Comparisons of residual lifetimes of coherent systems under dependence

J. Navarro¹, Universidad de Murcia (Spain).

¹Supported by Ministerio de Economía y Competitividad under Grant MTM2012-34023-FEDER.
Residual lifetimes

- $X_1, \ldots, X_n$ component lifetimes with RF
  \[ F_i(t) = \Pr(X_i > t). \]

- $T = \phi(X_1, \ldots, X_n)$ system lifetime with RF
  \[ F_T(t) = \Pr(T > t). \]

- We assume $F_i(t) = \Pr(X_i > t) > 0$ and $F_T(t) > 0$ for $t \geq 0$.

- Component residual lifetimes $X_{i,t} = (X_i - t|X_i > t)$ with RF:
  \[ F_{i,t}(x) = \Pr(X_{i,t} > x) = \Pr(X_i - t > x|X_i > t) = \frac{F_i(t + x)}{F_i(t)}. \]
Residual lifetimes

- $X_1, \ldots, X_n$ component lifetimes with RF
  $$\bar{F}_i(t) = \Pr(X_i > t).$$

- $T = \phi(X_1, \ldots, X_n)$ system lifetime with RF
  $$\bar{F}_T(t) = \Pr(T > t).$$

- We assume $\bar{F}_i(t) = \Pr(X_i > t) > 0$ and $\bar{F}_T(t) > 0$ for $t \geq 0$.

- Component residual lifetimes $X_{i,t} = (X_i - t|X_i > t)$ with RF:
  $$\bar{F}_{i,t}(x) = \Pr(X_{i,t} > x) = \Pr(X_i - t > x|X_i > t) = \frac{\bar{F}_i(t + x)}{\bar{F}_i(t)}.$$
Residual lifetimes

- $X_1, \ldots, X_n$ component lifetimes with RF
  \[
  \bar{F}_i(t) = \Pr(X_i > t).
  \]

- $T = \phi(X_1, \ldots, X_n)$ system lifetime with RF
  \[
  \bar{F}_T(t) = \Pr(T > t).
  \]

- We assume $\bar{F}_i(t) = \Pr(X_i > t) > 0$ and $\bar{F}_T(t) > 0$ for $t \geq 0$.

- Component residual lifetimes $X_{i,t} = (X_i - t|X_i > t)$ with RF:
  \[
  \bar{F}_{i,t}(x) = \Pr(X_{i,t} > x) = \Pr(X_i - t > x|X_i > t) = \frac{\bar{F}_i(t + x)}{\bar{F}_i(t)}.
  \]
Residual lifetimes

- $X_1, \ldots, X_n$ component lifetimes with RF

\[ \overline{F}_i(t) = \Pr(X_i > t). \]

- $T = \phi(X_1, \ldots, X_n)$ system lifetime with RF

\[ \overline{F}_T(t) = \Pr(T > t). \]

- We assume $\overline{F}_i(t) = \Pr(X_i > t) > 0$ and $\overline{F}_T(t) > 0$ for $t \geq 0$.

- Component residual lifetimes $X_{i,t} = (X_i - t|X_i > t)$ with RF:

\[ \overline{F}_{i,t}(x) = \Pr(X_{i,t} > x) = \Pr(X_i - t > x|X_i > t) = \frac{\overline{F}_i(t + x)}{\overline{F}_i(t)}. \]
We have two main options to define the system residual lifetime at time $t > 0$:

- The usual residual lifetime $T_t = (T - t | T > t)$ with RF

  $$F_t(x) = \Pr(T - t > x | T > t) = \frac{F_T(t + x)}{F_T(t)}.$$  

- The residual lifetime at the system level $T^*_t = (T - t | X_1 > t, \ldots, X_n > t)$ with RF

  $$F^*_t(x) = \Pr(T^*_t > x) = \frac{\Pr(T > t + x, X_1 > t, \ldots, X_n > t)}{\Pr(X_1 > t, \ldots, X_n > t)}$$

when $\Pr(X_1 > t, \ldots, X_n > t) > 0$. 

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
We have two main options to define the system residual lifetime at time $t > 0$:

- The usual **residual lifetime** $T_t = (T - t | T > t)$ with RF

$$
\overline{F}_t(x) = \Pr(T - t > x | T > t) = \frac{\overline{F}_T(t + x)}{\overline{F}_T(t)}.
$$

- The residual lifetime at the system level

$$
T^*_t = (T - t | X_1 > t, \ldots, X_n > t) \text{ with RF}
$$

$$
\overline{F}^*_t(x) = \Pr(T^*_t > x) = \frac{\Pr(T > t + x, X_1 > t, \ldots, X_n > t)}{\Pr(X_1 > t, \ldots, X_n > t)}
$$

when $\Pr(X_1 > t, \ldots, X_n > t) > 0$. 
We have two main options to define the system residual lifetime at time $t > 0$:

- The usual **residual lifetime** $T_t = (T - t \mid T > t)$ with RF

$$
\bar{F}_t(x) = \Pr(T - t > x \mid T > t) = \frac{\bar{F}_T(t + x)}{\bar{F}_T(t)}.
$$

- The **residual lifetime at the system level**

$T^*_t = (T - t \mid X_1 > t, \ldots, X_n > t)$ with RF

$$
\bar{F}^*_t(x) = \Pr(T^*_t > x) = \frac{\Pr(T > t + x, X_1 > t, \ldots, X_n > t)}{\Pr(X_1 > t, \ldots, X_n > t)}
$$

when $\Pr(X_1 > t, \ldots, X_n > t) > 0$. 

*SMART2016, Salerno*  
J. Navarro, E-mail: jorgenav@um.es
Which one is the best system?

Intuitively, it seems that $T^*_t$ should be always better than $T_t$.

It should be better to know that all the components are working at time $t$!

For $T = \min(X_1, \ldots, X_n)$, $T_t =_{ST} T^*_t$ (where $=_{ST}$ denotes equality in distribution) for all $t > 0$. 
Which one is the best system?

Intuitively, it seems that $T^*_t$ should be always better than $T_t$.

It should be better to know that all the components are working at time $t$!

For $T = \min(X_1, \ldots, X_n)$, $T_t =_{ST} T^*_t$ (where $=_{ST}$ denotes equality in distribution) for all $t > 0$. 
Which one is the best system?

Intuitively, it seems that $T_t^*$ should be always better than $T_t$.

It should be better to know that all the components are working at time $t$!

For $T = \min(X_1, \ldots, X_n)$, $T_t =_{ST} T_t^*$ (where $=_{ST}$ denotes equality in distribution) for all $t > 0$. 
Which one is the best system?

Intuitively, it seems that $T^*_t$ should be always better than $T_t$.

It should be better to know that all the components are working at time $t$!

For $T = \min(X_1, \ldots, X_n)$, $T_t =_{ST} T^*_t$ (where $=_{ST}$ denotes equality in distribution) for all $t > 0$. 

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
If $X_1, \ldots, X_n$ are independent, then

$$T_t = (T - t | T > t) \leq_{ST} T^*_t = (T - t | X_1 > t, \ldots, X_n > t);$$

(1)

see Pellerey and Petakos (IEEE Tr Rel, 2002) and Li and Lu (PEIS, 2003).

- Conditions on $(X_1, \ldots, X_n)$ to have (1) were given in Li, Pellerey and You (2013).
- They also proved that (1) is not necessarily true in the dependent (discrete) case.
If $X_1, \ldots, X_n$ are independent, then

$$T_t = (T - t | T > t) \leq_{ST} T^*_t = (T - t | X_1 > t, \ldots, X_n > t); \quad (1)$$

see Pellerey and Petakos (IEEE Tr Rel, 2002) and Li and Lu (PEIS, 2003).

Conditions on $(X_1, \ldots, X_n)$ to have (1) were given in Li, Pellerey and You (2013).

They also proved that (1) is not necessarily true in the dependent (discrete) case.
System residual lifetimes

- If $X_1, \ldots, X_n$ are independent, then

\[ T_t = (T - t | T > t) \leq_{ST} T^*_t = (T - t | X_1 > t, \ldots, X_n > t); \]

see Pellerey and Petakos (IEEE Tr Rel, 2002) and Li and Lu (PEIS, 2003).

- Conditions on $(X_1, \ldots, X_n)$ to have (1) were given in Li, Pellerey and You (2013).

- They also proved that (1) is not necessarily true in the dependent (discrete) case.
In this talk, conditions to get (1) are given when the dependence structure is known.

The conditions are based on distorted distribution representations.

Some conditions are obtained to get (1) for other usual stochastic orders.

These conditions can also be applied to the case of independent components.

Some illustrative examples are given.

They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.

Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!
In this talk, conditions to get (1) are given when the dependence structure is known.

The conditions are based on distorted distribution representations.

Some conditions are obtained to get (1) for other usual stochastic orders.

These conditions can also be applied to the case of independent components.

Some illustrative examples are given.

They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.

Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!
In this talk, conditions to get (1) are given when the dependence structure is known.

The conditions are based on distorted distribution representations.

Some conditions are obtained to get (1) for other usual stochastic orders.

These conditions can also be applied to the case of independent components.

Some illustrative examples are given.

They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.

Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!
In this talk, conditions to get (1) are given when the dependence structure is known.

The conditions are based on distorted distribution representations.

Some conditions are obtained to get (1) for other usual stochastic orders.

These conditions can also be applied to the case of independent components.

Some illustrative examples are given.

They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.

Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!
In this talk, conditions to get (1) are given when the dependence structure is known. The conditions are based on distorted distribution representations. Some conditions are obtained to get (1) for other usual stochastic orders. These conditions can also be applied to the case of independent components. Some illustrative examples are given. They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders. Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!
In this talk, conditions to get (1) are given when the dependence structure is known.

The conditions are based on distorted distribution representations.

Some conditions are obtained to get (1) for other usual stochastic orders.

These conditions can also be applied to the case of independent components.

Some illustrative examples are given.

They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.

Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!
In this talk, conditions to get (1) are given when the dependence structure is known.

The conditions are based on distorted distribution representations.

Some conditions are obtained to get (1) for other usual stochastic orders.

These conditions can also be applied to the case of independent components.

Some illustrative examples are given.

They show that (1) holds (or does not hold) for some copulas, system structures and stochastic orders.

Surprisingly, in some cases, the ordering in (1) does not hold or it can be reversed!
Generalized distorted distribution

The **generalized distorted distribution** (GDD) associated to \(n\) DF \(F_1, \ldots, F_n\) and to an increasing continuous **multivariate distortion function** \(Q : [0, 1]^n \rightarrow [0, 1]\) such that \(Q(0, \ldots, 0) = 0\) and \(Q(1, \ldots, 1) = 1\), is

\[
F_Q(t) = Q(F_1(t), \ldots, F_n(t)).
\]  
(2)

For the RF we have

\[
\overline{F}_Q(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t)),
\]  
(3)

where \(\overline{F} = 1 - F\), \(\overline{F}_Q = 1 - F_Q\) and \(\overline{Q}(u_1, \ldots, u_n) = 1 - Q(1 - u_1, \ldots, 1 - u_n)\) is the **multivariate dual distortion function**; see Navarro et al. (MCAP 2015).
The **generalized distorted distribution** (GDD) associated to \( n \) DF \( F_1, \ldots, F_n \) and to an increasing continuous **multivariate distortion function** \( Q : [0, 1]^n \rightarrow [0, 1] \) such that \( Q(0, \ldots, 0) = 0 \) and \( Q(1, \ldots, 1) = 1 \), is

\[
F_Q(t) = Q(F_1(t), \ldots, F_n(t)).
\]

For the RF we have

\[
\bar{F}_Q(t) = \overline{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)),
\]

where \( \bar{F} = 1 - F \), \( \bar{F}_Q = 1 - F_Q \) and

\( \overline{Q}(u_1, \ldots, u_n) = 1 - Q(1-u_1, \ldots, 1-u_n) \) is the **multivariate dual distortion function**; see Navarro et al. (MCAP 2015).
Coherent systems—GENERAL case

- **A path set** of $T$ is a set $P \subseteq \{1, \ldots, n\}$ such that if all the components in $P$ work, then the system works.

- **A minimal path set** of $T$ is a path set which does not contain other path sets.

- If $P_1, \ldots, P_m$ are the minimal path sets of $T$, then $T = \max_{j=1,\ldots,m} X_{P_j}$, where $X_P = \min_{i \in P} X_i$ and

\[
\overline{F}_T(t) = \Pr \left( \max_{j=1,\ldots,m} X_{P_j} > t \right) = \Pr \left( \bigcup_{j=1}^m \{X_{P_j} > t\} \right)
\]

\[
= \sum_{i=1}^m \overline{F}_{P_i}(t) - \sum_{i \neq j} \overline{F}_{P_i \cup P_j}(t) + \cdots \pm \overline{F}_{P_1 \cup \cdots \cup P_m}(t)
\]

where $\overline{F}_P(t) = \Pr(X_P > t)$. 

Coherent systems—GENERAL case

- A **path set** of $T$ is a set $P \subseteq \{1, \ldots, n\}$ such that if all the components in $P$ work, then the system works.

- A **minimal path set** of $T$ is a path set which does not contain other path sets.

If $P_1, \ldots, P_m$ are the minimal path sets of $T$, then

$$T = \max_{j=1,\ldots,m} X_{P_j},$$

where $X_P = \min_{i \in P} X_i$ and

$$\overline{F}_T(t) = \Pr \left( \max_{j=1,\ldots,m} X_{P_j} > t \right) = \Pr \left( \bigcup_{j=1}^{m} \{X_{P_j} > t\} \right)$$

$$= \sum_{i=1}^{m} \overline{F}_{P_i}(t) - \sum_{i \neq j} \overline{F}_{P_i \cup P_j}(t) + \cdots \pm \overline{F}_{P_1 \cup \cdots \cup P_m}(t)$$

where $\overline{F}_P(t) = \Pr(X_P > t)$. 
A **path set** of $T$ is a set $P \subseteq \{1, \ldots, n\}$ such that if all the components in $P$ work, then the system works.

A **minimal path set** of $T$ is a path set which does not contains other path sets.

If $P_1, \ldots, P_m$ are the minimal path sets of $T$, then $T = \max_{j=1,\ldots,m} X_{P_j}$, where $X_P = \min_{i \in P} X_i$ and

$$
\overline{F}_T(t) = \Pr \left( \max_{j=1,\ldots,m} X_{P_j} > t \right) = \Pr \left( \bigcup_{j=1}^m \{X_{P_j} > t\} \right)
$$

$$
= \sum_{i=1}^m \overline{F}_{P_i}(t) - \sum_{i \neq j} \overline{F}_{P_i \cup P_j}(t) + \cdots \pm \overline{F}_{P_1 \cup \cdots \cup P_m}(t)
$$

where $\overline{F}_P(t) = \Pr(X_P > t)$. 

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
The copula representation for the RF of \((X_1, \ldots, X_n)\) is

\[
\bar{F}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = K(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),
\]

where \(\bar{F}_i(t) = \Pr(X_i > t)\) and \(K\) is the survival copula. Hence

\[
\bar{F}_{1:k}(t) = \Pr(X_1 > t, \ldots, X_k > t) = K(\bar{F}_1(t), \ldots, \bar{F}_r(t), 1, \ldots, 1).
\]

Then the system reliability can be written as

\[
\bar{F}_T(t) = Q_{\phi,K}(\bar{F}_1(t), \ldots, \bar{F}_n(t)).
\]

- If the components are ID, then \(\bar{F}_T(t) = \bar{q}_{\phi,K}(\bar{F}(t))\).
- If the components are IND, then \(Q_{\phi,K}\) is a multinomial.
- If the components are IID, then \(\bar{q}_{\phi,K}(u) = \sum_{i=1}^{n} a_i u^i\), where \((a_1, \ldots, a_n)\) is the minimal signature.
Coherent system representation

- The copula representation for the RF of \((X_1, \ldots, X_n)\) is
  \[
  \overline{F}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = K(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)),
  \]
  where \(\overline{F}_i(t) = \Pr(X_i > t)\) and \(K\) is the survival copula. Hence
  \[
  \overline{F}_{1:k}(t) = \Pr(X_1 > t, \ldots, X_k > t) = K(\overline{F}_1(t), \ldots, \overline{F}_r(t), 1, \ldots, 1).
  \]
- Then the system reliability can be written as
  \[
  \overline{F}_T(t) = \overline{Q}_{\phi,K}(\overline{F}_1(t), \ldots, \overline{F}_n(t)).
  \]
- If the components are ID, then \(\overline{F}_T(t) = \overline{q}_{\phi,K}(\overline{F}(t))\).
- If the components are IND, then \(\overline{Q}_{\phi,K}\) is a multinomial.
- If the components are IID, then \(\overline{q}_{\phi,K}(u) = \sum_{i=1}^n a_i u^i\), where \((a_1, \ldots, a_n)\) is the minimal signature.
Coherent system representation

- The copula representation for the RF of \((X_1, \ldots, X_n)\) is

\[
\bar{F}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = K(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),
\]

where \(\bar{F}_i(t) = \Pr(X_i > t)\) and \(K\) is the survival copula. Hence

\[
\bar{F}_{1:k}(t) = \Pr(X_1 > t, \ldots, X_k > t) = K(\bar{F}_1(t), \ldots, \bar{F}_r(t), 1, \ldots, 1).
\]

- Then the system reliability can be written as

\[
\bar{F}_T(t) = \bar{Q}_{\phi,K}(\bar{F}_1(t), \ldots, \bar{F}_n(t)).
\]

- If the components are ID, then \(\bar{F}_T(t) = \bar{q}_{\phi,K}(\bar{F}(t))\).
- If the components are IND, then \(\bar{Q}_{\phi,K}\) is a multinomial.
- If the components are IID, then \(\bar{q}_{\phi,K}(u) = \sum_{i=1}^n a_i u^i\), where \((a_1, \ldots, a_n)\) is the minimal signature.
The copula representation for the RF of \((X_1, \ldots, X_n)\) is
\[
\bar{F}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = K(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),
\]
where \(\bar{F}_i(t) = \Pr(X_i > t)\) and \(K\) is the survival copula. Hence
\[
\bar{F}_{1:k}(t) = \Pr(X_1 > t, \ldots, X_k > t) = K(\bar{F}_1(t), \ldots, \bar{F}_r(t), 1, \ldots, 1).
\]

Then the system reliability can be written as
\[
\bar{F}_T(t) = \bar{Q}_{\phi,K}(\bar{F}_1(t), \ldots, \bar{F}_n(t)).
\]

- If the components are ID, then \(\bar{F}_T(t) = \bar{q}_{\phi,K}(\bar{F}(t))\).
- If the components are IND, then \(\bar{Q}_{\phi,K}\) is a multinomial.
- If the components are IID, then \(\bar{q}_{\phi,K}(u) = \sum_{i=1}^{n} a_i u^i\), where \((a_1, \ldots, a_n)\) is the minimal signature.
Coherent system representation

- The copula representation for the RF of \((X_1, \ldots, X_n)\) is

\[
\overline{F}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = K(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)),
\]

where \(\overline{F}_i(t) = \Pr(X_i > t)\) and \(K\) is the survival copula. Hence

\[
\overline{F}_{1:k}(t) = \Pr(X_1 > t, \ldots, X_k > t) = K(\overline{F}_1(t), \ldots, \overline{F}_r(t), 1, \ldots, 1).
\]

- Then the system reliability can be written as

\[
\overline{F}_T(t) = \overline{Q}_{\phi,K}(\overline{F}_1(t), \ldots, \overline{F}_n(t)).
\]

- If the components are ID, then \(\overline{F}_T(t) = \overline{q}_{\phi,K}(\overline{F}(t))\).

- If the components are IND, then \(\overline{Q}_{\phi,K}\) is a multinomial.

- If the components are IID, then \(\overline{q}_{\phi,K}(u) = \sum_{i=1}^n a_i u^i\), where \((a_1, \ldots, a_n)\) is the minimal signature.
The RF of $T_t = (T - t \mid T > t)$ is

$$F_t(x) = \frac{F_T(t + x)}{F_T(t)} = \frac{Q(F_1(t + x), \ldots, F_n(t + x))}{Q(F_1(t), \ldots, F_n(t))}.$$

Then

$$F_t(x) = \frac{Q(F_1(t)F_{1,t}(x), \ldots, F_n(t)F_{n,t}(x))}{Q(F_1(t), \ldots, F_n(t))},$$

where $F_{i,t}(x) = F_i(t + x)/F_i(t)$.

Therefore

$$F_t(x) = Q_t(F_{1,t}(x), \ldots, F_{n,t}(x)),$$

where

$$Q_t(u_1, \ldots, u_n) = \frac{Q(F_1(t)u_1, \ldots, F_n(t)u_n)}{Q(F_1(t), \ldots, F_n(t))}.$$
The RF of $T_t = (T - t | T > t)$ is

$$F_t(x) = \frac{\overline{F}_T(t + x)}{\overline{F}_T(t)} = \frac{\overline{Q}(\overline{F}_1(t + x), \ldots, \overline{F}_n(t + x))}{\overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t))}.$$ 

Then

$$\overline{F}_t(x) = \frac{\overline{Q}(\overline{F}_1(t)\overline{F}_{1,t}(x), \ldots, \overline{F}_n(t)\overline{F}_{n,t}(x))}{\overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t))},$$

where $\overline{F}_{i,t}(x) = \overline{F}_i(t + x)/\overline{F}_i(t)$.

Therefore

$$F_t(x) = \overline{Q}_t(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x)),$$

where

$$\overline{Q}_t(u_1, \ldots, u_n) = \frac{\overline{Q}(\overline{F}_1(t)u_1, \ldots, \overline{F}_n(t)u_n)}{\overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t))}.$$
Representations for the system residual lifetimes

- The RF of $T_t = (T - t | T > t)$ is
  \[
  \overline{F}_t(x) = \frac{\overline{F}_T(t + x)}{\overline{F}_T(t)} = \frac{\overline{Q}(\overline{F}_1(t + x), \ldots, \overline{F}_n(t + x))}{\overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t))}.
  \]

- Then
  \[
  \overline{F}_t(x) = \frac{\overline{Q}(\overline{F}_1(t)\overline{F}_{1,t}(x), \ldots, \overline{F}_n(t)\overline{F}_{n,t}(x))}{\overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t))},
  \]
  where $\overline{F}_{i,t}(x) = \overline{F}_i(t + x)/\overline{F}_i(t)$.

- Therefore
  \[
  \overline{F}_t(x) = \overline{Q}_t(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x)),
  \]
  where
  \[
  \overline{Q}_t(u_1, \ldots, u_n) = \frac{\overline{Q}(\overline{F}_1(t)u_1, \ldots, \overline{F}_n(t)u_n)}{\overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t))}.
  \]
Representations for the system residual lifetimes

- The RF of \( T^*_t = (T - t | X_1 > t, \ldots, X_n > t) \) is

\[
\overline{F}^*_t(x) = \frac{\Pr(T > t + x, X_1 > t, \ldots, X_n > t)}{\Pr(X_1 > t, \ldots, X_n > t)}.
\]

- As \( T = \max_{j=1,\ldots,m} X_{P_j} \) for the minimal path sets \( P_1, \ldots, P_m \), then

\[
\overline{F}^*_t(x) = \frac{\Pr(\max_{j=1,\ldots,m} X_{P_j} > t + x, X_1 > t, \ldots, X_n > t)}{K(\overline{F}_1(t), \ldots, \overline{F}_n(t))}.
\]

- Therefore

\[
\overline{F}^*_t(x) = \overline{Q}^*_t(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x)),
\]

where \( \overline{F}_{i,t}(x) = \overline{F}_i(t + x)/\overline{F}_i(t) \).
Representations for the system residual lifetimes

- The RF of $T_t^* = (T - t | X_1 > t, \ldots, X_n > t)$ is

$$
\overline{F}_t^*(x) = \frac{\Pr(T > t + x, X_1 > t, \ldots, X_n > t)}{\Pr(X_1 > t, \ldots, X_n > t)}.
$$

- As $T = \max_{j=1,\ldots,m} X_{P_j}$ for the minimal path sets $P_1, \ldots, P_m$, then

$$
\overline{F}_t^*(x) = \frac{\Pr(\max_{j=1,\ldots,m} X_{P_j} > t + x, X_1 > t, \ldots, X_n > t)}{K(\overline{F}_1(t), \ldots, \overline{F}_n(t))}.
$$

- Therefore

$$
\overline{F}_t^*(x) = \overline{Q}_t^*(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x)),
$$
where $\overline{F}_{i,t}(x) = \overline{F}_i(t + x)/\overline{F}_i(t)$. 
Representations for the system residual lifetimes

- The RF of $T^*_t = (T - t | X_1 > t, \ldots, X_n > t)$ is

$$F^*_t(x) = \frac{\Pr(T > t + x, X_1 > t, \ldots, X_n > t)}{\Pr(X_1 > t, \ldots, X_n > t)}.$$ 

- As $T = \max_{j=1,\ldots,m} X_{P_j}$ for the minimal path sets $P_1, \ldots, P_m$, then

$$F^*_t(x) = \frac{\Pr(\max_{j=1,\ldots,m} X_{P_j} > t + x, X_1 > t, \ldots, X_n > t)}{K(F_1(t), \ldots, F_n(t))}.$$ 

- Therefore

$$F^*_t(x) = \overline{Q}_t(F_{1,t}(x), \ldots, F_{n,t}(x)),$$

where $\overline{F}_{i,t}(x) = F_i(t + x)/F_i(t)$. 

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
Parallel system with two components

- $T = \max(X_1, X_2)$.
- Minimal path sets $P_1 = \{1\}$ and $P_2 = \{2\}$.
- System reliability function:

\[
\overline{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \overline{F}_1(t) + \overline{F}_2(t) - \Pr(X_1 > t, X_2 > t).
\]

- Then:

\[
\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \overline{F}_2(t)),
\]

where

\[
\overline{Q}(u_1, u_2) = u_1 + u_2 - K(u_1, u_2).
\]
Parallel system with two components

- \( T = \max(X_1, X_2) \).
- Minimal path sets \( P_1 = \{1\} \) and \( P_2 = \{2\} \).
- System reliability function:
  \[
  \bar{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \bar{F}_1(t) + \bar{F}_2(t) - \Pr(X_1 > t, X_2 > t).
  \]
  Then:
  \[
  \bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t)),
  \]
  where
  \[
  \bar{Q}(u_1, u_2) = u_1 + u_2 - K(u_1, u_2).
  \]
Parallel system with two components

- \( T = \max(X_1, X_2) \).
- Minimal path sets \( P_1 = \{1\} \) and \( P_2 = \{2\} \).
- System reliability function:

\[
\overline{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \overline{F}_1(t) + \overline{F}_2(t) - \Pr(X_1 > t, X_2 > t).
\]

- Then:

\[
\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \overline{F}_2(t)),
\]

where

\[
\overline{Q}(u_1, u_2) = u_1 + u_2 - K(u_1, u_2).
\]
Parallel system with two components

- \( T = \max(X_1, X_2) \).
- Minimal path sets \( P_1 = \{1\} \) and \( P_2 = \{2\} \).
- System reliability function:
  \[
  \overline{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \overline{F}_1(t) + \overline{F}_2(t) - \Pr(X_1 > t, X_2 > t).
  \]
- Then:
  \[
  \overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \overline{F}_2(t)),
  \]
  where
  \[
  \overline{Q}(u_1, u_2) = u_1 + u_2 - K(u_1, u_2).
  \]
Parallel system with two components

The RF of $T_t = (T - t | T > t)$ is

$$
\overline{F}_t(x) = \overline{Q}_t(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x)),
$$

where

$$
\overline{Q}_t(u_1, u_2) = \frac{\overline{Q}(\overline{F}_1(t)u_1, \overline{F}_2(t)u_2)}{\overline{Q}(\overline{F}_1(t), \overline{F}_2(t))}.
$$

Then

$$
\overline{Q}_t(u_1, u_2) = \frac{\overline{F}_1(t)u_1 + \overline{F}_2(t)u_2 - K(\overline{F}_1(t)u_1, \overline{F}_2(t)u_2)}{\overline{F}_1(t) + \overline{F}_2(t) - K(\overline{F}_1(t), \overline{F}_2(t))}.
$$
Parallel system with two components

- The RF of $T_t = (T - t | T > t)$ is
  \[
  \overline{F}_t(x) = \overline{Q}(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x)),
  \]
  where
  \[
  \overline{Q}(u_1, u_2) = \frac{\overline{Q}(\overline{F}_1(t)u_1, \overline{F}_2(t)u_2)}{\overline{Q}(\overline{F}_1(t), \overline{F}_2(t))}.
  \]
- Then
  \[
  \overline{Q}(u_1, u_2) = \frac{\overline{F}_1(t)u_1 + \overline{F}_2(t)u_2 - K(\overline{F}_1(t)u_1, \overline{F}_2(t)u_2)}{\overline{F}_1(t) + \overline{F}_2(t) - K(\overline{F}_1(t), \overline{F}_2(t))}.
  \]
Parallel system with two components

- The RF of $T^*_t = (T - t | X_1 > t, X_2 > t)$ is

$$
\bar{F}^*_t(x) = \frac{\Pr(\max(X_1, X_2) > t + x, X_1 > t, X_2 > t)}{\Pr(X_1 > t, X_2 > t)}.
$$

- Hence

$$
\bar{F}^*_t(x) = \frac{K(\bar{F}_1(t + x), c_2) + K(c_1, \bar{F}_2(t + x)) - K(\bar{F}_1(t + x), \bar{F}_2(t + x))}{K(c_1, c_2)},
$$

where $c_1 = \bar{F}_1(t)$ and $c_2 = \bar{F}_2(t)$.

- Then $F_t(x) = Q^*_t(F_1, t(x), F_2, t(x))$, where

$$
Q^*_t(u_1, u_2) = \frac{K(c_1 u_1, c_2) + K(c_1, c_2 u_2) - K(c_1 u_1, c_2 u_2)}{K(c_1, c_2)}.
$$
Parallel system with two components

- The RF of $T^*_t = (T - t|X_1 > t, X_2 > t)$ is

  $$F^*_t(x) = \frac{\Pr(\max(X_1, X_2) > t + x, X_1 > t, X_2 > t)}{\Pr(X_1 > t, X_2 > t)}.$$

- Hence

  $$F^*_t(x) = \frac{K(\bar{F}_1(t + x), c_2) + K(c_1, \bar{F}_2(t + x)) - K(\bar{F}_1(t + x), \bar{F}_2(t + x))}{K(c_1, c_2)},$$

  where $c_1 = \bar{F}_1(t)$ and $c_2 = \bar{F}_2(t)$.

- Then $F_t(x) = Q^*_t(\bar{F}_1,t(x), \bar{F}_2,t(x))$, where

  $$Q^*_t(u_1, u_2) = \frac{K(c_1 u_1, c_2) + K(c_1, c_2 u_2) - K(c_1 u_1, c_2 u_2)}{K(c_1, c_2)}.$$

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
Parallel system with two components

- The RF of $T^*_t = (T - t | X_1 > t, X_2 > t)$ is

$$\overline{F}_t^*(x) = \frac{\Pr(\max(X_1, X_2) > t + x, X_1 > t, X_2 > t)}{\Pr(X_1 > t, X_2 > t)}.$$ 

- Hence

$$\overline{F}_t^*(x) = \frac{K(\overline{F}_1(t + x), c_2) + K(c_1, \overline{F}_2(t + x)) - K(\overline{F}_1(t + x), \overline{F}_2(t + x))}{K(c_1, c_2)},$$

where $c_1 = \overline{F}_1(t)$ and $c_2 = \overline{F}_2(t)$.

- Then $\overline{F}_t(x) = \overline{Q}_t^*(\overline{F}_1, t(x), \overline{F}_2, t(x))$, where

$$\overline{Q}_t^*(u_1, u_2) = \frac{K(c_1 u_1, c_2) + K(c_1, c_2 u_2) - K(c_1 u_1, c_2 u_2)}{K(c_1, c_2)}.$$
Parallel system with two IND components

- If \( X_1 \) and \( X_2 \) are IND, then \( K(u_1, u_2) = u_1 u_2 \) and

\[
\overline{Q}^*(u_1, u_2) = \frac{F_1(t)u_1F_2(t) + F_1(t)F_2(t)u_2 - F_1(t)u_1F_2(t)u_2}{F_1(t)F_2(t)}
\]

that is,

\[
\overline{Q}^*(u_1, u_2) = u_1 + u_2 - u_1 u_2 = \overline{Q}(u_1, u_2).
\]

- This is a general property, i.e., if \( X_1, \ldots, X_n \) are IND, then

\[
\overline{Q}^*(u_1, \ldots, u_n) = \overline{Q}(u_1, \ldots, u_n).
\]

- Some authors consider the system \( T_{t}^{**} \) with reliability function

\[
\overline{F}_t^{**}(x) = \overline{Q}(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x)).
\]

- The meaning in practice is not clear for me.
Parallel system with two IND components

- If \(X_1\) and \(X_2\) are IND, then \(K(u_1, u_2) = u_1 u_2\) and

\[
Q^*_t(u_1, u_2) = \frac{F_1(t)u_1F_2(t) + F_1(t)F_2(t)u_2 - F_1(t)u_1F_2(t)u_2}{F_1(t)F_2(t)},
\]

that is,

\[
Q^*_t(u_1, u_2) = u_1 + u_2 - u_1u_2 = Q(u_1, u_2).
\]

- This is a general property, i.e., if \(X_1, \ldots, X_n\) are IND, then

\[
Q^*_t(u_1, \ldots, u_n) = Q(u_1, \ldots, u_n).
\]

- Some authors consider the system \(T_{**}^t\) with reliability function

\[
F_{**}(x) = Q(F_{1,t}(x), F_{2,t}(x)).
\]

- The meaning in practice is not clear for me.
Parallel system with two IND components

- If $X_1$ and $X_2$ are IND, then $K(u_1, u_2) = u_1 u_2$ and

$$Q^*_t(u_1, u_2) = \frac{F_1(t)u_1F_2(t) + F_1(t)F_2(t)u_2 - F_1(t)u_1F_2(t)u_2}{F_1(t)F_2(t)},$$

that is,

$$Q^*_t(u_1, u_2) = u_1 + u_2 - u_1 u_2 = Q(u_1, u_2).$$

- This is a general property, i.e., if $X_1, \ldots, X_n$ are IND, then

$$Q^*_t(u_1, \ldots, u_n) = Q(u_1, \ldots, u_n).$$

- Some authors consider the system $T^{**}_t$ with reliability function

$$F^{**}_t(x) = Q(F_{1,t}(x), F_{2,t}(x)).$$

- The meaning in practice is not clear for me.
Parallel system with two IND components

- If $X_1$ and $X_2$ are IND, then $K(u_1, u_2) = u_1 u_2$ and
  \[
  \bar{Q}_t^*(u_1, u_2) = \frac{\bar{F}_1(t)u_1\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_2(t)u_2 - \bar{F}_1(t)u_1\bar{F}_2(t)u_2}{\bar{F}_1(t)\bar{F}_2(t)},
  \]
  that is,
  \[
  \bar{Q}_t^*(u_1, u_2) = u_1 + u_2 - u_1 u_2 = \bar{Q}(u_1, u_2).
  \]
- This is a general property, i.e., if $X_1, \ldots, X_n$ are IND, then
  \[
  \bar{Q}_t^*(u_1, \ldots, u_n) = \bar{Q}(u_1, \ldots, u_n).
  \]
- Some authors consider the system $T_{**}^t$ with reliability function
  \[
  \bar{F}_{t}^{**}(x) = \bar{Q}(\bar{F}_{1,t}(x), \bar{F}_{2,t}(x)).
  \]
- The meaning in practice is not clear for me.
Comparison results-DD

- If $q_1$ and $q_2$ are two DF,

\[
q_1(F) \leq_{ord} q_2(F) \text{ for all } F?
\]

- If $q$ is a DF,

\[
F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)?
\]

- If $Q_1$ and $Q_2$ are two MDF,

\[
Q_1(F_1, \ldots, F_n) \leq_{ord} Q_2(F_1, \ldots, F_n)\
\]

- If $Q$ is a MDF,

\[
F_i \leq_{ord} G_i, i = 1, \ldots, n, \Rightarrow Q(F_1, \ldots, F_n) \leq_{ord} Q(G_1, \ldots, G_n)\
\]

If $q_1$ and $q_2$ are two DF,

\[ q_1(F) \leq_{ord} q_2(F) \text{ for all } F? \]

If $q$ is a DF,

\[ F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)? \]

If $Q_1$ and $Q_2$ are two MDF,

\[ Q_1(F_1, \ldots, F_n) \leq_{ord} Q_2(F_1, \ldots, F_n)? \]

If $Q$ is a MDF,

\[ F_i \leq_{ord} G_i, i = 1, \ldots, n, \Rightarrow Q(F_1, \ldots, F_n) \leq_{ord} Q(G_1, \ldots, G_n)? \]

Comparison results-DD

- If \( q_1 \) and \( q_2 \) are two DF,

\[
q_1(F) \leq_{ord} q_2(F) \text{ for all } F?
\]

- If \( q \) is a DF,

\[
F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)\]

- If \( Q_1 \) and \( Q_2 \) are two MDF,

\[
Q_1(F_1, \ldots, F_n) \leq_{ord} Q_2(F_1, \ldots, F_n)\]

- If \( Q \) is a MDF,

\[
F_i \leq_{ord} G_i, i = 1, \ldots, n, \Rightarrow Q(F_1, \ldots, F_n) \leq_{ord} Q(G_1, \ldots, G_n)\]

Comparison results-DD

- If $q_1$ and $q_2$ are two DF,
  \[ q_1(F) \leq_{\text{ord}} q_2(F) \text{ for all } F? \]

- If $q$ is a DF,
  \[ F \leq_{\text{ord}} G \Rightarrow q(F) \leq_{\text{ord}} q(G)? \]

- If $Q_1$ and $Q_2$ are two MDF,
  \[ Q_1(F_1, \ldots, F_n) \leq_{\text{ord}} Q_2(F_1, \ldots, F_n)? \]

- If $Q$ is a MDF,
  \[ F_i \leq_{\text{ord}} G_i, i = 1, \ldots, n, \Rightarrow Q(F_1, \ldots, F_n) \leq_{\text{ord}} Q(G_1, \ldots, G_n)? \]

Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow F_X(t) \leq F_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all $t$.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$ for all $t$.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all $t$.

Then

\[
\begin{align*}
X \leq_{LR} Y & \Rightarrow X \leq_{HR} Y \Rightarrow X \leq_{MRL} Y \\
\Downarrow & \Downarrow \Downarrow \\
X \leq_{RHR} Y & \Rightarrow X \leq_{ST} Y \Rightarrow E(X) \leq E(Y)
\end{align*}
\]
Main stochastic orderings

- \( X \leq_{ST} Y \Leftrightarrow F_X(t) \leq F_Y(t) \), stochastic order.

- \( X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t) \), hazard rate order.

- \( X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t) \) for all \( t \).

- \( X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t) \) for all \( t \).

- \( X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t) \) is nondecreasing, likelihood ratio order.

- \( X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t) \) for all \( t \).

Then

\[
X \leq_{LR} Y \quad \Rightarrow \quad X \leq_{HR} Y \quad \Rightarrow \quad X \leq_{MRL} Y \\
\downarrow \quad \downarrow \quad \downarrow \\
X \leq_{RHR} Y \quad \Rightarrow \quad X \leq_{ST} Y \quad \Rightarrow \quad E(X) \leq E(Y)
\]
Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \bar{F}_X(t) \leq \bar{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all $t$.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$ for all $t$.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all $t$.

Then

$$
\begin{align*}
X & \leq_{LR} Y \\
\Rightarrow & \\
\Downarrow & \\
X & \leq_{RHR} Y \\
\Rightarrow & \\
X & \leq_{ST} Y \\
\Rightarrow & \\
E(X) & \leq E(Y)
\end{align*}
$$
Main stochastic orderings

- $X \leq_{ST} Y \iff \overline{F}_X(t) \leq \overline{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \iff h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \iff (X - t \mid X > t) \leq_{ST} (Y - t \mid Y > t)$ for all $t$.
- $X \leq_{MRL} Y \iff E(X - t \mid X > t) \leq E(Y - t \mid Y > t)$ for all $t$.
- $X \leq_{LR} Y \iff f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
- $X \leq_{RHR} Y \iff (t - X \mid X < t) \geq_{ST} (t - Y \mid Y < t)$ for all $t$.

Then

$$
X \leq_{LR} Y \quad \Rightarrow \quad X \leq_{HR} Y \quad \Rightarrow \quad X \leq_{MRL} Y \\
\downarrow \quad \downarrow \quad \downarrow \\
X \leq_{RHR} Y \quad \Rightarrow \quad X \leq_{ST} Y \quad \Rightarrow \quad E(X) \leq E(Y)
$$
Main stochastic orderings

- \( X \leq_{ST} Y \iff F_X(t) \leq F_Y(t) \), stochastic order.
- \( X \leq_{HR} Y \iff h_X(t) \geq h_Y(t) \), hazard rate order.
- \( X \leq_{HR} Y \iff (X - t|X > t) \leq_{ST} (Y - t|Y > t) \) for all \( t \).
- \( X \leq_{MRL} Y \iff E(X - t|X > t) \leq E(Y - t|Y > t) \) for all \( t \).
- \( X \leq_{LR} Y \iff f_Y(t)/f_X(t) \) is nondecreasing, likelihood ratio order.

- \( X \leq_{RHR} Y \iff (t - X|X < t) \geq_{ST} (t - Y|Y < t) \) for all \( t \).

Then

\[
\begin{align*}
X \leq_{LR} Y & \quad \Rightarrow \quad X \leq_{HR} Y & \quad \Rightarrow \quad X \leq_{MRL} Y \\
\downarrow & & \downarrow & & \downarrow \\
X \leq_{RHR} Y & \quad \Rightarrow \quad X \leq_{ST} Y & \quad \Rightarrow \quad E(X) \leq E(Y)
\end{align*}
\]
Main stochastic orderings

- \( X \leq_{ST} Y \Leftrightarrow F_X(t) \leq F_Y(t) \), stochastic order.
- \( X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t) \), hazard rate order.
- \( X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t) \) for all \( t \).
- \( X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t) \) for all \( t \).
- \( X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t) \) is nondecreasing, likelihood ratio order.
- \( X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t) \) for all \( t \).

Then

\[
\begin{align*}
X & \leq_{LR} Y \quad \Rightarrow \quad X \leq_{HR} Y \quad \Rightarrow \quad X \leq_{MRL} Y \\
\downarrow & \hspace{1cm} \downarrow & \hspace{1cm} \downarrow \\
X & \leq_{RHR} Y \quad \Rightarrow \quad X \leq_{ST} Y \quad \Rightarrow \quad E(X) \leq E(Y)
\end{align*}
\]
Main stochastic orderings

- \( X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t) \), stochastic order.
- \( X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t) \), hazard rate order.
- \( X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t) \) for all \( t \).
- \( X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t) \) for all \( t \).
- \( X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t) \) is nondecreasing, likelihood ratio order.
- \( X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t) \) for all \( t \).
- Then

\[
\begin{align*}
X \leq_{LR} Y & \quad \Rightarrow \quad X \leq_{HR} Y & & \Rightarrow \quad X \leq_{MRL} Y \\
\downarrow & & \downarrow & & \downarrow \\
X \leq_{RHR} Y & \quad \Rightarrow \quad X \leq_{ST} Y & & \Rightarrow \quad E(X) \leq E(Y)
\end{align*}
\]
If $T_i$ has the RF $\bar{q}_i(\bar{F}(t))$, $i = 1, 2$, then:

- $T_1 \leq_{ST} T_2$ for all $F$ if and only if $\bar{q}_2/\bar{q}_1 \geq 1$ in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all $F$ if and only if $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all $F$ if and only if $q_2/q_1$ increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all $F$ if and only if $\bar{q}'_2/\bar{q}'_1$ decreases.
- $T_1 \leq_{MRL} T_2$ for all $F$ such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$. 
Comparison results-DD

- If \( T_i \) has the RF \( \bar{q}_i(F(t)) \), \( i = 1, 2 \), then:
- \( T_1 \leq_{ST} T_2 \) for all \( F \) if and only if \( \bar{q}_2/\bar{q}_1 \geq 1 \) in \((0, 1)\).
- \( T_1 \leq_{HR} T_2 \) for all \( F \) if and only if \( \bar{q}_2/\bar{q}_1 \) decreases in \((0, 1)\).
- \( T_1 \leq_{RHR} T_2 \) for all \( F \) if and only if \( q_2/q_1 \) increases in \((0, 1)\).
- \( T_1 \leq_{LR} T_2 \) for all \( F \) if and only if \( \bar{q}'_2/\bar{q}'_1 \) decreases.
- \( T_1 \leq_{MRL} T_2 \) for all \( F \) such that \( E(T_1) \leq E(T_2) \) if \( \bar{q}_2/\bar{q}_1 \) is bathtub in \((0, 1)\).
If $T_i$ has the RF $\bar{q}_i(F(t))$, $i = 1, 2$, then:

- $T_1 \leq_{ST} T_2$ for all $F$ if and only if $\bar{q}_2 / \bar{q}_1 \geq 1$ in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all $F$ if and only if $\bar{q}_2 / \bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all $F$ if and only if $q_2 / q_1$ increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all $F$ if and only if $\bar{q}_2' / \bar{q}_1'$ decreases.
- $T_1 \leq_{MRL} T_2$ for all $F$ such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2 / \bar{q}_1$ is bathtub in $(0, 1)$. 
If $T_i$ has the RF $\bar{q}_i(\overline{F}(t))$, $i = 1, 2$, then:

- $T_1 \leq_{ST} T_2$ for all $F$ if and only if $\bar{q}_2/\bar{q}_1 \geq 1$ in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all $F$ if and only if $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all $F$ if and only if $q_2/q_1$ increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all $F$ if and only if $\bar{q}'_2/\bar{q}'_1$ decreases.
- $T_1 \leq_{MRL} T_2$ for all $F$ such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$.  

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
If \( T_i \) has the RF \( \bar{q}_i(\bar{F}(t)), \ i = 1, 2, \) then:

- \( T_1 \leq_{ST} T_2 \) for all \( F \) if and only if \( \bar{q}_2/\bar{q}_1 \geq 1 \) in \( (0, 1) \).
- \( T_1 \leq_{HR} T_2 \) for all \( F \) if and only if \( \bar{q}_2/\bar{q}_1 \) decreases in \( (0, 1) \).
- \( T_1 \leq_{RHR} T_2 \) for all \( F \) if and only if \( q_2/q_1 \) increases in \( (0, 1) \).
- \( T_1 \leq_{LR} T_2 \) for all \( F \) if and only if \( \bar{q}_2'/\bar{q}_1' \) decreases.
- \( T_1 \leq_{MRL} T_2 \) for all \( F \) such that \( E(T_1) \leq E(T_2) \) if \( \bar{q}_2/\bar{q}_1 \) is bathtub in \( (0, 1) \).
If $T_i$ has the RF $\bar{q}_i(F(t))$, $i = 1, 2$, then:

- $T_1 \leq_{ST} T_2$ for all $F$ if and only if $\bar{q}_2/\bar{q}_1 \geq 1$ in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all $F$ if and only if $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all $F$ if and only if $q_2/q_1$ increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all $F$ if and only if $\bar{q}'_2/\bar{q}'_1$ decreases.
- $T_1 \leq_{MRL} T_2$ for all $F$ such that $E(T_1) \leq E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$. 
If \( T_i \) has RF \( Q_i(F_1, \ldots, F_n) \), \( i = 1, 2 \), then:

- \( T_1 \leq_{ST} T_2 \) for all \( F_1, \ldots, F_n \) if and only if \( Q_1 \leq Q_2 \) in \((0, 1)^n\).
- \( T_1 \leq_{HR} T_2 \) for all \( F_1, \ldots, F_n \) if and only if \( Q_2/Q_1 \) is decreasing in \((0, 1)^n\).
- \( T_1 \leq_{RHR} T_2 \) for all \( F_1, \ldots, F_n \) if and only if \( Q_2/Q_1 \) is increasing in \((0, 1)^n\).
If $T_i$ has RF $Q_i(F_1, \ldots, F_n)$, $i = 1, 2$, then:

- $T_1 \leq_{ST} T_2$ for all $F_1, \ldots, F_n$ if and only if $Q_1 \leq Q_2$ in $(0, 1)^n$.
- $T_1 \leq_{HR} T_2$ for all $F_1, \ldots, F_n$ if and only if $Q_2/Q_1$ is decreasing in $(0, 1)^n$.
- $T_1 \leq_{RHR} T_2$ for all $F_1, \ldots, F_n$ if and only if $Q_2/Q_1$ is increasing in $(0, 1)^n$. 
If $T_i$ has RF $Q_i(F_1, \ldots, F_n)$, $i = 1, 2$, then:

- $T_1 \leq_{ST} T_2$ for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\overline{Q}_1 \leq \overline{Q}_2$ in $(0, 1)^n$.
- $T_1 \leq_{HR} T_2$ for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\frac{Q_2}{Q_1}$ is decreasing in $(0, 1)^n$.
- $T_1 \leq_{RHR} T_2$ for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $Q_2/Q_1$ is increasing in $(0, 1)^n$. 
Comparison results-GDD

- If $T_i$ has RF $\bar{Q}_i(\bar{F}_1, \ldots, \bar{F}_n)$, $i = 1, 2$, then:
  - $T_1 \leq_{ST} T_2$ for all $\bar{F}_1, \ldots, \bar{F}_n$ if and only if $\bar{Q}_1 \leq \bar{Q}_2$ in $(0,1)^n$.
  - $T_1 \leq_{HR} T_2$ for all $\bar{F}_1, \ldots, \bar{F}_n$ if and only if $\bar{Q}_2/\bar{Q}_1$ is decreasing in $(0,1)^n$.
  - $T_1 \leq_{RHR} T_2$ for all $\bar{F}_1, \ldots, \bar{F}_n$ if and only if $Q_2/Q_1$ is increasing in $(0,1)^n$. 
These results can be applied to compare $T_t$ and $T_t^*$. For example:

- $T_t \leq_{ST} T_t^*$ ($\geq_{ST}$) holds for all $F_1, \ldots, F_n$ if and only if $Q_t \leq Q_t^*$ ($\geq$) in $(0,1)^n$.

- $T_t \leq_{HR} T_t^*$ ($\geq_{HR}$) for all $F_1, \ldots, F_n$ if and only if $Q_t^*/Q_t$ is decreasing (increasing) in $(0,1)^n$.

- $T_t \leq_{RHR} T_t^*$ ($\geq_{RHR}$) for all $F_1, \ldots, F_n$ if and only if $Q_t^*/Q_t$ is increasing (decreasing) in $(0,1)^n$. 
These results can be applied to compare $T_t$ and $T_t^\ast$. For example:

- $T_t \leq_{ST} T_t^\ast \ (\geq_{ST})$ holds for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\overline{Q}_t \leq \overline{Q}_t^\ast \ (\geq)$ in $(0, 1)^n$.

- $T_t \leq_{HR} T_t^\ast \ (\geq_{HR})$ for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\overline{Q}_t^\ast / \overline{Q}_t$ is decreasing (increasing) in $(0, 1)^n$.

- $T_t \leq_{RHR} T_t^\ast \ (\geq_{RHR})$ for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $Q_t^\ast / Q_t$ is increasing (decreasing) in $(0, 1)^n$. 
These results can be applied to compare $T_t$ and $T_t^*$. For example:

- $T_t \leq_{ST} T_t^*$ ($\geq_{ST}$) holds for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\overline{Q}_t \leq \overline{Q}_t^*$ ($\geq$) in $(0, 1)^n$.
- $T_t \leq_{HR} T_t^*$ ($\geq_{HR}$) for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\overline{Q}_t^*/\overline{Q}_t$ is decreasing (increasing) in $(0, 1)^n$.
- $T_t \leq_{RHR} T_t^*$ ($\geq_{RHR}$) for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $Q_t^*/Q_t$ is increasing (decreasing) in $(0, 1)^n$. 
These results can be applied to compare $T_t$ and $T_t^*$. For example:

- $T_t \leq_{ST} T_t^*$ ($\geq_{ST}$) holds for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\overline{Q}_t \leq \overline{Q}_t^*$ ($\geq$) in $(0, 1)^n$.

- $T_t \leq_{HR} T_t^*$ ($\geq_{HR}$) for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $\overline{Q}_t^*/\overline{Q}_t$ is decreasing (increasing) in $(0, 1)^n$.

- $T_t \leq_{RHR} T_t^*$ ($\geq_{RHR}$) for all $\overline{F}_1, \ldots, \overline{F}_n$ if and only if $Q_t^*/Q_t$ is increasing (decreasing) in $(0, 1)^n$. 
Example 1: Parallel system with two ID components

- \( T = \max(X_1, X_2) \) where \( X_1 \) and \( X_2 \) have DF \( F \).
- Then \( F_T(t) = \overline{q}(\overline{F}(t)) \) where
  \[
  \overline{q}(u) = \overline{Q}(u, u) = 2u - K(u, u).
  \]
- The RF of \( T_t = (T - t | T > t) \) is \( \overline{F}_t(x) = \overline{q}_t(\overline{F}_t(x)) \) where
  \[
  \overline{q}_t(u) = \overline{Q}_t(u, u) = \frac{\overline{q}(cu)}{\overline{q}(c)} = \frac{2cu - K(cu, cu)}{2c - K(c, c)},
  \]
  \( c = \overline{F}(t) \) and \( \overline{F}_t(x) = \overline{F}(x + t)/c. \)
- The RF of \( T^*_t = (T - t | T > t) \) is \( \overline{F}^*_t(x) = \overline{q}^*_t(\overline{F}_t(x)) \) where
  \[
  \overline{q}^*_t(u) = \overline{Q}^*_t(u, u) = \frac{K(cu, c) + K(c, cu) - K(cu, cu)}{K(c, c)}.
  \]
Example 1: Parallel system with two ID components

- \( T = \max(X_1, X_2) \) where \( X_1 \) and \( X_2 \) have DF \( F \).

- Then \( \overline{F}_T(t) = \overline{q}(\overline{F}(t)) \) where

\[
\overline{q}(u) = \overline{Q}(u, u) = 2u - K(u, u).
\]

- The RF of \( T_t = (T - t | T > t) \) is \( \overline{F}_t(x) = \overline{q}_t(\overline{F}_t(x)) \) where

\[
\overline{q}_t(u) = \overline{Q}_t(u, u) = \frac{\overline{q}(cu)}{\overline{q}(c)} = \frac{2cu - K(cu, cu)}{2c - K(c, c)},
\]

\( c = \overline{F}(t) \) and \( \overline{F}_t(x) = \overline{F}(x + t)/c. \)

- The RF of \( T^*_t = (T - t | T > t) \) is \( \overline{F}^*_t(x) = \overline{q}^*_t(\overline{F}_t(x)) \) where

\[
\overline{q}^*_t(u) = \overline{Q}^*_t(u, u) = \frac{K(cu, c) + K(c, cu) - K(cu, cu)}{K(c, c)}.
\]
Example 1: Parallel system with two ID components

- $T = \max(X_1, X_2)$ where $X_1$ and $X_2$ have DF $F$.
- Then $\overline{F}_T(t) = \overline{q}(\overline{F}(t))$ where
  $$\overline{q}(u) = \overline{Q}(u, u) = 2u - K(u, u).$$

- The RF of $T_t = (T - t | T > t)$ is $\overline{F}_t(x) = \overline{q}_t(\overline{F}_t(x))$ where
  $$\overline{q}_t(u) = \overline{Q}_t(u, u) = \frac{\overline{q}(cu)}{\overline{q}(c)} = \frac{2cu - K(cu, cu)}{2c - K(c, c)},$$
  $$c = \overline{F}(t) \text{ and } \overline{F}_t(x) = \overline{F}(x + t)/c.$$  

- The RF of $T_t^* = (T - t | T > t)$ is $\overline{F}^*_t(x) = \overline{q}^*_t(\overline{F}_t(x))$ where
  $$\overline{q}^*_t(u) = \overline{Q}^*_t(u, u) = \frac{K(cu, c) + K(c, cu) - K(cu, cu)}{K(c, c)}.$$
Example 1: Parallel system with two ID components

- $T = \max(X_1, X_2)$ where $X_1$ and $X_2$ have DF $F$.
- Then $\overline{F}_T(t) = \overline{q}(\overline{F}(t))$ where
  \[
  \overline{q}(u) = \overline{Q}(u, u) = 2u - K(u, u).
  \]
- The RF of $T_t = (T - t \mid T > t)$ is $\overline{F}_t(x) = \overline{q}_t(\overline{F}_t(x))$ where
  \[
  \overline{q}_t(u) = \overline{Q}_t(u, u) = \frac{\overline{q}(cu)}{\overline{q}(c)} = \frac{2cu - K(cu, cu)}{2c - K(c, c)},
  \]
  \[c = \overline{F}(t)\text{ and } \overline{F}_t(x) = \overline{F}(x + t)/c.\]
- The RF of $T^*_t = (T - t \mid T > t)$ is $\overline{F}^*_t(x) = \overline{q}^*_t(\overline{F}_t(x))$ where
  \[
  \overline{q}^*_t(u) = \overline{Q}^*_t(u, u) = \frac{K(cu, c) + K(c, cu) - K(cu, cu)}{K(c, c)}.
  \]
Example 1: Parallel system with two ID components

- $T_t \leq_{ST} T^*_t$ for all $F$ if and only if $\bar{q}_t \leq \bar{q}^*_t$ in $(0, 1)$, that is,

$$\frac{2cu - K(cu, cu)}{2c - K(c, c)} \leq \frac{K(cu, c) + K(c, cu) - K(cu, cu)}{K(c, c)}. \quad (4)$$

- If $K$ is EXC, it is equivalent to

$$\Psi(u) = (c - K(c, c))[K(cu, c) - K(cu, cu)] + \nu[K(cu, c) - uK(c, c)] \geq 0. \quad (5)$$

- Condition (5) holds if

$$\psi(u) = K(cu, c) - uK(c, c) \geq 0$$

for all $u \in [0, 1]$. 

SMART2016, Salerno

J. Navarro, E-mail: jorgenav@um.es
Example 1: Parallel system with two ID components

- $T_t \leq_{ST} T^*_t$ for all $F$ if and only if $\bar{q}_t \leq \bar{q}^*_t$ in $(0, 1)$, that is,

$$\frac{2cu - K(cu, cu)}{2c - K(c, c)} \leq \frac{K(cu, c) + K(c, cu) - K(cu, cu)}{K(c, c)}.$$  \hspace{1cm} (4)

- If $K$ is EXC, it is equivalent to

$$\Psi(u) = [c - K(c, c)][K(cu, c) - K(cu, cu)] + v[K(cu, c) - uK(c, c)] \geq 0.$$  \hspace{1cm} (5)

- Condition (5) holds if

$$\psi(u) = K(cu, c) - uK(c, c) \geq 0$$

for all $u \in [0, 1]$. \hspace{1cm}
Example 1: Parallel system with two ID components

- $T_t \leq_{ST} T_t^*$ for all $F$ if and only if $\bar{q}_t \leq \bar{q}_t^*$ in $(0, 1)$, that is,

  \[
  \frac{2cu - K(cu, cu)}{2c - K(c, c)} \leq \frac{K(cu, c) + K(c, cu) - K(cu, cu)}{K(c, c)}. \tag{4}
  \]

- If $K$ is EXC, it is equivalent to

  \[
  \Psi(u) = [c - K(c, c)][K(cu, c) - K(cu, cu)] + \nu[K(cu, c) - uK(c, c)] \geq 0. \tag{5}
  \]

- Condition (5) holds if

  \[
  \psi(u) = K(cu, c) - uK(c, c) \geq 0
  \]

  for all $u \in [0, 1]$. 
Example 1: Clayton copula

- If $K$ is the Clayton copula
  
  $$K(u, v) = \left( u^{-\theta} + v^{-\theta} - 1 \right)^{-1/\theta}, \quad \theta > 0,$$

  then

  $$\psi(u) = \left( u^{-\theta} c^{-\theta} + c^{-\theta} - 1 \right)^{-1/\theta} - \left( u^{-\theta} c^{-\theta} + u^{-\theta} [c^{-\theta} - 1] \right)^{-1/\theta}.$$

- Since $\theta > 0$ and $u^{-\theta} \geq 1$ for $u \in (0, 1)$, $\psi$ is nonnegative in $(0, 1)$ for all $c$.

- Therefore $T_t \leq_{ST} T^*_t$ holds for all $F$ and all $t \geq 0$. 

SMART2016, Salerno   J. Navarro, E-mail: jorgenav@um.es
Example 1: Clayton copula

- If $K$ is the Clayton copula

  $$K(u, v) = \left( u^{-\theta} + v^{-\theta} - 1 \right)^{-1/\theta}, \quad \theta > 0,$$

  then

  $$\psi(u) = \left( u^{-\theta} c^{-\theta} + c^{-\theta} - 1 \right)^{-1/\theta} - \left( u^{-\theta} c^{-\theta} + u^{-\theta} [c^{-\theta} - 1] \right)^{-1/\theta}.$$

- Since $\theta > 0$ and $u^{-\theta} \geq 1$ for $u \in (0, 1)$, $\psi$ is nonnegative in $(0, 1)$ for all $c$.

- Therefore $T_t \leq_{ST} T_t^*$ holds for all $F$ and all $t \geq 0$. 
Example 1: Clayton copula

- If $K$ is the Clayton copula

$$K(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad \theta > 0,$$

then

$$\psi(u) = \left(u^{-\theta} c^{-\theta} + c^{-\theta} - 1\right)^{-1/\theta} - \left(u^{-\theta} c^{-\theta} + u^{-\theta}[c^{-\theta} - 1]\right)^{-1/\theta}.$$  

- Since $\theta > 0$ and $u^{-\theta} \geq 1$ for $u \in (0, 1)$, $\psi$ is nonnegative in $(0, 1)$ for all $c$.

- Therefore $T_t \leq_{ST} T_t^*$ holds for all $F$ and all $t \geq 0$. 
Figure: Reliability functions of $T_t$ (black) and $T_t^*$ (red) when $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 2$. 
Example 1: Gumbel-Barnett Archimedean copula

- If $K$ is the Gumbel-Barnett Archimedean copula

$$K(u, v) = uv \exp \left[ -\theta \ln u \ln v \right], \quad \theta \in (0, 1], \quad (6)$$

then by plotting $\Psi(u)$, we see that it takes positive and negative values in the set $[0, 1]$ when $\theta = 1$.

- Therefore $T_t$ and $T_t^*$ are not ST ordered (for all $F$ and $t$).

- These conditions lead to a Gumbel bivariate exponential with a negative correlation.

- For example it does not hold when $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 1$. 
Example 1: Gumbel-Barnett Archimedean copula

- If $K$ is the Gumbel-Barnett Archimedean copula

  $$K(u, v) = uv \exp \left[ -\theta (\ln u)(\ln v) \right], \quad \theta \in (0, 1], \quad (6)$$

  then by plotting $\Psi(u)$, we see that it takes positive and negative values in the set $[0, 1]$ when $\theta = 1$.

- Therefore $T_t$ and $T_t^*$ are not ST ordered (for all $F$ and $t$).

- These conditions lead to a Gumbel bivariate exponential with a negative correlation.

- For example it does not hold when $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 1$. 

SMART2016, Salerno  
J. Navarro, E-mail: jorgenav@um.es
Example 1: Gumbel-Barnett Archimedean copula

If $K$ is the Gumbel-Barnett Archimedean copula

$$K(u, v) = uv \exp \left[ -\theta (\ln u)(\ln v) \right], \quad \theta \in (0, 1],$$

(6)

then by plotting $\Psi(u)$, we see that it takes positive and negative values in the set $[0, 1]$ when $\theta = 1$.

Therefore $T_t$ and $T^*_t$ are not ST ordered (for all $F$ and $t$).

These conditions lead to a Gumbel bivariate exponential with a negative correlation.

For example it does not hold when $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 1$. 

SMART2016, Salerno J. Navarro, E-mail: jorgenav@um.es
Example 1: Gumbel-Barnett Archimedean copula

- If $K$ is the Gumbel-Barnett Archimedean copula

  $$K(u, v) = uv \exp[-\theta(\ln u)(\ln v)], \quad \theta \in (0, 1], \quad (6)$$

  then by plotting $\Psi(u)$, we see that it takes positive and negative values in the set $[0, 1]$ when $\theta = 1$.

- Therefore $T_t$ and $T^*_t$ are not ST ordered (for all $F$ and $t$).

- These conditions lead to a Gumbel bivariate exponential with a negative correlation.

- For example it does not hold when $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 1$. 
Figure: Reliability functions of $T_t$ (black) and $T_t^*$ (red) when $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 1$. 

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
Example 1: Gumbel-Barnett Archimedean copula

Now we can study if $T_t \leq_{MRL} T_t^*$ holds.

By plotting the ratio $g(u) = \overline{q}_t(u)/\overline{q}_t^*(u)$ for $t = 1$ we see that it is first decreasing in $(0, u_0)$ and then increasing in $(u_0, 1]$ for a $u_0 \in (0, 1)$. 
Example 1: Gumbel-Barnett Archimedean copula

- Now we can study if $T_t \leq_{MRL} T_t^*$ holds.
- By plotting the ratio $g(u) = \bar{q}_t(u)/\bar{q}_t^*(u)$ for $t = 1$ we see that it is first decreasing in $(0, u_0)$ and then increasing in $(u_0, 1]$ for a $u_0 \in (0, 1)$.
Figure: Ratio $g(u) = \frac{\bar{q}_t(u)}{\bar{q}_t^*(u)}$ for $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 0.1, 0.2, \ldots, 1$ (from the bottom to the top).
Example 1: Gumbel-Barnett Archimedean copula

- Now we can study if $T_t \leq_{\text{MRL}} T_t^*$ holds.
- By plotting the ratio $g(u) = \bar{q}_t(u)/\bar{q}_t^*(u)$ for $t = 1$ we see that it is first decreasing in $(0, u_0)$ and then increasing in $(u_0, 1]$ for a $u_0 \in (0, 1)$.
- Hence $T_t \geq_{\text{MRL}} T_t^*$ for all $F$ such that $E(T_t) \geq E(T_t^*)$.
- For example, if $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 1$, then

$$E(T_t) = 1.05615 > E(T_t^*) = 0.77366.$$ 

- So $T_t \geq_{\text{MRL}} T_t^*$ for $t = 1$ and $\bar{F}(x) = e^{-x}$!
Example 1: Gumbel-Barnett Archimedean copula

- Now we can study if $T_t \leq_{\text{MRL}} T_t^*$ holds.
- By plotting the ratio $g(u) = \bar{q}_t(u)/\bar{q}_t^*(u)$ for $t = 1$ we see that it is first decreasing in $(0, \bar{u}_0)$ and then increasing in $(\bar{u}_0, 1]$ for a $\bar{u}_0 \in (0, 1)$.
- Hence $T_t \geq_{\text{MRL}} T_t^*$ for all $F$ such that $E(T_t) \geq E(T_t^*)$.
- For example, if $t = 1$, $\bar{F}(x) = e^{-x}$ and $\theta = 1$, then
  $$E(T_t) = 1.05615 > E(T_t^*) = 0.77366.$$  
  So $T_t \geq_{\text{MRL}} T_t^*$ for $t = 1$ and $\bar{F}(x) = e^{-x}$!!
Example 1: Gumbel-Barnett Archimedean copula

Now we can study if \( T_t \leq_{MRL} T_{t^*} \) holds.

By plotting the ratio \( g(u) = \bar{q}_t(u)/\bar{q}_{t^*}(u) \) for \( t = 1 \) we see that it is first decreasing in \((0, u_0)\) and then increasing in \((u_0, 1]\) for a \( u_0 \in (0, 1) \).

Hence \( T_t \geq_{MRL} T_{t^*} \) for all \( F \) such that \( E(T_t) \geq E(T_{t^*}) \).

For example, if \( t = 1 \), \( \bar{F}(x) = e^{-x} \) and \( \theta = 1 \), then

\[
E(T_t) = 1.05615 > E(T_{t^*}) = 0.77366.
\]

So \( T_t \geq_{MRL} T_{t^*} \) for \( t = 1 \) and \( \bar{F}(x) = e^{-x} \).
Example 2: Parallel system with two INID components

- $T = \max(X_1, X_2)$, $X_1, X_2$ IND with DF $F_1$ and $F_2$.
- Then
  \[
  Q(u_1, u_2) = u_1 + u_2 - u_1 u_2 = Q^*_t(u_1, u_2),
  \]
  \[
  Q^*_t(u_1, u_2) = \frac{c_1 u_1 + c_2 u_2 - c_1 c_2 u_1 u_2}{c_1 + c_2 - c_1 c_2},
  \]
  where $c_1 = \overline{F}_1(t)$ and $c_2 = \overline{F}_2(t)$.
- $T_t \leq_{HR} T^*_t$ holds for all $F_1, F_2$ if and only if
  \[
  \frac{Q(u, v)}{Q^*_t(u, v)} = \frac{(u + v - uv)(c_1 + c_2 - c_1 c_2)}{c_1 u + c_2 - c_1 c_2 uv}
  \]
  is decreasing in $u$ and $v$ in the set $[0, 1]^2$.
- As this property is not true, they are not HR ordered.
- Therefore Theorem 3 in Li and Lu (PEIS, 2003) is not correct.
Example 2: Parallel system with two INID components

- \( T = \max(X_1, X_2) \), \( X_1, X_2 \) IND with DF \( F_1 \) and \( F_2 \).
- Then
  \[
  \overline{Q}(u_1, u_2) = u_1 + u_2 - u_1u_2 = \overline{Q}^*(u_1, u_2),
  \]
  \[
  \overline{Q}_t(u_1, u_2) = \frac{c_1u_1 + c_2u_2 - c_1c_2u_1u_2}{c_1 + c_2 - c_1c_2},
  \]
  where \( c_1 = \overline{F}_1(t) \) and \( c_2 = \overline{F}_2(t) \).
- \( T_t \leq_{HR} T_t^* \) holds for all \( F_1, F_2 \) if and only if
  \[
  \frac{\overline{Q}(u, v)}{\overline{Q}_t(u, v)} = \frac{(u + v - uv)(c_1 + c_2 - c_1c_2)}{c_1u + c_2 - c_1c_2uv}
  \]
  is decreasing in \( u \) and \( v \) in the set \([0, 1]^2\).
- As this property is not true, they are not HR ordered.
- Therefore Theorem 3 in Li and Lu (PEIS,2003) is not correct.
Example 2: Parallel system with two INID components

- \( T = \max(X_1, X_2) \), \( X_1, X_2 \) IND with DF \( F_1 \) and \( F_2 \).
- Then
  \[
  \bar{Q}(u_1, u_2) = u_1 + u_2 - u_1 u_2 = \bar{Q}_t^*(u_1, u_2),
  \]
  \[
  \bar{Q}_t(u_1, u_2) = \frac{c_1 u_1 + c_2 u_2 - c_1 c_2 u_1 u_2}{c_1 + c_2 - c_1 c_2},
  \]
  where \( c_1 = \bar{F}_1(t) \) and \( c_2 = \bar{F}_2(t) \).

- \( T_t \leq_{HR} T_t^* \) holds for all \( F_1, F_2 \) if and only if
  \[
  \frac{\bar{Q}(u, v)}{\bar{Q}_t(u, v)} = \frac{(u + v - uv) (c_1 + c_2 - c_1 c_2)}{c_1 u + c_2 - c_1 c_2 uv}
  \]
  is decreasing in \( u \) and \( v \) in the set \([0, 1]^2\).
- As this property is not true, they are not HR ordered.
- Therefore Theorem 3 in Li and Lu (PEIS, 2003) is not correct.
Example 2: Parallel system with two INID components

- \( T = \max(X_1, X_2), X_1, X_2 \) IND with DF \( F_1 \) and \( F_2 \).
- Then
  \[
  \overline{Q}(u_1, u_2) = u_1 + u_2 - u_1 u_2 = \overline{Q}^*(u_1, u_2),
  \]
  \[
  \overline{Q}_t(u_1, u_2) = \frac{c_1 u_1 + c_2 u_2 - c_1 c_2 u_1 u_2}{c_1 + c_2 - c_1 c_2},
  \]
  where \( c_1 = \overline{F}_1(t) \) and \( c_2 = \overline{F}_2(t) \).
- \( T_t \leq_{HR} T_t^* \) holds for all \( F_1, F_2 \) if and only if
  \[
  \frac{\overline{Q}(u, v)}{\overline{Q}_t(u, v)} = \frac{(u + v - uv)(c_1 + c_2 - c_1 c_2)}{c_1 u + c_2 - c_1 c_2 uv}
  \]
  is decreasing in \( u \) and \( v \) in the set \([0, 1]^2\).
- As this property is not true, they are not HR ordered.
- Therefore Theorem 3 in Li and Lu (PEIS,2003) is not correct.
Example 2: Parallel system with two INID components

- \( T = \max(X_1, X_2) \), \( X_1, X_2 \) IND with DF \( F_1 \) and \( F_2 \).
- Then
  \[
  \overline{Q}(u_1, u_2) = u_1 + u_2 - u_1 u_2 = \overline{Q}^*(u_1, u_2),
  \]
  \[
  \overline{Q}_t(u_1, u_2) = \frac{c_1 u_1 + c_2 u_2 - c_1 c_2 u_1 u_2}{c_1 + c_2 - c_1 c_2},
  \]
  where \( c_1 = \overline{F}_1(t) \) and \( c_2 = \overline{F}_2(t) \).
- \( T_t \leq_{HR} T_t^* \) holds for all \( F_1, F_2 \) if and only if
  \[
  \frac{\overline{Q}(u, v)}{\overline{Q}_t(u, v)} = \frac{(u + v - uv)(c_1 + c_2 - c_1 c_2)}{c_1 u + c_2 - c_1 c_2 u v}
  \]
  is decreasing in \( u \) and \( v \) in the set \([0, 1]^2\).
- As this property is not true, they are not HR ordered.
- Therefore Theorem 3 in Li and Lu (PEIS,2003) is not correct.
Figure: Ratio $\bar{F}_t^*/\bar{F}_t$ for $t = 1$, $\bar{F}_1(x) = e^{-x}$ and $\bar{F}_2(x) = e^{-x/2}$ (black) or $\bar{F}_2(x) = e^{-x}$ (red).
Example 2: Parallel system with two INID components

- If $X_1, X_2$ are IID with DF $F$, then $T_t \leq_{HR} T^*_t$ holds for all $F$ since
  \[
  \frac{q(u)}{q_t(u)} = \frac{2 - u}{2 - uF(t)} \left(2 - \bar{F}(t)\right)
  \]
  is decreasing in $u$ in the set $[0, 1]$.

- Even more, $T_t \leq_{LR} T^*_t$ holds for all $F$ since
  \[
  \frac{q'(u)}{q'_t(u)} = \frac{1 - u}{1 - uF(t)} \left(2 - \bar{F}(t)\right)
  \]
  is a decreasing function in $[0, 1]$ for all $t > 0$. 

SMART2016, Salerno  J. Navarro, E-mail: jorgenav@um.es
Example 2: Parallel system with two INID components

- If $X_1, X_2$ are IID with DF $F$, then $T_t \leq_{HR} T^*_t$ holds for all $F$ since
  \[
  \frac{\bar{q}(u)}{\bar{q}_t(u)} = \frac{2 - u}{2 - uF(t)} (2 - \bar{F}(t))
  \]
  is decreasing in $u$ in the set $[0, 1]$.

- Even more, $T_t \leq_{LR} T^*_t$ holds for all $F$ since
  \[
  \frac{\bar{q}'(u)}{\bar{q}'_t(u)} = \frac{1 - u}{1 - uF(t)} (2 - \bar{F}(t))
  \]
  is a decreasing function in $[0, 1]$ for all $t > 0$. 

Example 3: Coherent system with DID components

Figure: System in Example 3.
Example 3: Coherent system with DID components

- \( T = \max(X_1, \min(X_2, X_3)) \), \( X_1, X_2, X_3 \) DID with DF \( F \).
- Then \( P_1 = \{1\} \), \( P_2 = \{2, 3\} \) and
  \[
  \bar{q}(u) = u + K(1, u, u) - K(u, u, u).
  \]
- Therefore \( \bar{q}_t(u) = \bar{q}(cu)/\bar{q}(c) \) and
  \[
  \bar{q}^*_t(u) = \frac{K(cu, c, c) + K(c, cu, cu) - K(cu, cu, cu)}{K(c, c, c)},
  \]
  where \( c = \bar{F}(t) \).
- We assume a Farlie-Gumbel-Morgenstern (FGM) copula
  \[
  K(u, v, w) = uvw(1 + \theta(1 - u)(1 - v)(1 - w)), \quad \theta \in [-1, 1].
  \]
Example 3: Coherent system with DID components

- \( T = \max(X_1, \min(X_2, X_3)) \), \( X_1, X_2, X_3 \) DID with DF \( F \).
- Then \( P_1 = \{1\} \), \( P_2 = \{2, 3\} \) and

\[ q(u) = u + K(1, u, u) - K(u, u, u). \]

- Therefore \( q_t(u) = q(cu)/q(c) \) and

\[ q^*_t(u) = \frac{K(cu, c, c) + K(c, cu, cu) - K(cu, cu, cu)}{K(c, c, c)}, \]

where \( c = F(t) \).
- We assume a Farlie-Gumbel-Morgenstern (FGM) copula

\[ K(u, v, w) = uvw(1 + \theta(1 - u)(1 - v)(1 - w)), \quad \theta \in [-1, 1]. \]
Example 3: Coherent system with DID components

- \( T = \max(X_1, \min(X_2, X_3)), \) \( X_1, X_2, X_3 \) DID with DF \( F \).
- Then \( P_1 = \{1\}, P_2 = \{2, 3\} \) and
  \[
  \bar{q}(u) = u + K(1, u, u) - K(u, u, u).
  \]
- Therefore \( \bar{q}_t(u) = \bar{q}(cu)/\bar{q}(c) \) and
  \[
  \bar{q}^*_t(u) = \frac{K(cu, c, c) + K(c, cu, cu) - K(cu, cu, cu)}{K(c, c, c)},
  \]
  where \( c = \bar{F}(t) \).
- We assume a Farlie-Gumbel-Morgenstern (FGM) copula
  \[
  K(u, v, w) = uvw(1 + \theta(1 - u)(1 - v)(1 - w)), \quad \theta \in [-1, 1].
  \]
Example 3: Coherent system with DID components

- \( T = \max(X_1, \min(X_2, X_3)) \), \( X_1, X_2, X_3 \) DID with DF \( F \).
- Then \( P_1 = \{1\}, P_2 = \{2, 3\} \) and

\[
\overline{q}(u) = u + K(1, u, u) - K(u, u, u).
\]

- Therefore \( \overline{q}_t(u) = \overline{q}(cu)/\overline{q}(c) \) and

\[
\overline{q}^*_t(u) = \frac{K(cu, c, c) + K(c, cu, cu) - K(cu, cu, cu)}{K(c, c, c)},
\]

where \( c = \overline{F}(t) \).

- We assume a Farlie-Gumbel-Morgenstern (FGM) copula

\[
K(u, \nu, w) = uvw(1 + \theta(1 - u)(1 - \nu)(1 - w)), \quad \theta \in [-1, 1].
\]
Figure: Ratio $g(u) = \overline{q}_t^*(u)/\overline{q}_t(u)$ for $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = -1, -0.9, \ldots, 1$ (from the bottom to the top).
Example 3: Coherent system with DID components

- As \( g(u) = \frac{\bar{q}_t^*(u)}{\bar{q}_t(u)} \geq 1 \), then \( T_t \leq_{ST} T_t^* \).
- As \( g(u) = \frac{\bar{q}_t^*(u)}{\bar{q}_t(u)} \geq 1 \) is not monotone, then \( T_t \) and \( T_t^* \) are not HR ordered.
Example 3: Coherent system with DID components

- As $g(u) = \frac{\bar{q}_t^*(u)}{\bar{q}_t(u)} \geq 1$, then $T_t \leq_{ST} T_t^*$.
- As $g(u) = \frac{\bar{q}_t^*(u)}{\bar{q}_t(u)} \geq 1$ is not monotone, then $T_t$ and $T_t^*$ are not HR ordered.
Further results

- For more results see Navarro and Durante (2015).

- Case 3: \( T^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t} = (T - t | H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}) \) where the (past) history of the system can be represented as

\[
H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t} = \{X_{i_1} = t_1, \ldots, X_{i_r} = t_r, X_j > t \text{ for } j \notin \{i_1, \ldots, i_r\}\},
\]

where \( 0 < r < n, 0 < t_1 < \cdots < t_r < t, \Pr(H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}) > 0 \) and the event \( H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t} \) implies \( T > t \).

- This case can also be represented as

\[
\Pr(T - t > x | H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}) = Q^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}(F_1, t(x), \ldots, F_n, t(x)).
\]
Further results

- For more results see Navarro and Durante (2015).

- Case 3: $T^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t} = (T - t|H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t})$ where the (past) history of the system can be represented as

$$H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t} = \{X_{i_1} = t_1, \ldots, X_{i_r} = t_r, X_j > t \text{ for } j \not\in \{i_1, \ldots, i_r\}\},$$

where $0 < r < n$, $0 < t_1 < \cdots < t_r < t$, $\Pr(H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}) > 0$ and the event $H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}$ implies $T > t$.

- This case can also be represented as

$$\Pr(T - t > x|H^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}) = Q^{(i_1, \ldots, i_r)}_{t_1, \ldots, t_r, t}(F_1, t(x), \ldots, F_n, t(x)).$$
Further results

- For more results see Navarro and Durante (2015).
- Case 3:  
  \[ T_{t_1,\ldots,t_r,t}^{(i_1,\ldots,i_r)} = (T - t|H_{t_1,\ldots,t_r,t}^{(i_1,\ldots,i_r)}) \]
  where the (past) history of the system can be represented as

  \[ H_{t_1,\ldots,t_r,t}^{(i_1,\ldots,i_r)} = \{ X_{i_1} = t_1, \ldots, X_{i_r} = t_r, X_j > t \text{ for } j \notin \{i_1, \ldots, i_r\} \}, \]

  where 0 < r < n, 0 < t_1 < \cdots < t_r < t, \Pr(H_{t_1,\ldots,t_r,t}^{(i_1,\ldots,i_r)}) > 0
  and the event \( H_{t_1,\ldots,t_r,t}^{(i_1,\ldots,i_r)} \) implies \( T > t \).
- This case can also be represented as

  \[ \Pr(T - t > x|H_{t_1,\ldots,t_r,t}^{(i_1,\ldots,i_r)}) = \overline{Q}_{t_1,\ldots,t_r,t}^{(i_1,\ldots,i_r)}(\overline{F}_1,t(x), \ldots, \overline{F}_n,t(x)). \]

The references of this talk

Recent references on coherent systems


Recent references on Distorted distributions


For the more references, please visit my personal web page:

https://webs.um.es/jorgenav/

Thank you for your attention!!
References

- For the more references, please visit my personal web page:

  https://webs.um.es/jorgenav/

- Thank you for your attention!!