Mixture representations based on signatures for coherent systems with heterogeneous components

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Coherent systems and order statistics

- $X_1, \ldots, X_n$ (positive) random variables.
- $\overline{F}_i(t) = \Pr(X_i > t)$ reliability (survival) function.
- $X_1, \ldots, X_n$ exchangeable (EXC), i.e., for any σ

$$(X_1, \ldots, X_n) =_{ST} (X_{\sigma(1)}, \ldots, X_{\sigma(n)}).$$

- $X_1, \ldots, X_n$ IID.
- $X_{1:n}, \ldots, X_{n:n}$ the associated OS.
- $X_{k:n}$ represents the lifetime of the k-out-of-n:F system.
- $T = \phi(X_1, \ldots, X_n)$ lifetime of a coherent system.

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Mixture representations

- Samaniego (IEEE TR, 1985), IID and $\overline{F}_1$ continuous, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} p_i \overline{F}_{i:n}(t), \quad (1.1)$$

where $p_i = \Pr(T = X_{i:n})$.

- $\mathbf{p} = (p_1, \ldots, p_n)$ is the signature of the system.
- p_i does not depend on $\overline{F}_1$ and

$$p_i = \frac{|\{\sigma : \phi(x_1, \ldots, x_n) = x_{i:n}, \text{ when } x_{\sigma(1)} < \ldots < x_{\sigma(n)}\}|}{n!} \quad (1.2)$$

- Navarro and Rychlik (JMVA, 2007), (1.1) holds for EXC r.v. with absolutely continuous joint distribution.
- Navarro, Samaniego, Balakrishnan and Bhathacharya (NRL, 2008), (1.1) holds for EXC r.v. when $\mathbf{p}$ is given by (1.2).

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Mixed systems

- A **mixed system** of order n is a stochastic mixture of coherent systems of order n (Boland and Samaniego, 2004).
- From (1.1), in the EXC case, all the mixed systems of order n can be written as mixtures of $X_{1:n}, \ldots, X_{n:n}$.
- The vector with the coefficients in that representation is called the signature of the mixed system.
- Conversely, any probability vector in the simplex $\{\mathbf{c} \in [0,1]^n : \sum_{i=1}^{n} c_i = 1\}$ determines a mixed system with reliability

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The signature of order n

- Samaniego's representation was extended in Navarro, Samaniego, Balakrishnan and Bhattacharya (NRL, 2008) as follows.
- If $T = \phi(X_1, \ldots, X_k)$ and $(X_1, \ldots, X_n)$ is an EXC r.v. with $n \geq k$, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} p_i^{(n)} \overline{F}_{i:n}(t) \tag{1.3}$$

  for some coefficients $p_i^{(n)}$, $i = 1, ..., n$.
- $\mathbf{p}^{(n)} = (p_1^{(n)}, \ldots, p_n^{(n)})$ is called the signature of order n of T.
- Note that T is equal in law to a mixed system based on $(X_1, \ldots, X_n)$ with signature $\mathbf{p}^{(n)}$.
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Generalized mixture representations

- Navarro, Ruiz and Sandoval (CSTM, 2007), if T has EXC components, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t). \qquad (1.4)$$

- $\mathbf{a} = (a_1, \ldots, a_n)$ is the minimal signature of T.
- a_i only depends on ϕ but can be negative and so (1.4) is a generalized mixture.
- A similar representation holds in terms of parallel systems.
- In particular, in the IID case:

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}^i(t) = h(\overline{F}(t)), \qquad (1.5)$$

where $h(x) = \sum_{i=1}^{n} a_i x^i$ is the domination or reliability polynomial.

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Stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$ stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow E(X - t|X > t) \leq E(Y - t|Y > t)$, mean residual life order.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
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Stochastic orderings relationships

$$
\begin{array}{ccccc}
E(X_{s,t}) \leq E(Y_{s,t}) & \Rightarrow & E(X_t) \leq E(Y_t) & \Rightarrow & E(X) \leq E(Y) \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X \leq_{DTM} Y & \Rightarrow & X \leq_{MRL} Y & \Rightarrow & X \leq_{M} Y \\
\Uparrow & & \Uparrow & & \Uparrow \\
X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{ST} Y \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_{s,t} \leq_{ST} Y_{s,t} & \Rightarrow & X_t \leq_{ST} Y_t & \Rightarrow & \overline{F}_X \leq \overline{F}_Y
\end{array}
$$

where $Z_t = (Z - t | Z > t)$ and $Z_{s,t} = (Z | s < Z < t)$ (see Navarro, Belzunce and Ruiz, PEIS, 1997).

Ordering results for systems-IID case

Theorem (Kochar, Mukerjee and Samaniego, NRL 1999)

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the signatures of the two coherent systems of order n, both based on components with IID lifetimes with common continuous reliability $\overline{F}$. Let T_1 and T_2 be their respective lifetimes.
(i) If $\mathbf{p}_1 \leq_{ST} \mathbf{p}_2$, then $T_1 \leq_{ST} T_2$.
(ii) If $\mathbf{p}_1 \leq_{HR} \mathbf{p}_2$, then $T_1 \leq_{HR} T_2$.
(iii) If $\mathbf{p}_1 \leq_{LR} \mathbf{p}_2$, then $T_1 \leq_{LR} T_2$.

Ordering results for k-out-of$n$ systems

- For any $X_1, \ldots, X_n$, we have

$$X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}. \quad (1.6)$$

- However,

$$X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n} \quad (1.7)$$

does not necessarily hold (see Navarro and Shaked, JAP, 2006)

- Analogously

$$X_{1:n} \leq_{MRL} \cdots \leq_{MRL} X_{n:n}, \quad (1.8)$$

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Ordering results for systems-EXC case

Theorem (Navarro et al., NRL 2008)

If $T_1 = \phi_1(Y_1, \ldots, Y_{n_1})$ and $T_2 = \phi_2(Z_1, \ldots, Z_{n_2})$ have signatures of order n $\mathbf{p}^{(n)} = (p_1, \ldots, p_n)$ and $\mathbf{q}^{(n)} = (q_1, \ldots, q_n)$, $\{Y_1, \ldots, Y_{n_1}\}$ and $\{Z_1, \ldots, Z_{n_2}\}$ are contained in $\{X_1, \ldots, X_n\}$ and $(X_1, \ldots, X_n)$ is EXC, then:
(i) If $\mathbf{p}^{(n)} \leq_{ST} \mathbf{q}^{(n)}$, then $T_1 \leq_{ST} T_2$.
(ii) If $\mathbf{p}^{(n)} \leq_{HR} \mathbf{q}^{(n)}$ and (1.7) holds, then $T_1 \leq_{HR} T_2$.
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New results included in this talk

- Representations for systems with INID components.
- Representations for systems with DNID components.
- Ordering properties based on signatures.
- The results are included in the paper:
  J. Navarro, F.J. Samaniego, and N. Balakrishnan.
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General mixture representation

- If $X_{1:n} < \cdots < X_{n:n}$, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} p_i \Pr(X_{i:n} > t | T = X_{i:n}), \qquad (2.1)$$

  where $p_i = \Pr(T = X_{i:n})$ and
  $\Pr(X_{i:n} > t | T = X_{i:n}) \neq \overline{F}_{i:n}(t)$.
- We can define two signatures: $\mathbf{p} = (p_1, \dots, p_n)$ with $p_i = \Pr(T = X_{i:n})$ (called probability signature) and $\mathbf{s} = (s_1, \dots, s_n)$ with s_i given by (1.2) (called structure signature).
- Example 5.1 in Navarro, Samaniego, Balakrishnan and Bhathacharya NRL,2008) proves that

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Generalized mixture representation using path sets

- A set $P \subseteq \{1,\dots,n\}$ is a *path set* of a coherent system if the system works when all the components in P work.
- A path set P is a *minimal path set* if it does not contain other path sets.
- If T has minimal path sets $P_1,\dots,P_k$, then

$$T = \max_{j=1,\dots,k} X_{P_j}, \tag{2.2}$$

where $X_{P_j} = \min_{i\in P_j} X_i$, for $j = 1,\dots,k$ (see Barlow and Proschan, 1975).
- From the inclusion-exclusion formula we have

$$\overline{F}_T(t) = \sum_{j=1}^{k} \overline{F}_{P_j}(t) - \sum_{i<j} \overline{F}_{P_i\cup P_j}(t) + \cdots + (-1)^{k+1}\overline{F}_{P_1\cup\dots\cup P_k}(t), \tag{2.3}$$

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Reliability representations for the INID case

- If $X_1, \ldots, X_n$ are independent, then

$$\overline{F}_P(t) = \prod_{i \in P} \overline{F}_i(t).$$

- Therefore (2.3) can be written as

$$\overline{F}_T(t) = H(\overline{F}_1(t), \ldots, \overline{F}_n(t)), \qquad (2.4)$$

where H is a multivariate polynomial which only depends on $P_1, \ldots, P_k$.

- H is called *reliability structure function* in Esary and Proschan (Tech, 1963).
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Reliability representations for the independent case

Theorem

Let $T = \phi(X_1, \dots, X_n)$ be the lifetime of a system with independent components, signature vector $\mathbf{s} = (s_1, \dots, s_n)$ and reliability polynomial and reliability structure function h and H, respectively. Then

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i \overline{G}_{i:n}(t), \tag{2.5}$$

where $\overline{G}_{i:n}(t) = P(Y_{i:n} > t)$ and $Y_1, \dots, Y_n$ are IID with a common reliability function

$$\overline{G}(t) = h^{-1}(H(\overline{F}_1(t), \dots, \overline{F}_n(t))). \tag{2.6}$$

Reliability representations for the independent case

- T is equal in law to a system with the same structure and IID components with reliability $\overline{G}$.

Definition

If $\psi : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is a real valued function, a *mean function* of ψ in S is a function $m_\psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\psi(x_1, \ldots, x_n) = \psi(z, \ldots, z)$$

for all $(x_1, \ldots, x_n) \in S$, where $z = m_\psi(x_1, \ldots, x_n)$.

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- If m_H is the mean function of H, then

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Ordering results INID case

Theorem

Let T and T^* be the lifetimes of two coherent systems with signatures $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$ and both with independent component lifetimes. Let h and h^* be their reliability polynomials and let H and H^* be their structure reliability functions. Let $\overline{G}$ and $\overline{G}^*$ be the reliability functions defined in (2.6).

(i) If $\overline{G} \leq_{ST} \overline{G}^*$ and $\mathbf{s} \leq_{ST} \mathbf{s}^*$, then $T \leq_{ST} T^*$;

(ii) If $\overline{G} \leq_{HR} \overline{G}^*$, $\mathbf{s} \leq_{HR} \mathbf{s}^*$ and either $xh'(x)/h(x)$ or $x(h^*)'(x)/h(x)$ is decreasing, then $T \leq_{HR} T^*$;

(iii) If $\overline{G} \leq_{RHR} \overline{G}^*$, $\mathbf{s} \leq_{RHR} \mathbf{s}^*$ and either $(1-x)h'(x)/(1-h(x))$ or $(1-x)(h^*)'(x)/(1-h^*(x))$ is increasing, then $T \leq_{RHR} T^*$.

Example

- $T = \max(\min(X_1, X_2), \min(X_3, X_4))$
- $\mathbf{s} = (0, \frac{2}{3}, \frac{1}{3}, 0)$
- The minimal path sets are $\{1, 2\}$ and $\{3, 4\}$ and then

$$\overline{F}_T(t) = \overline{F}_1(t)\overline{F}_2(t) + \overline{F}_3(t)\overline{F}_4(t) - \overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\overline{F}_4(t).$$

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- Also, $h(p) = 2p^2 - p^4$ and $\mathbf{a} = (0, 2, 0, -1)$.
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$$\overline{G} = \sqrt{1 - \sqrt{1 - \overline{F}_1\overline{F}_2 - \overline{F}_3\overline{F}_4 + \overline{F}_1\overline{F}_2\overline{F}_3\overline{F}_4}}. \qquad (2.7)$$

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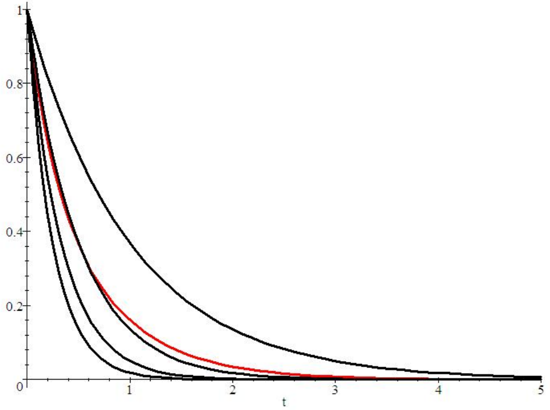

Figure: Mean reliability function $\overline{G}$ (red line) for exponential reliability functions $\overline{F}_i(t) = \exp(-it)$, $i = 1, 2, 3, 4$.

Reliability representations for the DNID case

- If $(X_1, ..., X_n)$ has joint reliability $\overline{F}$, then

$$\overline{F}_P(t) = \overline{\mathbf{F}}(\mathbf{t}_P),$$

where $\overline{\mathbf{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n)$ and $\mathbf{t}_P = (t_1, \ldots, t_n)$ with $t_i = t$ for $i \in P$ and $t_i = 0$ for $i \notin P$.

- Using Sklar's representation for $\overline{\mathbf{F}}$, we have

$$\overline{\mathbf{F}}(x_1, \ldots, x_n) = K(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)), \qquad (3.1)$$

where K is the survival copula. Then

$$\overline{F}_P(t) = K(\overline{F}_1(t_1), \ldots, \overline{F}_n(t_n)).$$

- Therefore (2.3) can be written as

$$\overline{F}_T(t) = W(\overline{F}_1(t), \ldots, \overline{F}_n(t)), \qquad (3.2)$$

where W only depends on K and $P_1, \ldots, P_k$ and is called *structure-dependence function*.

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where W only depends on K and $P_1, \ldots, P_k$ and is called *structure-dependence function*.

Reliability representations for the general case

Theorem

If $T = \phi(X_1, \ldots, X_n)$ *is the lifetime of a system with structure-dependence function* W *having right-continuous increasing mean function* m_W*, then*

$$T =_{ST} T^* = \phi(Y_1, \ldots, Y_n)$$

with ID component lifetimes $Y_1, \ldots, Y_n$ *with*

$$P(Y_1 > x_1, \ldots, Y_n > x_n) = K(\overline{G}_W(x_1), \ldots, \overline{G}_W(x_n)), \qquad (3.3)$$

where K *is the survival copula of* $(X_1, \ldots, X_n)$ *and*

$$\overline{G}_W(t) = m_W(\overline{F}_1(t), \ldots, \overline{F}_n(t)). \qquad (3.4)$$

Reliability representations for exchangeable copulas

Theorem

If T is the lifetime of a system with signature $\mathbf{s} = (s_1, \ldots, s_n)$, with structure-dependence function W having right-continuous increasing mean function m_W and with components having an exchangeable copula K, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i \overline{G}_{i:n}(t), \tag{3.5}$$

where $\overline{G}_{i:n}(t) = P(Y_{i:n} > t)$ and $Y_{1:n} < \cdots < Y_{n:n}$ are the order statistics obtained from the random variables $Y_1, \ldots, Y_n$ with joint reliability function as in (3.3).

Other reliability representation for general copulas

Theorem

If T is the lifetime of a system with signature $\mathbf{s} = (s_1, \ldots, s_n)$ and with component lifetimes $X_1, \ldots, X_n$ having structure-dependence function W, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i \overline{G}_{i:n}(t),$$

where $\overline{G}_{i:n}(t) = P(Y_{i:n} > t)$ and $Y_{1:n} < \cdots < Y_{n:n}$ are the order statistics obtained from IID r.v. $Y_1, \ldots, Y_n$ with common reliability function as

$$\widetilde{G}(t) = h^{-1}(W(\overline{F}_1(t), \ldots, \overline{F}_n(t))),$$

and h is the reliability polynomial.

Ordering results

Theorem

Let T and T^* be the lifetimes of two coherent systems with signatures $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$ and with components having the same exchangeable survival copula K. Let $\overline{G}_W$ and $\overline{G}_{W^*}$ be the reliability functions defined by (3.4). If $\mathbf{s} \leq_{ST} \mathbf{s}^*$ and $\overline{G}_W \leq \overline{G}_{W^*}$, then $T \leq_{ST} T^*$.

Theorem

Let T and T^* be the lifetimes of two coherent systems with signatures $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$ and T^* having IID component lifetimes with reliability function $\overline{R}$. If $\mathbf{s} \leq_{ST} \mathbf{s}^*$, and $\widetilde{G} \leq \overline{R}$, then $T \leq_{ST} T^*$.

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