New ordering results for coherent systems

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Coherent systems and order statistics

- $X_1, \ldots, X_n$ (positive) random variables.
- $X_1, \ldots, X_n$ IID.
- $X_1, \ldots, X_n$ exchangeable (EXC), i.e., for any σ

$$(X_1, \ldots, X_n) =_{ST} (X_{\sigma(1)}, \ldots, X_{\sigma(n)}).$$

- $\overline{F}(t) = \Pr(X_i > t)$ reliability (survival) function.
- $X_{1:n}, \ldots, X_{n:n}$ the associated OS.
- $X_{k:n}$ represents the lifetime of the k-out-of-n:F system.
- $T = \phi(X_1, \ldots, X_n)$ lifetime of a coherent system.

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Mixture representations

- Samaniego (IEEE TR, 1985), IID and $\overline{F}$ continuous, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i \overline{F}_{i:n}(t). \tag{1.1}$$

- $\mathbf{s} = (s_1, \ldots, s_n)$ is the signature of T, $s_i = \Pr(T = X_{i:n})$.
- s_i does not depend on $\overline{F}$ and

$$s_i = \frac{\left|\{\sigma : \phi(x_1, \ldots, x_n) = x_{i:n}, \text{ when } x_{\sigma(1)} < \ldots < x_{\sigma(n)}\}\right|}{n!} \tag{1.2}$$

- Navarro and Rychlik (JMVA, 2007), (1.1) holds for EXC r.v. with absolutely continuous joint distribution.
- Navarro, Samaniego, Balakrishnan and Bhathacharya (NRL, 2008), (1.1) holds for any EXC r.v. when $\mathbf{s}$ is computed from (1.2).

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Mixed systems

- A **mixed system** of order n is a stochastic mixture of coherent systems of order n (Boland and Samaniego 2004).
- From (1.1), all the mixed systems of order n can be written as mixtures of $X_{1:n}, \ldots, X_{n:n}$.
- The vector with the coefficients in that representation is called the signature of the mixed system.
- Conversely, any probability vector in the simplex $\{\mathbf{s} \in [0,1]^n : \sum_{i=1}^{n} s_i = 1\}$ determines a mixed system.

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The signature of order n

- Samaniego's representation was extended in Navarro, Samaniego, Balakrishnan and Bhattacharya (NRL, 2008) as follows.
- If $T = \phi(X_1, \ldots, X_k)$ and $(X_1, \ldots, X_n)$ is an EXC r.v. with $n \geq k$, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i^{(n)} \overline{F}_{i:n}(t). \quad (1.3)$$

- $\mathbf{s}^{(n)} = (s_1^{(n)}, \ldots, s_n^{(n)})$ is called the signature of order n of T.
- Note that T is equal in law to a mixed system based on $(X_1, \ldots, X_n)$ with signature $\mathbf{s}^{(n)}$.
- If $(X_1, \ldots, X_n)$ has an absolutely continuous joint distribution, then $\Pr(T = X_{i:n}) = s_i^{(n)}$ for $i = 1, \ldots, n$.
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Generalized mixture representations

- Navarro, Ruiz and Sandoval (CSTM, 2007), if T has EXC components, then

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t). \quad (1.4)$$

- $\mathbf{a} = (a_1, \ldots, a_n)$ is the minimal signature of T.
- a_i does not depend on F but can be negative.
- A similar representation holds in terms of parallel systems.
- In particular, in the IID case:

$$\overline{F}_T = \sum_{i=1}^{n} a_i \overline{F}^i(t) = p(\overline{F}(t)), \quad (1.5)$$

where $p(x) = \sum_{i=1}^{n} a_i x^i$ is the domination polynomial.

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Stochastic orderings

- $X \leq_{ST} Y \Leftrightarrow \overline{F}_X(t) \leq \overline{F}_Y(t)$ stochastic order.
- $X \leq_{HR} Y \Leftrightarrow h_X(t) \geq h_Y(t)$, hazard rate order.
- $X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for all t.
- $X \leq_{MRL} Y \Leftrightarrow m_X(t) \leq m_Y(t)$, mean residual life order.
- $X \leq_{LR} Y \Leftrightarrow f_Y(t)/f_X(t)$ is nondecreasing, likelihood ratio order.
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Stochastic orderings relationships

$$
\begin{array}{ccccc}
E(X_{s,t}) \le E(Y_{s,t}) & \Rightarrow & E(X_t) \le E(Y_t) & \Rightarrow & E(X) \le E(Y) \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X \le_{DTM} Y & \Rightarrow & X \le_{MRL} Y & \Rightarrow & X \le_{M} Y \\
\Uparrow & & \Uparrow & & \Uparrow \\
X \le_{LR} Y & \Rightarrow & X \le_{HR} Y & \Rightarrow & X \le_{ST} Y \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_{s,t} \le_{ST} Y_{s,t} & \Rightarrow & X_t \le_{ST} Y_t & \Rightarrow & \overline{F}_X \le \overline{F}_Y
\end{array}
$$

where $Z_t = (Z - t | Z > t)$ and $Z_{s,t} = (Z | s < Z < t)$ (see Navarro, Belzunce and Ruiz, PEIS, 1997).

Ordering results for systems-IID case

Theorem (Kochar, Mukerjee and Samaniego, NRL 1999)

Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the signatures of the two coherent systems of order n, both based on components with IID lifetimes with common continuous reliability $\overline{F}$. Let T_1 and T_2 be their respective lifetimes.
(i) If $\mathbf{s}_1 \leq_{ST} \mathbf{s}_2$, then $T_1 \leq_{ST} T_2$.
(ii) If $\mathbf{s}_1 \leq_{HR} \mathbf{s}_2$, then $T_1 \leq_{HR} T_2$.
(iii) If $\mathbf{s}_1 \leq_{LR} \mathbf{s}_2$, then $T_1 \leq_{LR} T_2$.

Ordering results for k-out-of$n$ systems

- For any $X_1, \dots, X_n$, we have

$$X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n}. \quad (1.6)$$

- However,

$$X_{1:n} \leq_{FR} \cdots \leq_{FR} X_{n:n} \quad (1.7)$$

does not necessarily hold (see Navarro and Shaked, JAP, 2006)

- Analogously

$$X_{1:n} \leq_{MRL} \cdots \leq_{MRL} X_{n:n}, \quad (1.8)$$

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Ordering results for systems-EXC case

Theorem (Navarro et al., NRL 2008)

If $T_1 = \phi_1(Y_1, \ldots, Y_{n_1})$ *and* $T_2 = \phi_2(Z_1, \ldots, Z_{n_2})$ *have signatures of order* n $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$, $\{Y_1, \ldots, Y_{n_1}\}$ *and* $\{Z_1, \ldots, Z_{n_2}\}$ *are contained in* $\{X_1, \ldots, X_n\}$ *and* $(X_1, \ldots, X_n)$ *is EXC, then:*

(*i*) *If* $\mathbf{p} \le_{ST} \mathbf{q}$, *then* $T_1 \le_{ST} T_2$.
(*ii*) *If* $\mathbf{p} \le_{FR} \mathbf{q}$ *and (1.7) holds, then* $T_1 \le_{FR} T_2$.
(*iii*) *If* $\mathbf{p} \le_{FR} \mathbf{q}$ *and (1.8) holds, then* $T_1 \le_{MRL} T_2$.
(*iv*) *If* $\mathbf{p} \le_{LR} \mathbf{q}$ *and (1.9) holds, then* $T_1 \le_{LR} T_2$.

New results included in this talk

- Extensions of the preceding ordering results for systems, in two ways:
- Necessary and sufficient conditions based on signatures for systems with EXC components.
- Sufficient conditions based on dispersion properties.
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Main result-exchangeable case

Theorem (Navarro and Rubio)

If $T_1 = \phi_1(X_1, \ldots, X_n)$ and $T_2 = \phi_2(X_1, \ldots, X_n)$ are two coherent (or mixed) systems with respective signatures $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ and $(X_1, \ldots, X_n)$ has a joint exchangeable distribution $\mathbf{F}$, then:

(i) $T_1 \leq_{ST} T_2$ holds for any $\mathbf{F}$ if, and only if $\mathbf{p} \leq_{ST} \mathbf{q}$ holds.

(ii) $T_1 \leq_{FR} T_2$ holds for any $\mathbf{F}$ satisfying (1.7) if, and only if $\mathbf{p} \leq_{FR} \mathbf{q}$ holds.

(iii) $T_1 \leq_{LR} T_2$ holds for any $\mathbf{F}$ satisfying (1.9) if, and only if $\mathbf{p} \leq_{LR} \mathbf{q}$ holds.

(iv) $T_1 \leq_{RFR} T_2$ holds for any $\mathbf{F}$ satisfying $X_{1:n} \leq_{RFR} \cdots \leq_{RFR} X_{n:n}$ if, and only if $\mathbf{p} \leq_{RFR} \mathbf{q}$ holds.

Stochastic comparisons of systems with 1-5 components

- The signatures of order 5 of the 208 coherent systems with 1-5 components were given in Table 1 of Navarro and Rubio (TEST, to appear), see also Navarro and Rubio (CSSC, 2010).
- There are 94 different signatures of order 5 since some systems have the same signatures.
- The systems with the same signatures are equal in law when the components are EXC.
- The ST ordering properties for the 1-50 systems are given in the next figure.
- The systems 51-94 are the dual systems of the systems 1-44 and their properties can be obtained from:

$$T_1^D \leq_{ST} T_2^D \Leftrightarrow T_1 \geq_{ST} T_2.$$

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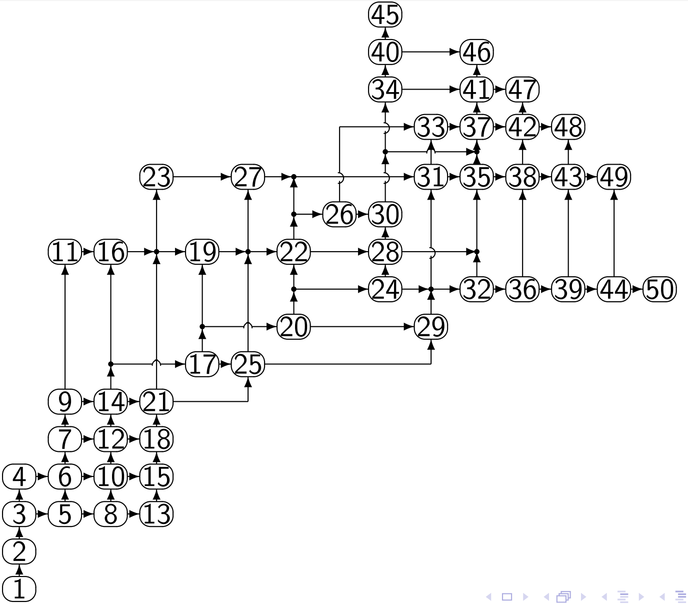
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A paradoxical example

- Let us consider $T_1 = \min(X_1, \max(X_2, X_3))$ and $T_2 = \max(X_1, \min(X_2, X_3, X_4))$.
- Let us assume that (X_1, X_2, X_3, X_4) has an exchangeable joint distribution function $\mathbf{F}$ satisfying (1.9).
- Their respective signatures of order 4 are $\mathbf{p} = (1/4, 5/12, 1/3, 0)$ and $\mathbf{q} = (0, 1/2, 1/4, 1/4)$.
- As

$$\frac{0}{1/4} < \frac{1/2}{5/12} > \frac{1/4}{1/3} < \frac{1/4}{0},$$

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- What is the mistake?
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- The answer is simple. They are ordered for any random vector $(X_1, X_2, X_3, X_4, X_5)$ with a 5 dimensional exchangeable joint distribution **F** satisfying (1.9).
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A paradoxical example: consequences

- This example shows three relevant facts.
- (1) Theorem 1.2 can prove ordering results for systems with n components whose signatures of order n are not ordered.
- (2) These new ordering results only holds for systems with components having exchangeable n-dimensional distributions which can be extended to exchangeable m-dimensional distributions (for an $m > n$).
- This property holds for the most relevant case, the systems with IID components. Hence, we obtain that $T_1 \leq_{LR} T_2$ whenever $X_1, \ldots, X_4$ are IID.

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- This extension property only depends on the copula of the random vector $(X_1, \ldots, X_n)$.
- For example, it holds under some conditions for Archimedean copulas.
- (3) If two systems have n IID components with a common reliability $\overline{F}$, lifetimes T_1 and T_2 and signatures $\mathbf{p}$ and $\mathbf{q}$, respectively, then $\mathbf{p} \leq_{LR} \mathbf{q}$ is not a necessary condition to have $T_1 \leq_{LR} T_2$ for any $\overline{F}$.
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Dispersion comparisons

- Convex order: $X \preceq_{CX} Y$ if $E(\phi(X)) \leq E(\phi(Y))$ for all convex functions ϕ such that the expectations exist.
- If $X \geq 0$, then

$$\mu = E(X) = \int_0^\infty \overline{F}(x)dx = \int_0^1 \overline{F}^{-1}(u)du, \qquad (3.1)$$

where $\overline{F}^{-1}(u) = \sup\{x : \overline{F}(x) \geq u\}$.

- Hence

$$h_F(u) = \overline{F}^{-1}(u)/\mu, \qquad 0 < u < 1, \qquad (3.2)$$

is a decreasing probability density function.

- Z_F will represent a r.v. having pdf h_F.
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- Convex order: $X \preceq_{CX} Y$ if $E(\phi(X)) \leq E(\phi(Y))$ for all convex functions ϕ such that the expectations exist.
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$$\mu = E(X) = \int_0^\infty \overline{F}(x)dx = \int_0^1 \overline{F}^{-1}(u)du, \qquad (3.1)$$

where $\overline{F}^{-1}(u) = \sup\{x : \overline{F}(x) \geq u\}$.

- Hence

$$h_F(u) = \overline{F}^{-1}(u)/\mu, \qquad 0 < u < 1, \qquad (3.2)$$

is a decreasing probability density function.

- Z_F will represent a r.v. having pdf h_F.
- If $E(X) = E(Y)$, we get

$$X \preceq_{CX} Y \Leftrightarrow Z_G \preceq_{ST} Z_F.$$

Main result

Theorem (Navarro and Rychlik, EJOR 2010))

Let $T_1 = \phi(X_1, \dots, X_n)$ *and* $T_2 = \phi(Y_1, \dots, Y_n)$ *with IID components having continuous reliability functions* $\overline{F}$ *and* $\overline{G}$*, respectively, and a common mean* $\mu = E(X_1) = E(Y_1)$*. Let* p *be the common domination polynomial. Then:*

(i) If p *is convex (concave) on* $(0, 1)$ *and* $X_1 \preceq_{CX} Y_1$*, then*

$$E(T_1) \geq E(T_2) \quad (\leq).$$

(ii) If p' *is convex (concave) on* $(0, 1)$ *and* $Z_F \preceq_{CX} Z_G$*, then*

$$E(T_1) \leq E(T_2) \quad (\geq).$$

Main result

Theorem (Navarro and Rychlik, EJOR 2010)

Let $T = \phi(X_1, \ldots, X_n)$ *with IID*$\sim F$ *components having mean* $\mu = E(X_i)$ *and domination polynomial* $p(x) = \sum_{i=1}^{n} a_i x^i$.
(i) If p *is convex (concave) on* $(0,1)$, *then*

$$\mu a_1 \leq E(T) \leq \mu \quad (\geq). \tag{4.1}$$

(ii) If p' *is convex (concave) on* $(0,1)$, *then*

$$\mu \inf_{x \in (0,1]} \frac{p(x)}{x} \leq E(T) \leq \mu \max(1, a_1) \tag{4.2}$$

$$\left(\mu \min(1, a_1) \leq E(T) \leq \mu \sup_{x \in (0,1]} \frac{p(x)}{x} \right). \tag{4.3}$$

Comments

Corollary (Navarro and Rychlik, EJOR 2010)

If p' is convex (concave) and $\alpha_F = E(Z_F)$, then

$$\mu \frac{p(2\alpha_F)}{2\alpha_F} \leq E(T) \leq \mu[(1-2\alpha_F)a_1 + 2\alpha_F] \quad (\geq). \tag{4.4}$$

- Parameter α_F is as a measure of concentration of F since

$$\alpha_F = E(Z_F) = \frac{1}{2\mu}\int_0^\infty \overline{F}^2(t)dt = \frac{E(X_{1:2})}{2\mu} = \frac{E(X_{1:2})}{E(X_{1:2}) + E(X_{2:2})}.$$

- Also, if γ_F is the Gini dispersion index of F, then

$$\alpha_F = \frac{1-\gamma_F}{2}.$$

- For more bounds see Navarro and Rychlik (EJOR, 2010).

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Table: Lower (L) and upper (U) bounds for $E(T)$ when $\mu = 1$ and the Gini index is 0.5. $E(T_{exp})$ gives the mean when F is exponential.

T	$p'(x)$	L	$E(T_{exp})$	U
$X_{1:2}$	linear	0.5	0.5	0.5
$X_{2:2}$	linear	1.5	1.5	1.5
$X_{1:3}$	cx	0.25	0.3333	0.5
$\min(X_1, \max(X_2, X_3))$	cv	0.5	0.6667	0.75
$X_{2:3}$	cv	0.5	0.8333	1
$\max(X_1, \min(X_2, X_3))$	cv	1	1.1667	1.25
$X_{3:3}$	cv	1.75	1.8333	2
$X_{1:4}$	cv	0.125	0.25	0.5
$\max(\min(X_1, X_2), \min(X_3, X_4))$	cv	0.5	0.75	0.875
Consecutive 2-out-of-4:G	cv	0.5	0.8333	1
$X_{4:4}$	cx	1.875	2.0833	2.5

Our references

- Jorge Navarro and Rafael Rubio (2010). Further ordering properties of coherent systems with exchangeable components based on signatures. Submitted.
- Jorge Navarro and Tomasz Rychlik (2010). Comparisons and bounds for expected lifetimes of reliability systems. European Journal of Operational Research 207, 309–317.

Our references

- Jorge Navarro and Rafael Rubio (2010). Further ordering properties of coherent systems with exchangeable components based on signatures. Submitted.
- Jorge Navarro and Tomasz Rychlik (2010). Comparisons and bounds for expected lifetimes of reliability systems. European Journal of Operational Research 207, 309–317.

Our references

- Navarro, J. and Shaked, M. (2006). Hazard Rate Ordering of Order Statistics and Systems. J. Appl. Probab. 43, 391-408.
- Navarro, J., Ruiz, J.M. and Sandoval, C.J. (2007). Properties of Coherent Systems with Dependent Components. Comm. Stat. Theory and Methods 36 (1), 1-17.
- Navarro, J. and Rychlik, T. (2007). Reliability and expectation bounds for coherent systems with exchangeable components. J. Multivariate Anal. 98, 102-113.
- Navarro, J., Rychlik, T. and Shaked, M. (2007). Are the Order Statistics Ordered? A Survey of Recent Results. Comm. Stat. Theory and Methods 36 (7), 1273-1290.

Our references

- Navarro, J. and Shaked, M. (2006). Hazard Rate Ordering of Order Statistics and Systems. J. Appl. Probab. 43, 391-408.
- Navarro, J., Ruiz, J.M. and Sandoval, C.J. (2007). Properties of Coherent Systems with Dependent Components. Comm. Stat. Theory and Methods 36 (1), 1-17.
- Navarro, J. and Rychlik, T. (2007). Reliability and expectation bounds for coherent systems with exchangeable components. J. Multivariate Anal. 98, 102-113.
- Navarro, J., Rychlik, T. and Shaked, M. (2007). Are the Order Statistics Ordered? A Survey of Recent Results. Comm. Stat. Theory and Methods 36 (7), 1273-1290.

Our references

- Navarro, J. and Shaked, M. (2006). Hazard Rate Ordering of Order Statistics and Systems. J. Appl. Probab. 43, 391-408.
- Navarro, J., Ruiz, J.M. and Sandoval, C.J. (2007). Properties of Coherent Systems with Dependent Components. Comm. Stat. Theory and Methods 36 (1), 1-17.
- Navarro, J. and Rychlik, T. (2007). Reliability and expectation bounds for coherent systems with exchangeable components. J. Multivariate Anal. 98, 102-113.
- Navarro, J., Rychlik, T. and Shaked, M. (2007). Are the Order Statistics Ordered? A Survey of Recent Results. Comm. Stat. Theory and Methods 36 (7), 1273-1290.

Our references

- Navarro, J. and Shaked, M. (2006). Hazard Rate Ordering of Order Statistics and Systems. J. Appl. Probab. 43, 391-408.
- Navarro, J., Ruiz, J.M. and Sandoval, C.J. (2007). Properties of Coherent Systems with Dependent Components. Comm. Stat. Theory and Methods 36 (1), 1-17.
- Navarro, J. and Rychlik, T. (2007). Reliability and expectation bounds for coherent systems with exchangeable components. J. Multivariate Anal. 98, 102-113.
- Navarro, J., Rychlik, T. and Shaked, M. (2007). Are the Order Statistics Ordered? A Survey of Recent Results. Comm. Stat. Theory and Methods 36 (7), 1273-1290.

- Navarro, J., Belzunce, F. and Ruiz, J.M. (1997). New stochastic orders based on double truncation. Probability in the Engineering and Informational Sciences 11, 395-402.
- Navarro, J. (2008). Likelihood ratio ordering of order statistics, mixtures and systems, J. Stat. Plann. Inference 138 (5), 1242-1257.
- Navarro, J. and Hernandez, P.J. (2008). Negative mixtures, order statistics and systems. In: *Advances in Mathematical and Statistical Modelling*. Series: Statistics for Industry and Technology. Arnold, B.C.; Balakrishnan, N.; Sarabia, J.M.; Minguez, R. (Eds.) 2008, Birkhauser.
- Navarro, J. and Hernandez, P.J. (2008). Mean residual life functions of finite mixtures and systems, Metrika 277-298.

- Navarro, J., Belzunce, F. and Ruiz, J.M. (1997). New stochastic orders based on double truncation. Probability in the Engineering and Informational Sciences 11, 395-402.
- Navarro, J. (2008). Likelihood ratio ordering of order statistics, mixtures and systems, J. Stat. Plann. Inference 138 (5), 1242-1257.
- Navarro, J. and Hernandez, P.J. (2008). Negative mixtures, order statistics and systems. In: *Advances in Mathematical and Statistical Modelling*. Series: Statistics for Industry and Technology. Arnold, B.C.; Balakrishnan, N.; Sarabia, J.M.; Minguez, R. (Eds.) 2008, Birkhauser.
- Navarro, J. and Hernandez, P.J. (2008). Mean residual life functions of finite mixtures and systems, Metrika 277-298.

- Navarro, J., Belzunce, F. and Ruiz, J.M. (1997). New stochastic orders based on double truncation. Probability in the Engineering and Informational Sciences 11, 395-402.
- Navarro, J. (2008). Likelihood ratio ordering of order statistics, mixtures and systems, J. Stat. Plann. Inference 138 (5), 1242-1257.
- Navarro, J. and Hernandez, P.J. (2008). Negative mixtures, order statistics and systems. In: *Advances in Mathematical and Statistical Modelling.* Series: Statistics for Industry and Technology. Arnold, B.C.; Balakrishnan, N.; Sarabia, J.M.; Minguez, R. (Eds.) 2008, Birkhauser.
- Navarro, J. and Hernandez, P.J. (2008). Mean residual life functions of finite mixtures and systems, Metrika 277-298.

- Navarro, J., Belzunce, F. and Ruiz, J.M. (1997). New stochastic orders based on double truncation. Probability in the Engineering and Informational Sciences 11, 395-402.
- Navarro, J. (2008). Likelihood ratio ordering of order statistics, mixtures and systems, J. Stat. Plann. Inference 138 (5), 1242-1257.
- Navarro, J. and Hernandez, P.J. (2008). Negative mixtures, order statistics and systems. In: *Advances in Mathematical and Statistical Modelling.* Series: Statistics for Industry and Technology. Arnold, B.C.; Balakrishnan, N.; Sarabia, J.M.; Minguez, R. (Eds.) 2008, Birkhauser.
- Navarro, J. and Hernandez, P.J. (2008). Mean residual life functions of finite mixtures and systems, Metrika 277-298.

- Navarro, J., Balakrishnan, N. and Samaniego, F.J. (2008). Mixture representations of residual lifetimes of used systems, Journal of Applied Probability 45 (4), 1097-1112.
- Navarro, J., Samaniego, F., Balakrishnan, N. and Bhattacharya, D. (2008). On the Application and Extension of System Signatures in Engineering Reliability, Naval Research Logistics 55, 313-327.
- Navarro, J. and Rubio, R. (2010). Computations of signatures of coherent systems with five components, Communications in Statistics Simulation and Computation 39 (1), 68-84.
- Navarro, J. and Rubio, R. (2010). Comparisons of coherent systems using stochastic precedence. To appear in TEST.
- Navarro, J. and Shaked, M. (2010). Some properties of the minimum and the maximum of random variables with joint logconcave distributions. Metrika 3, 313-317.

- Navarro, J., Balakrishnan, N. and Samaniego, F.J. (2008). Mixture representations of residual lifetimes of used systems, Journal of Applied Probability 45 (4), 1097-1112.
- Navarro, J., Samaniego, F., Balakrishnan, N. and Bhattacharya, D. (2008). On the Application and Extension of System Signatures in Engineering Reliability, Naval Research Logistics 55, 313-327.
- Navarro, J. and Rubio, R. (2010). Computations of signatures of coherent systems with five components, Communications in Statistics Simulation and Computation 39 (1), 68-84.
- Navarro, J. and Rubio, R. (2010). Comparisons of coherent systems using stochastic precedence. To appear in TEST.
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- Navarro, J., Balakrishnan, N. and Samaniego, F.J. (2008). Mixture representations of residual lifetimes of used systems, Journal of Applied Probability 45 (4), 1097-1112.
- Navarro, J., Samaniego, F., Balakrishnan, N. and Bhattacharya, D. (2008). On the Application and Extension of System Signatures in Engineering Reliability, Naval Research Logistics 55, 313-327.
- Navarro, J. and Rubio, R. (2010). Computations of signatures of coherent systems with five components, Communications in Statistics Simulation and Computation 39 (1), 68-84.
- Navarro, J. and Rubio, R. (2010). Comparisons of coherent systems using stochastic precedence. To appear in TEST.
- Navarro, J. and Shaked, M. (2010). Some properties of the minimum and the maximum of random variables with joint logconcave distributions. Metrika 3, 313-317.

- Navarro, J., Balakrishnan, N. and Samaniego, F.J. (2008). Mixture representations of residual lifetimes of used systems, Journal of Applied Probability 45 (4), 1097-1112.
- Navarro, J., Samaniego, F., Balakrishnan, N. and Bhattacharya, D. (2008). On the Application and Extension of System Signatures in Engineering Reliability, Naval Research Logistics 55, 313-327.
- Navarro, J. and Rubio, R. (2010). Computations of signatures of coherent systems with five components, Communications in Statistics Simulation and Computation 39 (1), 68-84.
- Navarro, J. and Rubio, R. (2010). Comparisons of coherent systems using stochastic precedence. To appear in TEST.
- Navarro, J. and Shaked, M. (2010). Some properties of the minimum and the maximum of random variables with joint logconcave distributions. Metrika 3, 313-317.

- Navarro, J., Balakrishnan, N. and Samaniego, F.J. (2008). Mixture representations of residual lifetimes of used systems, Journal of Applied Probability 45 (4), 1097-1112.
- Navarro, J., Samaniego, F., Balakrishnan, N. and Bhattacharya, D. (2008). On the Application and Extension of System Signatures in Engineering Reliability, Naval Research Logistics 55, 313-327.
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- Navarro, J. and Shaked, M. (2010). Some properties of the minimum and the maximum of random variables with joint logconcave distributions. Metrika 3, 313-317.