_1^P, \ldots, u_n^P)\) and \(u_i^P = u_i\) for \(i \in P\) and \(u_i^P = 1\) for \(i \notin P\).

Therefore, from the minimal path set repres., we get

\[
\bar{F}_T(t) = Q_{\phi,K}(\bar{F}_1(t), \ldots, \bar{F}_n(t)).
\]

In the ID case

\[
\bar{F}_T(t) = q_{\phi,K}(\bar{F}(t)).
\] (1.11)
The copula representation for the RF of \((X_1, \ldots, X_n)\) is

\[
F(x_1, \ldots, x_n) = K(F_1(x_1), \ldots, F(x_n)),
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where \(F_i(t) = \Pr(X_i > t)\) and \(K\) is the survival copula.

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Coherent systems-General case

- The copula representation for the RF of \((X_1, \ldots, X_n)\) is
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- Then
  \[
  \bar{F}_P(t) = Q_{P,K}(\bar{F}_1(t), \ldots, \bar{F}(t)),
  \]
  where \(Q_{P,K}(u_1, \ldots, u_n) = K(u^P_1, \ldots, u^P_n)\) and \(u^P_i = u_i\) for \(i \in P\) and \(u^P_i = 1\) for \(i \notin P\).

- Therefore, from the minimal path set repres., we get
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  \]
Example
Coherent system lifetime $T = \min(X_1, \max(X_2, X_3))$. 
Example

3! = 6 permutations.
Example

\[ X_1 < X_2 < X_3 \Rightarrow T = X_1 = X_{1:3} \]
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Example

IID \( F \) cont.: \( p = (2/6, 4/6, 0) = (1/3, 2/3, 0) \).
IID $\overline{F}$ cont.: $\overline{F}_{T}(t) = \frac{1}{3} \overline{F}_{1:3}(t) + \frac{2}{3} \overline{F}_{2:3}(t)$. 
Coherent system lifetime $T = \max(\min(X_1, X_2), \min(X_1, X_3))$.

Minimal path sets $P_1 = \{1, 2\}$ and $P_1 = \{1, 3\}$. 
$F_T(t) = \Pr(\{X_{\{1,3\}} > t\} \cup \{X_{\{1,2\}} > t\})$

$= \overline{F}_{\{1,2\}}(t) + \overline{F}_{\{1,3\}}(t) - \overline{F}_{\{1,2,3\}}(t)$. 
Example-general case

\[ F_{\{1,2\}}(t) = \bar{F}(t, t, 0) = K(\bar{F}_1(t), \bar{F}_2(t), 1), \ldots \]

\[ F_T(t) = Q_{\phi,K}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t)) \] where

\[ Q_{\phi,K}(u_1, u_2, u_3) = K(u_1, u_2, 1) + K(u_1, 1, u_3) - K(u_1, u_2, u_3). \]
Example-general case

EXC: $\bar{F}_T(t) = 2\bar{F}_{1:2}(t) - \bar{F}_{1:3}(t) = q_{\phi,K}(\bar{F}(t))$, where $q_{\phi,K}(u) = 2K(u, u, 1) - K(u, u, u)$. Minimal signature $\mathbf{a} = (0, 2, -1)$. 
Example-general case

\[ \text{IID: } \bar{F}_T(t) = 2\bar{F}^2(t) - \bar{F}^3(t) = q_\phi(\bar{F}(t)), \]
\[ \text{where } q_\phi(u) = 2u^2 - u^3. \]
The minimal signatures for \( n \leq 5 \) can be seen in: Navarro and Rubio (2010, Comm Stat Simul Comp 39, 68–84).
Generalized Order Statistics (GOS)

For an arbitrary DF $F$, GOS $X_{1:n}^{GOS}, \ldots, X_{n:n}^{GOS}$ based on $F$ can be obtained (Kamps, 1995, B. G. Teubner Stuttgart, p.49) via the quantile transformation

$$X_{r:n}^{GOS} = F^{-1}(U_{r:n}^{GOS}), \quad r = 1, \ldots, n,$$

where $(U_{1:n}^*, \ldots, U_{n:n}^*)$ has the joint PDF

$$g^{GOS}(u_1, \ldots, u_n) = k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} (1 - u_i)^{m_i} \right) (1-u_n)^{k-1}$$

for $0 \leq u_1 \leq \ldots \leq u_n < 1$, $n \geq 2$, $k \geq 1$, $\gamma_1, \ldots, \gamma_n > 0$ and $m_i = \gamma_i - \gamma_{i+1} - 1$. 
Generalized Order statistics (GOS)

If $\gamma_1, \ldots, \gamma_n$ are pairwise different, then

$$F_{r:n}^{GOS}(t) = 1 - c_{r-1} \sum_{i=1}^{r} a_{i,r} \left(1 - F(t)\right)^{\gamma_i} = q_{r:n}^{GOS}(F(t))$$

with the constants

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad a_{i,r} = \prod_{j=1}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n$$

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Then the GOS are DD from $F$. 

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Jorge Navarro, E-mail: jorgenav@um.es
Particular cases of GOS

- The GOS include:
  - OS, IID case \( m_1 = \cdots = m_{n-1} = 0 \) and \( k = 1 \).
  - kRV, k-th record values \( m_1 = \cdots = m_{n-1} = -1 \) and \( k = 1, 2, \ldots \).
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  - SOS, Sequential Order Statistics under the Proportional Hazard Rate (PHR) model, i.e., with \( F_r = F^{\alpha_r} \) for \( r = 1, \ldots, n \) \( (\gamma_r = (n - r + 1)\alpha_r \) and \( k = \alpha_n \).\)
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Preservation results

- If \( q_1 \) and \( q_2 \) are two DF,
  \[
  q_1(F) \leq_{ord} q_2(F) \text{ for all } F?
  \]

- If \( q \) is a DF,
  \[
  F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)\? 
  \]

- If \( Q_1 \) and \( Q_2 \) are two MDF,
  \[
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  \]

- If \( Q \) is a MDF,
  \[
  F_i \leq_{ord} G_i, \ i = 1, \ldots, n, \Rightarrow Q(F_1, \ldots, F_n) \leq_{ord} Q(G_1, \ldots, G_n)\?
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Main stochastic orderings

- \( X \leq_{ST} Y \iff F_X(t) \leq F_Y(t) \), stochastic order.
- \( X \leq_{HR} Y \iff h_X(t) \geq h_Y(t) \), hazard rate order.
- \( X \leq_{HR} Y \iff (X - t|X > t) \leq_{ST} (Y - t|Y > t) \) for all \( t \).
- \( X \leq_{MRL} Y \iff E(X - t|X > t) \leq E(Y - t|Y > t) \) for all \( t \).
- \( X \leq_{LR} Y \iff f_Y(t)/f_X(t) \) is nondecreasing, likelihood ratio order.
- \( X \leq_{RHR} Y \iff (t - X|X < t) \geq_{ST} (t - Y|Y < t) \) for all \( t \).

\[ \begin{align*}
X & \leq_{RHR} Y & \iff & X \leq_{MRL} Y & \Rightarrow & E(X) \leq E(Y) \\
\uparrow & & & \uparrow & & \uparrow \\
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Preservation of stochastic orders-DD

- If $T_i$ has the DD $q_i(F(t))$, $i = 1, 2$, then:
  - $T_1 \leq_{ST} T_2$ for all $F$ if and only if $q_1(u)/q_2(u) \geq 1$ in $(0, 1)$.
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  - $T_1 \leq_{LR} T_2$ ($\geq_{LR}$) for all $F$ if and only if $q_2(q_1^{-1}(u))$ is concave (convex) in $(0, 1)$.
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  - NEW $T_1 \leq_{MRL} T_2$ for all $F$ if $E(T_1) \leq E(T_2)$ and $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$.
Preservation of stochastic orders-DD

- If $T_i$ has the DD $q_i(F(t))$, $i = 1, 2$, then:
  - $T_1 \leq_{ST} T_2$ for all $F$ if and only if $q_1(u)/q_2(u) \geq 1$ in $(0, 1)$.
  - $T_1 \leq_{HR} T_2$ for all $F$ if and only if $\overline{q}_2/\overline{q}_1$ decreases in $(0, 1)$.
  - $T_1 \leq_{LR} T_2$ ($\geq_{LR}$) for all $F$ if and only if $q_2(q_1^{-1}(u))$ is concave (convex) in $(0, 1)$.
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  - $T_1 \leq_{LR} T_2$ ($\geq_{LR}$) for all $F$ if and only if $q_2(q_1^{-1}(u))$ is concave (convex) in $(0, 1)$.
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  - **NEW** $T_1 \leq_{MRL} T_2$ for all $F$ if $E(T_1) \leq E(T_2)$ and $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$. 

Preservation of stochastic orders-DD

- $F_1 \leq_{ST} F_2 \Rightarrow q(F_1) \leq_{ST} q(F_2)$.

- If $\alpha(u)$ is decreasing in $(0, 1)$, then

$$F_1 \leq_{HR} F_2 \Rightarrow q(F_1) \leq_{HR} q(F_2),$$

where $\alpha(u) = uq'(1 - u)/(1 - q(1 - u)) = u\bar{q}'(u)/\bar{q}(u)$.

- If $\beta_q(u)$ is decreasing and nonnegative in $(0, 1)$, then

$$F_1 \leq_{LR} F_2 \Rightarrow q(F_1) \leq_{LR} q(F_2),$$

where $\beta_q(u) = -uq''(1 - u)q'(1 - u) = u\bar{q}''(u)/\bar{q}'(u)$. 
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- $F_1 \leq_{ST} F_2 \Rightarrow q(F_1) \leq_{ST} q(F_2)$.

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- If $\beta_q(u)$ is decreasing and nonnegative in $(0, 1)$, then

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where $\beta_q(u) = -uq''(1-u)q'(1-u) = u\bar{q}''(u)/\bar{q}'(u)$.
F_1 \leq_{ST} F_2 \Rightarrow q(F_1) \leq_{ST} q(F_2).

If \alpha(u) is decreasing in (0, 1), then

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where \alpha(u) = uq'(1 - u)/(1 - q(1 - u)) = u\bar{q}'(u)/\bar{q}(u).

If \beta_q(u) is decreasing and nonnegative in (0, 1), then

F_1 \leq_{LR} F_2 \Rightarrow q(F_1) \leq_{LR} q(F_2),

where \beta_q(u) = -uq''(1 - u)q'(1 - u) = u\bar{q}''(u)/\bar{q}'(u).
Preservation of stochastic orders-GDD$^{NEW}$

- If $G_i = Q_i(F_1, \ldots, F_n)$, $i = 1, 2$, then:
  - $G_1 \leq_{ST} G_2$ for all $F_1, \ldots, F_n$ if and only if $Q_1/Q_2 \geq 1$ in $(0, 1)^n$.
  - $G_1 \leq_{HR} G_2$ for all $F_1, \ldots, F_n$ if and only if $\overline{Q}_2/\overline{Q}_1$ is decreasing in $(0, 1)^n$.
  - $G_1 \leq_{HR} G_2$ for all $F_1, \ldots, F_n$ if $\alpha_i^{Q_1} \geq \alpha_i^{Q_2}$ in $(0, 1)^n$ for $i = 1, \ldots, n$, where
    \[
    \alpha_i^{\Phi}(u_1, \ldots, u_n) = \frac{u_i D_i \Phi(u_1, \ldots, u_n)}{\Phi(u_1, \ldots, u_n)}
    \]  
    (2.1)
    and $D_i \overline{Q}(u_1, \ldots, u_n) = \frac{\partial}{\partial u_i} \overline{Q}(u_1, \ldots, u_n)$.
  - $G_1 \leq_{RHR} G_2$ for all $F_1, \ldots, F_n$ if and only if $Q_2/Q_1$ is increasing in $(0, 1)^n$.  

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- $G_1 \leq_{HR} G_2$ for all $F_1, \ldots, F_n$ if $\alpha_i^Q \geq \alpha_i^Q$ in $(0, 1)^n$ for $i = 1, \ldots, n$, where

$$\alpha_i^\Phi(u_1, \ldots, u_n) = \frac{u_i D_i \Phi(u_1, \ldots, u_n)}{\Phi(u_1, \ldots, u_n)} \quad (2.1)$$

and $D_i \overline{Q}(u_1, \ldots, u_n) = \frac{\partial}{\partial u_i} \overline{Q}(u_1, \ldots, u_n)$.

- $G_1 \leq_{RHR} G_2$ for all $F_1, \ldots, F_n$ if and only if $Q_2/Q_1$ is increasing in $(0, 1)^n$. 

References

Stochastic orders-DD
Stochastic orders-GDD
Stochastic aging classes
Examples

Preservation of stochastic orders-GDD^{NEW}
Preservation of stochastic orders-GDD$^{NEW}$

- If $G_i = Q_i(F_1, \ldots, F_n), i = 1, 2$, then:
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    \[
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    \]
    and $D_i\overline{Q}(u_1, \ldots, u_n) = \frac{\partial}{\partial u_i}\overline{Q}(u_1, \ldots, u_n)$.
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If $G_i = Q_i(F_1, \ldots, F_n)$, $i = 1, 2$, then:

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3. $G_1 \leq_{HR} G_2$ for all $F_1, \ldots, F_n$ if $\alpha_i^{\overline{Q}_1} \geq \alpha_i^{\overline{Q}_2}$ in $(0, 1)^n$ for $i = 1, \ldots, n$, where

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- $G_1 \leq_{HR} G_2$ for all $F_1, \ldots, F_n$ if $\alpha_i \frac{Q_1}{Q_i} \geq \alpha_i \frac{Q_2}{Q_i}$ in $(0, 1)^n$ for $i = 1, \ldots, n$, where

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If $F_Q = Q(F_1, \ldots, F_n)$ and $G_Q = Q(G_1, \ldots, G_n)$, then:

- $F_i \leq_{ST} G_i$ for $i = 1, \ldots, n \Rightarrow F_Q \leq_{ST} G_Q$.
- If $F_i \leq_{HR} G_i$ for $i = 1, \ldots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF $Q$ such that $\alpha_i^Q$ is decreasing in $(0, 1)^n$ for $i = 1, \ldots, n$.
- If $F_i \leq_{RHR} G_i$ for $i = 1, \ldots, n$, then $F_Q \leq_{RHR} G_Q$ for all MDF $Q$ such that $\alpha_i^Q$ is decreasing in $(0, 1)^n$ for $i = 1, \ldots, n$. 
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- If $F_i \leq_{HR} G_i$ for $i = 1, \ldots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF $Q$ such that $\alpha^Q_i$ is decreasing in $(0, 1)^n$ for $i = 1, \ldots, n$.
- If $F_i \leq_{RHR} G_i$ for $i = 1, \ldots, n$, then $F_Q \leq_{RHR} G_Q$ for all MDF $Q$ such that $\alpha^Q_i$ is decreasing in $(0, 1)^n$ for $i = 1, \ldots, n$. 


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  \[
  \beta Q = \frac{Q(u_1 v_1, \ldots, u_n v_n)}{Q(u_1, \ldots, u_n)}.
  \] (2.2)

  is decreasing in $u_1, \ldots, u_n$ and increasing in $v_1, \ldots, v_n$ in $(0, 1)^n \times (1, \infty)^n$.

- If $F_i \leq_{LR} G_i$ and $F_i$ is IHR (DHR) for $i = 1, \ldots, n$, then $F_Q \leq_{LR} G_Q$ for all MDF $Q$ such that
  \[
  \gamma Q = \frac{w_1 z_1 u_1 D_1 Q(u_1 v_1, \ldots, u_n v_n) + \cdots + w_n z_n u_n D_n Q(u_1 v_1, \ldots, u_n v_n)}{z_1 u_1 D_1 Q(u_1, \ldots, u_n) + \cdots + z_n u_n D_n Q(u_1, \ldots, u_n)}
  \]

  is decreasing in $u_1, \ldots, u_n$, increasing in $v_1, \ldots, v_n, w_1, \ldots, w_n$

  and increasing (decreasing) in $z_i$ in $(0, 1)^n \times (1, \infty) \times (0, \infty)^{2n}$. 
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- If $F_i \leq_{HR} G_i$ for $i = 1, \ldots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF $Q$ such that
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  is decreasing in $u_1, \ldots, u_n$ and increasing in $v_1, \ldots, v_n$ in $(0, 1)^n \times (1, \infty)^n$.

- If $F_i \leq_{LR} G_i$ and $F_i$ is IHR (DHR) for $i = 1, \ldots, n$, then $F_Q \leq_{LR} G_Q$ for all MDF $Q$ such that
  \[ \gamma Q = \frac{w_1z_1u_1D_1Q(u_1v_1, \ldots, u_nv_n) + \cdots + w_nz_nu_nD_nQ(u_1v_1, \ldots, u_nv_n)}{z_1u_1D_1Q(u_1, \ldots, u_n) + \cdots + z_nu_nD_nQ(u_1, \ldots, u_n)} \]
  is decreasing in $u_1, \ldots, u_n$, increasing in $v_1, \ldots, v_n, w_1, \ldots, w_n$ and increasing (decreasing) in $z_i$ in $(0, 1)^n \times (1, \infty) \times (0, \infty)^{2n}$.
Preservation of stochastic orders-GDD$^{NEW}$

- If $F_Q = Q(F_1, F_2, \ldots, F_n)$ and $G_Q = Q(G_1, F_2, \ldots, F_n)$, then:
- If $F_1 \leq_{HR} G_1$ and $F_1 \geq_{HR} F_i \ (\leq_{HR})$ for $i = 2, \ldots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF $Q$ such that

\[
\delta\overline{Q} = \frac{\overline{Q}(u_1v_1, u_1v_2, \ldots, u_1v_n)}{\overline{Q}(u_1, u_1v_2, \ldots, u_1v_n)}
\]

is decreasing in $u_1$ and decreasing (increasing) in $v_i$, $i = 1, \ldots, n$.

- If $F_1 \leq_{LR} G_1$ and $F_1 \leq_{LR} F_i \ (\geq_{LR})$ for $i = 2, \ldots, n$, then $F_Q \leq_{LR} G_Q$ for all MDF $Q$ such that

\[
\lambda\overline{Q} = \frac{w_1D_1\overline{Q}(u_1v_1, \ldots, u_1v_n) + \cdots + w_nD_n\overline{Q}(u_1v_1, \ldots, u_1v_n)}{D_1\overline{Q}(u_1, u_1v_2, \ldots, u_1v_n) + \cdots + D_n\overline{Q}(u_1, u_1v_2, \ldots, u_1v_n)}
\]

is decreasing in $u_1$, increasing in $v_1$ and increasing (decreasing) in $v_i$ and $w_i$ for $i = 2, \ldots, n$. 

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Preservation of stochastic orders-GDD

- If $F_Q = Q(F_1, F_2, \ldots, F_n)$ and $G_Q = Q(G_1, F_2, \ldots, F_n)$, then:
- If $F_1 \leq_{HR} G_1$ and $F_1 \geq_{HR} F_i$ ($\leq_{HR}$) for $i = 2, \ldots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF $Q$ such that

$$
\delta Q = \frac{Q(u_1 v_1, u_1 v_2, \ldots, u_1 v_n)}{Q(u_1, u_1 v_2, \ldots, u_1 v_n)}
$$

is decreasing in $u_1$ and decreasing (increasing) in $v_i$, $i = 1, \ldots, n$.

- If $F_1 \leq_{LR} G_1$ and $F_1 \leq_{LR} F_i$ ($\geq_{LR}$) for $i = 2, \ldots, n$, then $F_Q \leq_{LR} G_Q$ for all MDF $Q$ such that

$$
\lambda Q = \frac{w_1 D_1 Q(u_1 v_1, \ldots, u_1 v_n) + \cdots + w_n D_n Q(u_1 v_1, \ldots, u_1 v_n)}{D_1 Q(u_1, u_1 v_2, \ldots, u_1 v_n) + \cdots + D_n Q(u_1, u_1 v_2, \ldots, u_1 v_n)}
$$

is decreasing in $u_1$, increasing in $v_1$ and increasing (decreasing) in $v_i$ and $w_i$ for $i = 2, \ldots, n$. 
Preservation of stochastic orders-GDD^{NEW}

- If \( F_Q = Q(F_1, F_2, \ldots, F_n) \) and \( G_Q = Q(G_1, F_2, \ldots, F_n) \), then:
- If \( F_1 \leq_{HR} G_1 \) and \( F_1 \geq_{HR} F_i (\leq_{HR}) \) for \( i = 2, \ldots, n \), then \( F_Q \leq_{HR} G_Q \) for all MDF \( Q \) such that
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  \]
  is decreasing in \( u_1 \) and decreasing (increasing) in \( v_i \), \( i = 1, \ldots, n \).
- If \( F_1 \leq_{LR} G_1 \) and \( F_1 \leq_{LR} F_i (\geq_{LR}) \) for \( i = 2, \ldots, n \), then \( F_Q \leq_{LR} G_Q \) for all MDF \( Q \) such that
  \[
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Preservation results of aging classes

- Let $C$ be an aging class.
- If $q$ is a distorted function,

$$F \in C \Rightarrow q(F) \in C?$$

- If $Q$ is a multivariate distorted function,

$$F_i \in C, i = 1, \ldots, n, \Rightarrow Q(F_1, \ldots, F_n) \in C?$$

Preservation results of aging classes

- Let $\mathcal{C}$ be an aging class.
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  \[ F \in \mathcal{C} \implies q(F) \in \mathcal{C}? \]
- If $Q$ is a multivariate distorted function,
  \[ F_i \in \mathcal{C}, \ i = 1, \ldots, n, \implies Q(F_1, \ldots, F_n) \in \mathcal{C}? \]

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Stochastic aging classes

- $X$ is Increasing (Decreasing) Hazard Rate IHR (DHR) if $h$ is increasing (decreasing).
- $X$ is IHR $\iff (X - s | X > s) \geq_{ST} (X - t | X > t)$ for all $s < t$.
- $X$ is New Better (Worse) than Used NBU (NWU) if $X \geq_{ST} (X - t | X > t)$ ($\leq_{ST}$) for all $t > 0$.
- $X$ is Increasing (Decreasing) Likelihood Ratio ILR (DLR) if $f$ is log-concave (log-convex).
- $X$ is ILR $\iff (X - s | X > s) \geq_{LR} (X - t | X > t)$ for all $s < t$.
- $ILR \Rightarrow IHR \Rightarrow NBU$. 
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- \( X \) is Increasing (Decreasing) Hazard Rate IHR (DHR) if \( h \) is increasing (decreasing).

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- \( X \) is New Better (Worse) than Used NBU (NWU) if \( X \succeq_{ST} (X - t | X > t) (\preceq_{ST}) \) for all \( t > 0 \).

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- $X$ is Increasing (Decreasing) Hazard Rate IHR (DHR) if $h$ is increasing (decreasing).
- $X$ is IHR $\iff (X - s|X > s) \geq_{ST} (X - t|X > t)$ for all $s < t$.
- $X$ is New Better (Worse) than Used NBU (NWU) if $X \geq_{ST} (X - t|X > t)$ ($\leq_{ST}$) for all $t > 0$.
- $X$ is Increasing (Decreasing) Likelihood Ratio ILR (DLR) if $f$ is log-concave (log-convex).
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- $ILR \Rightarrow IHR \Rightarrow NBU$. 
Let $F_q = q(F)$ and $\alpha(u) = u \frac{q'(u)}{q(u)}$. Then:

- The IHR class is preserved (i.e. $F_q$ is IHR for all $F$ IHR) if and only if $\alpha$ is decreasing in $(0, 1)$.
- The DHR class is preserved if and only if $\alpha$ is increasing in $(0, 1)$.
- The IHR and DHR classes are preserved if and only if the PHR holds ($\alpha$ is constant).
- The NBU (NWU) class is preserved if and only if

$$\overline{q}(uv) \leq \overline{q}(u) \overline{q}(v) \quad (\geq), \quad 0 \leq u, v \leq 1. \quad (2.4)$$

- The NBU (NWU) class is preserved if the IHR (DHR) class is preserved.
Preservation of Stochastic aging classes DD

Let $F_q = q(F)$ and $\alpha(u) = u \bar{q}'(u)/\bar{q}(u)$. Then:

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Preservation of Stochastic aging classes

- In the IID case:
  - The IHR class and the HR order are preserved for $X_{i:n}$ since $\alpha_{i:n}(u)$ is decreasing (Esary and Proschan 1963, Tech.).
  - The DHR class is not necessarily preserved for $X_{i:n}$! It is only preserved for $X_{1:n}$ since $\alpha_{1:n}(u)$ is constant.
  - The IHR and DHR classes are not necessarily preserved under the formation of coherent systems! It depends on the system structure.
  - In the ID case the IHR class is not necessarily preserved for $X_{i:n}$! It depends on the copula (dependence).
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Preservation of Stochastic aging classes DD

Let $F_q = q(F)$ and let

$$\beta(u) = \frac{uq''(u)}{q'(u)},$$

and

$$\overline{\beta}(u) = \frac{(1 - u)q''(u)}{q'(u)}.$$

Then:

- If $F$ is ILR and there exists $a \in [0, 1]$ such that $\beta$ is non-negative and decreasing in $(0, a)$ and $\overline{\beta}$ is non-positive and decreasing in $(a, 1)$, then $F_q$ is ILR.
- If $F$ is DLR with support $(l, \infty)$ ($l \geq 0$), $\beta$ is non-negative and increasing in $(0, 1)$, then $F_q$ is DLR.
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Preservation of Stochastic aging classes GDD

Let $\bar{F}_Q = \overline{Q}(\bar{F}_1, \ldots, \bar{F}_n)$ and

$$\alpha_i(u_1, \ldots, u_n) = \frac{u_i D_i \overline{Q}(u_1, \ldots, u_n)}{\overline{Q}(u_1, \ldots, u_n)}.$$

Then:

- The IHR (DHR) class is preserved if $\alpha_i$ is decreasing (increasing) in $(0, 1)^n$ for $i = 1, \ldots, n$.
- The NBU (NWU) class is preserved if

$$\overline{Q}(u_1 v_1, \ldots, u_n v_n) \leq \overline{Q}(u_1, \ldots, u_n) \overline{Q}(v_1, \ldots, v_n) \quad (\geq)$$

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Preservation of Stochastic aging classes GDD

If $X_1, \ldots, X_n$ are independent, then:

- The NBU class is preserved under the formation of coherent systems (Esary, Marshall and Proschan, 1970, SIAM J Appl Math).
- The IHR class is not preserved under the formation of coherent systems (order statistics) in the independent case.
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Example-system IID case

Coherent system lifetime $T = \min(X_1, \max(X_2, X_3))$.

In the IID case: $q(u) = u + u^2 - u^3$ and $\bar{q}(u) = 2u^2 - 3u^3$.

Then $\alpha(u) = \frac{4 - 3u}{2 - u}$ is strictly decreasing.

The HR order is preserved.

The IHR class is preserved and the DHR is not always preserved.
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Example - paradoxical system IID case

- Coherent system lifetime $T = \max(X_1, \min(X_2, X_3))$.
- In the IID case: $\bar{q}(u) = u + u^2 - u^3$ and $q(u) = 2u^2 - 3u^3$.
- Then $\alpha(u) = \frac{1 + 2u - 3u^2}{1 + u - u^2}$ is strictly increasing in $(0, u_0)$ and strictly decreasing in $(u_0, 1)$, with $u_0 = \sqrt{5} - 2 = 0.236068$.
- The HR order is not necessarily preserved.
- Neither the IHR class nor the DHR are preserved.
Coherent system lifetime $T = \max(X_1, \min(X_2, X_3))$.

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Figure: HR (left) and RF (left) of the residual lifetimes \((T - t | T > t)\) of the system \(T = \max(X_1, \min(X_2, X_3))\) when \(X_i\) are IID \(\sim \text{Exp}(\mu = 1)\) with \(t = 0, 1, 2, 3\) (black, blue, red, green).
Figure: HR $X_1$ (left) and $T = \max(X_1, \min(X_2, X_3))$ (right) when $X_i$ are IID with $\bar{F}(t) = 1 - (1 - e^{-t})^a$ for $t > 0$ and $a = 2, 5$ (blue, black).
Example-DID case

- Series system \( X_{1:n} = \min(X_1, \ldots, X_n) \) with ID components having a Clayton-Oakes survival copula

\[
K(u_1, \ldots, u_n) = \left( \sum_{i=1}^{n} u_i^{1-\theta} - (n-1) \right)^{1/(1-\theta)}, \quad \theta > 1.
\]

- Then

\[
\bar{q}(u) = K(u, \ldots, u) = (nu^{1-\theta} - n + 1)^{1/(1-\theta)}.
\]

- As \( \alpha(u) = \frac{\theta}{n-(n-1)u^{\theta-1}} \) is a strictly increasing function for all \( \theta > 1 \), the DHR class is preserved for all \( n \).
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- However, the IHR class is not necessarily preserved.
- The HR order is not necessarily preserved.
Figure: HR of $T = \min(X_1, X_2)$ when $(X_1, X_2)$ has a C-O survival copula with $\theta = 2$ and $\bar{F}_i(t) = \exp(-t^a)$, $t > 0$, $i = 1, 2$ with $a = 1$ (black, Exponential), $a = 1.1, 1.2, 1.3, 1.4$ (blue, red, green, purple, IHR Weibull).
Example: Parallel system IND case

- Parallel system \( X_{1:2} = \max(X_1, X_2) \) with IND components.
- Then \( Q_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2 \).
- As \( \alpha_1^Q(u_1, u_2) = (u_1 - u_1 u_2)/(u_1 + u_2 - u_1 u_2) \) is increasing in \( u_1 \) and decreasing in \( u_2 \), then the IHR and DHR classes are not necessarily preserved.
- For the series system \( Q_{1:2}(u) = u_1 u_2 \) and as

\[
\frac{Q_{2:2}(u_1, u_2)}{Q_{1:2}(u_1, u_2)} = \frac{1}{u_1} + \frac{1}{u_2} - 1
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is decreasing, then \( X_{1:2} \leq_{HR} X_{2:2} \).
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Example-Parallel system IND case

- Parallel system $X_{1:2} = \max(X_1, X_2)$ with IND components.
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- As $\alpha_{Q}(u_1, u_2) = (u_1 - u_1 u_2) / (u_1 + u_2 - u_1 u_2)$ is increasing in $u_1$ and decreasing in $u_2$, then the IHR and DHR classes are not necessarily preserved.
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Figure: HR of $X_i$ (red), $X_{1:2}$ (blue) and $X_{2:2}$ (black) when $X_i \sim \text{Exp}(\mu = 1/i), i = 1, 2$. $X_i$ are IHR and DHR but $X_{2:2}$ is neither IHR nor DHR.
Parrondo’s paradox series systems-IID case

- Parrondo’s paradox shows (Game Theory) how, in some games, a random strategy might be better than any deterministic strategy.
- The same paradox holds for coherent systems.
- Let us assume that we have two kind of units with reliability functions $\overline{F}_1 \geq \overline{F}_2$ (in a similar number) to build series systems with two independent units.
- Let $T = \min(X_1, X_2)$ be the system obtained when $\overline{F}_i(t) = \Pr(X_i > t)$, $i = 1, 2$.
- Let $S$ be the system obtained when the units are chosen randomly.
- Then $T \leq_{ST} S$ since
  \[ \overline{F}_T(t) = \overline{F}_1(t)\overline{F}_2(t) \leq (0.5\overline{F}_1(t) + 0.5\overline{F}_1(t))^2 = \overline{F}_S(t). \]
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Figure: Reliability functions of systems $T$ (black) and $S$ (blue) when the units have exponential distributions with means 5 and 1.
Parrondo’s paradox in other systems

- The same happen with series systems of size $n$ with independent components.
- The orderings are reversed for parallel systems.
- In both cases, we compare the GDD $Q(F_1, \ldots, F_n)$ and $Q(G, \ldots, G)$, where $G = F_1 + \cdots + F_n)/n$.
- A function $g : \mathbb{R}^n \to \mathbb{R}$ is weakly Schur-concave (convex) if

$$g(u_1, u_2, \ldots, u_n) \leq g(\bar{u}, \bar{u}, \ldots, \bar{u}) \quad (\geq)$$

for all $(u_1, u_2, \ldots, u_n)$, where $\bar{u} = (u_1 + u_2 + \ldots + u_n)/n$. 
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Theorem (Navarro and Spizzichino, ASMBI 2010)

If \((X_1, X_2, \ldots, X_n)\) and \((Y_1, Y_2, \ldots, Y_n)\) have the same copula,

\[
\overline{F}_i(t) = \Pr(X_i > t) \quad \text{and} \quad \overline{F}(t) = (\overline{F}_1(t) + \ldots + \overline{F}_n(t))/n = \Pr(Y_i > t) \quad \text{for } i = 1, \ldots, n, \text{ and}
\]

\(Q_{\phi,K}\) is weakly Schur-concave (convex), then

\[
T = \phi(X_1, \ldots, X_n) \leq_{ST} S = \phi(Y_1, \ldots, Y_n) \quad (\geq_{ST}).
\]
Parrondo’s paradox in other systems

- This theorem can be applied to GDD.
- For $X_{1:n}$ with independent components
  $$\bar{Q}_{1:n}(u_1,\ldots,u_n) = u_1 \ldots u_n$$
  which is Schur-concave and so Parrondo’s paradox holds.
- For $X_{1:n}$ with dependent components
  $$\bar{Q}_{1:n,K}(u_1,\ldots,u_n) = K(u_1,\ldots,u_n).$$
- Many copulas are Schur-concave (e.g. Archimedean copulas) and so Parrondo’s paradox holds in many series systems.
- However there are copulas which are weakly Schur-convex and hence the ordering can be reversed for series systems (see Navarro and Spizzichino, ASMBI 2010).
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If $\bar{Q}$ is a GDF, we consider the GDD with RF

$$F_k(t) = \bar{Q}(\underbrace{F_X(t), \ldots, F_X(t)}_{k\text{-times}}, \underbrace{F_Y(t), \ldots, F_Y(t)}_{(n-k)\text{-times}}), \quad k = 0, \ldots, n$$

(3.1)

Here, e.g., we can assume $X \succeq_{ST} Y$.

The randomized GDD is obtained when the number $k$ of “god components” is chosen randomly according to a discrete random variable $K$ with support included in $\{0, \ldots, n\}$.

It is represented by the random variable $T_K$. 

Randomized GDD

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- Here, e.g., we can assume \( X \geq_{st} Y \).
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Proposition (Navarro, Pellerey and Di Crecenzo, 2014)

If $k$ is chosen randomly according to $K_1$ or $K_2$ and

$$
\varphi(k) = \overline{Q}(u, \ldots, u, v, \ldots, v)
$$

$k$–times $(n-k)$–times

is convex (concave) in $\{0, 1, \ldots, m\}$ for all $u, v \in (0, 1)$, then:

(i) $K_1 \leq_{CX} K_2$ implies $T_{K_1} \leq_{ST} T_{K_2}$ ($\geq_{st}$).

(ii) $X \geq_{ST} Y$ and $K_1 \leq_{ICX} K_2$ ($\leq_{ICV}$) imply $T_{K_1} \leq_{ST} T_{K_2}$. 
Figure: Reliability functions of systems $T$ (black) and $S$ (blue) when the units have exponential distributions with means 1 and 5.
Parrondo paradox example

- \( T = \min(X_1, X_2) \) with \( \overline{Q}(u, v) = uv \).
- It is obtained with \( K_1 \) such that \( \Pr(K_1 = 1) = 1 \).
- \( S \) is obtained with \( K_2 \) such that \( \Pr(K_2 = 1) = 1/2 \) and \( \Pr(K_2 = 0) = \Pr(K_2 = 2) = 1/4 \).
- Another reasonable option is obtained with \( K_3 \) such that \( \Pr(K_3 = i) = 1/3 \) for \( i = 0, 1, 2 \).
- The green line is obtained with \( K_4 \) such that \( \Pr(K_4 = 0) = \Pr(K_4 = 2) = 1/2 \).
- Note that \( E(K_i) = 1 \) for \( i = 1, 2, 3, 4 \).
- As \( \varphi(k) = u^k v^{1-k} \) is convex and \( K_1 \leq_C X K_2 \leq_C X K_3 \leq_C X K_4 \), then
  \[ \overline{F}_{K_1} \leq_{ST} \overline{F}_{K_2} \leq_{ST} \overline{F}_{K_3} \leq_{ST} \overline{F}_{K_4}. \]
- Actually, \( K_4 \) is the best option (the most convex) whenever \( E(K) = 1 \).
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  \[
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- Note that $E(K_i) = 1$ for $i = 1, 2, 3, 4$.
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  $$\overline{F}_{K_1} \leq_{ST} \overline{F}_{K_2} \leq_{ST} \overline{F}_{K_3} \leq_{ST} \overline{F}_{K_4}.$$  
- Actually, $K_4$ is the best option (the most convex) whenever $E(K) = 1$. 
Parrondo paradox example

- \( T = \min(X_1, X_2) \) with \( \bar{Q}(u, v) = uv \).
- It is obtained with \( K_1 \) such that \( \Pr(K_1 = 1) = 1 \).
- \( S \) is obtained with \( K_2 \) such that \( \Pr(K_2 = 1) = 1/2 \) and \( \Pr(K_2 = 0) = \Pr(K_2 = 2) = 1/4 \).
- Another reasonable option is obtained with \( K_3 \) such that \( \Pr(K_3 = i) = 1/3 \) for \( i = 0, 1, 2 \).
- The green line is obtained with \( K_4 \) such that \( \Pr(K_4 = 0) = \Pr(K_4 = 2) = 1/2 \).
- Note that \( E(K_i) = 1 \) for \( i = 1, 2, 3, 4 \).
- As \( \varphi(k) = u^k v^{1-k} \) is convex and \( K_1 \leq_{cx} K_2 \leq_{cx} K_3 \leq_{cx} K_4 \), then
  \[
  \overline{F}_{K_1} \leq_{ST} \overline{F}_{K_2} \leq_{ST} \overline{F}_{K_3} \leq_{ST} \overline{F}_{K_4}.
  \]
- Actually, \( K_4 \) is the best option (the most convex) whenever \( E(K) = 1 \).
Parrondo paradox example

- \( T = \min(X_1, X_2) \) with \( Q(u, v) = uv \).
- It is obtained with \( K_1 \) such that \( \Pr(K_1 = 1) = 1 \).
- \( S \) is obtained with \( K_2 \) such that \( \Pr(K_2 = 1) = 1/2 \) and \( \Pr(K_2 = 0) = \Pr(K_2 = 2) = 1/4 \).
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- Note that \( E(K_i) = 1 \) for \( i = 1, 2, 3, 4 \).
- As \( \varphi(k) = u^k v^{1-k} \) is convex and \( K_1 \leq_{cx} K_2 \leq_{cx} K_3 \leq_{cx} K_4 \), then
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- Note that \( E(K_i) = 1 \) for \( i = 1, 2, 3, 4 \).
- As \( \varphi(k) = u^k v^{1-k} \) is convex and
  \( K_1 \leq_{CX} K_2 \leq_{CX} K_3 \leq_{CX} K_4 \), then
  \[
  \overline{F}_{K_1} \leq_{ST} \overline{F}_{K_2} \leq_{ST} \overline{F}_{K_3} \leq_{ST} \overline{F}_{K_4}.
  \]
- Actually, \( K_4 \) is the best option (the most convex) whenever
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Parrondo paradox example

- \( T = \min(X_1, X_2) \) with \( \overline{Q}(u, v) = uv \).
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- As \( \varphi(k) = u^k v^{1-k} \) is convex and
  \( K_1 \leq_{C_X} K_2 \leq_{C_X} K_3 \leq_{C_X} K_4 \), then
  \[ \overline{F}_{K_1} \leq_{ST} \overline{F}_{K_2} \leq_{ST} \overline{F}_{K_3} \leq_{ST} \overline{F}_{K_4}. \]
- Actually, \( K_4 \) is the best option (the most convex) whenever
  \( E(K) = 1 \).
**Figure:** Reliability functions of systems $T = T_{K_1}$ (black), $S = T_{K_2}$ (blue), $T_{K_3}$ (purple) and $T_{K_4}$ (green) when the units have exponential distributions with means 5 and 1.
Our Main References


Ng, Navarro and Balakrishnan (2012). Parametric inference from system lifetime data under a proportional hazard rate model. Metrika 75, 367–388.
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