# Stochastic comparisons of coherent systems and order statistics

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# Outline

1. Distortion Functions
   - Proportional hazard rate model
   - Order statistics
   - Coherent systems
2. Comparison results and bounds
   - Comparison results
   - Distorted Distributions
   - Bounds
3. Residual lifetimes
   - Representations
   - Comparison results
   - Examples

# Distortion functions

- The distorted distributions were introduced in Yaari's dual theory of choice under risk (Econometrica 55 (1987):95–115).
- The **distorted distribution** (DD) associated to a distribution function (DF) $F$ and to an increasing continuous **distortion function** $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, is

$$F_q(t) = q(F(t)). \qquad (1.1)$$

- For the reliability functions (RF) $\overline{F} = 1 - F$, $\overline{F}_q = 1 - F_q$, we have

$$\overline{F}_q(t) = \overline{q}(\overline{F}(t)), \qquad (1.2)$$

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# Multivariate distortion functions

- The **generalized distorted distribution** (GDD) associated to $n$ DF $F_1, \ldots, F_n$ and to an increasing continuous **multivariate distortion function** $Q : [0,1]^n \to [0,1]$ such that $Q(0,\ldots,0) = 0$ and $Q(1,\ldots,1) = 1$, is

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \quad (1.3)$$

- For the RF we have

$$\overline{F}_Q(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t)), \quad (1.4)$$

where $\overline{F} = 1 - F$, $\overline{F}_Q = 1 - F_Q$ and $\overline{Q}(u_1,\ldots,u_n) = 1 - Q(1-u_1,\ldots,1-u_n)$ is the **multivariate dual distortion function**; see Navarro et al. (JAP, 2011).

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# Proportional hazard rate (PHR) model

- The PHR (Cox) model associated to a RF $\overline{F}$ is

$$\overline{F}_{\alpha}(t)=\left(\overline{F}(t)\right)^{\alpha}=\overline{q}\left(\overline{F}(t)\right)$$

for $\alpha>0$. $F_{\alpha}$ is a DD with $\overline{q}(u)=u^{\alpha}$ and $q(u)=1-(1-u)^{\alpha}$.

- The proportional reversed hazard rate (PRHR) model is

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# Order statistics (OS)

- $X_1, \ldots, X_n$ IID$\sim F$ random variables.
- Let $X_{1:n}, \ldots, X_{n:n}$ be the associated OS.
- Let $F_{i:n}(t) = \Pr(X_{i:n} \le t)$ be the DF, then

$$F_{i:n}(t) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} F_{j:j}(t) = q_{i:n}(F(t)), \quad (1.5)$$

(see David and Nagaraja 2003, p. 46) where

$$F_{j:j}(t) = \Pr(X_{j:j} \le t) = \Pr(\max(X_1, \ldots, X_j) \le t) = F^j(t)$$

and

$$q_{i:n}(u) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} u^j$$

is a strictly increasing polynomial in $[0, 1]$.

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# Coherent systems- IID case

- Samaniego (IEEE TR, 1985), IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i \overline{F}_{i:n}(t), \quad (1.6)$$

where $s_i = \Pr(T = X_{i:n})$.

- $\mathbf{s} = (s_1, \ldots, s_n)$ is the signature of the system.
- Then $T$ has a DD from $F$ with

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t) = \sum_{i=1}^{n} a_i \overline{F}^i(t) = \overline{q}(\overline{F}(t)), \quad (1.7)$$

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# Coherent systems- INID case

- Consider an $n$-component system with components of $r$ different types. Suppose that the system has $m_k$ components of type $k$, where $k = 1, \ldots, r$. Assume that the lifetimes of components of the same type are exchangeable and that the lifetimes of components of different types are independent. Then the *survival signature* of the system is a nonnegative function $\phi$ of $r$ variables, where $\phi(i_1, \ldots, i_r)$ for $i_k = 0, \ldots, m_k$ and $k = 1, \ldots, r$, represents the probability that the system works when precisely $i_k$ components of type $k$ are working for $k = 1, \ldots, r$.
- Coolen and Coolen-Maturi (2012), IND components:

$$\overline{F}_T(t) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} \phi(i_1, \ldots, i_r) \prod_{k=1}^{r} \binom{m_k}{i_k} F_k^{m_k - i_k}(t) \overline{F}_k^{i_k}(t). \tag{1.8}$$

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- Then
$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_r(t)) \quad (1.9)$$
where $\overline{Q}$ is a multinomial.
- If $r = 1$, $\overline{Q}$ is the domination polynomial.
- If $r = n$, then $\overline{Q}$ is the **reliability function of the structure**; see Barlow and Proschan (1975,p. 21).

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# Coherent systems-GENERAL case

- A **path set** of $T$ is a set $P \subseteq \{1, \dots, n\}$ such that if all the components in $P$ work, then the system works.
- A **minimal path set** of $T$ is a path set which does not contains other path sets.
- If $P_1, \dots, P_m$ are the minimal path sets of $T$, then $T = \max_{j=1,\dots,m} X_{P_j}$, where $X_P = \min_{i \in P} X_i$ and

$$\overline{F}_T(t) = \Pr\left(\max_{j=1,\dots,m} X_{P_j} > t\right) = \Pr\left(\cup_{j=1}^m \{X_{P_j} > t\}\right)$$
$$= \sum_{i=1}^m \overline{F}_{P_i}(t) - \sum_{i \neq j} \overline{F}_{P_i \cup P_j}(t) + \cdots \pm \overline{F}_{P_1 \cup \cdots \cup P_m}(t)$$

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- The copula representation for the RF of $(X_1, \ldots, X_n)$ is
$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = K(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)),$$
where $\overline{F}_i(t) = \Pr(X_i > t)$ and $K$ is the survival copula.
- Then
$$\overline{F}_P(t) = \overline{Q}_{P,K}(\overline{F}_1(t), \ldots, \overline{F}_n(t)),$$
where $\overline{Q}_{P,K}(u_1, \ldots, u_n) = K(u_1^P, \ldots, u_n^P)$ and $u_i^P = u_i$ for $i \in P$ and $u_i^P = 1$ for $i \notin P$.
- Therefore, from the minimal path set repres., we get
$$\overline{F}_T(t) = \overline{Q}_{\phi,K}(\overline{F}_1(t), \ldots, \overline{F}_n(t)).$$
- In the ID case $\overline{F}_T(t) = \overline{q}_{\phi,K}(\overline{F}(t))$.
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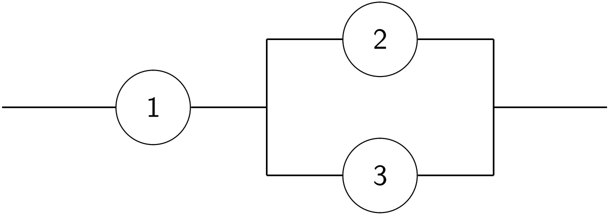


Coherent system lifetime $T = \min(X_1, \max(X_2, X_3))$.

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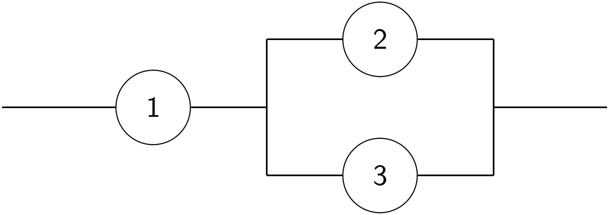
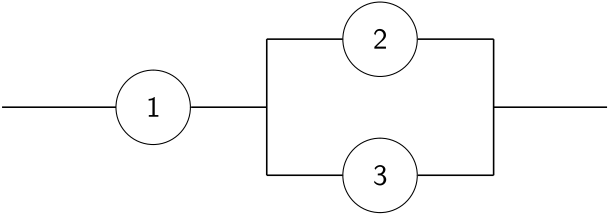


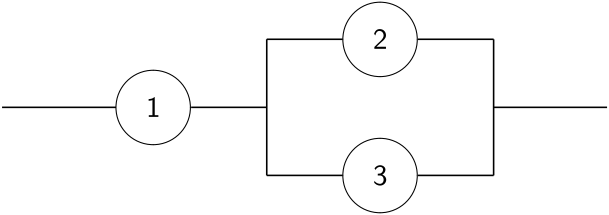

IID $\overline{F}$ cont.: $\mathbf{s} = (2/6, 4/6, 0) = (1/3, 2/3, 0)$.

# Example



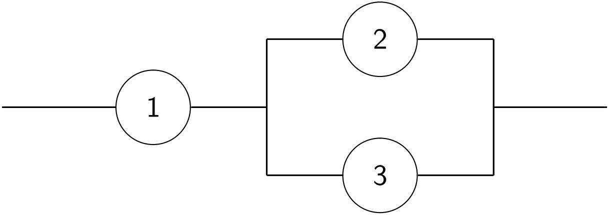

IID $\overline{F}$ cont.: $\overline{F}_T(t) = \frac{1}{3}\overline{F}_{1:3}(t) + \frac{2}{3}\overline{F}_{2:3}(t)$.

# Example-general case



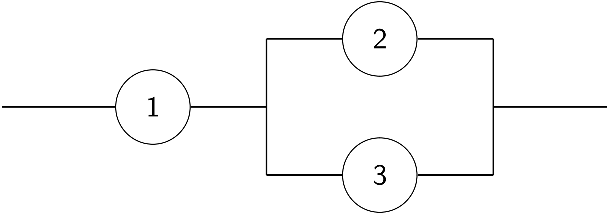

Coherent system lifetime $T = \max(\min(X_1, X_2), \min(X_1, X_3))$
Minimal path sets $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$.

# Example-general case



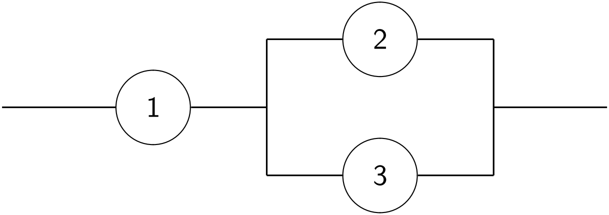

$$\overline{F}_T(t) = \Pr(\{X_{\{1,2\}} > t\} \cup \{X_{\{1,3\}} > t\})$$
$$= \overline{F}_{\{1,2\}}(t) + \overline{F}_{\{1,3\}}(t) - \overline{F}_{\{1,2,3\}}(t).$$

# Example-general case



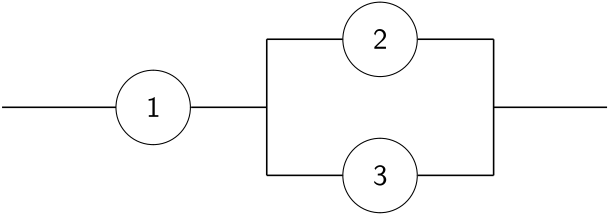

$\overline{F}_{\{1,2\}}(t) = \overline{\mathbf{F}}(t, t, 0) = K(\overline{F}_1(t), \overline{F}_2(t), 1),\ldots$
$\overline{F}_T(t) = \overline{Q}_{\phi,K}(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t))$ where
$\overline{Q}_{\phi,K}(u_1, u_2, u_3) = K(u_1, u_2, 1) + K(u_1, 1, u_3) - K(u_1, u_2, u_3).$

# Example-general case




ID: $\overline{F}_T(t) = \overline{q}_{\phi,K}(\overline{F}(t))$,
where $\overline{q}_{\phi,K}(u) = K(u,u,1) + K(u,1,u) - K(u,u,u)$.

# Example-general case




IID: $\overline{F}_T(t) = 2\overline{F}^2(t) - \overline{F}^3(t) = q_\phi(\overline{F}(t))$,
where $\overline{q}_\phi(u) = 2u^2 - u^3$ and $\mathbf{a} = (0, 2, -1)$.

# Example IND components

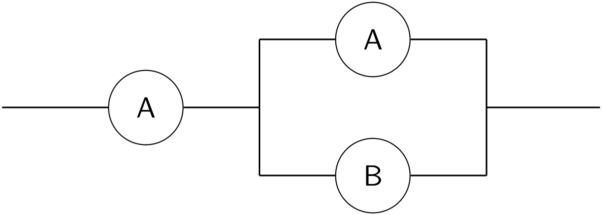



Figure: System 1.

$$\phi(0,0) = \phi(1,0) = \phi(0,1) = 0$$

# Example IND components

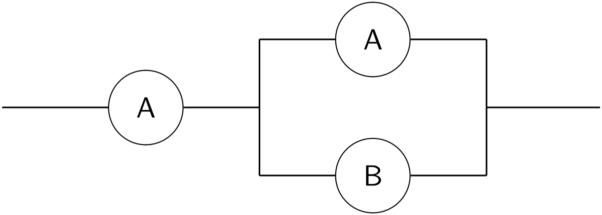



Figure: System 1.

$\phi(1,1) = 1/2$

# Example IND components

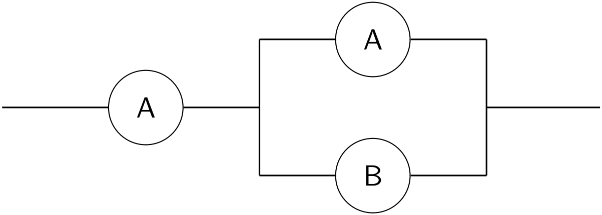



Figure: System 1.

$$\phi(2,0) = \phi(2,1) = 1$$

# Example IND components

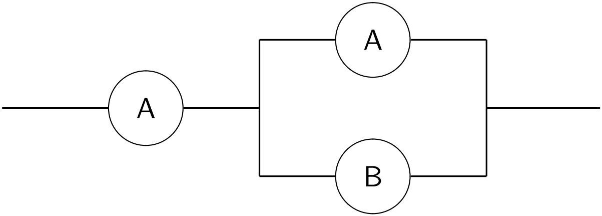



Figure: System 1.

$$\overline{F}_{T_1}(t) = \phi(1,1)\binom{2}{1}F_A^{2-1}(t)\overline{F}_A^1(t)\binom{1}{1}F_B^{1-1}(t)\overline{F}_B^1(t) + \dots$$

# Example IND components

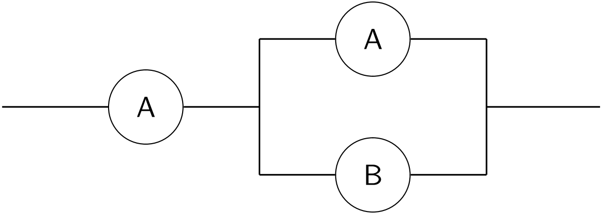



Figure: System 1.

$$\overline{F}_{T_1}(t) = F_A(t)\overline{F}_A(t)\overline{F}_B(t) + \overline{F}_A^2(t)F_B(t) + \overline{F}_A^2(t)\overline{F}_B(t), \text{ i.e.,}$$

$$\overline{F}_{T_1}(t) = \overline{F}_A^2(t) + \overline{F}_A(t)\overline{F}_B(t) - \overline{F}_A^2(t)\overline{F}_B(t)$$

# Example IND components

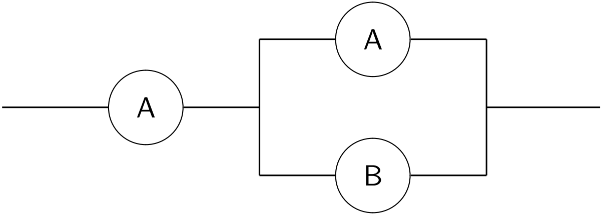



Figure: System 1.

$$\overline{F}_{T_1}(t) = \overline{Q}_{\phi,K}(\overline{F}_A(t), \overline{F}_A(t), \overline{F}_B(t)),$$ where

$$\overline{Q}_{\phi,K}(u_1, u_2, u_3) = K(u_1, u_2, 1) + K(u_1, 1, u_3) - K(u_1, u_2, u_3).$$

# Example IND components

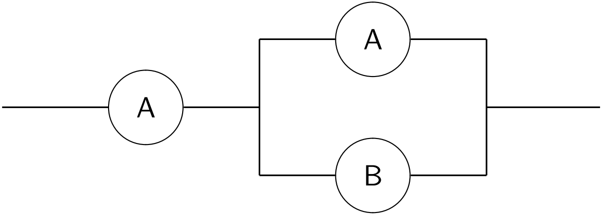



Figure: System 1.

INID: $\overline{F}_{T_1}(t) = \overline{Q}_1(\overline{F}_A(t), \overline{F}_B(t))$, where

$$\overline{Q}_1(u_1, u_2) = u_1^2 + u_1 u_2 - u_1^2 u_2.$$

# Example IND components

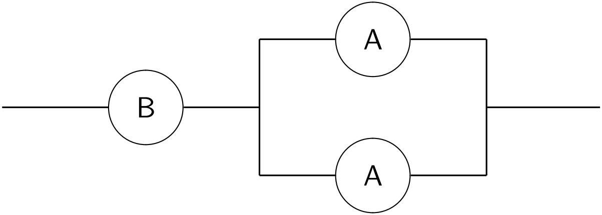



Figure: System 2.

$$\overline{F}_{T_2}(t) = \overline{Q}_{\phi,K}(\overline{F}_B(t), \overline{F}_A(t), \overline{F}_A(t)), \text{ where}$$

$$\overline{Q}_{\phi,K}(u_1, u_2, u_3) = K(u_1, u_2, 1) + K(u_1, 1, u_3) - K(u_1, u_2, u_3).$$

# Example IND components

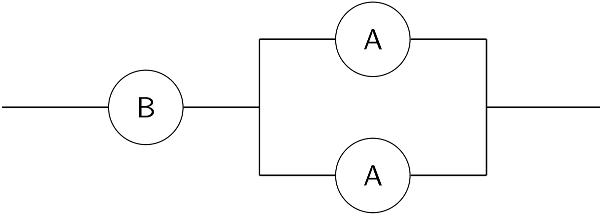



Figure: System 2.

INID: $\overline{F}_{T_2}(t) = \overline{Q}_2(\overline{F}_A(t), \overline{F}_B(t))$, where

$$\overline{Q}_2(u_1, u_2) = 2u_1u_2 - u_1^2 u_2.$$

# Comparison results-DD

- If $q_1$ and $q_2$ are two DF,

$$q_1(F) \leq_{ord} q_2(F) \text{ for all } F?$$

- If $q$ is a DF,

$$F \leq_{ord} G \Rightarrow q(F) \leq_{ord} q(G)?$$

- If $Q_1$ and $Q_2$ are two MDF,

$$Q_1(F_1, \dots, F_n) \leq_{ord} Q_2(F_1, \dots, F_n)?$$

- If $Q$ is a MDF,

$$F_i \leq_{ord} G_i, i = 1, \dots, n, \Rightarrow Q(F_1, \dots, F_n) \leq_{ord} Q(G_1, \dots, G_n)?$$

- Navarro, del Aguila, Sordo and Suárez-Llorens (2013, ASMBI) and (2015, MCAP) and Navarro and Gomis (2015, ASMBI).

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# Main stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$, stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all $t$.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$ for all $t$.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
- $X \leq_{RHR} Y \Leftrightarrow (t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for all $t$.
- Then

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y & \Rightarrow & E(X) \leq E(Y) \end{array}$$

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# Preservation of stochastic orders-DD

- If $T_i$ has the RF $\overline{q}_i(\overline{F}(t))$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all $F$ if and only if $\overline{q}_2 - \overline{q}_1 \geq 0$ in $(0, 1)$.
- $T_1 \leq_{HR} T_2$ for all $F$ if and only if $\overline{q}_2/\overline{q}_1$ decreases in $(0, 1)$.
- $T_1 \leq_{RHR} T_2$ for all $F$ if and only if $q_2/q_1$ increases in $(0, 1)$.
- $T_1 \leq_{LR} T_2$ for all $F$ if and only if $\overline{q}'_2/\overline{q}'_1$ decreases in $(0, 1)$.
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# Preservation of stochastic orders-GDD

- If $T_i$ has RF $\overline{Q}_i(\overline{F}_1, \ldots, \overline{F}_r)$, $i = 1, 2$, then:
- $T_1 \leq_{ST} T_2$ for all $\overline{F}_1, \ldots, \overline{F}_r$ if and only if $\overline{Q}_2 - \overline{Q}_1 \geq 0$ in $(0,1)^r$.
- $T_1 \leq_{HR} T_2$ for all $\overline{F}_1, \ldots, \overline{F}_r$ if and only if $\overline{Q}_2 / \overline{Q}_1$ is decreasing in $(0,1)^r$.
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# Open questions

- If $T_i$ has RF $\overline{Q}_i(\overline{F}_1,\dots,\overline{F}_r)$, $i=1,2$, then:
- $T_1 \leq_{MRL} T_2$ for all $\overline{F}_1,\dots,\overline{F}_r$ if and only if ???? (also in the case $r=1$).
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# Mixture representation for the case $r = 2$.

- Recently, Serkan Eryilmaz (2015, NAVAL RESEARCH LOGISTICS 62, 388–394) proved that if $r = 2$ and the components are independent, then

$$\overline{F}(t) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} c(i,j) \overline{F}^A_{m_1-i+1:m_1}(t) \overline{F}^B_{m_2-1+1:m_2}(t),$$

where $X^A_{m_1-i+1:m_1}(t)$ is the $m_1 - i + 1$ order statistics between the components of type 1, $X^B_{m_2-j+1:m_2}(t)$ is the $m_2 - j + 1$ order statistics between the components of type 2 and

$$c(i,j) = \phi(i,j) - \phi(i,j-1) - \phi(i-1,j) + \phi(i-1,j-1)$$

(by convention $\phi(i,j) = 0$ if $i < 0$ or $j < 0$).

# Residual lifetimes

- $X_1, \dots, X_n$ component lifetimes with RF

$$\overline{F}_i(t) = \Pr(X_i > t).$$

- $T = \phi(X_1, \dots, X_n)$ system lifetime with RF

$$\overline{F}_T(t) = \Pr(T > t).$$

- We assume $\overline{F}_i(t) = \Pr(X_i > t) > 0$ and $\overline{F}_T(t) > 0$ for $t \geq 0$.
- Component residual lifetimes $X_{i,t} = (X_i - t | X_i > t)$ with RF:

$$\overline{F}_{i,t}(x) = \Pr(X_{i,t} > x) = \Pr(X_i - t > x | X_i > t) = \frac{\overline{F}_i(t+x)}{\overline{F}_i(t)}.$$


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# System residual lifetimes

- We have two main options to define the system residual lifetime at time $t > 0$:
- The usual **residual lifetime** $T_t = (T - t | T > t)$ with RF

$$\overline{F}_t(x) = \Pr(T - t > x | T > t) = \frac{\overline{F}_T(t + x)}{\overline{F}_T(t)}.$$

- The **residual lifetime at the system level** $T_t^* = (T - t | X_1 > t, \dots, X_n > t)$ with RF

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# System residual lifetimes

- Which one is the best system?
- Intuitively, it seems that $T_t^*$ should be always better than $T_t$.
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- For $T = \min(X_1, \dots, X_n)$, $T_t =_{ST} T_t^*$ (where $=_{ST}$ denotes equality in distribution) for all $t > 0$.

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- If $X_1, \ldots, X_n$ are independent, then

$$T_t = (T - t | T > t) \leq_{ST} T_t^* = (T - t | X_1 > t, \ldots, X_n > t); \tag{3.1}$$

see Pellerey and Petakos (IEEE Tr Rel, 2002) and Li and Lu (PEIS, 2003).

- Conditions on $(X_1, \ldots, X_n)$ to have (3.1) were given in Li, Pellerey and You (2013).
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# Representations for the system residual lifetimes

- The RF of $T_t = (T - t | T > t)$ is

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$$\overline{F}_t(x) = \frac{\overline{Q}(\overline{F}_1(t)\overline{F}_{1,t}(x), \dots, \overline{F}_n(t)\overline{F}_{n,t}(x))}{\overline{Q}(\overline{F}_1(t), \dots, \overline{F}_n(t))},$$

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$$\overline{F}_t^*(x) = \frac{\Pr(\max_{j=1,\dots,m} X_{P_j} > t + x, X_1 > t, \dots, X_n > t)}{K(\overline{F}_1(t), \dots, \overline{F}_n(t))}.$$

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# Parallel system with two components

- $T = \max(X_1, X_2)$.
- Minimal path sets $P_1 = \{1\}$ and $P_2 = \{2\}$.
- System reliability function:

$$\overline{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \overline{F}_1(t) + \overline{F}_2(t) - \Pr(X_1 > t, X_2 > t).$$

- Then:

$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \overline{F}_2(t)),$$

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$$\overline{F}_T(t) = \Pr(\max(X_1, X_2) > t) = \overline{F}_1(t) + \overline{F}_2(t) - \Pr(X_1 > t, X_2 > t).$$

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# Comparison results-System residual lifetimes

- The results for GDD can be applied to compare $T_t$ and $T_t^*$. For example:
- $T_t \leq_{ST} T_t^*$ $(\geq_{ST})$ holds for all $\overline{F}_1, \dots, \overline{F}_n$ if and only if $\overline{Q}_t \leq \overline{Q}_t^*$ $(\geq)$ in $(0,1)^n$.
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# Example 1: Parallel system with two ID components

- $T = \max(X_1, X_2)$ where $X_1$ and $X_2$ have DF $F$.
- Then $\overline{F}_T(t) = \overline{q}(\overline{F}(t))$ where

$$\overline{q}(u) = \overline{Q}(u,u) = 2u - K(u,u).$$

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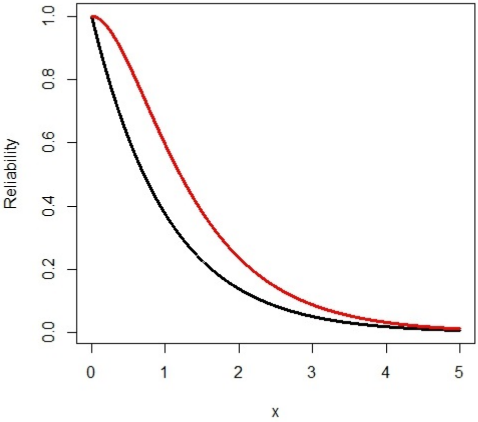


Figure: Reliability functions of $T_t$ (black) and $T_t^*$ (red) when $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = 2$.

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- If $K$ is the Gumbel-Barnett Archimedean copula

$$K(u,v) = uv \exp\left[-\theta(\ln u)(\ln v)\right], \quad \theta \in (0,1], \qquad (3.4)$$

then by plotting $\Psi(u)$, we see that it takes positive and negative values in the set $[0,1]$ when $\theta = 1$.

- Therefore $T_t$ and $T_t^*$ are not ST ordered (for all $F$ and $t$).
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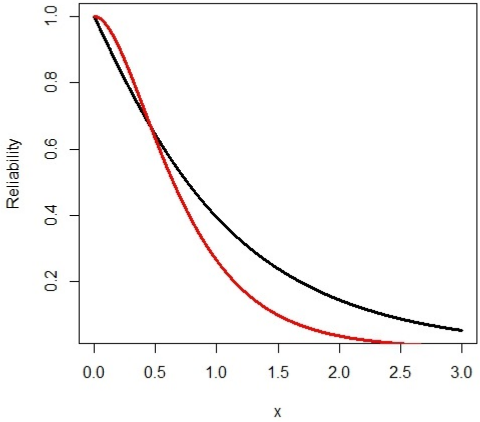


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- Now we can study if $T_t \leq_{MRL} T_t^*$ holds.
- By plotting the ratio $g(u) = \overline{q}_t(u)/\overline{q}_t^*(u)$ for $t = 1$ we see that it is first decreasing in $(0, u_0)$ and then increasing in $(u_0, 1]$ for a $u_0 \in (0, 1)$.

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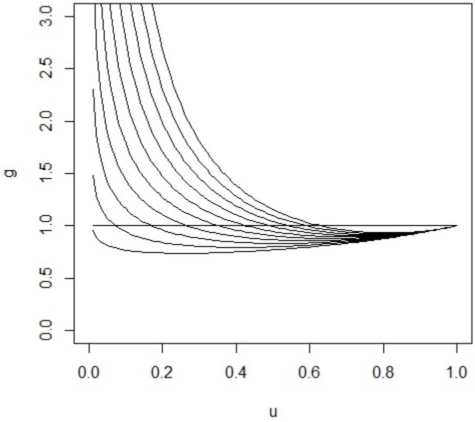


Figure: Ratio $g(u) = \overline{q}_t(u)/\overline{q}^*_t(u)$ for $t = 1$, $\overline{F}(x) = e^{-x}$ and $\theta = 0.1, 0.2, \ldots, 1$ (from the bottom to the top).

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$$E(T_t) = 1.05615 > E(T_t^*) = 0.77366.$$

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# Example 2: Parallel system with two INID components

- $T = \max(X_1, X_2)$, $X_1, X_2$ IND with DF $F_1$ and $F_2$.
- Then
$$\overline{Q}(u_1,u_2) = u_1 + u_2 - u_1u_2 = \overline{Q}_t^*(u_1,u_2),$$
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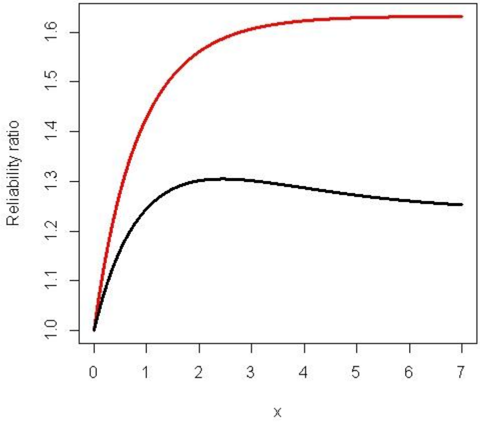


Figure: Ratio $\overline{F}_t^*/\overline{F}_t$ for $t = 1$, $\overline{F}_1(x) = e^{-x}$ and $\overline{F}_2(x) = e^{-x/2}$ (black) or $\overline{F}_2(x) = e^{-x}$ (red).

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- If $X_1, X_2$ are IID with DF $F$, then $T_t \leq_{HR} T_t^*$ holds for all $F$ since
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# Example 3: Coherent system with DID components

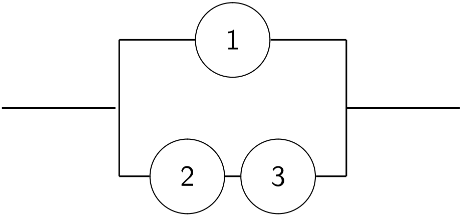


Figure: System in Example 3.

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- $T = \max(X_1, \min(X_2, X_3))$, $X_1, X_2, X_3$ DID with DF $F$.
- Then $P_1 = \{1\}$, $P_2 = \{2, 3\}$ and

$$q(u) = u + K(1, u, u) - K(u, u, u).$$

- Therefore $q_t(u) = q(cu)/q(c)$ and

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where $c = \overline{F}(t)$.

- We assume a Farlie-Gumbel-Morgenstern (FGM) copula

$$K(u, v, w) = uvw(1 + \theta(1-u)(1-v)(1-w)), \quad \theta \in [-1, 1].$$

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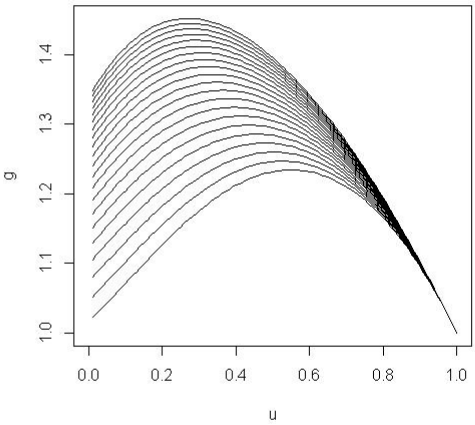


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# Open questions

- Representations and comparisons for

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